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CONVOLUTIONS WITH PROBABILITY DENSITIES AND APPLICATIONS TO PDES

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Abstract. In this article we introduce several new convolution operators, generated by some known probability densities. By using the inverse Fourier transform and taking inverse steps (in the analogues of the classical procedures used for example in the heat or Laplace equations), we deduce the initial and final value problems satisfied by the new convolution integrals.

1. Introduction

It is well known that the classical Gauss-Weierstrass, Poisson-Cauchy and Picard convolution singular integrals are based on convolutions with the standard normal density function $e^{-x^2}/\sqrt{\pi}$, standard Cauchy density function $\frac{1}{\pi(1+x^2)}$ and Laplace density function $e^{-|x|}/2$, respectively. Their approximation properties are studied, for example, in [1, 3]. Also, by using the Fourier transform method, it is known that the solutions of the initial value problems for the heat equation and Laplace equation are exactly the Gauss-Weierstrass and Poisson-Cauchy convolution singular integrals, respectively, see, e.g., [6, p. 23]. On the other hand, in our best knowledge, the initial value problem and the partial differential equation corresponding to the Picard singular integral, is missing from mathematical literature. The main aim of the present paper is somehow inverse: introducing convolution singular integrals based on some known probability densities, we use the inverse Fourier transform in order to find the partial differential equations (initial and final value problems) satisfied by these integrals, including the Picard singular integral.

2. Definitions of convolution operators

In this section we introduce several convolution operators, based on some well-known densities of probability.

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If $d(t,x) \geq 0$ for $t > 0$ and $x \in \mathbb{R}$ is a probability density, that is $\int_{-\infty}^{+\infty} d(t,x)dx = 1$, then our definitions are based on the general known formula

$$\begin{aligned} O_t(f)(x) &= d(t,\cdot) * f(\cdot) = \int_{-\infty}^{+\infty} f(u)d(t,x-u)du \\ &= \int_{-\infty}^{+\infty} f(x-v)d(t,v)\,dv. \end{aligned} \tag{2.1}$$

Definition 2.1. (i) For the Maxwell-Boltzmann type probability density (see, e.g., [8, pp. 104, 148-149])

$$d(t,x) = \frac{1}{\sqrt{2\pi}} \frac{x^2 e^{-x^2/(2t^2)}}{t^3}, \quad x \in \mathbb{R},\ t > 0$$

and $f : \mathbb{R} \to \mathbb{R}$, we can formally define the Maxwell-Boltzmann convolution operator

$$S_t(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-v) \frac{v^2 e^{-v^2/(2t^2)}}{t^3}\,dv, \quad t > 0,\ x \in \mathbb{R}. \tag{2.2}$$

(ii) For the Laplace type probability density (see, e.g., [2])

$$d(t,x) = \frac{1}{2t} e^{-|x|/t}, \quad t > 0,\ x \in \mathbb{R}$$

and $f : \mathbb{R} \to \mathbb{R}$, we can formally define the classical Picard convolution operator

$$P_t(f)(x) = \frac{1}{2t} \int_{-\infty}^{+\infty} f(x-v) \cdot e^{-|v|/t}\,dv, \quad t > 0,\ x \in \mathbb{R}. \tag{2.3}$$

(iii) For the exponential probability density (see, e.g., [2, 7])

$$d(t,x) = \frac{te^{-t|x|}}{2}, \quad x \in \mathbb{R},\ t > 0$$

and $f : \mathbb{R} \to \mathbb{R}$, we can formally define the exponential convolution operator

$$E_t(f)(x) = \int_{-\infty}^{+\infty} f(x-v) \frac{te^{-t|v|}}{2}\,dv, \quad t > 0,\ x \in \mathbb{R}. \tag{2.4}$$

(iv) For any $n \in \mathbb{N}$, $P_t(f)(x)$ can be generalized to the so called Jackson type generalization of the Picard singular integral defined by (see, e.g., [3])

$$\begin{aligned} P_{n,t}(f)(x) &= -\frac{1}{2t} \int_{-\infty}^{+\infty} \Big(\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} f(x+kv) e^{-|v|/t} \Big) dv \\ &= \int_{-\infty}^{+\infty} f(x-u) \Big[\sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} (\frac{1}{k}) \frac{e^{-|u|/(kt)}}{2t} \Big]\,du, \end{aligned}$$

for $t > 0$ and $x \in \mathbb{R}$.

(v) Starting from the well known Gauss-Weierstrass operator

$$W_t(f)(x) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(x-v) e^{-v^2/t}\,dv,$$

we can define its Jackson type generalization by (see, e.g., [3])

$$W_{n,t}(f)(x) = -\frac{1}{2C^*(t)} \int_{-\infty}^{+\infty} \Big(\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} f(x+kv) e^{-v^2/t} \Big) dv$$

$$= \int_{-\infty}^{+\infty} f(x-u)\Big[\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}(\frac{1}{k})\frac{e^{-u^2/(kt)}}{2C^*(t)}\Big]\,du,$$

for $t>0$ and $x\in\mathbb{R}$, where $C^*(t)=\int_0^\infty e^{-u^2/t}du=\frac{\sqrt{t\pi}}{2}$. Therefore, for any $n\in\mathbb{N}$, we can write

$$W_{n,t}(f)(x) = \int_{-\infty}^{+\infty} f(x-u)\Big[\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}(\frac{1}{k})\frac{e^{-u^2/(kt)}}{\sqrt{\pi t}}\Big]\,du.$$

3. Applications to PDEs

Concerning the convolution operators defined in Section 2, we can state the following applications to PDEs.

Theorem 3.1. *(i) Suppose that $f, f', f'', f''', f^{(4)}:\mathbb{R}\to\mathbb{R}$ are bounded and uniformly continuous on $\mathbb{R}$. Then the solution of the initial value problem*

$$\frac{\partial u}{\partial t}(x,t) = t^3\frac{\partial^4 u}{\partial x^4}(x,t) - t^2\frac{\partial^3 u}{\partial x^2\partial t}(x,t) + 3t\frac{\partial^2 u}{\partial x^2}(x,t),$$
$$\lim_{s\searrow 0} u(x,s) = f(x), \quad t>0,\ x\in\mathbb{R},$$

is $u(x,t) := S_t(f)(x)$.

(ii) Suppose that $f, f', f'':\mathbb{R}\to\mathbb{R}$ are bounded and uniformly continuous on $\mathbb{R}$. Then the solution of the initial value problem

$$\frac{\partial u}{\partial t}(x,t) = t^2\frac{\partial^3 u}{\partial x^2\partial t}(x,t) + 2t\frac{\partial^2 u}{\partial x^2}(x,t),$$
$$\lim_{s\searrow 0} u(x,s) = f(x), \quad t>0,\ x\in\mathbb{R}$$

is $u(x,t) := P_t(f)(x)$.

(iii) Suppose that $f, f', f'':\mathbb{R}\to\mathbb{R}$ are bounded and uniformly continuous on $\mathbb{R}$. Then the solution of the final value problem

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{t^2}\frac{\partial^3 u}{\partial x^2\partial t}(x,t) - \frac{2}{t^3}\frac{\partial^2 u}{\partial x^2}(x,t),$$
$$\lim_{s\to\infty} u(x,s) = f(x), \quad t>0,\ x\in\mathbb{R}$$

is $u(x,t) := E_t(f)(x)$.

(iv) Suppose that $f, f', f'':\mathbb{R}\to\mathbb{R}$ are bounded and uniformly continuous on $\mathbb{R}$. Then we have

$$P_{n,t}(f)(x) = \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}u_k(x,t),$$

where

$$u_k(x,t) = P_{kt}(f)(x) = \frac{1}{2kt}\int_{-\infty}^{+\infty} f(x-u)e^{-|u|/(kt)}du, \quad k=1,\dots,n+1$$

are solutions of the initial value problems (for $t>0$ and $x\in\mathbb{R}$)

$$\frac{\partial u_k}{\partial t}(x,t) = k^2t^2\frac{\partial^3 u_k}{\partial x^2\partial t}(x,t) + 2k^2t\frac{\partial^2 u_k}{\partial x^2}(x,t), \quad \lim_{s\searrow 0} u_k(x,s) = f(x).$$

(v) Suppose that $f, f', f'' : \mathbb{R} \to \mathbb{R}$ are bounded and uniformly continuous on $\mathbb{R}$. Then we have

$$W_{n,t}(f)(x) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \cdot u_k(x,t),$$

where

$$u_k(x,t) = \frac{1}{\sqrt{k}} W_{kt}(f)(x) = \frac{1}{k\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(x-u) e^{-u^2/(kt)} du, \quad k = 1, \dots, n+1$$

are solutions of the initial value problems

$$\frac{\partial u_k}{\partial t}(x,t) = \frac{k}{4} \frac{\partial^2 u_k}{\partial x^2}(x,t),$$
$$\lim_{s \searrow 0} u_k(x,s) = \frac{1}{\sqrt{k}} f(x), \quad t > 0, \ x \in \mathbb{R}.$$

Proof. Since for the convolution operator given by (2.1), in general we have $d(t,v) \geq 0$, for all $t > 0$ and $v \in \mathbb{R}$, by the standard method we easily obtain

$$\begin{aligned} |O_t(f)(x) - f(x)| &\leq \int_{-\infty}^{+\infty} |f(x-v) - f(x)| d(t,v)\, dv \\ &\leq \int_{-\infty}^{+\infty} \omega_1(f; |v|)_{\mathbb{R}} d(t,v)\, dv \\ &\leq 2\omega_1(f; \varphi(t))_{\mathbb{R}}, \end{aligned}$$

where $\omega_1(f;\delta)_{\mathbb{R}} = \sup\{|f(x) - f(y)|; x, y \in \mathbb{R}, |x-y| \leq \delta\}$ and $\varphi(t) = \int_{-\infty}^{+\infty} |v| \cdot d(t,v) dv$. Evidently this method is useful only if $\varphi(t) < +\infty$ for all $t > 0$.

To deduce the PDEs satisfied by various convolution operators, we need the concepts of Fourier transform of a function g. It is defined by

$$F(g)(\xi) = \hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x) e^{-i\xi x} dx, \quad \text{if} \int_{-\infty}^{+\infty} |g(x)| dx < +\infty,$$

and the inverse Fourier transform is defined by

$$F^{-1}(\hat{g})(x) = g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{g}(\xi) e^{i\xi x} d\xi.$$

(i) By making the change of variable $v = \sqrt{2} t s$, we obtain

$$\begin{aligned} \varphi(t) &= \frac{1}{t^3} \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} v^3 e^{-v^2/(2t^2)} dv \\ &= \frac{1}{t^3} \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} (2\sqrt{2} t^3 s^3) e^{-s^2} (\sqrt{2} t) ds \\ &= \frac{4\sqrt{2}}{\sqrt{\pi}} t \int_0^{\infty} s^3 e^{-s^2} ds \\ &= \frac{2\sqrt{2}}{\sqrt{\pi}} t < 2t, \end{aligned}$$

which implies

$$|S_t(f)(x) - f(x)| \leq 4\omega_1(f; t)_{\mathbb{R}}, \quad t > 0, \ x \in \mathbb{R}.$$

Taking into account the uniform continuity of f, the above inequality implies that $\lim_{t\searrow 0} S_t(f)(x) = f(x)$, for all $x \in \mathbb{R}$. Therefore we may take, by convention, $S_0(f)(x) = f(x)$, for all $x \in \mathbb{R}$.

Now, to deduce the PDE satisfied by $S_t(f)(x)$, we write it in the form

$$\begin{aligned} S_t(f)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \frac{(x-y)^2 e^{-(x-y)^2/(2t^2)}}{t^3}\, dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \cdot \hat{g}_t(y-x) dy. \end{aligned}$$

Here, by using standard reasoning and calculation (or WolframAlpha software), we obtain

$$g_t(\xi) = F_w^{-1}[w^2 e^{-w^2/(2t^2)}/t^3](\xi, t) = e^{-t^2\xi^2/2}(1 - t^2\xi^2),$$

which implies

$$\begin{aligned} S_t(f)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(y) \Big[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i(y-x)\xi} e^{-t^2\xi^2/2}(1-t^2\xi^2) d\xi\Big] dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Big[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iy\xi} f(y) dy\Big] e^{ix\xi} e^{-t^2\xi^2/2}(1-t^2\xi^2) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ix\xi} \hat{f}(\xi) e^{-t^2\xi^2/2}(1-t^2\xi^2) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ix\xi} \hat{u}(\xi, t) d\xi =: u(x, t), \end{aligned}$$

where

$$\hat{u}(\xi, t) = \hat{f}(\xi) \cdot e^{-t^2\xi^2/2}(1 - t^2\xi^2).$$

This is equivalent to $\hat{u}(\xi, t)\frac{e^{t^2\xi^2/2}}{1-t^2\xi^2} = \hat{f}(\xi)$, which is equivalent to

$$\frac{\partial}{\partial t}\Big[\hat{u}(\xi, t)\frac{e^{t^2\xi^2/2}}{1-t^2\xi^2}\Big] = \frac{\partial \hat{u}}{\partial t}(\xi, t)\frac{e^{t^2\xi^2/2}}{1-t^2\xi^2} + \hat{u}(\xi, t)\Big(\frac{e^{t^2\xi^2/2}}{1-t^2\xi^2}\Big)'_t = 0.$$

Note that the above relation can be evidently written under the form

$$\frac{\partial}{\partial t}\Big[\hat{u}(\xi, t)\frac{1}{F_w^{-1}(d(t, w))(\xi, t)}\Big] = 0,$$

where $d(t, x)$ is the Maxwell-Boltzmann type probability density in Definition 2.1, (i), entering in the formula for $S_t(f)(x)$.

After simple calculations, the above formula is formally equivalent to (of course for $1 \neq t^2\xi^2$)

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) + t^2\Big(-\xi^2\frac{\partial \hat{u}}{\partial t}(\xi, t)\Big) - 3t[-\xi^2\hat{u}(\xi, t)] - t^3 \cdot [\xi^4\hat{u}(\xi, t)] = 0.$$

Now, taking into account that

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) = \widehat{\frac{\partial u}{\partial t}}(\xi, t), \quad \widehat{\frac{\partial^2 u}{\partial x^2}}(\xi, t) = -\xi^2\hat{u}(\xi, t), \quad \widehat{\frac{\partial^4 u}{\partial x^4}}(\xi, t) = \xi^4\hat{u}(\xi, t),$$

and replacing the above, we obtain

$$F\Big(\frac{\partial u}{\partial t} + t^2\frac{\partial^3 u}{\partial x^2 \partial t} - 3t\frac{\partial^2 u}{\partial x^2} - t^3\frac{\partial^4 u}{\partial x^4}\Big)(\xi, t) = 0,$$

that is
$$\frac{\partial u}{\partial t}(x,t)=t^3\frac{\partial^4 u}{\partial x^4}(x,t)-t^2\frac{\partial^3 u}{\partial x^2\partial t}(x,t)+3t\frac{\partial^2 u}{\partial x^2}(x,t).$$
Finally, following the above steps in inverse order, we arrive at the conclusion in the statement.

(ii) By [1, p. 142, Corollary 3.4.2], we have
$$|f(x)-P_t(f)(x)|\le C\omega_2(f;t)_{\mathbb{R}}.$$
Therefore, it is immediate that $\lim_{t\searrow 0}P_t(f)(x)=f(x)$, for all $x\in\mathbb{R}$.

To deduce the PDE satisfied by $P_t(f)(x)$, we reason exactly as in case (i). Indeed, by standard calculations (or using WolframAlpha software), we obtain
$$F_w^{-1}[e^{-|w|/t}/(2t)](\xi,t)=\frac{1}{\sqrt{2\pi}}\frac{1}{1+t^2\xi^2}$$
and similar reasoning as in case (i) leads to
$$P_t(f)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{ix\xi}\hat{u}(\xi,t)d\xi:=u(x,t),$$
where
$$\hat{u}(\xi,t)=\hat{f}(\xi)\frac{1}{t^2\xi^2+1}.$$
In fact, directly as in the proof of Theorem 3.1, (i), we can write
$$\frac{\partial}{\partial t}\Big[\hat{u}(\xi,t)\frac{1}{F_w^{-1}(d(t,w))(\xi,t)}\Big]=0,$$
where $d(t,x)$ is the Laplace type probability density in Definition 2.1, (ii), entering in the formula for $P_t(f)(x)$. Therefore,
$$\frac{\partial}{\partial t}\left[\hat{u}(\xi,t)\cdot(1+t^2\xi^2)\right]=\frac{\partial\hat{u}}{\partial t}(\xi,t)+t^2\xi^2\frac{\partial\hat{u}}{\partial t}(\xi,t)+2t\xi^2\hat{u}(\xi,t)=0,$$
which leads to
$$\frac{\partial u}{\partial t}(x,t)=t^2\frac{\partial^3 u}{\partial x^2\partial t}(x,t)+2t\frac{\partial^2 u}{\partial x^2}(x,t).$$
Following the above steps, now from the end to the beginning, we arrive at the conclusion in the statement.

(iii) Firstly, we observe that $E_t(f)(x)=P_{1/t}(f)(x)$, for all $t>0$ and $x\in\mathbb{R}$. The, by (ii) we immediately get
$$|E_t(f)(x)-f(x)|=|P_{1/t}(f)(x)-f(x)|\le 2\omega_1\big(f;\frac{1}{t}\big)_{\mathbb{R}}.$$
Then, again by standard calculations (or using WolframAlpha), we have
$$F_w^{-1}(e^{-|w|t})(\xi,t)=\frac{\sqrt{2}}{\sqrt{\pi}}\frac{t}{t^2+\xi^2}.$$
This implies
$$F^{-1}(d(t,w))(\xi,t)=\frac{t}{2}F_w^{-1}(e^{-|w|t})(\xi,t)=\frac{1}{\sqrt{2\pi}}\frac{t^2}{t^2+\xi^2}.$$
It follows that
$$\frac{1}{F^{-1}(d(t,w))(\xi,t)}=\sqrt{2\pi}\frac{t^2+\xi^2}{t^2}=\sqrt{2\pi}\big(1+\frac{\xi^2}{t^2}\big).$$

Therefore, denoting $u(x,t) = E_t(f)(x)$, by the method used at the above points, we arrive at the PDE

$$\frac{\partial}{\partial t}\Big(\hat{u}(\xi,t)\big(1+\frac{\xi^2}{t^2}\big)\Big) = \frac{\hat{u}}{\partial t}(\xi,t)\big(1+\frac{\xi^2}{t^2}\big) + \hat{u}(\xi,t)\big(-\frac{2\xi^2}{t^3}\big) = 0.$$

This leads to the following PDE, satisfied by $u(x,t) = E_t(f)(x)$,

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{t^2}\frac{\partial^3 u}{\partial x^2 \partial t}(x,t) - \frac{2}{t^3}\frac{\partial^2 u}{\partial x^2}(x,t).$$

Since $E_t(f)(x) = P_{1/t}(f)(x)$, it follows that $\lim_{t\nearrow\infty} E_t(f)(x) = f(x)$, for all $x \in \mathbb{R}$.

Following the above steps in inverse order, we arrive at the conclusion in the statement.

(iv) Concerning the approximation properties of $P_{n,t}(f)(x)$, the following estimate was obtained in [3],

$$|f(x) - P_{n,t}(f)(x)| \leq \sum_{k=1}^{n+1} k!\binom{n+1}{k} \cdot \omega_{n+1}(f;t)_{\mathbb{R}},$$

where $\omega_{n+1}(f;\delta) = \sup_{0\leq h\leq\delta}\{|\Delta_h^{n+1} f(x); x \in \mathbb{R}\}$, with

$$\Delta_h^{n+1} = \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j} f(x+jh).$$

This implies $\lim_{t\searrow 0} P_{n,t}(f)(x) = f(x)$, for all $x \in \mathbb{R}$.

To deduce the PDE satisfied by $P_{n,t}(f)(x)$, since F_w^{-1} is linear operator, we use a calculation (or WolframAlpha software), $F_w^{-1}(e^{-|w|/t})(\xi,t) = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{t}{t^2\xi^2+1}$, replacing here t by kt, we easily obtain

$$\begin{aligned}
&F_w^{-1}\Big[\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\frac{1}{2tk}e^{-|w|/(kt)}\Big](\xi,t)\\
&= \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\frac{1}{kt}\frac{kt}{k^2t^2\xi^2+1}\\
&= \frac{1}{\sqrt{2\pi}}\cdot\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\frac{1}{k^2t^2\xi^2+1}.
\end{aligned}$$

Therefore, denoting $u(x,t) := P_{n,t}(f)(x)$ we obtain the differential equation

$$\frac{\partial}{\partial t}\Big[\hat{u}(\xi,t)\frac{1}{\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\frac{1}{k^2t^2\xi^2+1}}\Big] = 0,$$

which is equivalent to

$$\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\Big[\frac{\partial\hat{u}}{\partial t}(\xi,t)\frac{1}{k^2t^2\xi^2+1} + \hat{u}(\xi,t)\frac{2k^2t\xi^2}{(k^2t^2\xi^2+1)^2}\Big] = 0.$$

It is worth noting that letting

$$u_k(x,t) = P_{kt}(f)(x) = \frac{1}{2kt}\int_{-\infty}^{+\infty} f(x-u)e^{-|u|/(kt)}du,$$

we can write

$$P_{n,t}(f)(x)=\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\cdot u_k(x,t),$$

where reasoning as above for $P_t(f)(x)$, we easily obtain

$$\frac{\partial\widehat{u_k}}{\partial t}(\xi,t)=k^2t^2\frac{\partial^3\widehat{u_k}}{\partial x^2\partial t}(\xi,t)+2k^2t\frac{\partial^2\widehat{u_k}}{\partial x^2}(\xi,t)$$

and which implies

$$\frac{\partial u_k}{\partial t}(x,t)=k^2t^2\frac{\partial^3 u_k}{\partial x^2\partial t}(x,t)+2k^2t\frac{\partial^2 u_k}{\partial x^2}(x,t),\quad x\in\mathbb{R},\ t>0,\ k=1,\dots,n+1,$$

with $u_k(x,0)=f(x)$, for all $x\in\mathbb{R}$, $k=1,\dots,n+1$.

(v) Concerning the approximation properties of $W_{n,t}(f)(x)$, reasoning as in [3], we obtain the estimate

$$|f(x)-W_{n,t}(f)(x)|\le C_n\omega_{n+1}(f;\sqrt{t})_{\mathbb{R}},$$

where $C_n>0$ is a constant independent of f, t and x. This immediately implies that $\lim_{t\searrow 0}W_{n,t}(f)(x)=f(x)$, for all $x\in\mathbb{R}$.

To deduce the PDE satisfied by $W_{n,t}(f)(x)$, since F_w^{-1} is linear operator, using WolframAlpha software we have $F_w^{-1}(e^{-w^2/t})(\xi,t)=\sqrt{t}e^{-t\xi^2/4}/\sqrt{2}$. Replacing t by kt, we easily obtain $F_w^{-1}(e^{-w^2/(kt)})(\xi,t)=\sqrt{kt}e^{-kt\xi^2/4}/\sqrt{2}$ and

$$\begin{aligned}
&F_w^{-1}\Big[\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\frac{1}{\sqrt{\pi tk}}e^{-w^2/(kt)}\Big](\xi,t)\\
&=\frac{1}{\sqrt{\pi}}\cdot\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\frac{1}{k\sqrt{t}}\frac{\sqrt{kt}e^{-kt\xi^2/4}}{\sqrt{2}}\\
&=\frac{1}{\sqrt{2\pi}}\cdot\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\frac{1}{\sqrt{k}}\cdot e^{-kt\xi^2/4}.
\end{aligned}$$

Therefore, denoting $u(x,t):=W_{n,t}(f)(x)$ we obtain the differential equation

$$\frac{\partial}{\partial t}\Big[\hat{u}(\xi,t)\frac{1}{\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\frac{1}{\sqrt{k}}e^{-kt\xi^2/4}}\Big]=0,$$

which is equivalent to

$$\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\frac{1}{\sqrt{k}}\Big[\frac{\partial\hat{u}}{\partial t}(\xi,t)e^{-kt\xi^2/4}+\hat{u}(\xi,t)\frac{k\xi^2}{4}e^{-kt\xi^2/4}\Big]=0.$$

It is worth noting that letting

$$u_k(x,t)=\frac{1}{\sqrt{k}}W_{kt}(f)(x)=\frac{1}{k\sqrt{\pi t}}\int_{-\infty}^{+\infty}f(x-u)e^{-u^2/(kt)}du,$$

we can write

$$W_{n,t}(f)(x)=\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\cdot u_k(x,t).$$

Reasoning as above but for $W_t(f)(x)$, we easily obtain

$$\frac{\partial\widehat{u_k}}{\partial t}(\xi,t)=\frac{k}{4}\frac{\partial^2\widehat{u_k}}{\partial x^2}(\xi,t)$$

which implies

$$\frac{\partial u_k}{\partial t}(x,t) = \frac{k}{4}\frac{\partial^2 u_k}{\partial x^2}(x,t), \quad x \in \mathbb{R},\ t > 0,\ k = 1, \dots, n+1,$$

with $u_k(x,0) = f(x)$, for all $x \in \mathbb{R}$, $k = 1, \dots, n+1$. $\square$

The methods in this paper could be used to make analogous studies for the convolutions with other known probability densities, like the Rayleigh probability density (abbreviated p.d.), Gumbel p.d., logistic p.d., Johnson p.d., Fréchet p.d., Gompetz p.d., Lévy p.d., Lomax p.d. and so on.

It would be also of interest to use the methods in this paper for complex convolutions, based on the ideas and results in the books [4, 5].

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