SOLVABILITY OF SOME NEUMANN-TYPE BOUNDARY VALUE PROBLEMS FOR BIHARMONIC EQUATIONS

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Abstract. We study some boundary-value problems for inhomogeneous biharmonic equation with periodic boundary conditions. These problems are generalization to periodic data of the Neumann-type boundary-value problems considered before by the authors. We obtain existence and uniqueness of solutions for the problems under consideration.

1. Introduction

Many stationary processes occurring in physics and mechanics are described by equations of elliptic type. One of the important particular cases of fourth-order elliptic equations is the biharmonic equation. Solution of the plane deformation problems in elasticity theory in many cases can be reduced to the integration of biharmonic equations under corresponding boundary conditions. In addition, many problems of continuous media mechanics can be reduced to the solution of harmonic and biharmonic equations. However, convenient analytic expressions for the solutions of these problems are obtained only for domains of particular forms.

Applications of biharmonic problems in mechanics and physics are described in numerous investigations (see, for example, [1, 7, 28]). Applications of boundary value problems for biharmonic equations in mechanics and physics stimulate the study of various boundary value problems for biharmonic equations. One of the well known boundary value problems for biharmonic equations is the Dirichlet problem [3, 6, 11, 13, 16, 17, 30]. Recently other types of boundary value problems for biharmonic equation such as Riquier problem [8, 19, 29], Neumann problem [5, 7, 10, 19, 20, 21, 22, 24, 30, 31], spectral Steklov problem [11], Robin problem [13], generalized Robin problem [23], as well as fractional analogous of Neumann problem [4, 32, 33, 34] are begun to investigate intensively.

The theory of polyharmonic (biharmonic) equations and various boundary value problems for them was described in great detail in [12]. Conditions for the solvability of boundary value problems for elliptic equations and systems of equations contain the so-called complementing conditions. It was established that all problems of the given type are Fredholm-type problems. Therefore, the solvability of
these problems for homogeneous boundary conditions is guaranteed by the orthogonality of the right-hand sides of the equation to all solutions of the corresponding homogeneous conjugate equation. In the considered below particular case (biharmonic equation) of the common problem more detailed results can be obtained. In the present paper a new class of boundary value problems for inhomogeneous biharmonic equation $\Delta^2 u(x) = f(x)$ in the unit ball with periodic boundary conditions is studied.

This article is organized as follows. In Section 2 the statement of the main problem (2.1)-(2.4) is given. Some preliminary results are cited in Section 3. The necessary and sufficient conditions for solvability of the Neumann-type boundary value problems (3.1)-(3.3) are given in Theorems 3.1,3.3. Auxiliary integral equalities are derived in Lemmas 4.1–4.7. In Section 5 uniqueness conditions for the main problem (2.1)-(2.4) are given. Some preliminary results are cited in Section 3. The problem (2.1)-(2.4) belonging to the class $C^4(\Omega)$ is studied. Further, let $\nu$ be the unit normal to $\partial \Omega$ and $D^m_\nu = \frac{\partial^m}{\partial \nu^m}$ ($m \geq 1$) be the normal derivative of order $m$. In the domain $\Omega$ for $k = 1,2$ consider the following boundary value problems:

\begin{align}
\Delta^2 u(x) &= f(x), \quad x \in \Omega, \tag{2.1} \\
D^m_\nu u(x) &= g(x), \quad x \in \partial\Omega, \tag{2.2} \\
D^k_\nu u(x) - (-1)^k D^k_\nu u(x^*) &= g_1(x), \quad x \in \partial\Omega_+, \tag{2.3} \\
D^k_\nu u(x) + (-1)^k D^k_\nu u(x^*) &= g_2(x), \quad x \in \partial\Omega_+, \tag{2.4}
\end{align}

where $1 \leq m \leq 3, 1 \leq \ell_1 < \ell_2 \leq 3, \ell_1 \neq m, \ell_2 \neq m$.

By a solution of the problem (2.1)-(2.4) we mean a function $u(x) \in C^4(\Omega) \cap C^3(\overline{\Omega})$, which satisfies the conditions (2.1)-(2.4) in the classical sense.

Let $\partial^\beta = \frac{\partial^{\beta_1}\partial^{\beta_2}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}}$, where $\beta = (\beta_1, \ldots, \beta_n)$ is the multi-index, $|\beta| = \beta_1 + \ldots + \beta_n$ and $\partial^0 = I$ is the unit operator.

It is obvious that the necessary condition for existence of the solution to the problem (2.1)-(2.4) belonging to the class $C^4(\Omega)$ are the following compatibility conditions:

\begin{align}
\partial^\beta g_1(\hat{x}, 0) + (-1)^k \partial^\beta g_1(\hat{\alpha} \hat{x}, 0) &= 0, \quad |\beta| \leq p, \tag{2.5} \\
\partial^\beta g_2(\hat{x}, 0) - (-1)^k \partial^\beta g_2(\hat{\alpha} \hat{x}, 0) &= 0, \quad |\beta| \leq q, \tag{2.6}
\end{align}

where $\hat{x} = (x_1, \ldots, x_{n-1}), \hat{\alpha} = (\alpha_1, \ldots, \alpha_{n-1}), p$ and $q$ take the values $0,1,2,3$ depending on the order of the boundary operators $D_\nu^1$ and $D_\nu^2$. Note that analogous problems for the Poisson equation were investigated in [23] [26] [27] [34].
3. Preliminary results

In this section we consider the following Neumann-type problems:

\[ \Delta^2 u(x) = f(x), \quad x \in \Omega, \]  
\[ D_{\nu}^{m_1} u(x) = g_1(x), \quad x \in \partial\Omega, \]  
\[ D_{\nu}^{m_2} u(x) = g_2(x), \quad x \in \partial\Omega, \]  

where \( 1 \leq m_1 < m_2 \leq 3. \)

Problems (3.1)-(3.3) for different values of \( m_1 \) and \( m_2 \) are studied in \([5, 9, 10, 19, 20, 21, 22, 24, 30, 31]\). The case \( f(x) = 0, \; m_1 = 1, \; m_2 = 2 \) was considered by Bitsadze in \([5]\). It was established that the necessary and sufficient condition for solvability of the problem (3.1)-(3.3) have the form

\[ \int_{\partial\Omega} [g_2(x) - g_1(x)] dS_x = 0. \]

Further in [19] the following statement is established.

**Theorem 3.1.** Let \( m_1 = 1, \; m_2 = 2, \; f(x) \in C(\Omega), \; g_1(x) \in C^1(\partial\Omega), \; g_2(x) \in C(\partial\Omega). \) Then for solvability of the problem (3.1)-(3.3) it is necessary and sufficient that the following condition be fulfilled

\[ \int_{\partial\Omega} [f_2(x) - f_1(x)] dS_x = \frac{1}{2} \int_{\Omega} (1 - |x|^2) f(x) dx. \]  

If a solution of the problem exists then it is unique up to a constant term.

The case \( m_1 = 2, \; m_2 = 3 \) is investigated in \([30]\). The following statement is proved.

**Theorem 3.2.** Let \( m_1 = 2, \; m_2 = 3, \; f(x) \in C^{\lambda+1}(\Omega), \; g_1(x) \in C^{\lambda+2}(\partial\Omega), \; g_2(x) \in C^{\lambda+1}(\partial\Omega). \) Then for solvability of problem (3.1)-(3.3) it is necessary and sufficient that the following conditions be fulfilled

\[ \int_{\partial\Omega} g_2(x) dS_x = \frac{n-1}{2} \int_{\Omega} |x|^2 f(x) dx - \frac{n-3}{2} \int_{\Omega} f(x) dx, \]  
\[ \int_{\partial\Omega} x_j[g_2(x) - g_1(x)] dS_x = \frac{n-1}{2} \int_{\Omega} x_j |x|^2 f(x) dx - \frac{n-3}{2} \int_{\Omega} x_j f(x) dx \]

for \( j = 1, \ldots, n. \) If solution of the problem exists, then it is unique up to the first order polynomials.

In [21] the following statement is obtained.

**Theorem 3.3.** Let \( m_1 = 1, \; m_2 = 3, \; f(x) = C(\Omega), \; g_1(x) \in C(\partial\Omega), \; g_2(x) \in C(\partial\Omega). \) Then for solvability of problem (3.1)-(3.3) it is necessary and sufficient that the following condition be fulfilled

\[ \int_{\partial\Omega} g_2(x) dS_x = \frac{n-1}{2} \int_{\Omega} |x|^2 f(x) dx - \frac{n-3}{2} \int_{\Omega} f(x) dx. \]  

If a solution of the problem exists, then it is unique up to a constant term.
4. Auxiliary integral equalities

In what follows we need some integral equalities. Let \( f(x) \in C(\Omega) \), \( g_1(x) \in C(\partial\Omega) \), \( g_2(x) \in C(\partial\Omega) \). Denote
\[
\tilde{f}^\pm(x) = \frac{f(x) \pm f(x^*)}{2}, \quad g^\pm(x) = \frac{g(x) \pm g(x^*)}{2},
\]
(4.1)
\[
\tilde{g}^\pm(x) = \frac{1}{2} \begin{cases} g(x), & x \in \partial\Omega_+; \\ \pm g(x^*), & x \in \partial\Omega_. \end{cases}
\]

Lemma 4.1. Let \( f(x) \in C(\Omega) \) and \( g(x) \in C(\partial\Omega) \). Then
\[
\int_\Omega f(x^*) \, dx = \int_\Omega f(x) \, dx,
\]
(4.2)
\[
\int_{\partial\Omega} g(x^*) \, dx = \int_{\partial\Omega} g(x) \, dx.
\]
(4.3)

Proof. Let \( \tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), where \( \alpha_n = -1 \) and the other \( \alpha_j, \ j = 1, n-1 \) take one of the values \( \pm 1 \). Consider the matrix
\[
P = \begin{pmatrix} \alpha_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_n \end{pmatrix}.
\]

It is obvious that \( P^T = P \) and \( P \cdot P^T = E \). Consequently, \( P \) is an orthogonal matrix. It is known (see e.g. [2]), that if \( P \) is an orthogonal matrix, then
\[
\int_\Omega f(Px) \, dx = \int_\Omega f(x) \, dx, \quad \int_{\partial\Omega} g(Px) \, ds_x = \int_{\partial\Omega} g(x) \, ds_x.
\]

Since \( x^* = Px \) then we obtain (4.2) and (4.3). \( \square \)

Corollary 4.2. Let \( f(x) \in C(\Omega) \) and \( g(x) \in C(\partial\Omega) \). Then the following equalities hold:
\[
\int_\Omega f^+(x) \, dx = \int_\Omega f(x) \, dx, \quad \int_\Omega |x|^2 f^+(x) \, dx = \int_\Omega |x|^2 f(x) \, dx,
\]
(4.4)
\[
\int_\Omega f^-(x) \, dx = 0, \quad \int_\Omega |x|^2 f^-(x) \, dx = 0,
\]
(4.5)
\[
\int_{\partial\Omega} g^+(x) \, ds_x = \int_{\partial\Omega} g(x) \, ds_x,
\]
(4.6)
\[
\int_{\partial\Omega} g^-(x) \, ds_x = 0.
\]
(4.7)

Proof. By the definition of the functions \( f^\pm(x) \) we obtain
\[
\int_\Omega f^\pm(x) \, dx = \frac{1}{2} \int_\Omega f(x) \, dx \pm \frac{1}{2} \int_\Omega f(x^*) \, dx = \frac{1}{2} \int_\Omega f(x) \, dx \pm \frac{1}{2} \int_\Omega f(x) \, dx.
\]
Further, since \( |x| = |x^*| \) then (4.2) implies equalities (4.4) and (4.5). Equalities (4.6) and (4.7) can be proved similarly. \( \square \)

Lemma 4.3. Let \( g(x) \in C(\partial\Omega) \). Then the following equality holds
\[
\int_{\partial\Omega_+} g(x^*) \, ds_x = \int_{\partial\Omega_-} g(x) \, ds_x.
\]
(4.8)
Proof. To prove (4.8) we pass to the spherical coordinate system:

\[ x_1 = \cos \theta_1, \quad x_2 = \sin \theta_1 \cos \theta_2, \ldots, \quad x_{n-1} = \sin \theta_1 \ldots \sin \theta_2 \cos \theta_{n-1}, \quad x_n = \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \sin \theta_{n-1}, \]

where

\[ 0 \leq \theta_j \leq \pi, \quad j = 1, 2, \ldots, n-2, \quad 0 \leq \theta_{n-1} \leq 2\pi. \]

The Jacobian of this mapping has the form

\[ J(\theta) = \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \ldots \sin \theta_{n-1}. \]

Furthermore, we use the following elementary equalities:

\[ \cos(\pi \pm \theta) = -\cos \theta, \quad \sin(\pi \pm \theta) = \pm \sin \theta. \]

Since \( \partial \Omega_- = \partial \Omega \cap \{x \in \mathbb{R}^n : x_n \leq 0\} \) if and only if \( \pi \leq \theta_{n-1} \leq 2\pi, \ 0 \leq \theta_j \leq \pi, \ j = 1, 2, \ldots, n-2 \), then

\[
\int_{\partial \Omega_-} g(x^*) dS_x = \int_0^\pi \int_0^\pi \int_0^{2\pi} g(\alpha_1 \cos \theta_1, \alpha_2 \sin \theta_1 \cos \theta_2, \ldots, \sin \theta_{n-1}) J(\theta) d\theta_{n-1}.
\]

Let us make the change of variables in the last integral,

\[ \theta_j = \begin{cases} 
\pi - \xi_j, & \alpha_j = -1 \\
\xi_j, & \alpha_j = 1, \quad j = 1, 2, \ldots, n-2, 
\end{cases} \]

\[ \theta_{n-1} = \pi + \xi_{n-1}. \]

Note that under this change of variables we obtain the equality (if \( \alpha_j = -1, \ j = 1, 2, \ldots, n-2 \))

\[ J(\xi) = \sin^{n-2}(\pi - \xi_1) \sin^{n-3}(\pi - \xi_2) \ldots \sin(\pi - \xi_{n-2}) \]

\[ = \sin^{n-2} \xi_1 \sin^{n-3} \xi_2 \ldots \sin \xi_{n-2}, \]

i.e. the Jacobian’s sign is not changed. Further, since

\[ -\cos \theta_1 = -\cos(\pi - \xi_1) = \cos \xi_1, \]

\[ -\sin \theta_j = -\sin(\pi - \xi_j) = -\cos \pi \sin \xi_j = \sin \xi_j, \]

then after the change of variables we have

\[
\int_{\partial \Omega_-} g(x^*) dS_x = \int_0^\pi d\xi_1 \ldots \int_0^\pi d\xi_{n-2} \int_0^\pi g(\cos \xi_1, \sin \xi_1 \cos \xi_2, \ldots, \sin \xi_{n-1}) J(\xi) d\xi_{n-1} \]

\[ = \int_{\partial \Omega_+} g(x) dS_x. \]

\[ \square \]

**Corollary 4.4.** Let \( g(x) \in C(\partial \Omega) \). Then

\[
\int_{\partial \Omega} \tilde{g}^+(x) dS_x = \int_{\partial \Omega_+} g(x) dS_x, \quad (4.9)
\]

\[
\int_{\partial \Omega} \tilde{g}^-(x) dS_x = 0. \quad (4.10)
\]
Proof. Using definition of the function $g^+(x)$ and the equality (4.8), we have

$$\int_{\partial \Omega} g^+(x) dS = \frac{1}{2} \int_{\partial \Omega_+} g(x) dS_x + \frac{1}{2} \int_{\partial \Omega_-} g(x) dS_x.$$

Using (4.10) and according to definition (4.1) of function $g$, we have

$$\int_{\partial \Omega} g^-(x) dS = \frac{1}{2} \int_{\partial \Omega_+} g(x) dS_x - \frac{1}{2} \int_{\partial \Omega_-} g(x) dS_x.$$

Similarly, we obtain

$$\int_{\partial \Omega} g^-(x) dS = \frac{1}{2} \int_{\partial \Omega_+} g(x) dS_x - \frac{1}{2} \int_{\partial \Omega_-} g(x) dS_x = 0.$$

Lemma 4.5. Let $f(x) \in C(\partial \Omega)$. Then

$$\int_{\Omega} x_j f(x) dx = \alpha_j \int_{\Omega} x_j f(x) dx, \quad j = 1, 2, \ldots, n \quad (4.11)$$

Proof. Since $x^* = (\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_{n-1} x_{n-1}, -x_n)$, it follows that

$$\int_{\Omega} x_j f(x^*) dx = \int_{-1}^1 \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \cdots \int_{-\sqrt{1-x_n^2}}^{\sqrt{1-x_n^2}} x_j f(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_{n-1} x_{n-1}, -x_n) dx_n \cdots dx_1.$$

In the above integral we make the change of variables $y_k = \alpha_k x_k, \quad k = 1, 2, \ldots, n$, where $\alpha_n = -1$. Then

$$\int_{\Omega} x_j f(x^*) dx = \alpha_j \int_{\Omega} x_j f(x) dx.$$

Corollary 4.6. If $f(x) \in C(\bar{\Omega})$, then for all $j = 1, 2, \ldots, n$ we have

$$\int_{\Omega} x_j f^+(x) dx = \frac{1 + \alpha_j}{2} \int_{\Omega} x_j f(x) dx, \quad (4.12)$$

$$\int_{\Omega} x_j f^-(x) dx = \frac{1 - \alpha_j}{2} \int_{\Omega} x_j f(x) dx. \quad (4.13)$$

Proof. Using (4.10) and according to definition (4.1) of function $f^+(x)$ we obtain

$$\int_{\Omega} x_j f^+(x) dx$$
Lemma 4.3). Then we obtain

\[ \int_{\Omega} x_j f(x) \, dx = \frac{1}{2} \int_{\Omega} x_j f(x) \, dx + \frac{1}{2} \int_{\Omega} x_j f(x^*) \, dx \]

\[ = \frac{1}{2} \int_{\Omega} x_j f(x) \, dx + \frac{\alpha_j}{2} \int_{\Omega} x_j f(x) \, dx \]

\[ = \frac{1 + \alpha_j}{2} \int_{\Omega} x_j f(x) \, dx. \]

Similarly we can obtain

\[ \int_{\Omega} x_j f^-(x) \, dx = \frac{1}{2} \int_{\Omega} x_j f(x) \, dx - \frac{1}{2} \int_{\Omega} x_j f(x^*) \, dx = \frac{1 - \alpha_j}{2} \int_{\Omega} x_j f(x) \, dx. \]

□

Lemma 4.7. Let \( g(x) \in C(\partial\Omega) \). Then for \( j = 1, 2, \ldots, n \),

\[ \int_{\partial\Omega} x_j g(x^*) \, dS_x = \alpha_j \int_{\partial\Omega} x_j g(x) \, dS_x, \quad (4.14) \]

\[ \int_{\partial\Omega^-} x_j g(x^*) \, dS_x = \alpha_j \int_{\partial\Omega} x_j g(x) \, dS_x. \quad (4.15) \]

Proof. To prove this statement we pass to the spherical coordinate system (see Lemma 4.3). Then we obtain

\[ \int_{\partial\Omega} x_j g(x^*) \, dS_x = \int_0^\pi d\theta_1 \cdots \int_0^\pi d\theta_{n-2} \int_0^\pi \sin \theta_1 \cdots \sin \theta_{j-1} \cos \theta_j \]

\[ \times g(\alpha_1 \cos \theta_1, \ldots, -\sin \theta_1 \cdots \sin \theta_{n-1}) J(\theta) d\theta_{n-1} \]

\[ + \int_0^\pi d\theta_1 \cdots \int_0^\pi d\theta_{n-2} \int_\pi^{2\pi} \sin \theta_1 \cdots \sin \theta_{j-1} \cos \theta_j \]

\[ \times g(\alpha_1 \cos \theta_1, \ldots, -\sin \theta_1 \ldots \sin \theta_{n-1}) J(\theta) d\theta_{n-1}. \]

For the first integral we make the change of variables

\[ \theta_j = \begin{cases} \pi - \xi_j, & \alpha_j = -1 \\ \xi_j, & \alpha_j = 1, \quad j = 1, 2, \ldots, n-2, \\ \theta_{n-1} = \xi_{n-1} - \pi, \end{cases} \]

and use the equality \( \theta_{n-1} = \xi_{n-1} + \pi \) for the second integral. Note that under these changes of variables we have

\[ \sin \theta_k = \sin(\pi - \xi_k) = \sin \pi \cos \xi_k - \cos \pi \sin \xi_k = -\cos \xi_k, \quad k \leq n-2, \]

\[ \sin \theta_{n-1} = \sin(\xi_{n-1} - \pi) = \sin \xi_{n-1} \cos \pi - \cos \xi_{n-1} \sin \pi = -\sin \xi_{n-1}, \]

or

\[ \sin \theta_{n-1} = \sin(\xi_{n-1} + \pi) = \sin \xi_{n-1} \cos \pi + \cos \xi_{n-1} \sin \pi = -\sin \xi_{n-1}, \]

\[ \cos \theta_k = \cos(\pi - \xi_k) = \cos \pi \cos \xi_k - \sin \pi \sin \xi_k = -\cos \xi_k. \]

Consequently for the monomial \( x_j \) we obtain:

(a) if \( \alpha_j = -1 \), then

\[ x_j \rightarrow \sin \theta_1 \cdots \sin \theta_{j-1} \cos \theta_j \rightarrow \sin \xi_1 \cdots \sin \xi_{j-1} (-\cos \xi_j) \rightarrow \alpha_j x_j; \]
(b) if $\alpha_j = +1$, then
\[ x_j \rightarrow \alpha_j \sin \xi_1 \ldots \sin \xi_{j-1} \cos \xi_j \rightarrow \alpha_j x_j. \]

Thus we have the equality
\[
\int_{\partial \Omega} x_j g(x^*) \, dS_x
= \int_0^\pi d\theta_1 \ldots \int_0^\pi d\theta_{n-2} \int_0^\pi \sin \theta_1 \ldots \sin \theta_{j-1} \cos \theta_j
\]
\[ \times g(\alpha_1 \cos \theta_1, \ldots, -\sin \theta_1 \ldots \sin \theta_n) J(\theta) d\theta_{n-1} \]
\[ + \int_0^\pi d\theta_1 \ldots \int_0^\pi d\theta_{n-2} \int_0^{2\pi} \sin \theta_1 \ldots \sin \theta_{j-1} \cos \theta_j
\]
\[ \times g(\alpha_1 \cos \theta_1, \ldots, -\sin \theta_1 \ldots \sin \theta_n) J(\theta) d\theta_{n-1} \]
\[ = \int_0^\pi d\xi_1 \ldots \int_0^\pi d\xi_{n-2} \int_0^{2\pi} \sin \xi_1 \ldots \sin \xi_{j-1} \cos \xi_j
\]
\[ \times g(\cos \xi_1, \ldots, \sin \xi_1 \ldots \sin \xi_{n-1}) J(\xi) d\xi_{n-1} \]
\[ + \alpha_j \int_0^\pi d\xi_1 \ldots \int_0^\pi d\xi_{n-2} \int_0^{2\pi} \sin \xi_1 \ldots \sin \xi_{j-1} \cos \xi_j
\]
\[ \times g(\alpha_1 \cos \xi_1, \ldots, -\sin \xi_1 \ldots \sin \xi_{n-1}) J(\xi) d\xi_{n-1} = \alpha_j \int_{\partial \Omega^+} x_j g(x) \, dS_x. \]

Thus the equality (4.14) is proved. Consider the equality (4.15). In this case we have
\[
\int_{\partial \Omega^-} x_j g(x^*) \, dS_x
= \int_0^\pi d\theta_1 \ldots \int_0^\pi d\theta_{n-2} \int_0^\pi \sin \theta_1 \ldots \sin \theta_{j-1} \cos \theta_j
\]
\[ \times g(\alpha_1 \cos \theta_1, \ldots, -\sin \theta_1 \ldots \sin \theta_n) J(\theta) d\theta_{n-1} \]
\[ = \alpha_j \int_0^\pi d\xi_1 \ldots \int_0^\pi d\xi_{n-2} \int_0^{2\pi} \sin \xi_1 \ldots \sin \xi_{j-1} \cos \xi_j
\]
\[ \times g(\alpha_1 \cos \xi_1, \ldots, -\sin \xi_1 \ldots \sin \xi_{n-1}) J(\xi) d\xi_{n-1} = \alpha_j \int_{\partial \Omega^-} x_j g(x) \, dS_x. \]

\[ \square \]

**Corollary 4.8.** Let $g(x) \in C(\partial \Omega)$. Then for $j = 1, 2, \ldots, n$ the following equalities hold:
\[
\int_{\partial \Omega^-} x_j g^+(x) \, dS_x = \frac{1 \pm \alpha_j}{2} \int_{\partial \Omega} x_j g(x) \, dS_x, \quad (4.16)
\]
\[
\int_{\partial \Omega^+} x_j g^+(x) \, dS_x = \frac{1 \pm \alpha_j}{2} \int_{\partial \Omega} x_j g(x) \, dS_x. \quad (4.17)
\]

**Proof.** According to Lemma 4.7 we have
\[
\int_{\partial \Omega^-} x_j g(x^*) \, dS_x = \alpha_j \int_{\partial \Omega} x_j g(x) \, dS_x.
\]
Therefore, using (4.1) we obtain
\[
\int_{\partial \Omega} x_j g^\pm (x) dS = \frac{1}{2} \left[ \int_{\partial \Omega} x_j g(x) dS \pm \alpha_j \int_{\partial \Omega} x_j g(x) dS \right] = \frac{1}{2} \alpha_j \int_{\partial \Omega} x_j g(x) dS.
\]

Similarly we can get
\[
\int_{\partial \Omega} x_j \tilde{g}^\pm (x) dS = \frac{1}{2} \left[ \int_{\partial \Omega^+} x_j g(x) dS \pm \alpha_j \int_{\partial \Omega^+} x_j g(x) dS \right] = \frac{1}{2} \alpha_j \int_{\partial \Omega^+} x_j g(x) dS.
\]

Remark 4.9. Since \(\alpha_n = -1\), it follows that (4.16) and (4.17) imply
\[
\int_{\partial \Omega} x_n g^+(x) dS = 0, \quad \int_{\partial \Omega} x_n g^-(x) dS = \int_{\partial \Omega} x_n g(x) dS,
\]
\[
\int_{\partial \Omega} x_n \tilde{g}^+(x) dS = 0, \quad \int_{\partial \Omega} x_n \tilde{g}^-(x) dS = \int_{\partial \Omega} x_n g(x) dS.
\]

5. Uniqueness conditions

In this section we study uniqueness of solutions of the problems (2.1)-(2.4).

Theorem 5.1. Let \(k = 1\) and solution of problem (2.1)-(2.4) exist. Then
(1) in the case \(m = 1, \ell_1 = 2, \ell_2 = 3\), the solution of problem (2.1)-(2.4) is unique up to constant term;
(2) if \(m = 2, \ell_1 = 1, \ell_2 = 3\), or \(m = 3, \ell_1 = 1, \ell_2 = 2\), then the following cases are possible:
(a) if for all \(1 \leq j \leq 1\), \(\alpha_j = -1\), then the solution of homogeneous problem (2.1)-(2.4) is a function of the form
\[
u(x) = c_0 + \sum_{j=1}^n c_j x_j;
\]
(b) if for some \(j_0 \in \{1, 2, \ldots, n-1\}\) the equality \(\alpha_{j_0} = 1\) holds, then solution of the homogeneous problem (2.1)-(2.4) is a function of the form
\[
u(x) = c_0 + \sum_{j=1, j \neq j_0}^n c_j x_j.
\]

In particular, if \(\alpha_j = 1, 1 \leq j \leq n - 1\) then \(u(x) = c_0 + c_n x_n\).

Proof. Let \(k = 1\) and function \(u(x)\) is a solution of the homogeneous problem (2.1)-(2.4). Then \(u(x)\) is a biharmonic function that satisfies boundary conditions:
\[
D^\nu_m u(x) = 0, \quad x \in \partial \Omega,
\]
Further, if for all \(1 \leq j \leq n\) and therefore from the conditions (5.2) and (5.3) it follows that
\[
D^{\ell_1}_\nu u(x) = -D^{\ell_1}_\nu u(x^*), \quad x \in \partial \Omega_+, \quad D^{\ell_2}_\nu u(x) = D^{\ell_2}_\nu u(x^*), \quad x \in \partial \Omega_+.
\]

Then for all \(x \in \partial \Omega\) the following equalities hold
\[
D^{\ell_1}_\nu u(x) = -D^{\ell_1}_\nu u(x^*), \quad x \in \partial \Omega, \quad D^{\ell_2}_\nu u(x) = D^{\ell_2}_\nu u(x^*), \quad x \in \partial \Omega.
\]

On the other hand differentiating (5.4) along the normal \(\nu\) give us
\[
D^{\ell_2}_\nu u(x) = -D^{\ell_2}_\nu u(x^*), \quad x \in \partial \Omega.
\]

Then from the equalities (5.5) and (5.6) it follows that
\[
D^{\ell_2}_\nu u(x) = 0, \quad x \in \partial \Omega.
\]

Thereby the function \(u(x)\) is a solution of the problem
\[
\Delta^2 u(x) = 0, \quad x \in \Omega, \quad D^{\nu n}_\nu u(x)|_{\partial \Omega} = 0, \quad D^{\nu}_\nu u(x)|_{\partial \Omega} = 0.
\]

Furthermore when we use the results of Section 3. The following cases are possible:

(1) if \(m = 1, \ell_2 = 3\), then by Theorem 3.3 the function \(u(x) = c_0 \equiv const\) is a unique solution of the problem (5.7)-(5.8). Obviously, this function satisfies all conditions of the homogeneous problem (2.1)-(2.4) for \(k = 1\). Consequently, if \(m = 1, \ell_1 = 2, \ell_2 = 3\) then solution of the homogeneous problem (2.1)-(2.4) is a function \(u(x) = c_0\).

(2) if \(m = 2, \ell_2 = 3\) then according to Theorem 3.2, the unique solution of the homogeneous problem (5.7)-(5.8) is a function of the form:
\[
u(x) = c_0 + \sum_{j=1}^{n} c_j x_j,
\]
where \(c_j\) are constants, \(j = 0, 1, \ldots, n\). In this case \(\ell_1 = 1\) and for all \(x \in \partial \Omega\),
\[
D^{\nu 1}_\nu u(x)|_{\partial \Omega} = r \frac{\partial u(x)}{\partial r}|_{\partial \Omega} = \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}(x).
\]

Since \(u(x^*) = C_0 + \sum_{j=1}^{n} c_j \alpha_j x_j\), it follows that
\[
0 = D^{\nu}_\nu u(x) + D^{\nu}_\nu u(x^*)|_{\partial \Omega} = \sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} c_j \alpha_j x_j = \sum_{j=1}^{n} \left(1 + \alpha_j\right)c_j x_j.
\]

Further, if for all \(1 \leq j \leq n\): \(\alpha_j = -1\), then \(c_j\) are arbitrary numbers and if for some \(j_0 \in \{1, 2, \ldots, n - 1\}\), \(\alpha_{j_0} = 1\) then for the equality
\[
D^{\nu 1}_\nu u(x) + D^{\nu 1}_\nu u(x^*) = 0
\]
to be correct it is necessary that \( c_{j_0} = 0 \). Hence, if \( \alpha_j = -1, 1 \leq j \leq n \) then the function
\[
    u(x) = c_0 + \sum_{j=1}^{n} c_j x_j
\]
is a solution of the homogeneous problem (2.1)-(2.4). If for some \( j_0 \in \{1, 2, \ldots, n-1\} \), \( \alpha_{j_0} = 1 \), then the solution of the homogeneous problem (2.1)-(2.4) is a function of the form
\[
    u(x) = c_0 + \sum_{j=1, j \neq j_0}^{n} c_j x_j.
\]
In particular, if \( \alpha_j = 1 \) for all \( 1 \leq j \leq n - 1 \), then
\[
    u(x) = c_0 + c_n x_n.
\]

(3) Let \( m = 3, \ell_1 = 1, \ell_2 = 2 \). Then as in the case (2) the function of the form
\[
    u(x) = c_0 + \sum_{j=1}^{n} c_j x_j
\]
is a solution of the homogeneous problem (5.7)-(5.8). Making the same arguments as in the case \( m = 2, \ell_1 = 1, \ell_2 = 3 \) we obtain: If \( \alpha_j = -1, 1 \leq j \leq n \) then solution of the homogeneous problem (2.1)-(2.4) is the function
\[
    u(x) = c_0 + \sum_{j=1, j \neq j_0}^{n} c_j x_j.
\]
In particular, if \( \alpha_j = 1 \) for all \( 1 \leq j \leq n - 1 \) then the solution has the form
\[
    u(x) = c_0 + c_n x_n.
\]
\[\square\]

The following statement can be proved similarly.

**Theorem 5.2.** Let \( k = 2 \) and a solution of problem (2.1)-(2.4) exist. Then

(1) in the case \( m = 1, \ell_1 = 2, \ell_2 = 3 \) solution of the problem (2.1)-(2.4) is unique up to constant term;

(2) if \( m = 2, \ell_1 = 1, \ell_2 = 3 \) or \( m = 3, \ell_1 = 1, \ell_2 = 2 \), then the following cases are possible:

(a) if for all \( 1 \leq j \leq n - 1 \), \( \alpha_j = 1 \) then the solution of the homogeneous problem (2.1)-(2.4) is function of the form:
\[
    u(x) = c_0 + \sum_{j=1}^{n-1} c_j x_j;
\]

(b) if for some \( j_0 \in \{1, 2, \ldots, n\} \), \( \alpha_{j_0} = -1 \) then solution of the homogeneous problem (2.1)-(2.4) is a function of the form
\[
    u(x) = c_0 + \sum_{j=1, j \neq j_0}^{n} c_j x_j.
\]
In particular, if \( \alpha_j = -1, 1 \leq j \leq n, \) then \( u(x) = c_0. \)

6. Existence conditions

In this section we present results on existence of a solution of problem (2.1)-(2.4). Let

**Theorem 6.1.** Let \( k = 1 \) and functions \( f(x), \ g(x), \ g_j(x) \ j = 1, 2 \) be smooth enough and compatibility conditions (2.5), (2.6) be fulfilled. Then the necessary and sufficient conditions for solvability of problem (2.1)-(2.4) have the following form:

(1) if \( m = 1, \ell_1 = 2, \ell_2 = 3, \) then

\[
\frac{1}{2} \int_{\Omega} (1 - |x|^2) f(x) dx = \int_{\partial\Omega_+} g_1(x) dS_x - \int_{\partial\Omega} g(x) dS_x;
\]

(6.1)

(2) if \( m = 2, \ell_1 = 1, \ell_2 = 3, \) then

\[
\frac{1}{2} \int_{\Omega} (1 - |x|^2) f(x) dx
\]

\[
= \int_{\partial\Omega} g(x) dS_x - \int_{\partial\Omega_+} g_1(x) dS_x,
\]

\[
\frac{n - 1}{2} \int_{\Omega} x_j |x|^2 f(x) dx - \frac{n - 3}{2} \int_{\Omega} x_j f(x) dx
\]

\[
= \int_{\partial\Omega_+} x_j g_2(x) dS_x - \int_{\partial\Omega} x_j g(x) dS_x
\]

(6.2)

for all \( j \) such that \( \alpha_j = -1; \)

(3) if \( m = 3, \ell_1 = 1, \ell_2 = 2, \) then

\[
\frac{n - 1}{2} \int_{\Omega} |x|^2 f(x) dx - \frac{n - 3}{2} \int_{\Omega} f(x) dx
\]

\[
= \int_{\partial\Omega} g(x) dS_x,
\]

\[
\frac{n - 1}{2} \int_{\Omega} x_j |x|^2 f(x) dx - \frac{n - 3}{2} \int_{\Omega} x_j f(x) dx
\]

\[
= \int_{\partial\Omega_+} x_j g_2(x) dS_x - \int_{\partial\Omega} x_j g(x) dS_x,
\]

(6.3)

for all \( j \in \{1, 2, \ldots, n\} \) such that \( \alpha_j = -1. \)

**Proof.** We introduce two auxiliary functions

\[
v(x) = \frac{1}{2} [u(x) + u(x^*)], \quad w(x) = \frac{1}{2} [u(x) - u(x*)].
\]

It is obvious that \( u(x) = v(x) + w(x). \) It is easy to see that the functions \( v(x) \) and \( w(x) \) are solutions of the following Neumann-type problems:

\[
\Delta^2 v(x) = f^+(x), \ x \in \Omega, \ D_m^\nu v(x)|_{\partial\Omega} = g^+(x), \ D_m^\nu v(x)|_{\partial\Omega_+} = \tilde{g}^+_1(x),
\]

(6.4)

\[
\Delta^2 w(x) = f^-(x), \ x \in \Omega, \ D_m^\nu w(x)|_{\partial\Omega} = g^-(x), \ D_m^\nu w(x)|_{\partial\Omega_+} = \tilde{g}^-_2(x).
\]

(6.5)

Indeed applying the biharmonic operator \( \Delta^2 \) to the function \( v(x), \) we obtain

\[
\Delta^2 v(x) = \frac{1}{2} [\Delta^2 u(x) + \Delta^2 u(x^*)] = \frac{1}{2} [f(x) + f(x^*)] = f^+(x), \ x \in \Omega.
\]

Further, taking the boundary conditions (2.2), (2.3) into account, we have

\[
D_m^\nu v(x) = \frac{1}{2} [D_m^\nu u(x) + D_m^\nu u(x^*)] = \frac{1}{2} [g(x) + g(x^*)] = g^+(x), \ x \in \partial\Omega,
\]
D^2_0 v(x) = \frac{1}{2} [D^1_0 u(x) + D^1_0 u(x^*)] = \frac{1}{2} g(x), \quad x \in \partial \Omega_+,

D^2_0 v(x) = \frac{1}{2} [D^1_0 u(x) + D^1_0 u(x^*)] = \frac{1}{2} \bar{g}(x^*), \quad x \in \partial \Omega_-

Similarly, for function w(x) we obtain

\Delta^2 w(x) = \frac{1}{2} [\Delta^2 u(x) - \Delta^2 u(x^*)] = \frac{1}{2} [f(x) - f(x^*)] = f^- (x), \quad x \in \Omega,

D^m_0 w(x) = \frac{1}{2} [D^m_0 u(x) - D^m_0 u(x^*)] = \frac{1}{2} [g(x) - g(x^*)] = g^- (x), \quad x \in \partial \Omega,

D^2_0 w(x) = \frac{1}{2} [D^2_0 u(x) - D^2_0 u(x^*)] = \frac{1}{2} \bar{g}_2 (x), \quad x \in \partial \Omega_+,

D^2_0 w(x) = \frac{1}{2} [D^2_0 u(x) - D^2_0 u(x^*)] = \frac{1}{2} \bar{g}_2 (x^*), \quad x \in \partial \Omega_-

Note that if the function f(x) is a smooth enough function defined on the domain \( \Omega \), and function g(x) is defined on the sphere \( \partial \Omega \), then it is obvious that the functions \( f^\pm (x) \) and \( g^\pm (x) \) have the same properties. Moreover, if functions \( g_1(x) \) and \( g_2(x) \) are smooth on \( \partial \Omega \), then because of compatibility conditions (2.5), (2.6) the functions \( \bar{g}_1 (x) \) and \( \bar{g}_2 (x) \) have the same properties. Further, to study the solvability of the problems (6.6) and (6.7) we use the statements of Theorems 3.1 and 3.3.

1) if \( m = 1, \ell_1 = 2, \ell_2 = 3 \) then the necessary and sufficient conditions for solvability of the problems (6.6) and (6.7), respectively, are:

\left\{ \begin{array}{l}
\frac{1}{2} \int_\Omega (1 - |x|^2) f^+ (x) \, dx = \int_{\partial \Omega} \bar{g}_1^+ (x) - g^+ (x) \, dS_x, \\
\frac{1}{2} \int_\Omega ((n - 1)|x|^2 - (n - 3)) f^- (x) \, dx = \int_{\partial \Omega} \bar{g}_2^- (x) \, dS_x.
\end{array} \right. \tag{6.8}

From equalities (4.4), (4.6) and (4.9) we obtain

\left\{ \begin{array}{l}
\frac{1}{2} \int_\Omega (1 - |x|^2) f^+ (x) \, dx = \frac{1}{2} \int_\Omega (1 - |x|^2) f(x) \, dx.
\end{array} \right. \tag{6.9}

Similarly, from (4.4), (4.6) and (4.9), it follows that

\left\{ \begin{array}{l}
\frac{1}{2} \int_\Omega (1 - |x|^2) f^+ (x) \, dx = \frac{1}{2} \int_\Omega (1 - |x|^2) f(x) \, dx,
\int_{\partial \Omega} \bar{g}_1^+ (x) \, dS_x - \int_{\partial \Omega} \bar{g}_1^- (x) \, dS_x = \int_{\partial \Omega} g_1 (x) \, dS_x - \int_{\partial \Omega} g(x) \, dS_x.
\end{array} \right.

Consequently, condition (6.9) always holds and condition (6.8) can be rewritten in the form (6.1).

2) if \( m = 2, \ell_1 = 1, \ell_2 = 3 \), then the necessary and sufficient condition for solvability of the problem (6.6) has the form

\left\{ \begin{array}{l}
\frac{1}{2} \int_\Omega (1 - |x|^2) f^+ (x) \, dx = \int_{\partial \Omega} [g^+ (x) - \bar{g}_1^+ (x)] \, dS_x,
\end{array} \right.
which can be rewritten in the form
\[
\frac{1}{2} \int_{\Omega} (1 - |x|^2) f(x) \, dx = \int_{\partial \Omega} g(x) dS_x - \int_{\partial \Omega^+} g_1(x) dS_x.
\]
For problem (6.7) we obtain the conditions:
\[
\frac{1}{2} \int_{\Omega} [(n-1)|x|^2 - (n-3)] f^-(-x) \, dx = \int_{\partial \Omega} \tilde{g}_2^-(x) \, dS_x, \tag{6.10}
\]
\[
\frac{1}{2} \int_{\Omega} x_j [(n-1)|x|^2 - (n-3)] f^-(-x) \, dx = \int_{\partial \Omega} x_j \tilde{g}_2^-(x) - g_2^-(x) \, dS_x, \quad j = 1, 2, \ldots, n. \tag{6.11}
\]
From equalities (4.4) and (4.9), the condition (6.10) always holds. Further, using (4.12), (4.15) and (4.16) we have
\[
\frac{1}{2} \int_{\Omega} x_j [(n-1)|x|^2 - (n-3)] f^-(-x) \, dx = \frac{1 - \alpha_j}{2} \int_{\partial \Omega} x_j [(n-1)|x|^2 - (n-3)] f(x) \, dx,
\]
\[
\int_{\partial \Omega} x_j \tilde{g}_2^-(x) \, dS_x = \frac{1 - \alpha_j}{2} \int_{\partial \Omega^+} x_j g_2(x) \, dS_x,
\]
\[
\int_{\partial \Omega} x_j g^-(x) \, dS_x = \frac{1 - \alpha_j}{2} \int_{\partial \Omega} x_j g(x) \, dS_x.
\]
Then equality (6.11) can be rewritten in the form
\[
\frac{1 - \alpha_j}{4} \int_{\Omega} x_j [(n-1)|x|^2 - (n-3)] f(x) \, dx = \frac{1 - \alpha_j}{2} \left[ \int_{\partial \Omega^+} x_j g_2(x) \, dS_x - \int_{\partial \Omega} x_j g(x) \, dS_x \right], \quad j = 1, 2, \ldots, n. \tag{6.12}
\]
If for all \(1 \leq j \leq n-1\), \(\alpha_j = 1\) then condition (6.12) always holds for these indexes and in this case condition (6.11) for \(j = n\) can be rewritten in the form
\[
\frac{n-1}{2} \int_{\Omega} x_n |x|^2 f(x) \, dx - \frac{n-3}{2} \int_{\partial \Omega^+} x_n f(x) \, dx = \int_{\partial \Omega} x_n g_2(x) \, dS_x - \int_{\partial \Omega} x_n g(x) \, dS_x.
\]
If for some \(j_0 \in \{1, 2, \ldots, n\}\), \(\alpha_{j_0} = -1\), then for this \(j_0\), condition (6.11) can be rewritten in the form
\[
\frac{n-1}{2} \int_{\Omega} x_{j_0} |x|^2 f(x) \, dx - \frac{n-3}{2} \int_{\partial \Omega^+} x_{j_0} f(x) \, dx = \int_{\partial \Omega^+} x_{j_0} g_2(x) \, dS_x - \int_{\partial \Omega} x_{j_0} g(x) \, dS_x.
\]
(3) if \(m = 3, \ell_1 = 1, \ell_2 = 2\), then by Theorem 3.3 the problem’s solvability condition has the form
\[
\frac{1}{2} \int_{\Omega} [(n-1)|x|^2 - (n-3)] f^+(x) \, dx = \int_{\partial \Omega^+} g^+(x) \, dS_x.
\]
According to (4.3) and (4.5) the last condition can be rewritten in the form
\[
\frac{1}{2} \int_{\Omega} [(n-1)|x|^2 - (n-3)] f(x) \, dx = \int_{\partial \Omega} g(x) \, dS_x.
\]
Further, using Theorem 3.2 the solvability condition of problem (6.7) can be rewritten in the form
\[ \frac{1}{2} \int_{\Omega} [(n-1)|x|^2 - (n-3)]f^-(x)dx = \int_{\partial\Omega} g^-(x)dS_x, \]  
(6.13)
\[ \frac{1}{2} \int_{\Omega} x_j[(n-1)|x|^2 - (n-3)]f^-(x)dx 
= \int_{\partial\Omega} x_jg^-(x) - \tilde{g}_2(x)dS_x, \quad j = 1, 2, \ldots, n. \]  
(6.14)
From (4.4) and (4.6) it follows that condition (6.13) always holds. From (4.12), (4.15) and (6.14) we obtain
\[ \frac{1}{2} \int_{\Omega} x_j|\alpha_x|^2f(x)dx = -\frac{1-\alpha_j}{2} \int_{\Omega} x_j|\alpha|^2f(x)dx, \]
\[ \int_{\partial\Omega} x_jg^-(x)dS_x = -\frac{1-\alpha_j}{2} \int_{\partial\Omega} x_jg(x)dS_x, \]
\[ \int_{\partial\Omega} x_j\tilde{g}_2(x)dS_x = -\frac{1-\alpha_j}{2} \int_{\partial\Omega} x_jg_2(x)dS_x. \]
Then (6.14) can be rewritten as follows
\[ \frac{1}{4} \int_{\Omega} x_j[(n-1)|x|^2 - (n-3)]f(x)dx 
= \frac{1}{2} \left[ \int_{\partial\Omega} x_jg(x)dS_x - \int_{\partial\Omega} x_jg_2(x)dS_x \right], \quad j = 1, 2, \ldots, n. \]  
(6.15)
If \( \alpha_j = 1 \) then (6.15) holds, and if \( \alpha_j = -1 \) then this condition can be rewritten in the form
\[ \frac{n-1}{2} \int_{\Omega} x_j|\alpha_x|^2f(x)dx - \frac{n-3}{2} \int_{\Omega} x_jf(x)dx 
= \int_{\partial\Omega} x_jg(x)dS_x - \int_{\partial\Omega} x_jg_2(x)dS_x, \quad j = 1, 2, \ldots, n. \]
Thus equality (6.15) and, consequently, the theorem are proved. \[ \square \]

The following statement can be proved similarly to Theorem 6.1.

**Theorem 6.2.** Let \( k = 2 \) and the functions \( f(x), g(x), g_j(x), j = 1, 2 \) be smooth enough on the domains \( \bar{\Omega}, \partial\Omega \) and \( \partial\Omega_+ \), respectively, and the compatibility conditions (2.5) and (2.6) hold. Then the necessary and sufficient conditions for solvability of problem (2.7), (2.10) have the form:

1. If \( m = 1, \ell_1 = 2, \ell_2 = 3 \), then
\[ \frac{n-1}{2} \int_{\Omega} |x|^2f(x)dx - \frac{n-3}{2} \int_{\Omega} f(x)dx = \int_{\partial\Omega_+} g_2(x)dS_x; \]

2. If \( m = 2, \ell_1 = 1, \ell_2 = 3 \), then
\[ \frac{n-1}{2} \int_{\Omega} |x|^2f(x)dx - \frac{n-3}{2} \int_{\Omega} f(x)dx = \int_{\partial\Omega_+} g_2(x)dS_x, \]
and
\[ \frac{n-1}{2} \int_{\Omega} x_j|\alpha_x|^2f(x)dx - \frac{n-3}{2} \int_{\Omega} x_jf(x)dx = \int_{\partial\Omega_+} x_jg_2(x)dS_x - \int_{\partial\Omega} x_jg(x)dS_x. \]
for all $j \in \{1, 2, \ldots, n - 1\}$ for which $\alpha_j = 1$;

(3) if $m = 3$, $l_1 = 1$, $l_2 = 3$, then

$$\frac{n - 1}{2} \int_{\Omega} |x|^2 f(x) dx - \frac{n - 3}{2} \int_{\Omega} f(x) dx = \int_{\partial \Omega} g(x) dS_x,$$

and

$$\frac{n - 1}{2} \int_{\Omega} x_j |x|^2 f(x) dx - \frac{n - 3}{2} \int_{\Omega} x_j f(x) dx = \int_{\partial \Omega} x_j g(x) dS_x - \int_{\partial \Omega^+} x_j g_2(x) dS_x,$$

for all $j \in \{1, 2, \ldots, n - 1\}$ such that $\alpha_j = 1$.

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