LOCAL WELL-POSEDNESS FOR AN ERICKSEN-LESLIE’S PARABOLIC-HYPERBOLIC COMPRESSIBLE NON-ISOTHERMAL MODEL FOR LIQUID CRYSTALS

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Abstract. In this article we prove the local well-posedness for an Ericksen-Leslie’s parabolic-hyperbolic compressible non-isothermal model for nematic liquid crystals with positive initial density.

1. Introduction

We consider the following Ericksen-Leslie system modeling the hydrodynamic flow of compressible nematic liquid crystals \[1, 2, 3, 4, 5\]:

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla p(\rho, \theta) - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u &= -\nabla \cdot \left( \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 I_3 \right), \\
\partial_t (\rho e) + \text{div} (\rho e u) + p \text{div} u - \Delta \theta &= \frac{\mu}{2} |\nabla u + \nabla u^t|^2 + \lambda (\text{div} u)^2 + |\dot{d}|^2, \\
\ddot{d} - \Delta d &= d(|\nabla d|^2 - |d|^2), |d| = 1, \quad \text{in } \mathbb{R}^3 \times (0, \infty), \\
(\rho, u, \theta, d, d_1)(\cdot, 0) &= (\rho_0, u_0, \theta_0, d_0, d_1) \quad \text{in } \mathbb{R}^3, |d_0| = 1, \ d_0 \cdot d_1 = 0.
\end{align*}
\]

Here \(\rho, u, \theta\) is the density, velocity and temperature of the fluid, and \(d\) represents the macroscopic average of the nematic liquid crystals orientation field. \(e := C_V \theta\) is the internal energy and \(p := R \rho \theta\) is the pressure with positive constants \(C_V\) and \(R\). The viscosity coefficients \(\mu\) and \(\lambda\) of the fluid satisfy \(\mu > 0\) and \(\lambda + \frac{2}{3} \mu \geq 0\). The symbol \(\nabla d \odot \nabla d\) denotes a matrix whose \((i, j)\)th entry is \(\partial_i d \partial_j d\), \(I_3\) is the identity matrix of order 3, and it is easy to see that

\[
\text{div} \left( \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 I_3 \right) = -\sum_k \nabla d_k \Delta d_k, \quad \dot{d} := d_t + u \cdot \nabla d.
\]

\(u^t\) is the transpose of vector \(u\) and \(\partial_t u \equiv u_t\).

System \(1.1 - 1.3\) is the well-known full compressible Navier-Stokes-Fourier system. When \(u = 0\), \(1.4\) reduces to the wave maps system, which is one of the most beautiful and challenging nonlinear hyperbolic system. It has captured the

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attention of mathematicians for more than thirty years now. Moreover, the wave maps system is nothing other than the Euler-Lagrange system for the nonlinear sigma model, which is one of the fundamental problems in classical field theory.

When $\theta$ is a positive constant and the equation (1.4) is replaced by a harmonic heat flow

$$\dot{d} - \Delta d = d|\nabla d|^2,$$

(1.6)

this problem has received many studies. Huang, Wang and Wen [6, 7] (see also [8, 9]) show the local well-posedness of strong solutions with vacuum and prove some regularity criteria. Ding, Huang, Wen and Zi [10] (also see [11, 12]) studied the low Mach number limit. Jiang, Jiang and Wang [13] (see also [14]) proved the global existence of weak solutions in $\mathbb{R}^2$.

When the fluid is incompressible, i.e., $\text{div } u = 0$, the similar model has been studied in [15, 16].

The aim of this article is to prove a local-well posedness result when $\inf \rho_0 \geq 1/C$, we will prove the following result.

**Theorem 1.1.** Let $1/C \leq \rho_0 \leq C$, $0 \leq \theta_0$, $\nabla \rho_0 \in H^2$, $u_0, \theta_0, \dot{d}_0, \nabla d_0 \in H^3$, with $|d_0| = 1, d_0 \cdot d_1 = 0$. Then problem (1.1)-(1.5) has a unique strong solution $(\rho, u, \theta, d)$ satisfying

$$\frac{1}{C} \leq \rho \leq C, \quad 0 \leq \theta, \quad |d| = 1,$nabla\rho \in L^\infty(0, T; H^2), \quad u, \theta, \dot{d}, \nabla d \in L^\infty(0, T; H^3),$$

$$u, \theta \in L^2(0, T; H^4), \quad u_t, \theta_t \in L^2(0, T; H^2)$$

(1.7)

for some $T > 0$.

**Remark 1.2.** When $n = 2$ and taking $d := \begin{pmatrix} \cos \phi \\
\sin \phi \end{pmatrix}$, System (1.1)-(1.4) reduces to

$$\dot{\rho} = \text{div}(\rho u), \quad \dot{\theta} = \nabla p(\rho, \theta) - \mu \Delta u - (\lambda + \mu) \nabla \, \text{div } u$$

$$\dot{\phi} = \frac{\mu}{2} |\nabla u + \nabla u|^2 + \lambda (\text{div } u)^2 + |\dot{\phi}|^2,$$

And hence the well-known wave map

$$d_{tt} - \Delta d = d(|\nabla d|^2 - |d_t|^2)$$

reduces to the wave equation $\phi_{tt} - \Delta \phi = 0$.

**Remark 1.3.** Let $d$ be a smooth solution to the system (1.4) with the initial data $(d, d_t)(\cdot, 0) = (d_0, d_1)$, if the initial data $(d_0, d_1)$ obeys the conditions

$$|d_0| = 1, \quad d_0 \cdot d_1 = 0,$$

then we have $|d| = 1$ and $d \cdot d_t = 0$ for all times $t$.

**Proof of Remark 1.3** Denote $w := |d|^2 - 1$, multiplying (1.4) by $d$, we see that

$$\ddot{w} - \Delta w = 2w(|\nabla d|^2 - |d_t|^2).$$
Testing the above equation by \( \dot{w} \), we find that
\[
\frac{1}{2} \frac{d}{dt} \int (\dot{w}^2 + |\nabla w|^2) \, dx
= 2 \int w \dot{w}(|\nabla d|^2 - |\dot{d}|^2) \, dx + \int \Delta w (u \cdot \nabla w) \, dx - \int (u \cdot \nabla) \dot{w} \cdot \dot{w} \, dx
= 2 \int w \dot{w}(|\nabla d|^2 - |\dot{d}|^2) \, dx - \sum_i \int \partial_i u_i \partial_i w \, dx + \frac{1}{2} \int |\nabla w|^2 \text{div} u \, dx
+ \frac{1}{2} \int \dot{w}^2 \text{div} u \, dx
\leq C \int (w^2 + \dot{w}^2 + |\nabla w|^2) \, dx.
\]

On the other hand, we observe that
\[
\frac{1}{2} \frac{d}{dt} \int w^2 \, dx = \int (w \dot{w} - u \cdot \nabla w) \, dx \leq C \int (w^2 + \dot{w}^2 + |\nabla w|^2) \, dx.
\]

Combining the above two estimates and using the Gronwall inequality, we finish the proof. \( \Box \)

We denote
\[
M(t) := 1 + \sup_{0 \leq s \leq t} \left\{ \frac{1}{\rho} (s, s) \right\}_{L^\infty} + \|\rho(s, s)\|_{L^\infty} + \|\nabla \rho(s, s)\|_{H^2} + \|u(s, s)\|_{H^3}
+ \|\theta(s, s)\|_{H^3} + \|\dot{d}(s, s)\|_{H^3} + \|\nabla d(s, s)\|_{H^3}
+ \|u\|_{L^2(0, t; H^4)} + \|u_i\|_{L^2(0, t; H^2)} + \|\theta\|_{L^2(0, t; H^4)} + \|\theta_i\|_{L^2(0, t; H^2)}.
\]

**Theorem 1.4.** Let \( T^* \) be the maximal time of existence for problem (1.1)-(1.5) in the sense of Theorem 1.1. Then for any \( t \in [0, T^*) \), we have that
\[
M(t) \leq C_0 M(0) \exp(\sqrt{t} C(M(t)))
\]
for some given nondecreasing continuous functions \( C_0(\cdot) \) and \( C(\cdot) \).

It follows from (1.9) [17, 18, 19] that
\[
\sup_{0 \leq t \leq T} M(t) \leq C
\]
for some \( T \in (0, T^*) \).

In the proofs below, we will use the following bilinear commutator and product estimates due to Kato-Ponce [20]:
\[
\|D^s (fg) - fD^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}} \|D^{s-1} g\|_{L^{p_1}} + \|D^s f\|_{L^{p_2}} \|g\|_{L^{p_2}}),
\]
(1.11)
\[
\|D^s (fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}} \|D^s g\|_{L^{p_1}} + \|D^s f\|_{L^{p_2}} \|g\|_{L^{p_2}})
\]
(1.12)

with \( s > 0 \) and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( 1 < p < \infty \).

The proof of the uniqueness part is standard, we omit it here.

It is easy to prove Theorem 1.1 by the Galerkin method if we have (1.9) [6], thus we only need to show a priori estimates (1.9).
2. Proof of Theorem 1.4

Since the physical constants $C_V$ and $R$ do not bring any essential difficulties in our arguments, we shall take $C_V = R = 1$. First, it follows from (1.1) that

$$\rho(x, t) = \rho_0(y(0; x, t)) \exp \left\{ - \int_0^t \text{div} u(y(s; x, t), s) \, ds \right\}, \quad (2.1)$$

where $y(s; x, t)$ is the characteristic curve defined by

$$\frac{dy}{ds} = u(y, s), \quad y(t; x, t) = x.$$  

Then (2.1) gives

$$\rho, \frac{1}{\rho} \leq C_0 \exp(tC(M)). \quad (2.2)$$

Applying $\nabla$ to (1.1), testing by $\nabla \rho$, we see that

$$\frac{1}{2} \frac{d}{dt} \int |\nabla \rho|^2 \, dx = - \int \nabla \text{div}(\rho u) \nabla \rho \, dx \leq C(M),$$

which yields

$$||\nabla \rho(\cdot, t)||_{L^2} \leq C_0 + tC(M). \quad (2.3)$$

Applying $D^3$ to (1.1), testing by $D^3 \rho$, using (1.11) and (1.12), we find that

$$\frac{1}{2} \frac{d}{dt} \int (D^3 \rho)^2 \, dx$$

$$- \int (D^3(u\nabla \rho) - u \cdot \nabla D^3 \rho) D^3 \rho \, dx - \int u \cdot \nabla D^3 \rho \cdot D^3 \rho \, dx$$

$$\leq C(\|\nabla u\|_{L^\infty} \|D^3 \rho\|_{L^2} + \|\nabla \rho\|_{L^\infty} \|D^3 u\|_{L^2}) \|D^3 \rho\|_{L^2}$$

$$+ C(\|\rho\|_{L^\infty} \|D^3 \text{div} u\|_{L^2} + \|\text{div} u\|_{L^\infty} \|D^3 \rho\|_{L^2}) \|D^3 \rho\|_{L^2}$$

$$\leq C(M) + C(M) \|D^3 \text{div} u\|_{L^2},$$

which leads to

$$\|D^3 \rho(\cdot, t)\|_{L^2} \leq C + \sqrt{t}C(M). \quad (2.4)$$

It is easy to show that

$$\|u(\cdot, t)\|_{H^2} = \|u_0 + \int_0^t u_t \, ds\|_{H^2} \leq C_0 + \sqrt{t}C(M), \quad (2.5)$$

$$\|\theta(\cdot, t)\|_{H^2} \leq C_0 + \sqrt{t}C(M). \quad (2.6)$$

Testing (1.4) by $\dot{d}$ and using $d \cdot \dot{d} = 0$, we infer that

$$\frac{1}{2} \frac{d}{dt} \int (|\dot{d}|^2 + |\nabla d|^2) \, dx = \int u \cdot \nabla d \cdot \Delta d \, dx - \int (u \cdot \nabla) \dot{d} \cdot \dot{d} \, dx \leq C(M),$$

which implies

$$||\dot{d}(\cdot, t)||_{L^2} + ||\nabla d(\cdot, t)||_{L^2} \leq C_0 + tC(M). \quad (2.7)$$
Taking $D^3$ to \ref{E1.2}, testing by $D^3u$ and using \ref{E1.1}, we derive

\begin{align*}
\frac{1}{2} \frac{d}{dt} \int \rho |D^3u|^2 \, dx + \mu \int |\nabla D^3u|^2 \, dx + (\lambda + \mu) \int (\text{div}\, D^3u)^2 \, dx \\
= \int D^3 \rho \cdot \text{div}\, D^3u \, dx - \int (D^3(\rho u \cdot \nabla u) - \rho u \cdot \nabla D^3u) D^3u \, dx \\
- \int (D^3(\rho \partial_t u) - \rho D^3u_t) D^3u \, dx - \int (D^3(\nabla d \cdot \Delta d) - \nabla d \cdot \Delta D^3d) D^3u \, dx \\
= \int D^3u \cdot \nabla) d \cdot \Delta D^3d \, dx \\
=: I_1 + I_2 + I_3 + I_4 - I_5. \tag{2.8}
\end{align*}

Applying $D^3$ to \ref{E1.4} and testing by $D^3\dot{d}$, we obtain

\begin{align*}
\frac{1}{2} \frac{d}{dt} \int |D^3\dot{d}|^2 + |\nabla D^3d|^2 \, dx \\
= -\int (D^3(u \cdot \nabla \dot{d}) - u \cdot \nabla D^3d) D^3\dot{d} \, dx - \int (u \cdot \nabla) D^3\dot{d} \cdot D^3\dot{d} \, dx \\
+ \int D^3(d(|\nabla \dot{d}|^2 - |\dot{d}|^2)) D^3\dot{d} \, dx + \int \Delta D^3d \cdot (u \cdot \nabla D^3d) \, dx \\
+ \int \Delta D^3d \cdot (D^3(u \cdot \nabla d) - u \cdot \nabla D^3d - (D^3u \cdot \nabla d) \, dx + I_5 \\
=: \ell_1 + \ell_2 + \ell_3 + \ell_4 + \ell_5 + I_5. \tag{2.9}
\end{align*}

Summing \ref{E2.8} and \ref{E2.9}, we have

\begin{align*}
\frac{1}{2} \frac{d}{dt} \int (\rho |D^3u|^2 + |D^3\dot{d}|^2 + |\nabla D^3d|^2) \, dx \\
+ \mu \int |\nabla D^3u|^2 \, dx + (\lambda + \mu) \int (\text{div}\, D^3u)^2 \, dx \\
= \sum_{i=1}^{4} (I_i + \ell_i) + \ell_5. \tag{2.10}
\end{align*}

Using \ref{E1.11} and \ref{E1.12}, we bound $I_i$ ($i = 1, \ldots, 4$) and $\ell_i$ ($i = 1, \ldots, 5$) as follows.

\begin{align*}
I_1 &\leq C(\|\rho\|_{L^\infty} \|D^3\theta\|_{L^2} + \|	heta\|_{L^\infty} \|D^3\rho\|_{L^2}) \|\text{div}\, D^3u\|_{L^2} \leq C(M) \|\text{div}\, D^3u\|_{L^2}; \\
I_2 &\leq C(\|\nabla (\rho u)\|_{L^\infty} \|D^3u\|_{L^2} + \|\nabla u\|_{L^\infty} \|D^3(\rho u)\|_{L^2}) \|D^3u\|_{L^2} \leq C(M); \\
I_3 &\leq C(\|\nabla \rho\|_{L^\infty} \|D^2u\|_{L^2} + \|u_t\|_{L^\infty} \|D^3\rho\|_{L^2}) \|D^3u\|_{L^2} \leq C(M) \|u_t\|_{H^2}; \\
I_4 &\leq C(\|\nabla^2 d\|_{L^\infty} \|D^4d\|_{L^2}) \|D^3u\|_{L^2} \leq C(M); \\
\ell_1 &\leq C(\|\nabla u\|_{L^\infty} \|D^3\dot{d}\|_{L^2}^2 + C(\|\nabla \dot{d}\|_{L^\infty} \|D^3u\|_{L^2} \|D^3\dot{d}\|_{L^2} \leq C(M); \\
\ell_2 &\leq \frac{1}{2} \int |D^3\dot{d}|^2 \, dx \leq C(M); \\
\ell_3 &\leq C(\|d\|_{L^\infty} \|D^3(|\nabla d|^2 - |\dot{d}|^2)\|_{L^2} + (\|\nabla d\|_{L^\infty}^2 + \|d\|_{L^\infty}^2) \|D^3\dot{d}\|_{L^2} \|D^3\dot{d}\|_{L^2} \leq C(M); \\
\ell_4 &\leq \frac{1}{2} \int |D^3u|^2 \, dx \leq C(M); \\
\ell_5 &\leq C(\|\rho\|_{L^\infty} \|D^3u\|_{L^2} + \|ho\|_{L^\infty} \|D^3u\|_{L^2} \|D^3\dot{d}\|_{L^2} \|D^3\dot{d}\|_{L^2} \leq C(M). 
\end{align*}
\[ \ell_4 = \sum_{i, j} \int u_i \partial_i D^3 d \partial_j D^3 d \, dx \]
\[ = - \sum_{i, j} \int \partial_j u_i \partial_i D^3 d \partial_j D^3 d \, dx + \sum_{i, j} \frac{1}{2} \int \partial_j u_i \partial_j D^3 d \, dx \]
\[ \leq C \| \nabla u \|_{L^\infty} \| D^4 d \|_{L^2}^2 \leq C(M); \]
\[ \ell_5 = \int \Delta D^3 d (C_1 D^2 u \cdot \nabla Dd + C_2 D u \cdot \nabla D^2 d) \, dx \]
\[ = - \sum_i \int \partial_i D^3 d \partial_i (C_1 D^2 u \nabla Dd + C_2 D u \cdot \nabla D^2 d) \, dx \]
\[ \leq C \| D^4 d \|_{L^2} (\| D^3 u \|_{L^2} \| \nabla^2 d \|_{L^\infty} + \| D^2 u \|_{L^3} \| D^3 d \|_{L^6} + \| \nabla u \|_{L^\infty} \| D^4 d \|_{L^2}) \]
\[ \leq C(M). \]

Inserting the above estimates into (2.10), we have
\[ \frac{1}{2} \frac{d}{dt} \int (\rho |D^3 u|^2 + |D^3 d|^2 + |\nabla D^3 d|^2) \, dx \]
\[ + \mu \int |\nabla D^3 u|^2 \, dx + (\lambda + \mu) \int (\text{div } D^3 u)^2 \, dx \]
\[ \leq C(M) \| \text{div } D^3 u \|_{L^2} + C(M) + C(M) \| u_t \|_{H^2}. \] (2.11)

Integrating the above estimates in \([0, t]\), we arrive at
\[ \| D^3 u(\cdot, t) \|_{L^2}^2 + \| D^3 d(\cdot, t) \|_{L^2}^2 + \| \nabla D^3 d(\cdot, t) \|_{L^2}^2 + \int_0^t \int |D^4 u|^2 \, dx \, ds \]
\[ \leq C_0 \exp(\sqrt{t} C(M)). \] (2.12)

On the other hand, it follows from (1.2) that
\[ u_t = -u \cdot \nabla u + \frac{1}{\rho} \left[ \mu \Delta u + (\lambda + \mu) \nabla \text{div } u - \nabla p - \nabla \cdot (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I) \right] \]
which easily implies
\[ \| u_t \|_{L^2([0, t]; H^2)} \leq C_0 \exp(\sqrt{t} C(M)). \] (2.13)

Applying \( D^3 \) to (1.3), testing by \( D^3 \theta \) and using (1.1), (1.11), and (1.12), we have
\[ \frac{1}{2} \frac{d}{dt} \int \rho (D^3 \theta)^2 \, dx + \int |\nabla D^3 \theta|^2 \, dx \]
\[ = - \int (D^3 (\rho u \cdot \nabla \theta) - \rho u \cdot \nabla D^3 \theta) D^3 \theta \, dx - \int D^3 (\rho \text{div } u) \cdot D^3 \theta \, dx \]
\[ - \int (D^3 (\rho \theta_t) - \rho D^3 \theta_t) D^3 \theta \, dx \]
\[ + \int D^3 \left( \frac{\mu}{2} |\nabla u + \nabla u^t|^2 + \lambda (\text{div } u)^2 + |\partial d|^2 \right) D^3 \theta \, dx \]
\[ \leq C(\| \nabla (\rho u) \|_{L^\infty} \| D^3 \theta \|_{L^2} + \| \nabla \theta \|_{L^\infty} \| D^3 (\rho u) \|_{L^2}) \| D^3 \theta \|_{L^2} \]
\[ + C(p) \| L^\infty \| D^3 \text{div } u \|_{L^2} + \| \text{div } u \|_{L^\infty} \| D^3 p \|_{L^2} \| D^3 \theta \|_{L^2} \]
\[ + C(\| \nabla \rho \|_{L^\infty} \| D^2 \theta_t \|_{L^2} + \| \theta_t \|_{L^\infty} \| D^3 \rho \|_{L^2}) \| D^3 \theta \|_{L^2} \]
\[ + C(\| \nabla u \|_{L^\infty} \| D^4 u \|_{L^2} + \| \theta \|_{L^\infty} \| D^3 \theta \|_{L^2}) \| D^3 \theta \|_{L^2} \].
\leq C(M) + C(M)\|D^3 \text{div } u\|_{L^2} + C(M)\|\theta_t\|_{H^2} + C(M)\|D^4 u\|_{L^2},

which gives

\|D^3 \theta(\cdot, t)\|_{L^2} + \int_0^t \int |D^4 \theta|^2 \, dx \, ds \leq C_0 \exp(\sqrt{t}C(M)). \tag{2.14}

On the other hand, from (1.3) if follows that

\theta_t = -u \cdot \nabla \theta - \frac{p}{\rho} \text{div } u + \frac{1}{\rho} \left[ \frac{\mu}{2} |\nabla u + \nabla u^t|^2 \right] + \lambda (\text{div } u)^2 + |d|^2, \tag{2.15}

which easily leads to

\|\theta_t\|_{L^2(0, t; H^2)} \leq C_0 \exp(\sqrt{t}C(M)). \tag{2.16}

This completes the proof.

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