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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR NONLINEAR EQUATIONS INVOLVING THE SQUARE ROOT OF THE LAPLACIAN

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Abstract. This paper deals with the existence and multiplicity results for fractional problem involving the square root of the Laplacian $A_{1/2}$ in a bounded domain with zero Dirichlet boundary conditions by Morse theory and critical groups for a C^1 functional at both isolated critical points and infinity.

1. Introduction

This article concerns the existence and multiplicity of nontrivial weak solutions to nonlinear equations involving a non-local positive operator, the square root of the Laplacian in a bounded domain with zero Dirichlet boundary condition. Precisely, we study the fractional problem

$$\begin{gathered} A_{1/2}u = f(u) \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geqslant 2$, and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies the condition

(A1) $f(0) \equiv 0$ and there exist $a > 0$ and $1 \leqslant p < 2^\sharp := \frac{2N}{N-1}$ such that

$$|f(t)| \leqslant a(1 + |t|^{p-1}) \quad \text{for all } t \in \mathbb{R}.$$

According to [12], the operator $A_{1/2}$ is regarded as the square root of the Laplacian operator $-\Delta$ and is defined as follows. Let $\{\lambda_j, \varphi_j\}_{j=1}^\infty$ be the eigenvalues and the corresponding eigenfunctions of the Laplacian operator $-\Delta$ in Ω with zero Dirichlet boundary data on $\partial\Omega$; that is, $\int_\Omega \varphi_j\varphi_k dx = \delta_{j,k}$ and

$$\begin{gathered} -\Delta\varphi_j = \lambda_j\varphi_j \quad x \in \Omega, \\ \varphi_j = 0 \quad x \in \partial\Omega. \end{gathered} \tag{1.2}$$

For $u \in H_0^1(\Omega)$ with

$$u(x) = \sum_{j=1}^\infty \alpha_j\varphi_j(x), \quad x \in \Omega,$$

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the operator $A_{1/2}$ is defined by

$$A_{1/2}u := \sum_{j=1}^{\infty} \alpha_j \lambda_j^{1/2} \varphi_j.$$

It is proved in [12] that the operator $A_{1/2}$ is self-adjoint and positive definite and $\{\lambda_j^{1/2}, \varphi_j\}_{j=1}^{\infty}$ are the eigenvalues and the corresponding eigenfunctions of $A_{1/2}$ on Ω. Precisely, one has that

$$\begin{gathered} A_{1/2}\varphi_j = \mu_j \varphi_j \quad x \in \Omega, \\ \varphi_j = 0 \quad x \in \partial\Omega, \end{gathered} \tag{1.3}$$

here and in the sequel we denote by

$$\mu_j := \lambda_j^{1/2}, \quad j \in \mathbb{N} \tag{1.4}$$

the j-th eigenvalue of the operator $A_{1/2}$. The precise mathematical description and basic properties of the operator $A_{1/2}$ will be recalled in the next section.

It should be pointed out that the operator $A_{1/2}$ is different from the integro-differential operator $(-\Delta)^s$ with $s = 1/2$, where $(-\Delta)^s (0 < s < 1)$ is defined, up to a constant, as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N$$

and is the infinitesimal generators of Lévy stable diffusion processes (see [6]). In [33] the authors showed that the operator $A_{1/2}$ depends on the domain Ω considered, since its eigenfunctions and eigenvalues depend on Ω, while the integral one $(-\Delta)^{1/2}$ evaluated at some point is independent of the domain in which the equation is set. Besides, the eigenvalues and eigenfunctions of these two fractional operators behave quite different.

The fractions of the Laplacian, such as the square root of the Laplacian $A_{1/2}$ considered in the present paper, appear in flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics, anomalous diffusions in plasmas, and American options in finances (see [3, 24, 38]).

Nonlinear equations involving the fractional Laplacian have attracted much attention in the recent years. A lot of interest has been devoted to the fractional Laplacian problems with various nonlinearities in getting the existence, non-existence and regularity results as well as the qualitative properties, see [1, 4, 5, 9, 10, 11, 12, 13, 14, 17, 21, 22, 34, 36, 37, 40] and the references therein.

Through the Dirichlet-to-Neumann map due to Stein([35]) on Ω, Cabré and Tan in their well-known work [12] constructed a framework by transforming the nonlocal problem (1.1) to a local problem equivalently in the cylinder $\mathcal{C} = \Omega \times (0, \infty)$ with mixed boundary data which has variational structure so that the classical variational methods work well. Under such a framework from [12], the existence of a positive solution of (1.1) for $f(u) = |t|^{q-1}t$ with $1 < q < \frac{N+1}{N-1}$ was obtained in [12] by constrained minimization method, Tan studied in [36] the existence of a positive solution of (1.1) with critical nonlinearity case of $f(t) = \mu t + |t|^{\frac{2}{N-1}} t$ by the mountain pass theorem, and in [40] Nehari manifold method was applied to get the the existence of solutions and multiple solutions of (1.1) for $f(t) = \mu t + b(x)|t|^{q-1}t$ with $0 < q < \frac{N+1}{N-1}$ and sign-changing weight $b(x)$.

The aim of the present paper is to establish the existence and multiplicity results for (1.1) for general nonlinear function f with subcritical growth. The problem (1.1) has admitted a trivial solution $u = 0$ because $f(0) \equiv 0$, we are interested in finding nontrivial weak solutions of (1.1). The existence of nontrivial weak solutions of (1.1) depends mainly upon the behaviors of the nonlinear term f or its primitive $F(t) = \int_0^t f(s)ds$ near infinity and near zero.

Now we state the assumptions on the nonlinearity f in our context. Near infinity we make the following assumptions.

(A2) There exist $R > 0$ and $\theta > 2$ such that
$$0 < \theta F(t) \leqslant f(t)t \quad \text{for } |t| \geqslant R. \tag{1.5}$$

(A3) For some eigenvalue μ_m of $A_{1/2}$ with $m > 1$ there exist the limits
$$\lim_{|t|\to\infty} \frac{f(t)}{t} = \mu_m, \tag{1.6}$$
$$\lim_{|t|\to\infty} \pm(f(t)t - 2F(t)) = +\infty. \tag{1.7}$$

(A4) There exist $\mu < \mu_1$ and $C > 0$ such that
$$F(t) \leqslant \frac{1}{2}\mu t^2 + C \quad \text{for all } t \in \mathbb{R}. \tag{1.8}$$

(A5) There exist the limits
$$\lim_{|t|\to\infty} \frac{2F(t)}{t^2} = \mu_1, \tag{1.9}$$
$$\lim_{|t|\to\infty} (2F(t) - \mu_1 t^2) = -\infty. \tag{1.10}$$

Near the origin we make the following assumptions.

(A6) There exist $\delta > 0$ and $\tau \in (1, 2)$ such that
$$f(t)t > 0 \quad \text{for } 0 < |t| \leqslant \delta, \tag{1.11}$$
$$\tau F(t) - f(t)t \geqslant 0 \quad \text{for } |t| \leqslant \delta. \tag{1.12}$$

(A7) There exist $\delta > 0$ and $k \geqslant 1$ such that for two different adjacent eigenvalues $\mu_k < \mu_{k+1}$ of $A_{1/2}$, it holds that
$$\mu_k t^2 \leqslant 2F(t) \leqslant \mu_{k+1} t^2 \quad \text{for } |t| \leqslant \delta, \tag{1.13}$$

(A8) There exist $\delta > 0$ such that
$$2F(t) \leqslant \mu_1 t^2 \quad \text{for } |t| \leqslant \delta. \tag{1.14}$$

The main results of this article are the following theorems for equations driven by the square root of the Laplacian. The first theorem is related to the existence of one nontrivial weak solution of (1.1).

Theorem 1.1. *Assume* (A1). *Then problem* (1.1) *admits at least one nontrivial weak solution in each of the following cases:*

(a) (A2) *and* (A7),
(b) (A2) *and* (A8),
(c) (A3) *and* (A6),
(d) (A3) *and* (A8),
(e) (A4) *and* (A6),
(f) (A5) *and* (A6).

In the second theorem we establish the multiplicity of nontrivial solutions of (1.1).

Theorem 1.2. *Assume* (A1). *Then problem* (1.1) *admits at least two nontrivial weak solutions in each of the following cases:*

(a) (A4) *and* (A7),
(b) (A5) *and* (A7).

Now we give some remarks about the conditions presented above.

We first look at the conditions near infinity. The condition (A2) is the well-known Ambrosetti-Rabinowitz superquadratic condition at infinity introduced in the pioneering paper [2] and has been used extensively in the literature in dealing with superlinear variational problems. We mention a famous work [39] by Wang where the condition (A2) was exploited to describe the topological property of the energy functional at infinity and a third nontrivial solution for superlinear elliptic equations was obtained via Morse theory. In the condition (A3), (1.6) means that problem (1.1) is completely resonant at the eigenvalue μ_m of $A_{1/2}$ near infinity, while (1.7) is so-called the non-quadratic conditions (see [23]). We regard the condition (A4) as a weak version of sub-quadratic condition since it includes $\lim_{|t|\to\infty} 2F(t)/t^2 = 0$ or $\lim_{|t|\to\infty} f(t)/t = 0$ as the special case. The condition (A5) means problem (1.1) is resonant near infinity at μ_1 from the left side. In this case we use (1.10) that is weaker than the case $+$ in (1.7).

Next we look at the conditions near zero. The conditions (A6) means that the function f is superlinear near zero as which implies $\lim_{t\to 0} 2F(t)/t^2 = \infty$. This condition was introduced in [30] and a similar case was seen in [31]. The condition (A7) means that problem (1.1) is resonant near zero between two consecutive eigenvalues of $A_{1/2}$. This condition was first introduced in [27]. The condition (A8) means that problem (1.1) is resonant near zero at μ_1 from the left side.

Theorems 1.1 and 1.2 will be proved by applying the infinite dimensional Morse theory to the fractional framework. Because of the nonlocal feature of problem (1.1) on Ω, it is difficult for us to apply Morse theory directly. Instead, we apply Morse theory to an extended local problem in the cylinder $\mathcal{C}$ which is equivalent to (1.1) according to the framework built in [12]. By studying the variational functional corresponding to the extended local problem in the cylinder $\mathcal{C}$ with Morse theory and critical groups at zero and at infinity, we prove Theorem 1.1 for the existence of one nontrivial weak solutions of (1.1). The multiplicity result in Theorem 1.2 will be obtained by applying a three critical point theorem in [29].

The article is organized as follows. In Section 2, we present the functional space related to problem (1.1) together with the basic properties about the operator $A_{1/2}$. Then we recall some abstract results about Morse theory and critical groups. In Section 3, we satisfy the compactness of the functional and give the computations of critical groups at infinity. In Section 4, we compute the critical groups at zero. In Section 5, we give the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

In this section we will give the preliminaries for the variational settings related to problem (1.1) and some abstract results in Morse theory.

2.1. **Functional spaces and the operator $A_{1/2}$.** We first recall briefly the functional framework built in [12]. Denote the upper half space in $\mathbb{R}^{N+1}$ by

$$\mathbb{R}^{N+1}_+ = \{(x,y):\ x\in\mathbb{R}^N,\ y>0\},$$

and the half cylinder standing on Ω by $\mathcal{C}=\Omega\times(0,+\infty)$ and its lateral boundary by $\partial_L\mathcal{C}=\partial\Omega\times(0,\infty)$. Consider the Sobolev space of functions with trace vanishing on $\partial_L\mathcal{C}$:

$$H^1_{0,L}(\mathcal{C})=\Big\{v\in L^2(\mathcal{C}): v=0 \text{ on } \partial_L\mathcal{C},\ \int_{\mathcal{C}}|\nabla v|^2\,dx\,dy<\infty\Big\}.$$

Then $H^1_{0,L}(\mathcal{C})$ is a Hilbert space with the scalar product

$$\langle v,w\rangle=\int_{\mathcal{C}}\nabla v\nabla w\,dx\,dy$$

and the norm

$$\|v\|=\Big(\int_{\mathcal{C}}|\nabla v|^2\,dx\,dy\Big)^{1/2}.$$

From [12, Lemmas 2.4 and 2.5] we get the following embedding results:

Proposition 2.1. *The embedding from $H^1_{0,L}(\mathcal{C})$ into $L^q(\Omega)$ is continuous for all $q\in[1,\frac{2N}{N-1}]$ and is compact for all $q\in[1,\frac{2N}{N-1})$. Moreover, there is $c_q>0$ such that*

$$\Big(\int_{\Omega\times\{0\}}|v(x,0)|^q dx\Big)^{1/q}\leqslant c_q\Big(\int_{\mathcal{C}}|\nabla v|^2\,dx\,dy\Big)^{1/2}\quad \textit{for all } v\in H^1_{0,L}(\mathcal{C}). \tag{2.1}$$

Denote by tr_Ω the trace operator on $\Omega\times\{0\}$ for functions in $H^1_{0,L}(\mathcal{C})$:

$$\mathrm{tr}_\Omega\, v:=v(\cdot,0),\ \text{ for } v\in H^1_{0,L}(\mathcal{C}).$$

Let $\mathcal{V}_0(\Omega)$ be the space of all traces on $\Omega\times\{0\}$ of functions in $H^1_{0,L}(\mathcal{C})$; that is,

$$\mathcal{V}_0(\Omega):=\big\{u=\mathrm{tr}_\Omega\, v: v\in H^1_{0,L}(\mathcal{C})\big\}.$$

Then by [12, Lemma 2.10], $\mathcal{V}_0(\Omega)$ can be characterized as

$$\mathcal{V}_0(\Omega)=\Big\{u\in L^2(\Omega): u=\sum_{j=1}^\infty\alpha_j\varphi_j \text{ satisfies } \sum_{j=1}^\infty\alpha_j^2\lambda_j^{1/2}<+\infty\Big\} \tag{2.2}$$

and the space $H^1_{0,L}(\mathcal{C})$ can be characterized as (see the proof of[12, Lemma 2.10])

$$H^1_{0,L}(\mathcal{C})=\Big\{v\in L^2(\mathcal{C}): v(x,y)=\sum_{j=1}^\infty\alpha_j\varphi_j\exp(-\lambda_j^{1/2}y) \text{ with } \sum_{j=1}^\infty\alpha_j^2\lambda_j^{1/2}<+\infty\Big\}.$$

Where the pair $\{\lambda_j,\varphi_j\}_{j\in\mathbb{N}}$ are the eigenvalue and the corresponding eigenfunction of $-\Delta$ on Ω with zero boundary value on $\partial\Omega$, as stated in (1.2).

For a given function $u\in\mathcal{V}_0(\Omega)$, its harmonic extension v to the cylinder $\mathcal{C}$ is the weak solution of the problem

$$\begin{gathered}-\Delta v=0\quad\text{in } \mathcal{C},\\ v=0\quad\text{on } \partial_L\mathcal{C},\\ v=u\quad\text{on } \Omega\times\{0\}.\end{gathered} \tag{2.3}$$

The idea of the harmonic extension was introduced in the pioneering work of Caffarelli-Silvestre[16] where the fractional Laplacian in the whole space was dealt with.

The definition and properties of the operator $A_{1/2}$ are stated as follows.

Proposition 2.2 ([12]). *For $u=\sum_{j=1}^{\infty}\alpha_j\varphi_j\in\mathcal{V}_0(\Omega)$, there exists a unique harmonic extension v in $\mathcal{C}$ of u such that $v\in H^1_{0,L}(\mathcal{C})$, and it is given by the expansion*

$$v(x,y)=\sum_{j=1}^{\infty}\alpha_j\varphi_j(x)\exp(-\lambda_j^{1/2}y),\quad \text{for all } (x,y)\in\mathcal{C}. \tag{2.4}$$

The operator $A_{1/2}:\mathcal{V}_0(\Omega)\to\mathcal{V}_0^*(\Omega)$ is given by the Dirichlet-to-Neumann map

$$A_{1/2}u:=\frac{\partial v}{\partial\nu}\Big|_{\Omega\times\{0\}}, \tag{2.5}$$

where $\mathcal{V}_0^*(\Omega)$ is the dual space of $\mathcal{V}_0(\Omega)$ and where ν is the unit outer normal to $\mathcal{C}$ at $\Omega\times\{0\}$. We have

$$A_{1/2}u=\sum_{j=1}^{\infty}\alpha_j\lambda_j^{1/2}\varphi_j, \tag{2.6}$$

and that $A_{1/2}\circ A_{1/2}$ is equal to $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$. The inverse $A_{1/2}^{-1}$ is the unique positive square root of the inverse Laplacian $(-\Delta)^{-1}$ in Ω with zero Dirichlet boundary values on $\partial\Omega$.

Now we consider the linear eigenvalue problem

$$\begin{gathered} A_{1/2}u=\mu u\quad\text{in }\Omega,\\ u=0\quad\text{on }\partial\Omega. \end{gathered} \tag{2.7}$$

By the definition of $A_{1/2}$, we see that a nontrivial function $u\in\mathcal{V}_0(\Omega)$ is an eigenfunction associated to the eigenvalue μ if and only if the harmonic extension v of u to the cylinder $\mathcal{C}$ satisfies

$$\begin{gathered} -\Delta v=0\quad\text{in }\mathcal{C},\\ v=0\quad\text{on }\partial_L\mathcal{C},\\ \frac{\partial v}{\partial\nu}=\mu u\quad\text{on }\Omega\times\{0\}. \end{gathered} \tag{2.8}$$

We have that $\{\lambda_j^{1/2},\varphi_j\}_{j\in\mathbb{N}}$ are the eigenvalues and the corresponding eigenfunctions of (2.7) (see [12, Lemma 2.13]). Setting

$$\mu_j=\lambda_j^{1/2}\quad\text{and}\quad e_j(x,y)=\varphi_j(x)\exp(-\mu_j y)\quad\text{for all } j\in\mathbb{N}. \tag{2.9}$$

Then all the pairs $\{\mu_j,e_j\}_{j\in\mathbb{N}}$ satisfy (2.8): for all $j\in\mathbb{N}$,

$$\begin{gathered} -\Delta e_j=0\quad\text{in }\mathcal{C},\\ e_j=0\quad\text{on }\partial_L\mathcal{C},\\ \frac{\partial e_j}{\partial\nu}=\mu_j e(\cdot,0)=\mu_j\varphi_j\quad\text{on }\Omega\times\{0\}, \end{gathered} \tag{2.10}$$

The eigenfunction sequence $\{e_j\}_{j\in\mathbb{N}}$ forms an orthogonal basis of $H^1_{0,L}(\mathcal{C})$. The eigenvalue sequence $\{\mu_j\}_{j\in\mathbb{N}}$ has the following variational characterizations:

$$\mu_1=\min_{v\in H^1_{0,L}(\mathcal{C})\setminus\{0\}}\frac{\int_{\mathcal{C}}|\nabla v|^2\,dx\,dy}{\int_\Omega|v(x,0)|^2dx}=\int_{\mathcal{C}}|\nabla e_1|^2\,dx\,dy, \tag{2.11}$$

and

$$\mu_j = \min_{v\in\mathbb{P}_j\setminus\{0\}} \frac{\int_{\mathcal{C}} |\nabla v|^2\,dx\,dy}{\int_\Omega |v(x,0)|^2 dx} = \int_{\mathcal{C}} |\nabla e_j|^2\,dx\,dy,$$

where

$$\mathbb{P}_j = \big\{v \in H^1_{0,L}(\mathcal{C}) : \langle v, e_i\rangle = 0 \text{ for } i = 1,2,\dots,j-1\big\}.$$

Moreover, μ_1 is simple and $0 < \mu_1 < \mu_2 \leqslant \dots \leqslant \mu_j \leqslant \dots \to \infty$ as $j \to \infty$, and that each μ_j has finite multiplicity. For $j \in \mathbb{N}$, let ℓ_j be the multiplicity of μ_j; that is,

$$\mu_{j-1} < \mu_j = \mu_{j+1} = \dots = \mu_{j+\ell_j-1} < \mu_{j+\ell_j}.$$

Set

$$\begin{gathered} H^-(\mu_j) = \operatorname{span}\{e_1,\dots,e_{j-1}\}, \quad H(\mu_j) = \operatorname{span}\{e_j,\dots,e_{j+\ell_j-1}\}, \\ H^+(\mu_j) = \overline{\operatorname{span}\{e_{j+\ell_j}, e_{j+\ell_j+1},\dots,\}} = \big[H^-(\mu_j)\oplus H(\mu_j)\big]^{\perp}. \end{gathered}$$

Then

$$H^1_{0,L}(\mathcal{C}) = H^-(\mu_j)\oplus H(\mu_j)\oplus H^+_j(\mu_j). \tag{2.12}$$

Proposition 2.3. *The following variational inequalities hold:*

$$\begin{gathered} \int_{\mathcal{C}} |\nabla v|^2\,dx\,dy \leqslant \mu_{j-1}\int_\Omega |v(x,0)|^2 dx \quad \textit{for all } v \in H^-(\mu_j), \\ \int_{\mathcal{C}} |\nabla v|^2\,dx\,dy = \mu_j\int_\Omega |v(x,0)|^2 dx, \quad \textit{for all } v \in H(\mu_j), \\ \int_{\mathcal{C}} |\nabla v|^2\,dx\,dy \geqslant \mu_{j+\ell_j}\int_\Omega |v(x,0)|^2 dx. \quad \textit{for all } v \in H^+(\mu_j). \end{gathered}$$

2.2. **Extended problem, weak solutions and variational formula.** With the preliminaries in the previous subsection at hand, we turn to problem (1.1). We say that a function $u \in \mathcal{V}_0(\Omega)$ is a *weak* solution of (1.1) if the function $v \in H^1_{0,L}(\mathcal{C})$ with $\operatorname{tr}_\Omega v = v(\cdot,0) = u$ *weakly* solves the extended problem

$$\begin{gathered} -\Delta v = 0 \quad \text{in } \mathcal{C}, \\ v = 0 \quad \text{on } \partial_L\mathcal{C}, \\ \frac{\partial v}{\partial\nu} = f(v(\cdot,0)) \quad \text{on } \Omega\times\{0\}, \end{gathered} \tag{2.13}$$

that is the function v satisfies the variational formula

$$\int_{\mathcal{C}} \nabla v\nabla\phi\,dx\,dy = \int_\Omega f(v(x,0))\phi(x,0)dx \quad \text{for all } \phi \in H^1_{0,L}(\mathcal{C}). \tag{2.14}$$

Observe that the extended problem (2.13) has a variational structure, indeed, it is the Euler-Lagrange equation of the functional $\mathcal{J} : H^1_{0,L}(\mathcal{C}) \to \mathbb{R}$ defined by

$$\mathcal{J}(v) = \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2\,dx\,dy - \int_\Omega F(v(x,0))dx, \quad v \in H^1_{0,L}(\mathcal{C}). \tag{2.15}$$

Since the nonlinear function f satisfies the subcritical growth condition (A1), by Proposition 2.1, the functional $\mathcal{J}$ is well-defined on $H^1_{0,L}(\mathcal{C})$ and is of class C^1 with derivative given by

$$\langle \mathcal{J}'(v),\phi\rangle = \int_{\mathcal{C}} \nabla v\nabla\phi\,dx\,dy - \int_\Omega f(v(x,0))\phi(x,0)dx. \tag{2.16}$$

Therefore critical points of $\mathcal{J}$ are exactly weak solutions of (2.13) and then the traces of critical points of $\mathcal{J}$ are exactly weak solutions to problem (1.1).

We will apply Morse theory and critical groups to find critical points of $\mathcal{J}$.

2.3. **Preliminaries about Morse theory.** In this subsection we collect some results on Morse theory for a C^1 functional $\mathcal{J}$ defined on a Hilbert space E.

Let $\mathcal{J} \in C^1(E,\mathbb{R})$. Denote for $c \in \mathbb{R}$

$$\mathcal{J}^c = \{z \in E : \mathcal{J}(z) \leqslant c\}, \quad \mathcal{K}_c = \{z \in E : \mathcal{J}'(z) = 0,\ \mathcal{J}(z) = c\}.$$

We say that the functional $\mathcal{J}$ possesses the deformation property at the level $c \in \mathbb{R}$ if for any $\bar{\epsilon} > 0$ and any neighborhood $\mathcal{N}$ of $\mathcal{K}_c$, there are $\epsilon > 0$ and a continuous deformation $\zeta : [0,1] \times E \to E$ such that

(i) $\zeta(z,t) = z$ for either $t = 0$ or $z \notin \mathcal{J}^{-1}([c-\bar{\epsilon}, c+\bar{\epsilon}])$;
(ii) $\mathcal{J}(\zeta(t,z))$ is nonincreasing in t for any $z \in E$;
(iii) $\zeta(\mathcal{J}^{c+\epsilon} \setminus \mathcal{N}) \subset \mathcal{J}^{c-\epsilon}$.

We say that $\mathcal{J}$ possesses the deformation property if $\mathcal{J}$ possesses the deformation property at all $c \in \mathbb{R}$.

We say that $\mathcal{J}$ satisfies the Palais-Smale condition at the level $c \in \mathbb{R}$ if any sequence $\{z_n\} \subset E$ satisfying $\mathcal{J}(z_n) \to c$ and $\mathcal{J}'(z_n) \to 0$ as $n \to \infty$ has a convergent subsequence. We say that $\mathcal{J}$ satisfies the the Palais-Smale condition condition if $\mathcal{J}$ satisfies the Palais-Smale condition at each $c \in \mathbb{R}$.

We say that $\mathcal{J}$ satisfies the Cerami condition at the level $c \in \mathbb{R}$ if any sequence $\{z_n\} \subset E$ such that $\mathcal{J}(z_n) \to c$ and $(1+\|z_n\|)\|\mathcal{J}'(z_n)\| \to 0$ as $n \to \infty$ has a convergent subsequence. We say that $\mathcal{J}$ satisfies the Cerami condition if $\mathcal{J}$ satisfies the Cerami condition at any $c \in \mathbb{R}$.

We note that if $\mathcal{J}$ satisfies the Palais-Smale condition or the Cerami condition then $\mathcal{J}$ possesses the deformation property (see [20, 8]).

Let $\mathcal{K} = \{z \in E : \mathcal{J}'(z) = 0\}$. Assume that $\mathcal{J}(\mathcal{K})$ is bounded from below by $a \in \mathbb{R}$ and $\mathcal{J}$ possesses the deformation property at all $c \leqslant a$. The group $C_q(\mathcal{J},\infty) := H_q(E, \mathcal{J}^a), q \in \mathbb{Z}$, is called the q-th critical group of $\mathcal{J}$ at infinity ([8]), where $H_*(A,B)$ denotes a singular relative homology group of the pair (A,B) with integer coefficients.

Let z_0 be an isolated critical point of $\mathcal{J}$ with $\mathcal{J}(z_0) = c \in \mathbb{R}$, and U be a neighborhood of z_0 such that $U \cap \mathcal{K} = \{z_0\}$. The group $C_q(\mathcal{J}, z_0) := H_q(\mathcal{J}^c \cap U, \mathcal{J}^c \cap U \setminus \{z_0\}), q \in \mathbb{Z}$, is called the q-th critical group of $\mathcal{J}$ at z_0.

Assume that $\mathcal{J}$ possesses the deformation property and $\mathcal{K}$ is a finite set. We have the following basic facts from Morse theory (see [19, 32, 8]). If $\mathcal{K} = \emptyset$ then $C_q(\mathcal{J},\infty) \cong 0$ for all $q \in \mathbb{Z}$. Thus if $C_q(\mathcal{J},\infty) \ncong 0$ for some $q \in \mathbb{Z}$ then $\mathcal{K} \neq \emptyset$. Assume that $0 \in \mathcal{K}$. If $\mathcal{K} = \{0\}$ then $C_q(\mathcal{J},\infty) \cong C_q(\mathcal{J},0)$ for all $q \in \mathbb{Z}$. Thus if $C_q(\mathcal{J},\infty) \ncong C_q(\mathcal{J},0)$ for some $q \in \mathbb{Z}$ then $\mathcal{J}$ must have a critical point differing from 0. Therefore the basic idea in applying Morse theory to find nonzero critical points of $\mathcal{J}$ is to compute critical groups both at infinity and at 0.

The critical group $C_q(\mathcal{J},\infty)$ can be computed partially when $\mathcal{J}$ has a saddle point geometry at infinity.

Proposition 2.4 ([8]). *Let E be a Hilbert space such that $E = V_\infty \oplus W_\infty$ with $\ell = \dim V_\infty < \infty$. Let $\mathcal{J} \in C^1(E,\mathbb{R})$ possess the deformation property. Suppose that $\mathcal{J}$ satisfies*

(i) $\inf_{z \in W_\infty} \mathcal{J}(u) > -\infty$;
(ii) $\mathcal{J}(z) \to -\infty$ *as* $\|z\| \to \infty$, $z \in V_\infty$.

Then $C_\ell(\mathcal{J},\infty) \ncong 0$.

The critical group $C_q(\mathcal{J},0)$ can be computed partially when $\mathcal{J}$ has a local linking structure at zero.

Proposition 2.5 ([28]). *Let E be a Hilbert space such that $E = V_0 \oplus W_0$ with $\ell_0 = \dim V_0 < \infty$. Let $\mathcal{J} \in C^1(E,\mathbb{R})$ possess the deformation property. Assume that $\mathcal{J}$ has an isolated critical point $z = 0$ with $\mathcal{J}(0) = 0$. If $\mathcal{J}$ has a local linking at 0 with respect to $E = V_0 \oplus W_0$ where $\ell_0 = \dim V_0 < \infty$ i.e., there exists $\rho > 0$ small such that*

$$\mathcal{J}(z) \leqslant 0, \quad z \in V_0,\ \|z\| \leqslant \rho, \quad \mathcal{J}(z) > 0,\ z \in W_0,\ 0 < \|z\| \leqslant \rho. \tag{2.17}$$

Then $C_{\ell_0}(\mathcal{J},0) \ncong 0$.

In this subsection we recall a very general version of the famous three critical point theorem.

Proposition 2.6 ([29]). *Let E be a Hilbert space and let $\mathcal{J} \in C^1(E,\mathbb{R})$ possess the deformation property and be bounded from below. Assume that $\mathcal{J}$ has an isolated critical point $z_* \in E$ such that*

(i) *z_* is homological nontrivial, i.e., $C_q(\mathcal{J},z_*) \ncong 0$ for some $q \in \mathbb{Z}$,*
(ii) *z_* is not the global minimizer of $\mathcal{J}$.*

Then $\mathcal{J}$ has at least three critical points.

We point out that the all above results on Morse theory are valid for E being a Banach space.

3. Compactness and critical groups at infinity

We will prove Theorems 1.1 and 1.2 by studying the functional $\mathcal{J}$ defined by (2.15):

$$J(v) = \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2\,dx\,dy - \int_{\Omega} F(v(x,0))dx, \quad v \in H^1_{0,L}(\mathcal{C}).$$

We will use Morse theory and critical groups computations to get the existence of nontrivial critical points of $\mathcal{J}$. First of all we study the bounded compactness of $\mathcal{J}$. We have the following result.

Lemma 3.1. *Let f satisfy* (A1). *Then any bounded sequence $\{v_n\} \subset H^1_{0,L}(\mathcal{C})$ such that*

$$\mathcal{J}'(v_n) \to 0 \quad \text{in } (H^1_{0,L}(\mathcal{C}))^* \text{ as } n \to \infty \tag{3.1}$$

has a convergent subsequence.

Proof. Let $\{v_n\} \subset H^1_{0,L}(\mathcal{C})$ be bounded and satisfy (3.1). Since $H^1_{0,L}(\mathcal{C})$ is a Hilbert space and then is reflexive, there is a subsequence of $\{v_n\}$, it is still denoted by $\{v_n\}$, and there exists $v^* \in H^1_{0,L}(\mathcal{C})$, such that

$$v_n \rightharpoonup v^* \text{ weakly in } H^1_{0,L}(\mathcal{C}) \text{ as } n \to \infty. \tag{3.2}$$

By Proposition 2.1, up to a subsequence, it holds

$$\begin{gathered} \operatorname{tr}_\Omega v_n \to \operatorname{tr}_\Omega v^* \quad \text{strongly in } L^q(\Omega) \quad \forall q \in [1, 2^\sharp), \\ v_n(x,0) \to v^*(x,0) \quad \text{a.e. in } \Omega \end{gathered} \tag{3.3}$$

as $n \to \infty$, and there exists $\kappa_q \in L^q(\Omega)$ such that

$$|v_n(x,0)| \leqslant \kappa_q(x) \quad \text{a.e. in } \Omega \text{ for any } n \in \mathbb{N}. \tag{3.4}$$

By (A1), (3.3), (3.4) and the Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_\Omega f(v_n(x,0))v_n(x,0)dx = \int_\Omega f(v^*(x,0))v^*(x,0)dx, \tag{3.5}$$

$$\lim_{n\to\infty}\int_\Omega f(v_n(x,0))v^*(x,0)dx = \int_\Omega f(v^*(x,0))v^*(x,0)dx. \tag{3.6}$$

Since $\{v_n\}$ is bounded, by (3.1) we have

$$\langle \mathcal{J}'(v_n), v_n\rangle = \|v_n\|^2 - \int_\Omega f(v_n(x,0))v_n(x,0)dx \to 0 \quad \text{as } n\to\infty. \tag{3.7}$$

Consequently, from (3.5)and (3.7) we deduce that

$$\lim_{n\to\infty}\|v_n\|^2 = \int_\Omega f(v^*(x,0))v^*(x,0)dx. \tag{3.8}$$

Furthermore, using (3.1) again, we have

$$\langle \mathcal{J}'(v_n), v^*\rangle = \langle v_n, v^*\rangle - \int_\Omega f(v_n(x,0))v^*(x,0)dx \to 0, \quad \text{as } n\to\infty. \tag{3.9}$$

By (3.2), (3.6)–(3.9) we obtain

$$\|v^*\|^2 = \int_\Omega f(v^*(x,0))v^*(x,0)dx. \tag{3.10}$$

Thus, (3.8) and (3.10) give that

$$\lim_{n\to\infty}\|v_n\|^2 = \|v^*\|^2.$$

Finally we have that

$$\|v_n - v^*\|^2 = \|v_n\|^2 + \|v^*\|^2 - 2\langle v_n, v^*\rangle \to 0 \ \text{ as } n\to\infty.$$

The proof is complete. □

Next we prove the compactness of the functional $\mathcal{J}$ and compute the critical groups of $\mathcal{J}$ at infinity. We will use $C_i > 0$ to denote various constants independent of the functions in $H^1_{0,L}(\Omega)$.

Lemma 3.2. *Assume* (A1) *and* (A2).

(i) *The functional $\mathcal{J}$ satisfies the Palais-Smale condition.*
(ii) *$C_q(\mathcal{J},\infty)\cong 0$ for all $q\in\mathbb{Z}$.*

Proof. (i) Let $\{v_n\}\subset H^1_{0,L}(\mathcal{C})$ be such that $\{\mathcal{J}(v_n)\}$ is bounded from above by some $C_1 > 0$ for all $n\in\mathbb{N}$ and

$$\mathcal{J}'(v_n)\to 0 \quad \text{in } (H^1_{0,L}(\mathcal{C}))^* \text{ as } n\to\infty. \tag{3.11}$$

By Lemma 3.1 we only need to show that $\{v_n\}$ is bounded in $H^1_{0,L}(\mathcal{C})$. Now it follows from (A1) and (A2) that for n large,

$$
\begin{aligned}
&\theta C_1 + \|v_n\| \\
&\geqslant \theta\mathcal{J}(v_n) - \langle \mathcal{J}'(v_n), v_n\rangle \\
&= \frac{\theta-2}{2}\|v_n\|^2 - \int_\Omega \Big(\theta F(v_n(x,0)) - f(v_n(x,0))v_n(x,0)\Big)dx \\
&= \frac{\theta-2}{2}\|v_n\|^2 - \int_{\{|v_n(x,0)|\geqslant R\}} \Big(\theta F(v_n(x,0)) - f(v_n(x,0))v_n(x,0)\Big)dx \\
&\quad - \int_{\{|v_n(x,0)|<R\}} \Big(\theta F(v_n(x,0)) - f(v_n(x,0))v_n(x,0)\Big)dx \\
&\geqslant \frac{\theta-2}{2}\|v_n\|^2 - \int_{\{|v_n(x,0)|\leqslant R\}} |\theta F(v_n(x,0)) - f(v_n(x,0))v_n(x,0)|dx \\
&\geqslant \frac{\theta-2}{2}\|v_n\|^2 - C_2
\end{aligned}
\tag{3.12}
$$

where

$$C_2 = |\Omega| \sup_{|t|\leqslant R} |\theta F(t) + f(t)t|.$$

Since $\theta > 2$, it follows from (3.12) that $\{v_n\}$ is bounded in $H^1_{0,L}(\mathcal{C})$. By Lemma 3.1 one sees that $\{v_n\}$ has a convergent subsequence .

(ii) Denote $B_1 = \{v \in H^1_{0,L}(\mathcal{C}) : \|v\| \leqslant 1\}$. By (1.5), we deduce that there is $C_3 > 0$ such that

$$F(t) \geqslant C_3|t|^\theta, \quad \text{for all } |t| \geqslant R. \tag{3.13}$$

For $v \in \partial B_1 = \{v \in H^1_{0,L}(\mathcal{C}) : \|v\| = 1\}$ and $\eta > 0$, we have

$$
\begin{aligned}
\mathcal{J}(\eta v) &= \frac{1}{2}\eta^2 \int_{\mathcal{C}} |\nabla v|^2\,dx\,dy - \int_\Omega F(\eta v(x,0))dx \\
&= \frac{1}{2}\eta^2 - C_3\int_{\{|\eta v(x,0)|\geqslant R\}} |\eta v(x,0)|^\theta dx + \int_{|\eta v(x,0)|<R} |F(\eta v(x,0))|dx \\
&\leqslant \frac{1}{2}\eta^2 - C_3\int_\Omega |\eta v(x,0)|^\theta dx + C_3\int_{\{|\eta v(x,0)|<R\}} |\eta v(x,0)|^\theta dx \\
&\quad + \int_{\{|\eta v(x,0)|<R\}} |F(\eta v(x,0))|dx \\
&\leqslant \frac{1}{2}\eta^2 - C_3\eta^\theta \|\operatorname{tr}_\Omega v\|^\theta_{L^\theta(\Omega)} + C_4
\end{aligned}
$$

where

$$C_4 = |\Omega|(C_3R^\theta + \sup_{|t|\leqslant R}|F(t)|).$$

Since $\theta > 2$, it follows that

$$\lim_{\eta\to+\infty} \mathcal{J}(\eta v) = -\infty. \tag{3.14}$$

For $v \in \partial B_1$ and $\eta > 0$, by (1.5) we have

$$\frac{d}{d\eta}\mathcal{J}(\eta v) = \langle \mathcal{J}'(\eta v), v\rangle$$

$$
\begin{aligned}
&= \eta \int_{\mathcal{C}} |\nabla v|^2 \, dx \, dy - \int_{\Omega} f(\eta v(x,0)) v(x,0) dx \\
&= \frac{1}{\eta}\Big(2\mathcal{J}(\eta v) + \int_{\Omega} (2F(\eta v(x,0)) - f(\eta v(x,0))\eta v(x,0)) dx\Big) \\
&\leqslant \frac{1}{\eta}\Big(2\mathcal{J}(\eta v) + \int_{\{|\eta v(x,0)|\leqslant R\}} (2F(\eta v(x,0)) - f(\eta v(x,0))\eta v(x,0)) dx\Big) \\
&\leqslant \frac{1}{\eta}(2\mathcal{J}(\eta v) + C_5),
\end{aligned}
$$

where

$$
C_5 = |\Omega| \sup_{|t|\leqslant R} (2|F(t)| + R|f(t)|).
$$

Therefore, for any a fixed $a < -C_5/2$,

$$
\mathcal{J}(\eta v) \leqslant a \;\Rightarrow\; \frac{d}{d\eta}\mathcal{J}(\eta v) < 0. \tag{3.15}
$$

Since $\mathcal{J}(0) = 0$, it follows from (3.14) and (3.15) that for any $v \in \partial B_1$, there is a unique $\eta(v) > 0$ such that

$$
\mathcal{J}(\eta(v)v) = a, \quad v \in \partial B_1. \tag{3.16}
$$

By (3.16) and the Implicit Function Theorem we have that $\eta \in C(\partial B_1, \mathbb{R})$. Now we define

$$
\pi(v) = \begin{cases} 1, & \text{if } \mathcal{J}(v) \leqslant a, \\ \|v\|^{-1}\eta(\|v\|^{-1}v), & \text{if } \mathcal{J}(v) > a, \ v \neq 0. \end{cases}
$$

Then $\pi \in C(H^1_{0,L}(\mathcal{C}) \setminus \{0\}, \mathbb{R})$. Define the mapping $\xi : [0,1] \times H^1_{0,L}(\mathcal{C}) \setminus \{0\} \to H^1_{0,L}(\mathcal{C}) \setminus \{0\}$ by

$$
\xi(\sigma, v) = (1-\sigma)v + \sigma\pi(v)v.
$$

It is easy to see that ξ is continuous. For all $v \in H^1_{0,L}(\mathcal{C}) \setminus \{0\}$ with $\mathcal{J}(v) > a$, by (3.16),

$$
\mathcal{J}(\xi(1,v)) = \mathcal{J}(\pi(v)v) = \mathcal{J}(\eta(\|v\|^{-1}v)\|v\|^{-1}v) = a.
$$

Therefore $\xi(1,v) \in \mathcal{J}^a$ for all $v \in H^1_{0,L}(\mathcal{C}) \setminus \{0\}$, and $\xi(\sigma, v) = v$ for all $\sigma \in [0,1]$, $v \in \mathcal{J}^a$. Then $\mathcal{J}^a$ is a strong deformation retract of $H^1_{0,L}(\mathcal{C}) \setminus \{0\}$. It follows that

$$
\begin{aligned}
C_q(\mathcal{J}, \infty) &= H_q(H^1_{0,L}(\mathcal{C}), \mathcal{J}^a) \\
&\cong H_q(H^1_{0,L}(\mathcal{C}), H^1_{0,L}(\mathcal{C}) \setminus \{0\}) \\
&\cong H_q(B_1, \partial B_1) \cong 0, \quad q \in \mathbb{Z},
\end{aligned}
$$

since ∂B_1 is contractible which follows from $\dim H^1_{0,L}(\mathcal{C}) = \infty$. The proof is complete. □

We remark here that the idea for computing $C_q(\mathcal{J}, \infty) \cong 0$ is essentially from the famous paper [39] where superlinear Laplacian problems were studied. We use this idea for superlinear problems involved with the square root of Laplacian.

Lemma 3.3. *Assume* (A3). *Then*

(i) *the functional $\mathcal{J}$ satisfies the Cerami condition.*
(ii) (A3) *with $+$ in* (1.7) *implies $C_{\ell_\infty}(\mathcal{J}, \infty) \ncong 0$, where $\ell_\infty = \dim H^-(\mu_m)$.*
(iii) (A3) *with $-$ in* (1.7) *implies $C_{\ell^*_\infty}(\mathcal{J}, \infty) \ncong 0$, where $\ell^*_\infty = \dim \big[H^-(\mu_m) \oplus H(\mu_m)\big]$.*

Proof. Denote
$$g(t) = f(t) - \mu_m t, \quad 2G(t) = 2F(t) - \mu_m t^2.$$
We first note that (1.6) implies (A1) and
$$\lim_{|t|\to\infty} \frac{2G(t)}{t^2} = \lim_{|t|\to\infty} \frac{g(t)}{t} = 0. \tag{3.17}$$
By (1.7) we have
$$\lim_{|t|\to\infty} \pm(g(t)t - 2G(t)) = \lim_{|t|\to\infty} \pm(f(t)t - 2F(t)) = +\infty. \tag{3.18}$$
It follows from (3.17) that for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that
$$|g(t)| \leqslant \epsilon|t| + C_\epsilon \quad \text{for all } t \in \mathbb{R}. \tag{3.19}$$
We rewrite the functional $\mathcal{J}$ defined by (2.15) as
$$\mathcal{J}(v) = \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2 \,dx\,dy - \frac{\mu_m}{2}\int_\Omega |v(x,0)|^2 dx - \int_\Omega G(v(x,0))dx, \tag{3.20}$$
and rewrite the derivative of $\mathcal{J}$ as
$$\begin{aligned}\langle \mathcal{J}'(v), \phi\rangle = \int_{\mathcal{C}} \nabla v \nabla\phi \,dx\,dy - \mu_m \int_\Omega v(x,0)\phi(x,0)dx \\ - \int_\Omega g(v(x,0))\phi(x,0)dx.\end{aligned} \tag{3.21}$$

(i) Now we begin to satisfy the Cerami condition. Let $\{v_n\} \subset H^1_{0,L}(\mathcal{C})$ be such that
$$\mathcal{J}(v_n) \to c \in \mathbb{R} \quad \text{as } n \to \infty \tag{3.22}$$
$$(1 + \|v_n\|)\|\mathcal{J}'(v_n)\|_* \to 0 \quad \text{as } n \to \infty. \tag{3.23}$$
We first show that $\{v_n\}$ is bounded in $H^1_{0,L}(\mathcal{C})$. By the way of contradiction, we assume that
$$\|v_n\| \to \infty, \quad n \to \infty. \tag{3.24}$$
Set $w_n = \frac{v_n}{\|v_n\|}$. Then $\|w_n\| \equiv 1$ for all $n \in \mathbb{N}$. By Proposition 2.1, up to a subsequence if necessary, there is some $w^* \in H^1_{0,L}(\mathcal{C})$ satisfying
$$\begin{gathered} w_n \rightharpoonup w^*, \quad \text{weakly in } H^1_{0,L}(\mathcal{C}) \\ \operatorname{tr}_\Omega w_n \to \operatorname{tr}_\Omega w^* \quad \text{strongly in } L^q(\Omega) \quad \forall q \in [1, 2^\sharp), \\ w_n(x,0) \to w^*(x,0) \quad \text{a.e. in } \Omega \end{gathered} \tag{3.25}$$
as $n \to \infty$, and there exists $\psi \in L^q(\Omega)$ such that
$$|w_n(x,0)| \leqslant \psi(x) \quad \text{a.e. in } \Omega \text{ for any } n \in \mathbb{N}. \tag{3.26}$$
By (3.21) and (3.23), we have that for any $\phi \in H^1_{0,L}(\mathcal{C})$,
$$\begin{aligned}\langle \mathcal{J}'(v_n), \phi\rangle = \int_{\mathcal{C}} \nabla v_n \nabla\phi \,dx\,dy - \mu_m \int_\Omega v_n(x,0)\phi(x,0)dx \\ - \int_\Omega g(v_n(x,0))\phi(x,0)dx \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{3.27}$$

Take $\phi = w_n - w^*$ in (3.27), and divide it by $\|v_n\|$, we get

$$\begin{aligned}
&\int_{\mathcal{C}} \nabla w_n \nabla (w_n - w^*) dx - \mu_m \int_\Omega w_n(x,0)(w_n(x,0) - w^*(x,0)) dx \\
&- \int_\Omega \frac{g(v_n(x,0))}{\|v_n\|}(w_n(x,0) - w^*(x,0)) dx \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.28}$$

By (3.19), (3.25), Proposition 2.1 and the Hölder inequality, we have

$$\begin{aligned}
&\Big| \int_\Omega \frac{g(v_n(x,0))}{\|v_n\|}(w_n(x,0) - w^*(x,0)) dx \Big| \\
&\leqslant \frac{1}{\|v_n\|} \int_\Omega (\epsilon |v_n(x,0)| + C_\epsilon) |w_n(x,0) - w^*(x,0)| dx \\
&\leqslant \epsilon \| \operatorname{tr}_\Omega w_n\|_{L^2(\Omega)} \| \operatorname{tr}_\Omega w_n - \operatorname{tr}_\Omega w^*\|_{L^2(\Omega)} \\
&\quad + \frac{C_\epsilon}{\|v_n\|} \int_\Omega |w_n(x,0) - w^*(x,0)| dx \\
&\leqslant \epsilon c_2 \| \operatorname{tr}_\Omega w_n - \operatorname{tr}_\Omega w^*\|_{L^2(\Omega)} + C_\epsilon \frac{\| \operatorname{tr}_\Omega w_n - \operatorname{tr}_\Omega w^*\|_{L^1(\Omega)}}{\|v_n\|} \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{3.29}$$

where c_2 is the embedding constant of $H^1_{0,L}(\mathcal{C}) \hookrightarrow L^2(\Omega)$. Moreover, (3.25) implies that

$$\int_\Omega w_n(x,0)(w_n(x,0) - w^*(x,0)) dx \to 0 \quad \text{as } n \to \infty. \tag{3.30}$$

It follows from (3.28), (3.29) and (3.30) that

$$\langle w_n, w_n - w^* \rangle = \int_{\mathcal{C}} \nabla w_n \nabla (w_n - w^*) \, dx \, dy \to 0 \quad \text{as } n \to \infty.$$

By (3.25), it is clear that

$$\langle w^*, w_n - w^* \rangle = \int_{\mathcal{C}} \nabla w^* \nabla (w_n - w^*) \, dx \, dy \to 0 \quad \text{as } n \to \infty.$$

Thus

$$\|w_n - w^*\|^2 = \langle w_n - w^*, w_n - w^* \rangle \to 0 \quad \text{as } n \to \infty.$$

This proves

$$w_n \to w^* \quad \text{strongly in } H^1_{0,L}(\mathcal{C}) \tag{3.31}$$

and $\|w^*\| = 1$. Now dividing by $\|v_n\|$ in (3.27), we deduce that for all $\phi \in H^1_{0,L}(\mathcal{C})$,

$$\int_{\mathcal{C}} \nabla w_n \nabla \phi \, dx \, dy - \mu_m \int_\Omega w_n(x,0)\phi(x,0) dx - \int_\Omega \frac{g(v_n(x,0))}{\|v_n\|} \phi(x,0) dx \to 0 \tag{3.32}$$

as $n \to \infty$. Since for each $\phi \in H^1_{0,L}(\mathcal{C})$, by (3.19) we have

$$\begin{aligned}
&\Big| \int_\Omega \frac{g(v_n(x,0))}{\|v_n\|} \phi(x,0) dx \Big| \\
&\leqslant \epsilon \| \operatorname{tr}_\Omega w_n\|_{L^2(\Omega)} \| \operatorname{tr}_\Omega \phi\|_{L^2(\Omega)} + \frac{C_\epsilon \| \operatorname{tr}_\Omega \phi\|_{L^1(\Omega)}}{\|v_n\|} \\
&\leqslant \epsilon c_2 \| \operatorname{tr}_\Omega \phi\|_{L^2(\Omega)} + \frac{C_\epsilon \| \operatorname{tr}_\Omega \phi\|_{L^1(\Omega)}}{\|v_n\|},
\end{aligned}$$

it follows that

$$\lim_{n\to\infty}\int_\Omega \frac{g(v_n(x,0))}{\|v_n\|}\phi(x,0)dx = 0, \quad \forall \phi \in H^1_{0,L}(\mathcal{C}). \tag{3.33}$$

By (3.31), (3.32) and (3.33), setting $n \to \infty$, we have

$$\int_{\mathcal{C}} \nabla w^* \nabla \phi dx = \mu_m \int_\Omega w^*(x,0)\phi(x,0)dx, \quad \forall \phi \in H^1_{0,L}(\mathcal{C}).$$

Therefore w^* weakly solves the linear elliptic equation in the cylinder $\mathcal{C}$,

$$\begin{gathered}
-\Delta w^* = 0 \quad \text{in } \mathcal{C},\\
w^* = 0 \quad \text{on } \partial_L \mathcal{C},\\
\frac{\partial w^*}{\partial \nu} = \mu_m w^*(x,0) \quad \text{on } \Omega \times \{0\}.
\end{gathered}$$

This means that $w^*(\cdot,0)$ is an eigenfunction corresponding to the eigenvalue μ_m of the operator $A_{1/2}$ therefore an eigenfunction corresponding to λ_m of the operator $-\Delta$. By the unique continuity property of the eigenfunctions of $-\Delta$, we have that $w^*(x,0) \neq 0$ a.e in Ω. Thus by (3.24) and (3.25) we obtain

$$|v_n(x,0)| = \|v_n\||w_n(x,0)| \to \infty \quad \text{uniformly for a.e. } x \in \Omega.$$

It follows from (3.18) that

$$\lim_{n\to\infty} \big(f(v_n(x,0))v_n(x,0) - 2F(v_n(x,0))\big) = \pm\infty \quad \text{uniformly for a.e. } x \in \Omega.$$

Then Fatou's lemma gives

$$\int_\Omega \Big(f(v_n(x,0))v_n(x,0) - 2F(v_n(x,0))\Big)dx \to \pm\infty. \tag{3.34}$$

On the other hand, it follows from (3.22) and (3.23) that

$$2\mathcal{J}(v_n) - \langle \mathcal{J}'(v_n), v_n\rangle \to 2c,$$

therefore

$$\int_\Omega \Big(f(v_n(x,0))v_n(x,0) - 2F(v_n(x,0))\Big)dx = 2\mathcal{J}(v_n) - \langle \mathcal{J}'(v_n), v_n\rangle \to 2c,$$

which contradicts (3.34). Thus $\{v_n\}$ is bounded and then the Cerami condition follows from Lemma 3.1.

(ii) We will prove that the functional $\mathcal{J}$ has the geometric feature required by Proposition 2.4 with respect to the orthogonal splitting (see (2.12))

$$H^1_{0,L}(\mathcal{C}) = H^-(\mu_m) \oplus \big[H(\mu_m) \oplus H^+(\mu_m)\big] := V_\infty \oplus W_\infty.$$

By (3.17), for $\epsilon > 0$ small, there is $M_\epsilon > 0$ such that

$$G(t) \geqslant -\frac{1}{2}\epsilon t^2 - M_\epsilon \quad \text{for all } t \in \mathbb{R}. \tag{3.35}$$

Thus

$$\int_\Omega G(v(x,0))dx \geqslant -\frac{1}{2}\epsilon \int_\Omega |v(x,0)|^2 dx - M_\epsilon|\Omega|.$$

For $v \in H^-(\mu_m)$, by (3.35) and Propositon 2.3, we have

$$\begin{aligned}
\mathcal{J}(v) &= \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2\,dx\,dy - \frac{\mu_m}{2}\int_\Omega |v(x,0)|^2 dx - \int_\Omega G(v(x,0))dx\\
&\leqslant \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2\,dx\,dy - \frac{1}{2}(\mu_m - \epsilon)\int_\Omega |v(x,0)|^2 dx + M_\epsilon|\Omega|
\end{aligned}$$

$$\leqslant \frac{1}{2}\Big(1-\frac{\mu_m-\epsilon}{\mu_{m-1}}\Big)\int_{\mathcal{C}}|\nabla v|^2\,dx\,dy+M_\epsilon|\Omega|$$

Let $\epsilon\in(0,\mu_m-\mu_{m-1})$ be fixed. Then we have that

$$\mathcal{J}(v)\to-\infty \quad \text{for } v\in H^-(\mu_m) \text{ with } \|v\|\to\infty. \tag{3.36}$$

It follows from (3.18) and (A3) with $+$ in (1.7), that for every $T>0$, there is $M>0$ such that

$$g(t)t-2G(t)\geqslant T \quad \text{for all } |t|\geqslant M.$$

For $t>0$, we have

$$\frac{d}{dt}\Big[\frac{G(t)}{t^2}\Big]=\frac{g(t)t-2G(t)}{t^3}. \tag{3.37}$$

Integrating (3.37) over $[t,s]\subset[M,\infty)$, we obtain

$$\frac{G(s)}{s^2}-\frac{G(t)}{t^2}\geqslant\frac{T}{2}\big(\frac{1}{t^2}-\frac{1}{s^2}\big).$$

Letting $s\to+\infty$ and using (3.17), we see that

$$G(t)\leqslant-\frac{T}{2} \quad \text{for } t\geqslant M.$$

A similar process shows that

$$G(t)\leqslant-\frac{T}{2} \quad \text{for all } t\leqslant -M.$$

Hence

$$\lim_{|t|\to\infty}G(t)=-\infty. \tag{3.38}$$

For $v\in H(\mu_m)\oplus H^+(\mu_m)=\big[H^-(\mu_m)\big]^\perp$, we write $v=\bar v+\tilde v$, $\bar v\in H(\mu_m),\tilde v\in H^+(\mu_m)$. Then by Propostion 2.3 we have

$$\begin{aligned}\mathcal{J}(v)&=\frac{1}{2}\int_{\mathcal{C}}|\nabla v|^2\,dx\,dy-\frac{\mu_m}{2}\int_\Omega|v(x,0)|^2dx-\int_\Omega G(v(x,0))dx\\&=\frac{1}{2}\int_{\mathcal{C}}|\nabla \tilde v|^2\,dx\,dy-\frac{\mu_m}{2}\int_\Omega|\tilde v(x,0)|^2dx-\int_\Omega G(v(x,0))dx\\&\geqslant\frac{1}{2}\Big(1-\frac{\mu_m}{\mu_{m+\ell_m}}\Big)\int_{\mathcal{C}}|\nabla\tilde v|^2\,dx\,dy-\int_\Omega G(v(x,0))dx\end{aligned}$$

By (3.38), we see that for some $K>0$ it holds that

$$G(t)\leqslant K \quad \text{for all } t\in\mathbb{R}.$$

Therefore

$$\mathcal{J}(v)\geqslant\frac{1}{2}\Big(1-\frac{\mu_m}{\mu_{m+\ell_m}}\Big)\int_{\mathcal{C}}|\nabla\tilde v|^2\,dx\,dy-K|\Omega|.$$

It follows that

$$\mathcal{J}(v)=\mathcal{J}(\tilde v+\bar v)\to\infty \quad \text{as } \|\tilde v\|\to\infty. \tag{3.39}$$

Now we show that

$$\|\tilde v\| \text{ bounded and } \|\bar v\|\to\infty \implies \mathcal{J}(v)=\mathcal{J}(\tilde v+\bar v)\to\infty. \tag{3.40}$$

We only need to show that for any $\{v_n=\tilde v_n+\bar v_n\}$ such that $\{\|\tilde v_n\|\}$ is bounded and $\|\bar v_n\|\to\infty$ implies $\mathcal{J}(\tilde v_n+\bar v_n)\to\infty$ as $n\to\infty$.

Set $w_n = v_n/\|v_n\|$ and write $w_n = \tilde{w}_n + \bar{w}_n$, $\tilde{w}_n = \frac{\tilde{v}_n}{\|v_n\|}, \bar{w}_n = \frac{\bar{v}_n}{\|v_n\|}$. We have

$$\|\tilde{w}_n\| \to 0 \quad \text{and} \quad \|\bar{w}_n\| \to 1 \quad \text{as } n \to \infty. \tag{3.41}$$

By Proposition 2.1, up to a subsequence if necessary, there is some $w_* \in H(\mu_m) \oplus H^+(\mu_m)$ such that

$$\begin{gathered} w_n \rightharpoonup w_*, \quad \text{weakly in } H^1_{0,L}(\mathcal{C}) \\ \operatorname{tr}_\Omega w_n \to \operatorname{tr}_\Omega w_* \quad \text{strongly in } L^2(\Omega), \\ w_n(x,0) \to w_*(x,0) \quad \text{a.e. in } \Omega \end{gathered} \tag{3.42}$$

as $n \to \infty$. From (3.41) and (3.42) we deduce that $w_* \in E(\mu_m)$ and $\|w_*\| = 1$. Therefore w_* weakly solves the linear elliptic equation in the cylinder $\mathcal{C}$,

$$\begin{gathered} -\Delta w_* = 0 \quad \text{in } \mathcal{C}, \\ w_* = 0 \quad \text{on } \partial_L \mathcal{C}, \\ \frac{\partial w}{\partial \nu} = \mu_m w_*(x,0) \quad \text{on } \Omega \times \{0\}. \end{gathered}$$

This means that $w_*(\cdot,0)$ is an eigenfunction corresponding to the eigenvalue μ_m of $A_{1/2}$. Therefore $w(x,0) \neq 0$ for a.e. $x \in \Omega$. It follows that

$$|v_n(x,0)| = \|v_n\||v_n(x,0)| \to \infty, \quad \text{for a.e. } x \in \Omega, \text{ as } n \to \infty. \tag{3.43}$$

Now by (3.38), (3.43) and Fatou's lemma, we obtain

$$\mathcal{J}(v_n) \geqslant \frac{1}{2}\Big(1 - \frac{\mu_m}{\mu_{m+\ell_m}}\Big) \int_{\mathcal{C}} |\nabla \tilde{v}_n|^2 \,dx\,dy - \int_\Omega G(v_n(x,0))dx \to \infty$$

as $n \to \infty$. This proves (3.40). Now (3.39) and (3.40) imply

$$\mathcal{J}(v) \to \infty \quad \text{for } v \in H(\mu_m) \oplus H^+(\mu_m) \text{ with } \|v\| \to \infty. \tag{3.44}$$

It follows from the fact of $\mathcal{J}$ being weakly lower semicontinuous on $H(\mu_m) \oplus H^+(\mu_m)$ and (3.44) that $\mathcal{J}$ is bounded from below on $H(\mu_m) \oplus H^+(\mu_m)$. Finally by Proposition 2.4 we get $C_{\ell_\infty}(J,\infty) \ncong 0$, where $\ell_\infty = \dim H^-(\mu_m)$.

(iii) In a similar way we can prove that the functional $\mathcal{J}$ has the geometric feature required by Proposition 2.4 with respect to the orthogonal splitting (see (2.12))

$$H^1_{0,L}(\mathcal{C}) = \big[H^-(\mu_m) \oplus H(\mu_m)\big] \oplus H^+(\mu_m) := V_\infty \oplus W_\infty.$$

Therefore, $C_{\ell^*_\infty}(J,\infty) \ncong 0$, where $\ell^*_\infty = \dim H^-(\mu_m) \oplus H(\mu_m)$. The proof is complete. $\square$

Lemma 3.4. *Assume* (A1) *and* (A4).

(i) *The functional $\mathcal{J}$ is coercive on $H^1_{0,L}(\mathcal{C})$.*
(ii) *The functional $\mathcal{J}$ satisfies the Palais-Smale condition.*
(iii) $C_q(\mathcal{J},\infty) \cong \delta_{q,0}\mathbb{Z}$.

Proof. (i) For $v \in H^1_{0,L}(\mathcal{C})$, we have by (A4) that

$$\begin{aligned} \mathcal{J}(v) &= \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2 \,dx\,dy - \int_\Omega F(v(x,0))dx \\ &\geqslant \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2 \,dx\,dy - \frac{1}{2}\mu \int_\Omega |v(x,0)|^2 dx - C|\Omega| \\ &\geqslant \frac{1}{2}\Big(1 - \frac{\mu}{\mu_1}\Big)\|v\|^2 - C|\Omega|. \end{aligned}$$

Since $\mu < \mu_1$, we have that $\mathcal{J}(v) \to \infty$ as $\|v\| \to \infty$. This proves that $\mathcal{J}$ is coercive.

(ii) Let $\{v_n\} \subset H^1_{0,L}(\mathcal{C})$ be a Palais-Smale sequence at $c \in \mathbb{R}$. By the coerciveness of $\mathcal{J}$, $\{v_n\}$ is bounded and then by Lemma 3.1 it contains convergent subsequence.

(iii) Since $\mathcal{J}$ is coercive and is weakly lower semicontinuous on $H^1_{0,L}(\mathcal{C})$, $\mathcal{J}$ attains its global minima $\inf \mathcal{J}$ at some v_*:

$$\mathcal{J}(v_*) = \min_{v \in H^1_{0,L}(\mathcal{C})} \mathcal{J}(v).$$

Take $a < \mathcal{J}(v_*)$. Then

$$C_q(\mathcal{J}, \infty) = H_q(H^1_{0,L}(\mathcal{C}), \mathcal{J}^a) \cong H_q(\{v_*\}, \emptyset) \cong \delta_{q,0}\mathbb{Z}.$$

The proof is complete. □

Lemma 3.5. *Assume* (A5). *Then*

(i) *the functional $\mathcal{J}$ is coercive on $H^1_{0,L}(\mathcal{C})$;*
(ii) *the functional $\mathcal{J}$ satisfies the Palais-Smale condition;*
(iii) $C_q(\mathcal{J}, \infty) \cong \delta_{q,0}\mathbb{Z}$.

Proof. (i) We first prove that under the condition (A5) the functional $\mathcal{J}$ is coercive on $H^1_{0,L}(\mathcal{C})$. Denote $2G(t) = 2F(t) - \mu_1 t^2$. Then (1.10) implies

$$\lim_{|t| \to \infty} G(t) = -\infty. \tag{3.45}$$

Rewrite $\mathcal{J}$ as

$$\mathcal{J}(v) = \frac{1}{2} \int_{\mathcal{C}} |\nabla v|^2 \,dx\,dy - \frac{\mu_1}{2} \int_{\Omega} |v(x,0)|^2 dx - \int_{\Omega} G(v(x,0))dx,$$

for $v \in H^1_{0,L}(\mathcal{C})$. Assume that $\mathcal{J}$ is not coercive on $H^1_{0,L}(\mathcal{C})$, then there is a sequence $\{v_n\} \subset H^1_{0,L}(\mathcal{C})$ such that

$$\|v_n\| \to \infty \quad \text{as } n \to \infty \tag{3.46}$$

and

$$\mathcal{J}(v_n) \leqslant C \quad \text{for all } n \in \mathbb{N}. \tag{3.47}$$

for some $C \in \mathbb{R}$. Set $w_n = \frac{v_n}{\|v_n\|}$ then $\|w_n\| \equiv 1$ for all $n \in \mathbb{N}$. By Proposition 2.1, up to a subsequence if necessary, there is a $w^* \in H^1_{0,L}(\mathcal{C})$ satisfying

$$\begin{gathered} w_n \rightharpoonup w^* \quad \text{weakly in } H^1_{0,L}(\mathcal{C}), \\ \operatorname{tr}_\Omega w_n \to \operatorname{tr}_\Omega w^* \quad \text{strongly in } L^2(\Omega), \\ w_n(x,0) \to w^*(x,0) \quad \text{a.e. in } \Omega. \end{gathered} \tag{3.48}$$

By (3.45) we see that $G(t)$ is bounded from above by some constant $K > 0$ for all $t \in \mathbb{R}$. Now from (3.47) we deduce that

$$\frac{C}{\|v_n\|^2} \geqslant \frac{\mathcal{J}(v_n)}{\|v_n\|^2} \geqslant \frac{1}{2} \int_{\mathcal{C}} |\nabla w_n|^2 \,dx\,dy - \frac{\mu_1}{2} \int_{\Omega} |w_n(x,0)|^2 dx - \frac{K|\Omega|}{\|v_n\|^2}. \tag{3.49}$$

It follows from (3.46), (3.48) and (3.49) that

$$\limsup_{n \to \infty} \int_{\mathcal{C}} |\nabla w_n|^2 \,dx\,dy \leqslant \mu_1 \int_{\Omega} |w^*(x,0)|^2 dx. \tag{3.50}$$

On the other hand, by the variational characterization of μ_1 and the lower semicontinuity of the norm, we have

$$\mu_1 \int_\Omega |w^*(x,0)|^2 dx \leqslant \int_{\mathcal{C}} |\nabla w^*|^2 \,dx\,dy \leqslant \liminf_{n\to\infty} \int_{\mathcal{C}} |\nabla w_n|^2 \,dx\,dy. \tag{3.51}$$

From (3.50) and (3.51) we have that

$$\lim_{n\to\infty} \|w_n\|^2 = \|w^*\|^2, \tag{3.52}$$

$$\int_{\mathcal{C}} |\nabla w^*|^2 \,dx\,dy = \mu_1 \int_\Omega |w^*(x,0)|^2 dx. \tag{3.53}$$

Since $H^1_{0,L}(\mathcal{C})$ is a Hilbert space, we have by (3.48) and (3.52) that

$$w_n \to w^* \quad \text{strongly in } H^1_{0,L}(\mathcal{C}) \text{ as } n \to \infty.$$

Hence $\|w^*\| = 1$ and by (3.53) we see that $w^*(x,0) = \pm(\mu_1)^{-\frac{1}{2}}\varphi_1(x)$. This implies

$$|v_n(x,0)| \to \infty \quad \text{uniformly for a.e. } x \in \Omega. \tag{3.54}$$

Now by (3.45), (3.47), (3.54) and the Fatou's lemma we have that

$$\begin{aligned} C &\geqslant \frac{1}{2}\int_{\mathcal{C}} |\nabla v_n|^2 \,dx\,dy - \frac{\mu_1}{2}\int_\Omega |v_n(x,0)|^2 dx - \int_\Omega G(v_n(x,0))dx \\ &\geqslant -\int_\Omega G(v_n(x,0))dx \to \infty \quad \text{as } n \to \infty. \end{aligned}$$

This contradiction shows that $\mathcal{J}$ is coercive on $H^1_{0,L}(\mathcal{C})$.

(ii) Let $\{v_n\} \subset H^1_{0,L}(\mathcal{C})$ be a Palais-Smale sequence at $c \in \mathbb{R}$. By the coerciveness of $\mathcal{J}$, $\{v_n\}$ is bounded and then by Lemma 3.1 it contains a convergent subsequence.

(iii) Since $\mathcal{J}$ is coercive and is weakly lower semicontinuous on $H^1_{0,L}(\mathcal{C})$, $\mathcal{J}$ attains its global minima $\inf \mathcal{J}$ at some v_*:

$$\mathcal{J}(v_*) = \min_{v\in H^1_{0,L}(\mathcal{C})} \mathcal{J}(v).$$

Take $a < \mathcal{J}(v_*)$. Then

$$C_q(\mathcal{J},\infty) = H_q(H^1_{0,L}(\mathcal{C}), \mathcal{J}^a) \cong H_q(\{v_*\}, \emptyset) \cong \delta_{q,0}\mathbb{Z}.$$

The proof is complete. □

4. Critical groups at zero

In this section we compute the critical groups of the functional $\mathcal{J}$ at zero. We will use $C_i > 0$ to denote various constants independent of the functions in $H^1_{0,L}(\mathcal{C})$. We also make a convention that problem (1.1) has finitely many weak solutions and so the trivial solution is an isolated critical point of $\mathcal{J}$.

Lemma 4.1. *Assume* (A1) *and* (A6). *Then* $C_q(\mathcal{J},0) \cong 0$ *for all* $q \in \mathbb{Z}$.

Proof. By the definition of critical groups, we write

$$C_q(\mathcal{J},0) := H_q(B_\rho(0) \cap \mathcal{J}^0, (B_\rho(0) \cap \mathcal{J}^0) \setminus \{0\}),$$

where $B_\rho(0) = \{v \in H^1_{0,L}(\mathcal{C}) : \|v\| \leqslant \rho\}$, and $\rho > 0$ is to be chosen suitable for use. We will construct a deformation mapping for the topological pairs $(B_\rho(0), B_\rho(0) \setminus \{0\})$ and $(B_\rho(0) \cap \mathcal{J}^0, (B_\rho(0) \cap \mathcal{J}^0) \setminus \{0\})$.

A direct calculation by using (1.11) and (1.12) shows that there exists a constant $C_1 > 0$ such that

$$F(t) \geqslant C_1|t|^\tau \quad \text{for all } |t| \leqslant \delta.$$

By (A1), there exists a constant $C_2 > 0$ such that for some $\max\{2, p\} < \gamma < 2^\sharp$,

$$|F(t)| \leqslant C_2|t|^\gamma, \quad |f(t)t| \leqslant C_2|t|^\gamma \quad \text{for all } |t| > \delta. \tag{4.1}$$

Therefore,

$$F(t) \geqslant C_1|t|^\tau - C_2|t|^\gamma \quad \text{for all } t \in \mathbb{R}.$$

Take a function $v \in H^1_{0,L}(\mathcal{C})$ with $v \neq 0$, then for $\eta > 0$ we have

$$\begin{aligned}
\mathcal{J}(\eta v) &= \frac{1}{2}\int_{\mathcal{C}} |\nabla(\eta v)|^2 \,dx\,dy - \int_\Omega F(\eta v(x,0))dx \\
&\leqslant \frac{1}{2}\eta^2\|v\|^2 - C_1\int_\Omega |\eta v(x,0)|^\tau dx + C_2\int_\Omega |\eta v(x,0)|^\gamma dx \\
&\leqslant \frac{1}{2}\eta^2\|v\|^2 - C_1\eta^\tau\|\operatorname{tr}_\Omega v\|^\tau_{L^\tau(\Omega)} + C_2\eta^\gamma\|\operatorname{tr}_\Omega v\|^\gamma_{L^\gamma(\Omega)}.
\end{aligned} \tag{4.2}$$

Since $1 < \tau < 2 < \gamma < 2^\sharp$, one sees from (4.2) that for given $v \in H^1_{0,L}(\mathcal{C})$ with $v \neq 0$, there exists $\eta_0 = \eta_0(v) > 0$ such that

$$\mathcal{J}(\eta v) < 0 \quad \text{for all } 0 < \eta < \eta_0. \tag{4.3}$$

Let $v \in H^1_{0,L}(\mathcal{C})$ be such that

$$\mathcal{J}(v) = \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2 \,dx\,dy - \int_\Omega F(v(x,0))dx = 0.$$

It follows from (1.12), (A1) and the continuous embedding $H^1_{0,L}(\mathcal{C}) \hookrightarrow L^q(\Omega)$ for any $q \in [1, 2^\sharp]$ that

$$\begin{aligned}
\frac{d}{d\eta}\mathcal{J}(\eta v)\big|_{\eta=1} &= \int_{\mathcal{C}} |\nabla v|^2 \,dx\,dy - \int_\Omega f(v(x,0))v(x,0)dx \\
&= \frac{2-\tau}{2}\|v\|^2 + \int_\Omega \big(\tau F(v(x,0)) - f(v(x,0))v(x,0)\big)dx \\
&\geqslant \frac{2-\tau}{2}\|v\|^2 + \int_{\{|v(x,0)|>\delta\}} \big(\tau F(v(x,0)) - f(v(x,0))v(x,0)\big)dx \\
&\geqslant \frac{2-\tau}{2}\|v\|^2 - \int_{\{|v(x,0)|>\delta\}} \big(|2F(v(x,0))| + |f(v(x,0)v(x,0)|\big)dx \\
&\geqslant \frac{2-\tau}{2}\|v\|^2 - C_3\int_{\{|v(x,0)|>\delta\}} |v(x,0)|^\gamma dx \\
&\geqslant \frac{2-\tau}{2}\|v\|^2 - C_4\|v\|^\gamma
\end{aligned}$$

Thus we can find some $\rho > 0$ such that

$$\frac{d}{d\eta}\mathcal{J}(\eta v)\big|_{\eta=1} > 0, \quad \text{for } v \in H^1_{0,L}(\mathcal{C}) \text{ with } \mathcal{J}(v) = 0 \quad \text{and} \quad 0 < \|v\| \leqslant \rho. \tag{4.4}$$

By (4.3) and (4.4), one sees that for each $v \in B_\rho(0) \setminus \{0\}$ with $\mathcal{J}(v) > 0$, there exists a unique $\eta_0 = \eta_0(v) > 0$ such that

$$\mathcal{J}(\eta v) < 0 \quad \text{for all } 0 < \eta < \eta_0. \tag{4.5}$$

From now on we fix $\rho > 0$. We claim that if $v \in B_\rho(0) \setminus \{0\}$ and $\mathcal{J}(v) < 0$ then

$$\mathcal{J}(\eta v) < 0 \quad \text{for all } \eta \in (0,1). \tag{4.6}$$

Let $v \in B_\rho(0)$ and $\mathcal{J}(v) < 0$. By the continuity of $\mathcal{J}$, there exists $\vartheta \in (0,1]$ such that

$$\mathcal{J}(\eta v) < 0 \quad \text{for all } \eta \in (1-\vartheta, 1).$$

We show (4.6) by proving $\vartheta = 1$. Suppose that there is some $\eta^* \in (0, 1-\vartheta]$ such that

$$\mathcal{J}(\eta^* v) = 0, \quad \mathcal{J}(\eta v) < 0 \quad \text{for all } \eta \in (\eta^*,\ 1).$$

Denote $v^* = \eta^* v$. Then by (4.4), we have

$$\frac{d}{d\eta}\mathcal{J}(\eta v^*)\big|_{\eta=1} > 0. \tag{4.7}$$

But $\eta > \eta^*$ implies

$$\mathcal{J}(\eta v) - \mathcal{J}(\eta^* v) < 0,$$

which implies

$$\frac{d}{d\eta}\mathcal{J}(\eta v^*)\big|_{\eta=1} = \lim_{\eta \to \eta^*_+} \frac{\mathcal{J}(\eta v) - \mathcal{J}(\eta v^*)}{\eta - \eta^*} \leqslant 0.$$

This contradicts (4.7). Hence $\vartheta = 1$ and (4.6) holds.

Now we define a mapping $\eta : B_\rho(0) \to [0,1]$ by

$$\eta(v) = \begin{cases} 1, & \text{for } v \in B_\rho(0) \text{ with } \mathcal{J}(v) \leqslant 0, \\ \eta, & \text{for } v \in B_\rho(0) \text{ with } \mathcal{J}(v) > 0, \mathcal{J}(\eta v) = 0, \eta < 1. \end{cases}$$

By (4.4), (4.5) and (4.6), the mapping η is well-defined and if $\mathcal{J}(v) > 0$ then there exists a unique $\eta(v) \in (0,1)$ such that

$$\begin{gathered} \mathcal{J}(\eta(v)v) = 0, \\ \mathcal{J}(\eta v) < 0, \quad \forall \eta \in (0, \eta(v)) \\ \mathcal{J}(\eta v) > 0, \quad \forall \eta \in (\eta(v), 1) \end{gathered} \tag{4.8}$$

It follows from (4.4), (4.8) and the Implicit Function Theorem that the mapping η is continuous in v. Define a mapping $h : [0,1] \times B_\rho(0) \to B_\rho(0)$ by

$$h(t,v) = (1-t)v + t\eta(v)v,\ t \in [0,1],\ v \in B_\rho(0).$$

It is easy to see that the mapping h is a continuous deformation from $(B_\rho(0), B_\rho(0)\setminus \{0\})$ to $(B_\rho(0) \cap \mathcal{J}^0, (B_\rho(0) \cap \mathcal{J}^0) \setminus \{0\})$. By the homotopy invariance of homology group, we have for all $q \in \mathbb{Z}$,

$$C_q(\mathcal{J}, 0) = H_q(B_\rho(0) \cap \mathcal{J}^0, (B_\rho(0) \cap \mathcal{J}^0) \setminus \{0\}) \cong H_q(B_\rho(0), B_\rho(0) \setminus \{0\}) \cong 0.$$

since $B_\rho(0) \setminus \{0\}$ is contractible. The proof is complete. $\square$

We remark that the idea for computing critical groups at zero is essentially from [30] where Laplacian equations with superlinear at zero was studied. The similar idea was presented in [31] to deal with also the same problem as in [30] using a global sign condition $2F(t) - f(t)t > 0$ for all $t \neq 0$. In [25] this idea was used for studying p-Laplacian problems.

Lemma 4.2. *Assume that* (A1) *and* (A7) *hold. Then* $C_{\ell_0}(\mathcal{J}, 0) \ncong 0$ *where* $\ell_0 = \dim H^-(\mu_{k+1})$.

Proof. We will prove that the functional $\mathcal{J}$ has a local linking structure at 0 with respect to the orthogonal splitting (see (2.12))

$$H^1_{0,L}(\mathcal{C}) = H^-(\mu_{k+1}) \oplus \big[H(\mu_{k+1}) \oplus H^+(\mu_{k+1})\big] := V_0 \oplus W_0.$$

(i) For $v \in H^-(\mu_{k+1})$, by Proposition 2.1 we have

$$\| \operatorname{tr}_\Omega v\|_{L^2(\Omega)} \leqslant C_2\|v\|.$$

Note that $\operatorname{tr}_\Omega(H^-(\mu_{k+1})) = \operatorname{span}\{\varphi_1, \dots, \varphi_k\} \subset L^\infty(\Omega)$ is finite dimensional and all norms are equivalent, we can find a positive constant $\rho > 0$ such that

$$\|v\| \leqslant \rho \Rightarrow |v(x,0)| \leqslant \| \operatorname{tr}_\Omega v\|_{L^\infty(\Omega)} \leqslant \delta.$$

It follows from (1.13) and Propostion 2.3 that for any $v \in H^-(\mu_{k+1})$ with $\|v\| \leqslant \rho$, we have

$$\begin{aligned}\mathcal{J}(v) &= \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2\, dx\, dy - \int_\Omega F(v(x,0))dx \\ &\leqslant \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2\, dx\, dy - \frac{\mu_k}{2}\int_\Omega |v(x,0)|^2 dx \leqslant 0.\end{aligned} \tag{4.9}$$

(ii) For $v \in \big[H(\mu_{k+1}) \oplus H^+(\mu_{k+1})\big] = \overline{\operatorname{span}\{e_{k+1}, \dots\}}$, we write $v = \bar v + \tilde v$, where $\bar v \in H(\mu_{k+1})$, $\tilde v \in H^+(\mu_{k+1})$. Then by Propostion 2.3 we have

$$\begin{aligned}\mathcal{J}(v) &= \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2\, dx\, dy - \int_\Omega F(v(x,0))dx \\ &= \frac{1}{2}\int_{\mathcal{C}} |\nabla \tilde v|^2\, dx\, dy - \frac{\mu_{k+1}}{2}\int_\Omega |\tilde v(x,0)|^2 dx \\ &\quad + \int_\Omega \Big(\frac{\mu_{k+1}}{2}|v(x,0)|^2 - F(v(x,0))\Big)dx \\ &\geqslant \frac{1}{2}\Big(1 - \frac{\mu_{k+1}}{\mu_{k+1+\ell_{k+1}}}\Big)\|\tilde v\|^2 + \int_\Omega \Big(\frac{\mu_{k+1}}{2}|v(x,0)|^2 - F(v(x,0))\Big)dx.\end{aligned} \tag{4.10}$$

For $|v(x,0)| \leqslant \delta$, by (1.13) we get that

$$\int_{\{|v(x,0)| \leqslant \delta\}} \Big(\frac{1}{2}\mu_{k+1}|v(x,0)|^2 - F(v(x,0))\Big)dx \geqslant 0. \tag{4.11}$$

Since $\operatorname{tr}_\Omega H(\mu_{k+1})$ is finite dimensional, there exists $\rho > 0$ such that

$$\|v\| \leqslant \rho \;\Rightarrow\; \| \operatorname{tr}_\Omega \bar v\|_{L^\infty(\Omega)} \leqslant \frac{1}{3}\delta.$$

For $\|v\| \leqslant \rho$ and $|v(x,0)| > \delta$,

$$|\tilde v(x,0)| \geqslant |v(x,0)| - |\bar v(x,0)| > \frac{2}{3}|v(x,0)|.$$

By (A1), take $\max\{2,p\} < \gamma < 2^\sharp$, there is $C_5 > 0$ such that

$$\Big|\frac{1}{2}\mu_{k+1}t^2 - F(t)\Big| \leqslant C_5|t|^\gamma \quad \text{for all } |t| \geqslant \delta.$$

By Proposition 2.1, we have

$$\begin{aligned}
&\int_{\{|v(x,0)|>\delta\}} \big|\frac{1}{2}\mu_{k+1}|v(x,0)|^2 - F(v(x,0))\big|dx \\
&\leqslant C_5 \int_{\{|v(x,0)|>\delta\}} |v(x,0)|^\gamma dx \\
&\leqslant C_5 (3/2)^\gamma \int_\Omega |\tilde{v}(x,0)|^\gamma dx \\
&\leqslant C_5 (3/2)^\gamma C_\gamma^\gamma \|\tilde{v}\|^\gamma := C_6 \|\tilde{v}\|^\gamma.
\end{aligned} \tag{4.12}$$

Then by (4.10), (4.11) and (4.12) we get

$$\mathcal{J}(v) \geqslant \frac{1}{2}\Big(1 - \frac{\mu_{k+1}}{\mu_{k+1+\ell_{k+1}}}\Big)\|\tilde{v}\|^2 - C_6\|\tilde{v}\|^\gamma. \tag{4.13}$$

Since $\gamma > 2$, we see from (4.13) that for $\rho > 0$ small

$$\mathcal{J}(v) > 0 \quad \text{for } \|v\| \leqslant \rho \text{ and } \tilde{v} \neq 0. \tag{4.14}$$

On the other hand, we conclude that for $\|v\| \leqslant \rho$ with $\tilde{v} = 0$ and $\bar{v} \neq 0$,

$$\mathcal{J}(v) = \mathcal{J}(\bar{v}) = \int_\Omega \Big(\frac{1}{2}\mu_{k+1}\bar{v}^2(x,0) - F(\bar{v}(x,0))\Big)dx > 0. \tag{4.15}$$

Otherwise, if for some $\bar{v}_* \neq 0$ and $\|\bar{v}_*\| \leqslant \rho$ such that $\mathcal{J}(\bar{v}_*) = 0$, then by (A7) we have

$$F(\bar{v}_*(x,0)) = \frac{1}{2}\mu_{k+1}\bar{v}_*^2(x,0) \quad \text{for a.e.} x \in \Omega. \tag{4.16}$$

As $v_* \in H(\mu_{k+1})$, all σv_* for $\sigma \in [-1,1]$ are critical points of $\mathcal{J}$ and so 0 is not isolated. It is a contradiction. Therefore we get the conclusion that

$$\mathcal{J}(v) > 0 \quad \text{for } v \in H(\mu_{k+1}) \oplus H^+(\mu_{k+1}) \text{ with } 0 < \|v\| \leqslant \rho. \tag{4.17}$$

Now by (4.9), (4.17) and Proposition 2.5, $C_{\ell_0}(J,0) \ncong 0$, where $\ell_0 = \dim H^-(\mu_{k+1})$. The proof is complete. □

Lemma 4.3. *Assume* (A1) *and* (A8)*. Then we have*

$$C_q(\mathcal{J},0) \cong \delta_{q,0}\mathbb{Z}, \quad q \in \mathbb{Z}.$$

Proof. We will show that 0 is a strictly local minimizer of $\mathcal{J}$. For $v \in H^1_{0,L}(\mathcal{C})$, we $v = \bar{v} + \tilde{v}$ where $\bar{v} \in H(\mu_1)$ and $\tilde{v} \in H^+(\mu_1)$. Take $\rho > 0$ small such that

$$\|v\| \leqslant \rho \ \Rightarrow \ \|\operatorname{tr}_\Omega \bar{v}\|_\infty \leqslant \frac{1}{3}\delta.$$

Then

$$|v(x,0)| > \delta \ \Rightarrow \ |v(x,0)| < \frac{3}{2}|\tilde{v}(x,0)|. \tag{4.18}$$

By (A1), take $\max\{2,p\} < \gamma < 2^\sharp$, there is $C_7 > 0$ such that

$$\frac{1}{2}\mu_1 t^2 + |F(t)| \leqslant C_7|t|^\gamma \quad \text{for } |t| > \delta. \tag{4.19}$$

By (4.18), (4.19) and Proposition 2.1, we have

$$\int_{\{|v(x,0)|>\delta\}} \big|\frac{1}{2}\mu_1|v(x,0)|^2 - F(v(x,0))\big|dx \leqslant C_8\|\tilde{v}\|^\gamma. \tag{4.20}$$

Now for $\|v\| \leqslant \rho$, by (A8), (4.20) and (2.3), we have

$$\begin{aligned}
\mathcal{J}(v) &= \frac{1}{2}\int_{\mathcal{C}} |\nabla v|^2\,dx\,dy - \int_\Omega F(v(x,0))dx \\
&= \frac{1}{2}\|\tilde{v}\|^2 - \frac{\mu_1}{2}\int_\Omega |\tilde{v}(x,0)|^2 dx \\
&\quad + \int_{\{|v(x,0)|\leqslant\delta\}} \Big(\frac{1}{2}\mu_1|v(x,0)|^2 - F(v(x,0))\Big)dx \\
&\quad + \int_{\{|v(x,0)|>\delta\}} \Big(\frac{\mu_1}{2}|v(x,0)|^2 - F(v(x,0))\Big)dx \\
&\geqslant \frac{1}{2}\Big(1-\frac{\mu_1}{\mu_2}\Big)\|\tilde{v}\|^2 - C_8\|\tilde{v}\|^\gamma.
\end{aligned} \tag{4.21}$$

Arguing in the same way as that in the proof of Lemma 4.2 we can prove that $v = 0$ is a strictly local minimizer of $\mathcal{J}$. Thus $C_q(\mathcal{J}, 0) \cong \delta_{q,0}\mathbb{Z}$, $q \in \mathbb{Z}$. The proof is complete. □

We remark here that (A8) includes the nonresonance case $2F(t) \leqslant \mu t^2$ with $\mu < \mu_1$ for $|t| \leqslant \delta$ as a special case.

5. Proofs of main results

In this section we give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. (a) By Lemma 3.2, the functional J satisfies the Palais-Smale condition and

$$C_q(\mathcal{J}, \infty) \cong 0 \quad \text{for all } q \in \mathbb{Z}. \tag{5.1}$$

By Lemma 4.2, we have that

$$C_{\ell_0}(\mathcal{J}, 0) \ncong 0. \tag{5.2}$$

It follows that

$$C_{\ell_0}(\mathcal{J}, \infty) \ncong C_{\ell_0}(\mathcal{J}, 0). \tag{5.3}$$

Therefore $\mathcal{J}$ has at least one nontrivial critical point.

(c) By Lemma 3.3, the functional J satisfies the Cerami condition and

$$C_\ell(\mathcal{J}, \infty) \ncong 0 \quad \text{for } \ell = \ell_\infty \text{ or } \ell = \ell_\infty^*. \tag{5.4}$$

By Lemma 4.1, we have

$$C_q(\mathcal{J}, 0) \cong 0 \quad \text{for all } q \in \mathbb{Z}. \tag{5.5}$$

It follows that

$$C_\ell(\mathcal{J}, \infty) \ncong C_\ell(\mathcal{J}, 0). \tag{5.6}$$

Therefore $\mathcal{J}$ has at least one nontrivial critical point.

The other cases are proved in a similar way. The proof is complete. □

Proof of Theorem 1.2. We give the proof for the case (b). By Lemma 3.4, $\mathcal{J}$ is coercive on $H^1_{0,L}(\Omega)$ and satisfies the Palais-Smale condition. Thus $\mathcal{J}$ is bounded from below and has a global minimizer. By Lemma 4.2, we have that

$$C_{\ell_0}(\mathcal{J}, 0) \ncong 0. \tag{5.7}$$

Since $\ell_0 \geqslant 1$, the trivial critical point 0 is homological nontrivial and is not a minimizer of $\mathcal{J}$. It follows from Proposition 2.6 that $\mathcal{J}$ has at least two nontrivial critical points. The proof is complete. □

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