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UPPER SEMICONTINUITY OF ATTRACTORS AND CONTINUITY OF EQUILIBRIUM SETS FOR PARABOLIC PROBLEMS WITH DEGENERATE p-LAPLACIAN

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Abstract. In this work we obtain some continuity properties on the parameter q at $p = q$ for the Takeuchi-Yamada problem which is a degenerate p-laplacian version of the Chafee-Infante problem. We prove the continuity of the flows and the equilibrium sets, and the upper semicontinuity of the global attractors.

1. Introduction

The inspiration for this study arose from the description by Chafee and Infante of the bifurcation scheme and stability properties of the equilibrium solutions for the semilinear problem

$$\begin{gathered} u_t = \lambda u_{xx} + u - u^3, \quad (x,t) \in (0,1) \times (0,+\infty) \\ u(0,t) = u(1,t) = 0, \quad 0 \le t < +\infty \\ u(x,0) = u_0(x), \quad x \in (0,1), \end{gathered} \tag{1.1}$$

where λ is a positive parameter and the initial data are sufficiently smooth [4]. Using a time-map method that adjusts the initial speed of a Cauchy problem to ensure that the desired boundary conditions are satisfied, Chafee and Infante proved that for fixed values of $\lambda > 0$ there are a finite number of stationary solutions to the problem (1.1), which bifurcate in pairs from the null solution at each point of a decreasing sequence $\{\lambda_n\}$, each new pair being symmetrical with respect to the abscissa axis and containing one more zero than the prior pair in such way that when $\lambda_n \to 0$, the number of stationary solutions of (1.1) tends to infinity.

Since this problem belongs to a class of problems in which trajectories asymptotically tend to the equilibrium points when $t \to \infty$, and also when $t \to -\infty$ in the case of complete trajectories, detailed knowledge of the stationary solutions is useful in understanding the attractor structure, which for gradient systems, is the set of all equilibrium solutions with their connecting trajectories, [6].

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Subsequently Takeuchi and Yamada published a detailed description of the bifurcation diagram for equilibria of the quasilinear problem

$$
\begin{gathered}
u_t = \lambda(|u_x|^{p-2}u_x)_x + |u|^{q-2}u(1-|u|^r), \quad (x,t) \in (0,1) \times (0,+\infty) \\
u(0,t) = u(1,t) = 0, \quad 0 \le t < +\infty \\
u(x,0) = u_0(x), \quad x \in (0,1),
\end{gathered} \tag{1.2}
$$

where $p > 2$, $q \ge 2$, $r > 0$ and $\lambda > 0$, taking into consideration the relations between p and q, [13]. Denoting by $E_\lambda = E_\lambda(p,q)$ the set of equilibria of problem (1.2), which describes a gradient system, we find that:

- If $p > q$, $E_\lambda = \{0\} \cup_{n=0}^{\infty} \pm E_\lambda^n$, where $E_\lambda^n = E_\lambda^n(p,q)$ is the set of stationary solutions ϕ_n with n zeros in $(0,1)$ and the sign $\pm$ indicates the sign of $(\phi_n)_x(0)$, any equilibrium $\phi \in +E_\lambda = E_\lambda$ has positive initial condition $\phi_x(0)$ and $-E_\lambda$ is the set of the opposites. In this case, there is a relevant sequence $\{\lambda_n\}$ such that, if $n \ge 1$ and $\lambda \ge \lambda_n$, then E_λ^n is a single set. E_λ^0 is always a single set.
- if $p = q$, E_λ changes depending on where the parameter λ is located with respect to two sequences, $\{\lambda_n^*\}$ and $\{\lambda_n\}$. The first sequence sets the maximum number of zeros allowed to an equilibrium. The second sequence, as in the prior case, states for each $n > 0$, if E_λ^n is a single or a continuum set. If $\lambda \ge \lambda_0^*$, then $E_\lambda = \{0\}$. If $\lambda_{M+1}^* \le \lambda < \lambda_M^*$, then $E_\lambda = \{0\} \cup_{n=0}^{M} \pm E_\lambda^n$. E_λ^0 is always a single set.
- if $p < q$, E_λ also changes according the position of λ with respect to two sequences $\{\lambda_n^*(p,q)\}$ and $\{\lambda_n\}$. Again, if $\lambda \ge \lambda_0^*$, $E_\lambda = \{0\}$. If $\lambda_{M+1}^* < \lambda \le \lambda_M^*$, then $E_\lambda = \{0\} \cup_{n=0}^{M} (\pm\{F_\lambda^n\} \cup \pm E_\lambda^n)$, where $F_\lambda^n = \{\psi_n\}$ and, if $\lambda = \lambda_n^*(p,q)$, $E_\lambda^n = \emptyset$, $E_\lambda^n = \{\phi_n\}$ if $\lambda_n \le \lambda < \lambda_n^*(p,q)$. Here ψ_n and ϕ_n are equilibria with n zeros in $(0,1)$, $|\psi_n(x)| < |\phi_n(x)|$ for all $x \in (0,1)$ except for zero points of ψ_n and ϕ_n.

In any case, if $n \ge 1$ and $\lambda < \lambda_n$, then E_λ^n is diffeomorphic to $[0,1]^n$.

When $p = q$ there are notable similarities between problems (1.1) and (1.2), particularly in regarding the stability properties of the equilibria. The trivial solution in each case is asymptotically stable for large values of the diffusion parameter λ and becomes unstable when the first pair of nontrivial equilibria bifurcates from null solution. These, in turn, remain asymptotically stable, while other stationary solutions are unstable.

The principal difference between problems (1.1) and (1.2) lies in the following. In the former, semilinear problem, although the number of elements in the equilibrium set tends to infinity when the diffusion goes to zero, it remains discrete, because the equilibria bifurcate from the trivial solution in pairs. In the later, quasilinear problem, however, the equilibrium set can contain continuum components if the diffusion coefficient is insufficiently large since the stationary solutions can reach their extremes at 1 and -1, which are zeros on the right side of the equation. Thus, stationary solutions can form flat cores when attaining these values and, although the sum of the lengths of all flat cores must be constant, it can be freely distributed among them. Accordingly there is a continuum of equilibrium solutions with the same number of zeros. This situation does not occur in the semilinear problem, as the "x-time" required for equilibria to achieve their extremes in 1 or -1 is infinite.

Nevertheless, in regard to problem (1.2), for each fixed value of λ, there are finite connected components of $E_\lambda(p,p)$, each composed of solutions containing the same number of zeros and bifurcating from the trivial solution. The attractor is the finite union of the unstable set of the connected components of E_λ. In this case, the attractor is the union of E_λ with the complete trajectories joining their connected parties, [2, 13]. It was proved in [2] that the problem (1.1) can be attained as a limit of (1.2) when $p \downarrow 2$ and, for each fixed value of λ, E_λ behaves continuously with relation to p, becoming discrete when p is located in some positive distance of 2.

Only the case $p = q$ is considered in [2], as in other cases, the complex configuration of the equilibrium sets diverges significantly. The purpose here is to prove the continuity of the equilibrium set of problem (1.2) with respect to q when $q \to p$. To this end, the following questions must be considered. For λ fixed, when $q \uparrow p$, even when q is close to p, given $n > 0$ there are at least two solutions in $E_\lambda(p, q)$ having n zeros in $(0, 1)$. When $p = q$, however, there exists a maximum value M such that any solution in $E_\lambda(p, p)$ have a number of zeros less than or equal to M. The value of M is determined as a function of the position of λ with respect to the points of the sequence $\{\lambda_n\}$. When $q \downarrow p$, the number of zeros in $(0, 1)$ of equilibria is bounded if $p = q$ or $p < q$ but the sequences that determine the maximum value of zeros for a stationary solution are distinct, being $\{\lambda_n^*\}$ in the first case and $\{\lambda_n^*(p, q)\}$ in the latter. Further, given n, $E_\lambda(p, q)$ can contain two entirely distinct equilibria with n zeros, $\pm\psi_\lambda^n$, that do not appear in the configuration of $E_\lambda(p, p)$. As will be shown in Section 4 these unanticipated equilibria, i.e., the stationary solutions which are not supposed to exist in case $p = q$, converges to the trivial solution when $q \to p$ despite the value of λ.

The lower continuity of attractors is not an easy problem and there is no much we know about. In the specific case when $p = q$ and the diffusion parameter λ is such that $\lambda_1^* \le \lambda \le \lambda_0^*$, then we can say that the attractors $\mathcal{A}_p$ of the problem (1.2) are lower semicontinuous at $p = 2$. This follows from the fact that, in this case, there exists only two complete trajectories for the Chafee-Infante problem (case $p = 2$), (see [8], p126), and then we can combine the continuity of the semigroups on p with the continuity of the equilibrium set to verify that each point on those complete trajectories can be reached as a limit of points on complete trajectories inside the attractors $\mathcal{A}_p, p > 2$.

Regarding the diffusion parameter λ, once λ_n depends on (p, q), the question arises if the connected components of $E_\lambda(p, q)$ and $E_\lambda(p, p)$ have similar cardinality properties when q is close to p, whether when $p \ne q$ the equilibrium components of $E_\lambda(p, q)$ that have natural correspondence with some component $E_\lambda^n(p, p)$ when $p = q$ are discrete or continuum according to the respective cardinality of $E_\lambda^n(p, p)$. The answer to this query is no, as detailed in Section 4. There is but one situation described in Case 3, Section 4, in which this fact must be addressed, but the continuity of $E_\lambda(p, q)$ is not affected. Similarly, despite the fact that sequence $\lambda_n^*(p, q)$ depends on q, the same maximum value M for the amount of zeros allowed to an equilibrium in $E_\lambda(p, p)$ and $E_\lambda(p, q)$, $p < q$, is found consistently.

Based on the preceding, the continuity of the sets $E_\lambda(p, q)$ is studied via the continuity properties of equilibria. In Section 4, the ordinary differential equation, which describes the stationary solutions of (1.2), is reviewed and its dependence on initial conditions and parameters is analyzed. Section 2 presents the required uniform estimates and locates the dynamics of problem (1.2) in $W_0^p(0, 1)$. Subsequently it is proven that the family of attractors $\mathcal{A}_{pq}$ is upper semicontinuous with respect to (p,q) in $C[0,1]$ topology. Additionally, if p remains fixed, $\mathcal{A}_{pq}$ is upper semicontinuous with respect to q in $W_0^{1,p}(0,1)$.

2. Uniform estimates and asymptotic properties

The asymptotic behavior of solutions of problem (1.2) is a well known issue, and it is not difficult to prove that (1.2) defines a semigroup which has a global attractor when set in $L^2(0,1)$ or even in $W_0^{1,p}(0,1)$, since it enjoys good properties of compactness and dissipativity. In this section we list all the necessary estimates to guarantee the existence of these attractors. Most of the results below is shown in [2], so we will just explicit the uniformity of the upper bounds with respect to parameters p and q when (p,q) is in a bounded subset R of $(2,\infty)\times[2,\infty)$. We will denote by u_{pq} a solution of (1.2).

We first need to obtain estimates for the $L^2(0,1)$ norm of solutions. This is done exactly as in [2, Lemma 2.1], whose statement is repeated here properly fitted to the context of this work.

Lemma 2.1. *Let u_{pq} be a solution of (1.2) with $u_{pq}(0)=u_0\in L^2(0,1)$. Given $T_0>0$ there exists $\widetilde{K}_1>0$ such that $\|u_{pq}(t)\|_2<\widetilde{K}_1$ for $t\ge T_0$ and $(p,q)\in R$. Furthermore, given $B\subset L^2(0,1)$, B bounded, there exists $K_1>0$ such that $\|u_{pq}(t)\|_2<K_1$ for $t\ge 0$, $(p,q)\in R$ and $u_0\in B$. The positive constants $\widetilde{K}_1, K_1$ are independent of $(p,q)\in R$, $r>0$ and $\lambda>0$.*

Remark 2.2. We note that the constant $\widetilde{K}_1$ gives us a $L^2(0,1)$ estimate after some time has elapsed from the origin, and it is uniform on $(p,q)\in R$, completely independent of the initial data and uniform on bounded sets with respect to the parameter r. The constant K_1, which estimates solutions since the origin, carries, as expected, a dependence on the initial data which is uniform however on bounded subsets of $L^2(0,1)$.

To establish the estimates on $W_0^{1,p}(0,1)$ we introduce the following notation: $\varphi^1_{pq},\varphi^2_q : L^2(0,1)\to\mathbb{R}$ given by

$$\varphi^1_{pq}(u)\doteq\begin{cases}\frac{\lambda}{p}\int_0^1|u_x(x)|^p dx+\frac{1}{q+r}\int_0^1|u(x)|^{q+r}dx, & u\in W_0^{1,p}(0,1)\\ +\infty, & \text{otherwise,}\end{cases}$$

and

$$\varphi^2_q(u)\doteq\begin{cases}\frac{1}{q}\int_0^1|u(x)|^q dx, & u\in L^q(0,1)\\ +\infty, & \text{otherwise}\end{cases}$$

It is advantageous to rewrite the equation in (1.2) in an abstract way involving the difference of two subdifferential operators. Thus the existence of global solutions is easily obtained as a consequence of [11] and new estimates can be obtained, this time in a stronger norm.

$$\frac{du}{dt}(t)+\partial\varphi^1_{pq}(u(t))-\partial\varphi^2_q(u(t))=0 \tag{2.1}$$

where $\partial\varphi^1_{pq}$ and $\partial\varphi^2_q$ are subdifferential of φ^1_{pq} and φ^2_q respectively.

Remark 2.3. Given c_0 and q_M, $0<c_0<1$, $q_M>2$, there exists $c>0$ depending only on r and c_0 such that

$$\varphi^2_q(u)\le c_0\varphi^1_{pq}(u)+c$$

for each $u \in W_0^{1,p}(0,1)$, $\lambda > 0$ and $2 \le q \le q_M$. In fact, if $\eta > 0$,

$$\begin{aligned}\varphi_q^2(u) &= \frac{1}{q}\|u\|_{L^q(0,1)}^q \\ &\le \frac{r}{(q+r)(\eta q)^{\frac{q+r}{r}}} + q\eta^{\frac{q+r}{q}}\Big(\frac{\lambda}{p}\|u\|_{W_0^{1,p}(0,1)}^p + \frac{1}{q+r}\|u\|_{L^{q+r}(0,1)}^{q+r}\Big).\end{aligned}$$

Let $g(q) = (\frac{c_0}{q})^{\frac{q}{q+r}} = e^{\frac{q}{q+r}\ln(c_0/q)}$, then $g'(q) < 0$ and it is enough to choose η such that $0 < \eta < (\frac{c_0}{q_M})^{\frac{q_M}{q_M+r}}$ for $2 \le q \le q_M$. Then $q\eta^{\frac{q+r}{q}} \le c_0$ and

$$c \doteq \frac{r}{(2+r)2^{\frac{2+r}{2}}\eta^{\frac{q_M+r}{r}}}\,.$$

The following lemmas show the estimates we have in $W_0^{1,p}(0,1)$ norm. Note that, even if the initial data are taken into $L^2(0,1)$, since the flow is governed by a subdifferential (so it has good smoothing properties), we can ensure strong estimates in $W_0^{1,p}(0,1)$ from any positive time elapsed from the origin.

Lemma 2.4. *Given $\delta > 0$ there exists $\tilde{K}_2 > 0$ such that $\|u_{pq}(t)\|_{W_0^{p,q}(0,1)} \le \tilde{K}_2$ for $t \ge \delta$ and for all initial data u_0 in $L^2(0,1)$*

Remark 2.5. The above lemma is a direct consequence of [13, Lemma 2.1] and the first assertion of Lemma 2.1. The constant $\tilde{K}_2$ carries the same dependence of $\tilde{K}_1$, that means, it is uniform on $(p,q) \in R$, completely independent of the initial data and uniform on bounded sets with respect to the parameter r.

However, if we are interested in estimates since the beginning of evolution, so we naturally find upper bounds dependent on the initial data. The demonstration is exactly the same as [2, Lemma 2.2].

Lemma 2.6. *Let u_{pq} be a solution of (1.2) with $u_{pq}(0) = u_0 \in W_0^{1,p}(0,1)$. Given $M > 0$ there exists a positive constant $K_2 > 0$ such that $\|u_{pq}(t)\|_{W_0^{1,p}(0,1)} < K_2$ for $t \ge 0$ and $(p,q) \in R$. Furthermore, the positive constant K_2 can be uniformly chosen for $(p,q) \in R$, and $\|u_0\|_{W_0^{1,p}(0,1)} \le M$.*

Finally, from the above lemma we conclude our set of uniform estimates of $\{u_{pq}\}$, giving bounds to the solutions of the problem (1.2) in $L^\infty(0,1)$.

Lemma 2.7. *Let u_{pq} be a solution of (1.2) with $u_{pq}(0) = u_0 \in W_0^{1,p}(0,1)$ and $\|u_0\|_{W_0^{1,p}(0,1)} \le M$. From Lemma 2.6 we obtain*

$$\|u_{pq}(t)\|_\infty \le K_3(M), \quad t \ge 0.$$

Remark 2.8. If the initial data are in $L^2(0,1) - W_0^{1,p}(0,1)$, for each $\delta > 0$ we find $\tilde{K}_3$ depending on δ and p, with

$$\|u_{pq}(t)\|_\infty \le \tilde{K}_3(\delta,p), \quad t \ge \delta.$$

The existence of the global attractor in $L^2(0,1)$ is a simple consequence of Lemma 2.1 and Lemma 2.6, as it is claimed in [2, Corollary 2.3]. It is also very simple to obtain the existence of global attractor to the restriction of the semigroup to the space $W_0^{1,p}(0,1)$. In fact, for each $(p,q) \in R$, let us denote by $\{S_{pq}(t)\}$ the semigroup associated with problem (1.2) in $W_0^{1,p}(0,1)$. We prove below that $\{S_{pq}(t)\}$ is a continuous semigroup of compact operators. The following result will be necessary.

Lemma 2.9. *Let $B \subset W_0^{1,p}(0,1)$ be a bounded set and let $T > 0$, $q_M > 2$. There is a constant K_4 such $\|\frac{\partial}{\partial t}u_{pq}(t)\|_{L^2(0,1)} \le K_4$ for any $p > 2$, $2 < q < q_M$, $t \in [0,T]$ and $u_0 \in B$.*

Proof. Multiplying the equation in (1.2) by $\frac{\partial}{\partial t}u_{pq}(t)$ and integrating from 0 to T we obtain

$$\begin{aligned}\int_0^T \|\frac{\partial}{\partial t}u_{pq}(s)\|^2_{L^2(0,1)}ds + \varphi^1_{pq}(u_{pq}(T)) &\le \varphi^2_q(u_{pq}(T)) + \varphi^1_{pq}(u_{pq}(0)) \\ &\le c_0\varphi^1_{pq}(u_{pq}(T)) + \varphi^1_{pq}(u_{pq}(0)) + c\end{aligned} \tag{2.2}$$

where $c_0 < 1$ and c is the same of Remark 2.3. On the other hand, if we denote $f_q(s) \doteq |s|^{q-2}s(1-|s|^r)$ then

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_{pq}(t+h) - u_{pq}(t)\|^2_{L^2(0,1)} &= \langle \frac{\partial}{\partial t}u_{pq}(t+h) - \frac{\partial}{\partial t}u_{pq}(t), u_{pq}(t+h) - u_{pq}(t)\rangle \\ &\le \langle f_q u_{pq}(t+h) - f_q u_{pq}(t), u_{pq}(t+h) - u_{pq}(t)\rangle \\ &\le C\|u_{pq}(t+h) - u_{pq}(t)\|^2_{L^2(0,1)},\end{aligned}$$

where $C \doteq (q_M - 1)^{\frac{q_M+r-2}{r}}$. So by Gronwall we obtain

$$\|u_{pq}(t+h) - u_{pq}(t)\|^2_{L^2(0,1)} \le \|u_{pq}(s+h) - u_{pq}(s)\|^2_{L^2(0,1)}e^{2CT}.$$

Therefore,

$$T\|\frac{\partial}{\partial t}u_{pq}(t)\|^2_{L^2(0,1)} \le \int_0^T \|\frac{\partial}{\partial t}u_{pq}(s)\|^2_{L^2(0,1)}ds\, e^{2CT}$$

and from (2.2) we conclude that

$$\|\frac{\partial}{\partial t}u_{pq}(t)\|_{L^2(0,1)} \le K_4.$$

□

Theorem 2.10. *For each $t > 0$ the mapping $S_{pq}(t) : W_0^{1,p}(0,1) \to W_0^{1,p}(0,1)$ is continuous and compact.*

Proof. Let $T > 0$, $0 < t < T$ and $\{u_{0_n}\} \subset W_0^{1,p}(0,1)$ a sequence converging to u_0 in $W_0^{1,p}(0,1)$. Then $u_{0_n} \to u_0$ in $L^2(0,1)$ and, from [13, Lemma 2.1], $S_{pq}(\cdot)u_{0_n} \to S_{pq}(\cdot)u_0$ in $L^p(0,T;W_0^{1,p}(0,1))$. Therefore we can conclude that exists a subsequence denoted by $\{u_n(t)\} \subset \{S_{pq}(t)u_{0_n}\}$ which converges to $u(t) \doteq S_{pq}(t)u_0$ a.e. in $[0,T]$. Let $A \subset [0,T]$ the set where $\|u_n(\cdot) - u(\cdot)\|_{W_0^{1,p}(0,1)} \to 0$. Given an arbitrary $t \in (0,T]$ we claim that $\|u_n(t)\|_{W_0^{1,p}(0,1)} \to \|u(t)\|_{W_0^{1,p}(0,1)}$. In fact, for each $\theta \in A$

$$\begin{aligned}|\varphi^1_{pq}(u_n(t)) - \varphi^1_{pq}(u(t))| &\le |\varphi^1_{pq}(u_n(t)) - \varphi^1_{pq}(u_n(\theta))| + |\varphi^1_{pq}(u_n(\theta)) - \varphi^1_{pq}(u(\theta))| \\ &\quad + |\varphi^1_{pq}(u(\theta)) - \varphi^1_{pq}(u(t))|\end{aligned}$$

and

$$\begin{aligned}|\varphi^1_{pq}(u_n(t)) - \varphi^1_{pq}(u_n(\theta))| &\le \int_\theta^t \Big|\langle \partial\varphi^1_{pq}(u_n(s)), \frac{\partial}{\partial s}u_n(s)\rangle\Big| ds \\ &\le \frac{3}{2}\int_\theta^t \|\frac{\partial}{\partial s}u_n(s)\|^2_{L^2(0,1)}ds + \frac{1}{2}\int_\theta^t \|f_q(u_n(s))\|^2_{L^2(0,1)}ds,\end{aligned}$$

where $f_q(s) \doteq |s|^{q-2}s(1-|s|^r)$. We can obtain the same result changing u_n by u in the above inequality. So, it follows from Lemma 2.7 and Lemma 2.9 that, given

$\eta > 0$, we can choose $\theta \in A$ close to t and n large enough to obtain $|\varphi^1_{pq}(u_n(t)) - \varphi^1_{pq}(u(t))| \leq \eta$. Therefore we conclude that $u_n(t) \to u(t)$ in $W^{1,p}_0(0,1)$.

We observe that, in fact, this proof shows that $S_{pq}(t)$ is continuous from $L^2(0,1)$ to $W^{1,p}_0(0,1)$.

To prove the second statement, let $B \subset W^{1,p}_0(0,1)$ a bounded subset. Let us prove that $S_{pq}(t)B$ is relatively compact in $W^{1,p}_0(0,1)$. As $W^{1,p}_0(0,1)$ is compactly immersed in $L^2(0,1)$, given any sequence $\{u_{0_n}\} \subset B$, there is u_0 such that $u_{0_n} \to u_0 \in L^2(0,1)$ and so $S_{pq}(t)u_{0_n} \to S_{pq}(t)u_0$ in $W^{1,p}_0(0,1)$, which concludes the proof. □

The existence of a global attractor $\mathcal{A}_{pq}$ for $S_{pq}(t)$ in $W^{1,p}_0(0,1)$ is a consequence of Lemma 2.6, Theorem 2.10, and [9, Theorem 2.2].

Proposition 2.11. *Given $(p,q) \in R$, let $S_{pq}(t) : W^{1,p}_0(0,1) \to W^{1,p}_0(0,1)$ the semigroup determined by problem* (1.2). *Then $\{S_{pq}(t)\}$ has a global attractor, which is compact and invariant.*

3. Continuity of flows and upper semicontinuity of the attractors

In this section we proof that, given $T > 0$ and $(p_0, q_0) \in R$, the solutions $\{u_{pq}\}$ of (1.2) go to the solution $u_{p_0q_0}$ of (1.2) in $C([0,T]; L^2(0,1))$, when $p \to p_0$ and $q \to q_0$. After that, we will obtain the upper semicontinuity of the family of global attractors

$$\{\mathcal{A}_{pq} \subset W^{1,p}_0(0,1);\ (p,q) \in R\}$$

of (1.2) at (p_0, q_0) in the topologies of $L^2(0,1)$ and $C([0,1])$. Furthermore, when $p = p_0$ we will prove the upper semicontinuity in $W^{1,p_0}_0(0,1)$.

First of all we observe that from Section 2, there exists a positive constant M, independent of $t \geq 0$ and $(p,q) \in R$, such that

$$\|u_{pq}(t)\|_{W^{1,p}_0(0,1)} \leq M$$

for all $t \geq 0$ and $(p,q) \in R$. Following exactly the same steps in Section 3 of [2] we obtain an adapted version of Baras'Theorem, [15], as we state bellow.

Lemma 3.1. *Given $T > 0$, the set*

$$\begin{aligned} M^{pq} := \big\{ & u_{pq} \subset W^{1,p}_0(0,1) : (p,q) \in R,\ u_{pq} \text{ is a solution of } (1.2) \text{ with} \\ & u_{pq}(0) = u_{0\,pq} \in W^{1,p}_0(0,1),\ u_{0\,pq} \to u_0 \text{ as } (p,q) \to (p_0,q_0) \text{ in } L^2(0,1) \\ & \text{and } \|u_{0\,pq}\|_{W^{1,p}_0(0,1)} \leq M,\ \forall (p,q) \in R \big\}, \end{aligned}$$

is relatively compact in $C([0,T]; L^2(0,1))$.

Theorem 3.2. *For each $(p,q) \in R$, let $\{u_{pq}\} \subset W^{1,p}_0(0,1)$ be a solution of*

$$\begin{gathered} \frac{d}{dt} u_{pq}(t) - \lambda(|(u_{pq}(t))_x|^{p-2}(u_{pq}(t))_x)_x = |u_{pq}(t)|^{q-2}u_{pq}(t)(1 + |u(t)|^r_{pq}), \quad t > 0 \\ u_{pq}(0) = u_{0\,pq} \in W^{1,p}_0(0,1). \end{gathered}$$

Suppose that $\|u_{0\,pq}\|_{W^{1,p}_0(0,1)} \leq M$ for every $(p,q) \in R$ and $u_{0\,pq} \to u_0$ as $(p,q) \to (p_0,q_0)$ in $L^2(0,1)$. Then, for each $T > 0$, $u_{pq} \to u$ in $C([0,T]; L^2(0,1))$ as $(p,q) \to (p_0,q_0)$, where u is a solution of

$$\frac{d}{dt} u(t) - \lambda(|u_x(t)|^{p_0-2}u_x(t))_x = |u(t)|^{q_0-2}u(t)(1 + |u(t)|^r), \quad t > 0$$

$$u(0) = u_0 \in L^2(0,1).$$

Proof. Throughout this proof we denote

$$f_q(v(t)) = |v(t)|^{q-2}v(t)(1+|v(t)|^r).$$

Since

$$\|u_{pq}(t)\|_{W_0^{1,p}(0,1)} \leq M$$

for all $t > 0$, $(p,q) \in R$ with M independent of $t \geq 0$ and $(p,q) \in R$, we obtain that $\{u_{pq}(t)\}$ is uniformly bounded in $L^\infty(0,1)$ for $(p,q) \in R$ and $t \in [0,T]$. Furthermore, from Lemma 3.1, $\{u_{pq}\}$ converges in $C([0,T];L^2(0,1))$ to a function $u : [0,T] \to L^2(0,1)$, when $p \to p_0$ and $q \to q_0$. Since $f_q(u_{pq})$ is uniformly integrable in $L^1([0,T];L^2(0,1))$ and

$$\begin{aligned} \|f_q(u_{pq}(t)) - f_{q_0}(v(t))\| &\leq \|f_q(u_{pq}(t)) - f_{q_0}(u_{pq}(t))\| + \|f_{q_0}(u_{pq}(t)) - f_{q_0}(u(t))\| \\ &\leq \bar{K}|q-q_0| + \tilde{K}|u_{pq}(t) - u(t)|, \end{aligned}$$

we obtain $f_q(u_{pq}(t)) \to f_{q_0}(u(t))$ in $L^2(0,1)$ for each $t > 0$ when $p \to p_0$ and $q \to q_0$. Now, with the same arguments used in [2] we obtain that u is a weak solution of

$$\begin{aligned} \frac{d}{dt}u(t) - \lambda(|u_x(t)|^{p_0-2}u_x(t))_x = |u(t)|^{q_0-2}u(t)(1+|u(t)|^r), \quad t > 0 \\ u(0) = u_0 \in L^2(0,1). \end{aligned}$$

and we obtain the desired result. □

Corollary 3.3. *The family of global attractors $\{\mathcal{A}_{pq} \subset W_0^{1,p}(0,1)) : (p,q) \in R\}$ of problem* (1.2) *is upper semicontinuous at (p_0,q_0) in the $L^2(0,1)$ topology.*

Proof. The results in Section 2 imply that there exists a bounded set $\mathcal{B} \subset L^2(0,1)$ such that $\mathcal{A}_{pq} \subset \mathcal{B}$, for every $(p,q) \in R$. Since $\mathcal{A}_{p_0,q_0}$ attracts bounded sets of $L^2(0,1)$, for every $\delta > 0$, there is $T_1 > 0$ in such way that

$$\sup_{\psi_{pq} \in \mathcal{A}_{pq},\, (p,q) \in R} \operatorname{dist}_{L^2(\Omega)}(u_{p_0,q_0}(T_1;\psi_{pq}), \mathcal{A}_{p_0,q_0}) \leq \frac{\delta}{2},$$

where $u_{p_0q_0}(t;\psi_{pq})$ is a solution of problem (1.2) when $p = p_0$ and $q = q_0$ with initial condition ψ_{pq}.

Now, the previous results in this section imply that there exist $\delta_0 > 0$ and $\epsilon > 0$ such that

$$\|u_{pq}(t;\psi_{pq}) - u_{p_0q_0}(t;\psi_{pq})\|_{L^2(0,1)} < \frac{\delta}{2},$$

for $|p-p_0| < \delta_0$, $|q-q_0| < \epsilon$ and $T \geq t \geq T_1$.

Thus, for $|p-p_0| < \delta_0$, we obtain

$$\begin{aligned} &\operatorname{dist}_{L^2(0,1)}(u_{pq}(T_1;\psi_{pq}), \mathcal{A}_{p_0q_0}) \\ &\leq \|u_{pq}(T_1,\psi_{pq}) - u_{p_0q_0}(T_1;\psi_{pq})\|_{L^2(0,1)} + \operatorname{dist}_{L^2(0,1)}(u_{p_0q_0}(T_1,\psi_{pq}), \mathcal{A}_{p_0q_0}) < \delta. \end{aligned}$$

On the other hand, it follows from the invariance of the attractors that

$$\operatorname{dist}_{L^2(0,1)}(\mathcal{A}_{pq}, \mathcal{A}_{p_0q_0}) \leq \delta,$$

for every $|p-p_0| < \delta_0$ and q such that $|q-q_0| \leq \epsilon$ showing the upper semicontinuity desired. □

Remark 3.4. It follows from Theorem 2.6, Corollary 3.3, [3, Lemma 1.1] and the compact immersion of $W_0^{1,2}(0,1)$ in $C([0,1]$ that the family $\{\mathcal{A}_{pq}\}$ is upper semicontinuous at (p_0,q_0) in the topology of $C([0,1])$.

Now we are interested in obtaining the upper semicontinuity of global attractors of (1.2) in a stronger topology. To do that, we consider p fixed, and $q \to q_0$.

Theorem 3.5. *For each $(p_0,q) \in R$, let $\{u_{p_0q}\} \subset W_0^{1,p_0}(0,1)$ be a solution of*

$$\begin{gathered}
\frac{d}{dt}u_{p_0q}(t) - \lambda(|(u_{p_0q}(t))_x|^{p_0-2}(u_{p_0q}(t))_x)_x \\
= |u_{p_0q}(t)|^{q-2}u_{p_0q}(t)(1+|u(t)_{p_0q}|^r), \quad t>0 \\
u_{p_0q}(0) = u_{0\,p_0q} \in W_0^{1,p_0}(0,1).
\end{gathered}$$

Suppose that $\|u_{0\,p_0q}\|_{W_0^{1,p_0}(0,1)} \le M$ for every $(p_0,q) \in R$ and $u_{0\,p_0q} \to u_0$ in $L^2(0,1)$ as $q \to q_0$. Then, for each $T>0$, $u_{p_0q} \to u$ in $C([0,T];W_0^{1,p_0}(0,1))$ as $q \to q_0$, where u is a solution of

$$\begin{gathered}
\frac{d}{dt}u(t) - \lambda(|(u(t))_x|^{p_0-2}(u(t))_x)_x = |u(t)|^{q_0-2}u(t)(1+|u(t)|^r), \quad t>0 \\
u(0) = u_0 \in W_0^{1,p_0}(0,1).
\end{gathered}$$

The above theorem is a simple consequence of Tartar's Inequality, Lemma 2.7 and Lemma 2.9.

With the same argument as in the proof of Corollary 3.3 we can prove the next result.

Corollary 3.6. *The family of global attractors $\{\mathcal{A}_{pq} \subset W_0^{1,p_0}(0,1)) : (p_0,q) \in R\}$ of problem (1.2) is upper semicontinuous at (p_0,q_0) in the topology of $W_0^{1,p_0}(0,1)$.*

4. Continuity of equilibrium sets

In this section, considering p fixed, we prove the continuity of the family of equilibrium points of the equation (1.2) when q goes to p. To analyze the continuity of the equilibrium sets it is interesting to remember how the stationary solutions are obtained in [13].

Let $\phi_{\alpha q}$ be a solution of

$$\begin{gathered}
\lambda(\psi)_x + f_q(\phi_{\alpha q}) = 0, \quad \text{in } (0,\infty) \\
\phi_{\alpha q}(0) = 0, \\
\psi(0) = \alpha
\end{gathered} \tag{4.1}$$

where α is a parameter, $\psi = |(\phi_{\alpha q})_x|^{p-2}(\phi_{\alpha q})_x$ and $f_q(\phi) = |\phi|^{q-2}\phi(1-|\phi|^r)$. We observe that, in order to a solution of (4.1) be an equilibrium point of (1.2), α must be such that $\phi_{\alpha q}(1) = 0$.

We denote by $X(\alpha,p,q)$ the function that measure the x-time that the solution $\phi_{\alpha q}$ of (4.1) takes to reach the first maximum point. Because of the symmetry we have that $\phi(2X(\alpha,p,q)) = 0$ or, more generally, $\phi_{\alpha,q}(2kX(\alpha,p,q)) = 0$, $k = 1,2,\dots$. Also, we have that $2nX(\alpha,p,q) = 1$ is a sufficient condition to $\phi_{\alpha q}$ be an equilibrium point of (1.2) with $n-1$ zeros in $(0,1) \subset \mathbb{R}$. Due to the symmetry of the problem, we can only consider $\alpha > 0$.

The function X is

$$X(\alpha,p,q)=\Big(\frac{\lambda(p-1)}{p}\Big)^{1/p}I(p,q,\tilde{\phi}_{\alpha,q}),$$

where $\tilde{\phi}_{\alpha,q}$ is the maximum value of $\phi_{\alpha,q}$ and

$$I(p,q,a)=\int_0^a(F_q(a)-F_q(\phi))^{-1/p}d\phi,$$

with $F_q(\phi)=F(\phi,q)=\int_0^\phi f_q(s)ds=\frac{\phi^q}{q}-\frac{\phi^{q+r}}{q+r}\in C^1((0,\infty)\times(2,\infty))$.

In [13], the authors studied the behavior of the function $Y(p,q)$, which describes the distance between two consecutive zeros of an equilibrium and, analyzing their graphs for $p>q$, $p=q$ and $p<q$, they obtain that if $p>q$ there exists a decreasing sequence $\lambda_n(p,q)$, $\lambda_n(p,q)\to 0$ when $n\to\infty$ such that the equilibrium set $E=\{0\}\cup\cup_{i=0}^{\infty}E_i^{\pm}$ where $E_i^{\pm}$ denote the equilibrium sets within the equilibria with i zeros in $(0,1)$ and if $\lambda<\lambda_n(p,q)$, the set $E_i^{\pm}$ is diffeomorphic to $[0,1]^i$, for $1\le i\le n$. We observe that in this case there are equilibrium points with any amount of zeros in $(0,1)$.

If $p\le q$ there exist decreasing sequences $\lambda_n(p,q)\to 0$ and $\lambda_n^*(p,q)\to 0$ such that $\lambda_n^*(p,q)>\lambda_n(p,q)$. If $p=q$, for $\lambda_{M+1}\le\lambda<\lambda_M$, the equilibrium set is given by $E=\{0\}\cup\cup_{i=0}^{M}E_i^{\pm}$. If $p<q$, for $\lambda_{M+1}<\lambda\le\lambda_M$ the equilibrium set is given by $E=\{0\}\cup\cup_{i=0}^{M}(E_i^{\pm}\cup\{F_i^{\pm}\})$, where $E_i^{\pm}$ denote the equilibrium sets containing equilibria with i zeros in $(0,1)$ and $F_i^{\pm}=\{\psi_i^{\pm}\}$ also is equilibrium with i zeros in $(0,1)$. Furthermore, if $p\le q$ and $\lambda<\lambda_n(p,q)$, the set $E_i^{\pm}$ is diffeomorphic to $[0,1]^i$, for $1\le i\le n$. In any case, $E_0^{\pm}=\{\phi_0^{\pm}\}$ for $\lambda<\lambda_0(p,q)$.

About the stability of the equilibria, in [13, Theorems 4.2, 4.3], they obtain that 0 is asymptotically stable if $p=q$ and $\lambda\ge\lambda_0$ or if $p<q$, 0 is unstable for $p>q$ or $p=q$ and $\lambda<\lambda_0$. The equilibrium ϕ_0^{+} is asymptotically stable if $\lambda>\lambda_0^*$ and attractive for $\lambda\le\lambda_0^*$, and if $q>p$, ψ_0 is unstable for $\lambda\le\lambda_0$.

Since we deal with the dependence on the parameter q and there are qualitative changes in the equilibrium sets depending on the relation between p and q, if necessary, we will exhibit explicitly the parameters p and q.

To prove the continuity of the equilibrium set, we take a sequence of equilibria in $E_i^{\pm}$ with a fixed number of zeros and, analyzing the initial slopes of such stationary solutions, we conclude through the continuity properties of problem (4.1), that this sequence must converge to an equilibrium point of the limit problem with the same amount of zeros in $(0,1)$ or, when it is not possible, the sequence converges to the null stationary solution. We also prove that any sequence of equilibria taken in $\{\psi_i\}\subset F_i^{\pm}$ converges to zero.

We start with the analysis of the dependence of $\tilde{\phi}_{\alpha,q}$ on q and α. We know that $\tilde{\phi}_{\alpha,q}$ is strictly increasing and C^1 in α, $\alpha\in[0,\alpha_0)$ (see [2]). With respect to q, since $\tilde{\phi}_{\alpha,q}$ is the maximum value of $\phi_{\alpha,q}$, then $\tilde{\phi}_{\alpha,q}$ satisfies

$$F(\tilde{\phi}_{\alpha,q},q)=\lambda\frac{(p-1)}{p}|\alpha|^{\frac{p}{p-1}}.$$

Calculating

$$\frac{\partial}{\partial q}F(\phi,q)=\frac{\phi^q(q\ln\phi-1)}{q^2}-\frac{\phi^{q+r}((q+r)\ln\phi-1)}{(q+r)^2}=\beta(q)-\beta(q+r),$$

where $\beta(\theta)=\frac{\phi^\theta(\theta\ln\phi-1)}{\theta^2}$, for $\theta\ge 2$. As $\beta'(\theta)>0$, thus $\frac{\partial}{\partial q}F(\phi,q)<0$.

Also $\frac{\partial}{\partial\phi}F(\phi,q)=\phi^{q-1}-\phi^{q+r-1}>0$, if $\phi\in(0,1)$. Using the Implicit Function Theorem, we obtain that the map $\tilde{\phi}_{\alpha q}$ is C^1 on (α,q) . Also,

$$\frac{\partial}{\partial q}\tilde{\phi}_{\alpha q}=-\frac{\frac{\partial}{\partial q}F(\tilde{\phi}_{\alpha q},q)}{\frac{\partial}{\partial\phi}F(\tilde{\phi}_{\alpha q},q)}>0,$$

then $\tilde{\phi}_{\alpha q}$ is strictly increasing on q.

Now we analyze the function $I(p,q,a)$. In [13], the authors rewrite $I(p,q,a)$ as

$$I=I(p,q,a)=\int_0^a(F_q(a)-F_q(\phi))^{-1/p}d\phi=a^{1-q/p}\int_0^1\Phi_q(s,a)^{-1/p}ds,$$

where $\Phi_q(s,a)=\frac{1-s^q}{q}-\frac{1-s^{q+r}}{q+r}a^r$. Then we obtain $I(p,q,a)$ is C^2 on $(2,\infty)\times[2,\infty)\times(0,1]$. For each p fixed, we analyze the behavior of $I(p,q,a)$ with respect the parameter q. We study the behavior of $I(p,q,a)$ with respect to q for a close to zero because $I(p,q,a)$ is C^2 on $(2,\infty)\times[2,\infty)\times(0,1]$ and the major difference in the cases occurs close to zero. We prove that $I(p,q,a)$ is increasing with respect to q for a near to zero.

Lemma 4.1. *For $0\le a<e^{-1/2}$ fixed, $\frac{\partial}{\partial q}I(p,q,a)>0$, for $(p,q)\in(2,\infty)\times[2,\infty)$.*

Proof. In fact, since $I(p,q,a)=\int_0^a(F_q(a)-F_q(\phi))^{-1/p}d\phi$, it follows that

$$\begin{aligned}\frac{\partial}{\partial q}I(p,q,a)&=\int_0^a\frac{\partial}{\partial q}(F_q(a)-F_q(\phi))^{-1/p}d\phi\\&=\int_0^a-\frac{1}{p}(F_q(a)-F_q(\phi))^{-1/p-1}\frac{\partial}{\partial q}(F_q(a)-F_q(\phi))d\phi\end{aligned}$$

Since $(F_q(a)-F_q(\phi))^{-1/p-1}>0$, we only consider

$$\begin{aligned}\frac{\partial}{\partial q}(F_q(a)-F_q(\phi))&=\frac{a^q\ln(a)}{q}+\frac{-a^q}{q^2}-\frac{a^{q+r}\ln(a)}{(q+r)}+\frac{a^{q+r}}{(q+r)^2}\\&\quad-\Big[\frac{\phi^q\ln(\phi)}{q}+\frac{-\phi^q}{q^2}-\frac{\phi^{q+r}\ln(\phi)}{(q+r)}+\frac{\phi^{q+r}}{(q+r)^2}\Big]\end{aligned}$$

Now we define $\varphi(\theta)=\frac{a^\theta}{\theta^2}-\frac{\phi^\theta}{\theta^2}$. Then $\varphi'(\theta)\le0$, thus $\varphi(q+r)-\varphi(q)<0$. Define also $\psi(\theta)=\frac{\theta^q\ln(\theta)}{q}-\frac{\theta^{q+r}\ln(\theta)}{(q+r)}$. Then, for $\theta<e^{-1/2}$

$$\psi'(\theta)\le[\theta^{q-1}-\theta^{q+r-1}]\Big(\ln\theta+\frac{1}{q+r}\Big)<0,$$

thus $\psi(a)-\psi(\phi)<0$ for $0<\phi<a<e^{-1/2}$. Therefore,

$$\frac{\partial}{\partial q}(F_q(a)-F_q(\phi))=\varphi(q+r)-\varphi(q)+\psi(a)-\psi(\phi)<0 \tag{4.2}$$

Finally, we obtain

$$\frac{\partial I}{\partial q}(p,q,a)=\int_0^a-\frac{1}{p}(F_q(a)-F_q(\phi))^{-1/p-1}\frac{\partial}{\partial q}(F_q(a)-F_q(\phi))d\phi>0. \tag{4.3}$$

□

Now we consider $p > q$. In [13], the authors show that $\frac{\partial I}{\partial a}(p, q, a) > 0$ for $q < p$, then

$$\frac{\partial X}{\partial q}(\alpha, p, q) = \big(\frac{\lambda(p-1)}{p}\big)^{1/p}\Big(\frac{\partial I}{\partial q}(p, q, \tilde{\phi}_{\alpha q}) + \frac{\partial I}{\partial a}(p, q, \tilde{\phi}_{\alpha q})\frac{\partial \tilde{\phi}_{\alpha q}}{\partial q}\Big) > 0, \quad p > q.$$

Fixed p and n, for each $q < p$, we consider α_q^n the initial condition such that the x-time $X(\alpha_q^n, p, q)$ is kept constant and equal to $1/2n$. We have that

$$\begin{aligned} 0 = \frac{dX}{dq}(\alpha_q^n, p, q) &= \frac{\partial X}{\partial \alpha}(\alpha_q^n, p, q)\frac{d\alpha}{dq} + \frac{\partial X}{\partial q}(\alpha_q^n, p, q) \\ &= \big(\frac{\lambda(p-1)}{p}\big)^{1/p}\big[\frac{\partial I}{\partial a}(p, q, \tilde{\phi}_{\alpha q})\frac{\partial \tilde{\phi}_{\alpha q}}{\partial \alpha}\frac{d\alpha}{dq} + \frac{\partial I}{\partial q}(p, q, \tilde{\phi}_{\alpha q}]) + \frac{\partial I}{\partial a}(p, q, \tilde{\phi}_{\alpha q})\frac{\partial \tilde{\phi}_{\alpha q}}{\partial q}\big] \\ &= \big(\frac{\lambda(p-1)}{p}\big)^{1/p}\big[\frac{\partial I}{\partial a}(p, q, \tilde{\phi}_{\alpha q})(\frac{\partial \tilde{\phi}_{\alpha q}}{\partial \alpha}\frac{d\alpha}{dq} + \frac{\partial \tilde{\phi}_{\alpha q}}{\partial q}) + \frac{\partial I}{\partial q}(p, q, \tilde{\phi}_{\alpha q})\big]. \end{aligned}$$

Since $\frac{d}{dq}(\tilde{\phi}_{\alpha q}) = \frac{\partial \tilde{\phi}_{\alpha q}}{\partial \alpha}\frac{d\alpha}{dq} + \frac{\partial \tilde{\phi}_{\alpha q}}{\partial q}$, $\frac{\partial I}{\partial q} > 0$, $\frac{\partial I}{\partial a} > 0$ for $q < p$, and $\frac{\partial \tilde{\phi}_{\alpha q}}{\partial \alpha} > 0$, $\frac{\partial \tilde{\phi}_{\alpha q}}{\partial q} > 0$ then we conclude that $\frac{d\alpha}{dq} < 0$. We summarize the previous results in the following lemma.

Lemma 4.2. *If $p > q$, let $\alpha(q)$ be such that $X(\alpha(q), p, q)$ remains constant. Then $\alpha(q)$ is decreasing with respect to q.*

Now we can prove the following result.

Theorem 4.3. *Suppose $p > 2$ fixed. Let M be the maximum number of zeros of an equilibrium when $q = p$. Let $\phi_n(q) \in E_n^{\pm}$ for $p > q$. If $n \le M$, then $\phi_n(q)$ converges to another stationary solution, with the same amount of zeros when $q \to p^-$. If n is greater than M, then $\|\phi_n(q)\|_{C^1(0,1)}$ goes to zero when $q \to p^-$.*

Proof. We rewrite (4.1) in the form

$$\dot{z} = h(z, q), \tag{4.4}$$

where $z = [\phi, \psi]$ and $h((\phi, \psi), q) = (sign(\psi)|\psi|^{1/(p-1)}, -f_q(\phi)/\lambda)$. We have that the map h depends continuously on q and its local Lipschitz constant with respect to z is independent of q for $q \in (q_0, p]$, where q_0 is close enough to p. As it is done in [2], if α_q^n is such that $X(\alpha_q^n, p, q) = \frac{1}{2n}$, there is an open set $U \subset \mathbb{R}^2$ such that $(\alpha, q) \in U$ and α is a C^1 function of q. Then, once we have that the solution z_q of (4.4) depends continuously on q and on $(\phi(0), \psi(0)) = (0, \alpha_q)$, (see [7]), z_q converges to z_p when $q \to p$. If $n > M$, we obtain $\alpha_q^n \to 0$ when $q \to p^-$. In fact, since α_q^n is decreasing and bounded, given a sequence q_j, $q_j \to p^-$, and $\alpha_j^n = \alpha^n(q_j)$, there exists α^n such that $\alpha_j^n \to \alpha^n$. If $\alpha^n > 0$, by continuity, we obtain that when $p = q$ there exists an equilibrium point of (1.2) with n zeros in $(0, 1)$. Since $n > M$, it is not possible, then $\alpha^n = 0$. Therefore, from the continuous dependence of initial data and parameters, we obtain $\|\phi_j(q)\|_{C^1(0,1)}$ goes to zero when $q \to p^-$. $\square$

Regarding the case $q > p$, since $\frac{\partial I}{\partial a}(p, q, a) < 0$ when $q > p$ and a is close to zero it is not possible analyze the sign of $\frac{\partial X}{\partial q}$. In [13], it was proved that for each q, $q > p$ there is only one $a^*(q)$ such that $a^*(q)$ is the minimum point of $I(p, q, a)$, that means, $\frac{\partial I}{\partial a}(q, a^*(q)) = 0$ and $\frac{\partial^2 I}{\partial a^2}(q, a^*(q)) > 0$. We will prove that $a^*(q)$ goes to zero when q goes to p^+.

First of all, using the Implicit Function Theorem for $\frac{\partial I}{\partial a}(q,a)=0$, we obtain that $a^*(q)$ is a C^1 function. Then we have the following theorem.

Theorem 4.4. *Suppose $p>2$ fixed. Let $\phi_i(q)\in E_i^{\pm}$ for $q>p$. Then $\phi_i(q)$ converges to another stationary solution, with the same amount of zeros when $q\to p^+$. If $\psi_i(q)\in F_i^{\pm}$, then $\|\psi_i(q)\|_{C^1(0,1)}$ goes to zero when $q\to p^+$.*

Proof. The first part of the statement follows as in the previous theorem.

Let q_n be a sequence that $q_n\to p^+$ and $a_n^*=a^*(q_n)$. Since a_n^* is a bounded sequence it contains a convergent subsequence $a_{n_k}^*$. Suppose that $a_{n_k}^*\to a^*>0$. Then $I(p,p,a^*)=\lim_{k\to\infty} I(p,q_{n_k},a_{n_k}^*)$ and $\frac{\partial I}{\partial a}(p,p,a^*)=\lim_{k\to\infty}\frac{\partial I}{\partial a}(p,q_{n_k},a_{n_k}^*)=0$, that means, a^* is a critical point of $I(p,p,a)$.

But, in [13] the authors have proved that $I(p,p,a)$ is strictly increasing in $[0,1)$. Then, it is only possible $a^*=0$ for any sequence a_n^*. Thus, we conclude that $a^*(q_n)$ goes to 0 when $q_n\to p^+$. Therefore, since that each equilibrium point $\psi_i(q)$ of (1.2) is a solution of (4.1) with initial date $\phi(0)=0$ and $\psi(0)=\tilde\alpha_{nq}$, where $\tilde\alpha_{nq}$ is the α such that $\tilde\alpha_{nq}<a^*(q)$, from the continuous dependence with respect initial data and parameter q, we have that $\psi_i(q_n)$ converges to zero when $q_n\to p^+$ in $C^1[0,1]$. □

Now we join some results about $I(p,q,a)$ for $q>p$ in the following lemma.

Lemma 4.5. *If $q>p$, then*

(i) *$a^*(q)\to 0$ when $q\to p^+$,*
(ii) *$\tilde I(q)=I(p,q,a^*(q))$ is increasing with respect to q,*
(iii) *$\tilde I(q)\to I_0=I(p,p,0)$, when $q\to p^+$.*

Proof. Item (i) follows from the prior discussion.

(ii) $\tilde I(q)=I(p,q,a^*(q))$, with p fixed. We obtain

$$\frac{d\tilde I}{dq}(q)=\frac{\partial I}{\partial a}(q,a^*(q))\frac{da^*}{dq}(q)+\frac{\partial I}{\partial q}(q,a^*(q))=\frac{\partial I}{\partial q}(q,a^*(q))>0,$$

which means that the minimum value of I is increasing with q.

(iii) It follows by using (ii) and the continuity of $I(p,p,a)$ in $a=0$ and $I(p,q,a)$ for $a>0$. □

Since the sequences $\lambda_n(p,q)$ and $\lambda_n^*(p,q)$ depends on (p,q), even if λ is fixed it is possible to occur changes in the relation between λ and $\lambda_n^*(p,q)$ and $\lambda_n(p,q)$ when $q\to p$. Then, before proving the continuity of equilibrium sets $E(p,q)$ in $q=p$ we analyze that the possibilities among λ, $\lambda_n(p,q)$ and $\lambda_n^*(p,q)$.

Let $\{\lambda_n\}$, $\{\lambda_n^*\}$, $\{\lambda_n(p,q)\}$ and $\{\lambda_n^*(p,q)\}$ be defined as follows:

$$\lambda_n^*\doteq\frac{p}{p-1}(2(n+1)I_0)^{-p},$$
$$\lambda_n^*(p,q)\doteq\frac{p}{p-1}(2(n+1)I^*(q))^{-p}, q>p,$$

where $I^*(q)=I(p,q,a^*(q))$ denotes the minimum value of $I(p,q,a)$ with relation to a, $I_0=lim_{a\to0^+}I(p,p,a)$, and

$$\lambda_n\doteq\lambda_n(p,p)=\frac{p}{p-1}(2(n+1)I(p,p,1))^{-p};\lambda_n(p,q)=\frac{p}{p-1}(2(n+1)I(p,q,1))^{-p}.$$

Here $\{\lambda_n^*\}$, $\{\lambda_n^*(p,q)\}$ are the sequence that determine the number of zeros allowed to a stationary solution of (1.2) when $p=q$ and $p<q$ respectively, and $\{\lambda_n(p,q)\}$

determines the existence of continuum components in $E_\lambda(p,q)$. All details can be found in [13].

Now we fix p and λ. We observe that there are only four possibilities

(1) $\lambda \neq \lambda_i$ and $\lambda \neq \lambda_j^*$ for any i and any j;
(2) $\lambda \neq \lambda_i$ and $\lambda = \lambda_j^*$ for any i and for some j;
(3) $\lambda = \lambda_i$ and $\lambda \neq \lambda_j^*$ for some i and for any j;
(4) $\lambda = \lambda_i$ and $\lambda = \lambda_j^*$ for some i and some j, $i > j$.

We have the following:

Case 1. Let j_0 be the least index such that $\lambda > \lambda_{j_0}(p,p)$. Since $I(p,q,1)$ behaves continuously on q, if q is close enough to p, than $\lambda > \lambda_{j_0}(p,q)$. By Lemma 4.5, we also have that $I^*(q) = \min I(p,q,a)$ is increasing with q if $q > p$ and $I^*(q) \to I_0$ when $q \downarrow p$. So, if $\lambda > \lambda_{i_0}^*$ for some given i_0, then $\lambda > \lambda_{i_0}^*(p,q)$, if q is close enough to p. Therefore, if $p < q$, q can be chosen in a neighborhood of p in such way that the maximum number of zeros of any equilibrium in $E_\lambda(p,q)$ is M and, in both case $p > q$ or $p < q$, components having equilibrium with the same amount of zeros, namely k, are discrete or continuous according with the cardinality of $E_\lambda^k(p,p)$. Thus, in this case there is no additional qualitative differences between the sets of equilibrium beyond those which we deal in the prior discussion.

Case 2. To analyze this case, let us consider the variation of $\lambda_j^*(p,q)$ with respect to q, $q > p$. By Lemma 4.5, $I^*(q)$ is increasing with q if $q > p$ and $I^*(q) \to I_0$ when $q \downarrow p$, then $\lambda_j^*(p,q) < \lambda_j^*$. Thus, $\lambda = \lambda_j^*$ implies $\lambda > \lambda_j^*(p,q)$. This allows us to conclude that if there exist stationary solutions in $E_\lambda(p,q)$ having n zeros in $(0,1)$, then there is also solutions in $E_\lambda(p,p)$ having n zeros in $(0,1)$. In other words, once $\lambda = \lambda_j^*$ there is no solution with j zeros in $(0,1)$ and, as $\lambda_j^*(p,q) < \lambda_j^* = \lambda$ there is no solution with j zeros in $(0,1)$ for $q > p$. Finally, if $\lambda = \lambda_j^*$ then $\lambda < \lambda_k^*$, for $0 \leq k \leq j-1$ the analysis follows the Case 1, for solutions with k zeros in $(0,1)$.

Case 3. Once $\lambda_i(p,q) = \frac{p}{p-1}(2(i+1)I(p,q,1))^{-p}$, using the continuity of $I(p,q,1)$ we obtain $\lambda_i(p,q) \to \lambda_i$ when $q \to p$. If $I(p,q,1) < I(p,p,1)$ there exists a continuum of solutions with i zeros for (p,q). In despite of this, we know that, if $X_j(q)$ is the "x-time" that an equilibrium $\phi_j(q) \in E_\lambda^j(p,q)$ needs to reach its first maximum, then $X_j(q) \to \frac{1}{2(j-1)}$ as $q \to p$. So we obtain that all sequence of stationary solutions in the continuum sets $E_\lambda^j(p,q)$ converges to the same equilibrium in $E_\lambda^j(p,p)$, when $q \to p$. If $I(p,q,1) > I(p,p,1)$ the solutions with i zeros do not reach the maximum value equal 1 for (p,q). In this case, by the continuity of I the maximum value goes to 1.

Case 4. This case follows from Cases 2 and 3.

Remark 4.6. Regarding the equilibria $\pm\psi_n$ that appear when $q > p$ it is known that with respect to parameter λ they arise as spontaneous bifurcations, [13], but our analysis shows that with respect to q, $\pm\psi_n$ bifurcate from trivial solution.

Now we are ready to state our main result concerning to the continuity on q of the equilibrium sets $E(p,q)$. The upper semicontinuity in $L^2(0,1)$ and $W_0^{1,p}(0,1)$ follows easily from Theorem 3.2 and Corollaries 3.3 and 3.6. From the prior analysis presented in this section we can conclude the upper and lower semicontinuity in $C^1[0,1]$.

Theorem 4.7. *The family $E(p,q)$ is upper and lower semicontinuous on q as q goes to p in $C^1[0,1]$.*

Proof. If $q \downarrow p$, given any sequence $\{\varphi_q\}$, $\varphi_q \in E(p,q)$ for each q, there is a subsequence of $\{\varphi_q\}$ containing only equilibria with the same amount of zeros in $(0,1)$. Then we know from Theorem 4.4 that this subsequence converges to an equilibrium in $E(p,p)$.

If $q \uparrow p$, given a sequence $\{\phi_q\}$, $\phi_q \in E(p,q)$ for each q, which contains a subsequence with the same amount of zeros, then we know from Theorem 4.3 that this subsequence converges to an equilibrium in $E(p,p)$. But in this case it is also possible to find a sequence $\phi_q \in E(p,q)$ in such way that the number of zeros of ϕ_q goes to infinity with q. In this case, we observe that this sequence goes to the null solution.

So we conclude from [3, Lemma 1.1] that $E(p,q)$ is upper semicontinuous at $q = p$.

To prove the lower semicontinuity, let $\phi_p \in E(p,p)$. We have three possible situations. If the maximum value of ϕ_p is less than 1 and n is the amount of zeros of ϕ_p in $(0,1)$, the sequence $\phi_q \in E(p,q)$ containing only equilibria with n zeros converges to ϕ_p according with Theorems 4.3 and 4.4. If ϕ_p achieves 1 but does not have flat cores we can repeat the prior argument (observe that it is possible only if $\lambda = \lambda_n$ and this situation was discussed in the Case 3). When ϕ_p presents flat cores, then $\lambda < \lambda_n$ and, from the continuity of $\lambda_n(p,q)$ on q, we conclude that equilibria with n zeros in $E(p,q)$ present flat cores as well (we have used an analogous argument in Case 1). In this case, we construct the approaching sequence. Let f_i be the length of the i-th flat core, for $i = 1,\dots,n+1$. For q close to p, let $X(p,q)$ the x-time spent to an equilibrium in $E_n(p,q)$ achieve the maximum value equals to 1. If $X(p,q) > X(p,p)$ we pick in $E(p,q)$ an equilibrium ϕ_q with n zeros in $(0,1)$ such that the length of i-th flat core is $f_i - 2(X(p,q) - X(p,p))$. If $X(p,q) < X(p,p)$ we choose an equilibrium ϕ_q with n zeros in $(0,1)$ such that the length of i-th flat core is $f_i + 2(X(p,p) - X(p,q))$. In any case $\phi_q \to \phi_p$ as $q \to p$.

The lower semicontinuity follows from [3, Lemma 1.1]. □

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