Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 243, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

HIGHER ORDER MULTI-TERM TIME-FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS INVOLVING CAPUTO-FABRIZIO DERIVATIVE

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Communicated by Mokhtar Kirane

Abstract. In this work we discuss higher order multi-term partial differential equation (PDE) with the Caputo-Fabrizio fractional derivative in time. Using method of separation of variables, we reduce fractional order partial differential equation to the integer order. We represent explicit solution of formulated problem in particular case by Fourier series.

1. Introduction

Consideration of new fractional derivative with non-singular kernel was initiated by Caputo and Fabrizio in their work [1]. Motivation came from application. Precisely, new fractional derivatives can better describe material heterogeneities and structures with different scales. Special role of spatial fractional derivative in the study of the macroscopic behaviors of some materials, related with nonlocal interactions, which are prevalent in determining the properties of the material was also highlighted. In their next work [2], authors represented some applications of the introduced fractional derivative. Nieto and Losada [3] studied some properties of this fractional derivative naming it as Caputo-Fabrizio (CF) derivative. They introduced fractional integral associated with the CF derivative, applying it to the solution of linear and nonlinear differential equations involving CF derivative.

Later, many authors showed interest in the CF derivative and as a result, several applications were discovered. For instance, in groundwater modeling, in electrical circuits, in controlling the wave movement, in nonlinear Fisher's reaction-diffusion equation, in modeling of a mass-spring-damper system, etc. We note also some recent works related with CF derivative [4, 5, 6, 7, 8, 9].

Different methods were applied for solving differential equations involving CF derivative. Namely, Laplace transform, reduction to integral equations, and reduction to integer order differential equations. Last two methods were used in the works [10, 11].

In this paper, we aim to show an algorithm to reduce initial value problem (IVP) for multi-term fractional DE with CF derivative to the IVP for integer order DE and

2010 *Mathematics Subject Classification.* 33E12.
Key words and phrases. Fractional differential equations; separation of variables; Caputo-Fabrizio derivative.

Submitted August 10, 2017. Published October 5, 2017.

using this result to prove a unique solvability of a boundary-value problem (BVP) for PDE involving CF derivative on time-variable. First, we give preliminary information on CF derivative and then we formulate our main problem. Representing formal solution of the formulated problem by infinite series, in particular case, we prove uniform convergence of that infinite series.

2. Preliminaries

The fractional derivative of order $\alpha\,(0<\alpha<1)$ in CF sense [2] is defined as

$$ {}_{CF}D_{at}^{\alpha}g(t)=\frac{1}{1-\alpha}\int_a^t g'(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds\,. \tag{2.1}$$

This operator is well defined on the space

$$W^{\alpha,1}=\left\{g(t)\in L^1(a,\infty):(g(t)-g_a(s))e^{-\frac{\alpha}{1-\alpha}(t-s)}\in L^1(a,t)\times L^1(a,\infty)\right\},$$

whose norm, for $\alpha\neq 1$ is given by

$$\|g(t)\|_{W^{\alpha,1}}=\int_a^\infty |g(t)dt+\frac{\alpha}{1-\alpha}\int_a^\infty\int_{-\infty}^t |g_s(s)|e^{-\frac{\alpha}{1-\alpha}(t-s)}dsdt,$$

where $g_a(t)=g(t)$, $t\geq a$, $g_a(t)=0$, $-\infty<t<a$ [2]. The following equality is shown in [4],

$$ {}_{CF}D_{at}^{\alpha+n}g(t)={}_{CF}D_{at}^{\alpha}\left({}_{CF}D_{at}^{n}g(t)\right)\,.$$

3. Formulation of a problem and formal solution

Consider the time-fractional PDE

$$\sum_{n=0}^{k}\lambda_n\cdot{}_{CF}D_{0t}^{\alpha+n}u(t,x)-u_{xx}(t,x)=f(t,x) \tag{3.1}$$

in a domain $\Omega=\{(t,x):\,0<t<q,\,0<x<1\}$. Here λ_n are given real numbers, $f(t,x)$ is a given function, $k\in\mathbf{N}_0$, $q\in\mathbb{R}^+$.

Problem. Find a solution of (3.1) satisfying the following conditions:

$$u(t,x)\in C^2(\Omega),\quad \frac{\partial^n u(t,x)}{\partial t^n}\in W^{\alpha,1}(0,q), \tag{3.2}$$

$$u(t,0)=u(t,1)=0,\quad \frac{\partial^i u(t,x)}{\partial t^i}\big|_{t=0}=\tilde{C}_i,\quad i=0,1,2,\dots,k, \tag{3.3}$$

where $\tilde{C}_i$ are any real numbers.

3.1. Solution of higher order multi-term fractional ordinary differential equations.

We expand $u(t,x)$ as the Fourier series

$$u(t,x)=\sum_{m=1}^{\infty}T_m(t)\sin m\pi x, \tag{3.4}$$

where $T_m(t)$ are the Fourier coefficients of $u(t,x)$.

Substituting this representation in (3.1) and considering the initial conditions (3.3), we get the following IVP with respect to time-variable:

$$\begin{gathered}\sum_{n=0}^{k}\lambda_n\cdot{}_{CF}D_{0t}^{\alpha+n}T_m(t)+(m\pi)^2T_m(t)=f_m(t),\\ T_m^{(i)}(0)=C_i,\quad i=0,1,2,\dots,k\end{gathered} \tag{3.5}$$

where $f_m(t)$ are Fourier coefficients of $f(t,x)$.

Based on definition (2.1) and initial conditions (3.3), after integrating by parts, we rewrite the fractional derivatives as follows:

$$
\begin{aligned}
{}_{CF}D_{0t}^{\alpha}T_m(t) &= \frac{1}{1-\alpha}T_m(t) - \frac{\alpha}{(1-\alpha)^2}\int_0^t T_m(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds - \frac{T_m(0)}{1-\alpha}e^{-\frac{\alpha}{1-\alpha}t},\\
{}_{CF}D_{0t}^{\alpha+1}T_m(t) &= \frac{1}{1-\alpha}T'_m(t) - \frac{\alpha}{(1-\alpha)^2}T_m(t) + \frac{\alpha^2}{(1-\alpha)^3}\int_0^t T_m(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds\\
&\quad + \frac{\alpha T_m(0)}{(1-\alpha)^2}e^{-\frac{\alpha}{1-\alpha}t} - \frac{T'_m(0)}{1-\alpha}e^{-\frac{\alpha}{1-\alpha}t},\\
{}_{CF}D_{0t}^{\alpha+2}T_m(t) &= \frac{1}{1-\alpha}T''_m(t) - \frac{\alpha}{(1-\alpha)^2}T'_m(t) + \frac{\alpha^2}{(1-\alpha)^3}T_m(t)\\
&\quad - \frac{\alpha^3}{(1-\alpha)^4}\int_0^t T_m(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds - \frac{\alpha^2 T_m(0)}{(1-\alpha)^3}e^{-\frac{\alpha}{1-\alpha}t}\\
&\quad + \frac{\alpha T'_m(0)}{(1-\alpha)^2}e^{-\frac{\alpha}{1-\alpha}t} - \frac{T''_m(0)}{1-\alpha}e^{-\frac{\alpha}{1-\alpha}t}.
\end{aligned}
$$

Continuing this procedure, we find for $n \geq 1$ the formula

$$
\begin{aligned}
{}_{CF}D_{0t}^{\alpha+n}T_m(t) &= \frac{1}{1-\alpha}\Big\{\sum_{i=0}^{n}\big(-\frac{\alpha}{1-\alpha}\big)^i\big[T_m^{(n-i)}(t) - T_m^{(n-i)}(0)e^{-\frac{\alpha}{1-\alpha}t}\big]\\
&\quad + \big(-\frac{\alpha}{1-\alpha}\big)^{n+1}\int_0^t T_m(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds\Big\}.
\end{aligned} \tag{3.6}
$$

We substitute (3.6) into (3.5) and deduce

$$
\begin{aligned}
&\sum_{n=0}^{k}\frac{\lambda_n}{(1-\alpha)}\Big\{\sum_{i=0}^{n}\big(-\frac{\alpha}{1-\alpha}\big)^i\big[T_m^{(n-i)}(t) - T_m^{(n-i)}(0)e^{-\frac{\alpha}{1-\alpha}t}\big]\\
&+ \big(-\frac{\alpha}{1-\alpha}\big)^{n+1}\int_0^t T_m(s)e^{-\frac{\alpha}{1-\alpha}(t-s)}ds\Big\} + (m\pi)^2 T_m(t)\\
&= f_m(t).
\end{aligned}
$$

We multiply this equality by $(1-\alpha)e^{\frac{\alpha}{1-\alpha}t}$:

$$
\begin{aligned}
&\sum_{n=0}^{k}\lambda_n\Big\{\sum_{i=0}^{n}\big(-\frac{\alpha}{1-\alpha}\big)^i\big[T_m^{(n-i)}(t)e^{\frac{\alpha}{1-\alpha}t} - T_m^{(n-i)}(0)\big]\\
&+ \big(-\frac{\alpha}{1-\alpha}\big)^{n+1}\int_0^t T_m(s)e^{\frac{\alpha}{1-\alpha}s}ds\Big\} + (m\pi)^2(1-\alpha)T_m(t)e^{\frac{\alpha}{1-\alpha}t}\\
&= (1-\alpha)e^{\frac{\alpha}{1-\alpha}t}f_m(t).
\end{aligned} \tag{3.7}
$$

Introducing new function $\tilde{T}_m(t) = T_m(t)e^{\frac{\alpha}{1-\alpha}t}$, we rewrite some items of (3.7):

$$\begin{aligned} T'_m(t)e^{\frac{\alpha}{1-\alpha}t} &= \tilde{T}'_m(t) - \frac{\alpha}{1-\alpha}\tilde{T}_m(t), \\ T''_m(t)e^{\frac{\alpha}{1-\alpha}t} &= \tilde{T}''_m(t) - \frac{2\alpha}{1-\alpha}\tilde{T}'_m(t) + \big(\frac{\alpha}{1-\alpha}\big)^2\tilde{T}_m(t), \\ &\dots \\ T^{(n)}_m(t)e^{\frac{\alpha}{1-\alpha}t} &= \sum_{j=0}^{n}(-1)^{n-j}\frac{n!}{j!(n-j)!}\big(\frac{\alpha}{1-\alpha}\big)^{n-j}\tilde{T}^{(j)}_m(t). \end{aligned} \tag{3.8}$$

We note that $\tilde{T}^{(0)}_m(t) = \tilde{T}_m(t)$.

Considering (3.8), from (3.7) we deduce

$$\begin{aligned} &\sum_{n=0}^{k}\lambda_n\Big\{\sum_{i=0}^{n}\big(-\frac{\alpha}{1-\alpha}\big)^i\sum_{j=0}^{n-i}\frac{(n-i)!}{j!(n-i-j)!}\big(-\frac{\alpha}{1-\alpha}\big)^{n-i-j} \\ &\times\big[\tilde{T}^{(j)}_m(t) - \tilde{T}^{(j)}_m(0)\big] + \big(-\frac{\alpha}{1-\alpha}\big)^{n+1}\int_0^t\tilde{T}_m(s)ds\Big\} + (m\pi)^2(1-\alpha)\tilde{T}_m(t) \\ &= (1-\alpha)e^{\frac{\alpha}{1-\alpha}t}f_m(t). \end{aligned} \tag{3.9}$$

Differentiating (3.9) once by t, we will get $(k+1)$th order DE. Using its general solution and applying initial conditions, one can get explicit form of functions $T_m(t)$, consequently formal solution of the formulated problem is represented by infinite series (3.4). Imposing certain conditions to the given functions, we prove uniform convergence of infinite series, which will complete the proof of the unique solvability of the formulated problem.

In the next section we show the complete steps in a particular case. We note that even this particular case was not considered before.

4. Particular case

In this subsection we consider the case $k = 2$, to show the complete steps. In this case, after differentiating (3.9) once with respect to t, we get the following third order ordinary DE,

$$\tilde{T}'''_m(t) + A_1\tilde{T}''_m(t) + A_2\tilde{T}'_m(t) + A_3\tilde{T}_m = g_m(t) \tag{4.1}$$

where $\tilde{T}_m(t) = T_m(t)e^{\frac{\alpha}{1-\alpha}t}$,

$$\begin{gathered} A_1 = -\frac{3\alpha}{1-\alpha} + \frac{\lambda_1}{\lambda_2}, \\ A_2 = 3\big(-\frac{\alpha}{1-\alpha}\big)^2 + \frac{\lambda_0 - \frac{2\alpha\lambda_1}{1-\alpha} + (m\pi)^2(1-\alpha)}{\lambda_2}, \\ A_3 = \big(-\frac{\alpha}{1-\alpha}\big)^3 + \frac{\big(-\frac{\alpha}{1-\alpha}\big)^2\lambda_1 - \frac{\alpha}{1-\alpha}\lambda_0}{\lambda_2}, \\ g_m(t) = [\alpha f_m(t) + (1-\alpha){f_m}'(t)]e^{\frac{\alpha}{1-\alpha}t}. \end{gathered} \tag{4.2}$$

4.1. **General solution.** The characteristic equation of (4.1) is

$$\mu^3 + A_1\mu^2 + A_2\mu + A_3 = 0,$$

whose discriminant is

$$\Delta_m = -4A_1^3A_3 + A_1^2A_2^2 - 4A_2^3 + 18A_1A_2A_3 - 27A_3^2.$$

According to the general theory, form of solutions depends on the sign of the discriminant. Below we will give explicit forms of solutions case by case.

Case $\Delta_m > 0$. The characteristic equation has 3 different real roots (μ_1, μ_2, μ_3), hence based on general solution we can find explicit form of $T_m(t)$ as

$$\begin{aligned}
T_m(t) &= C_1 e^{(\mu_1 - \frac{\alpha}{1-\alpha})t} + C_2 e^{(\mu_2 - \frac{\alpha}{1-\alpha})t} + C_3 e^{(\mu_3 - \frac{\alpha}{1-\alpha})t} \\
&\quad + \frac{1}{(\mu_2\mu_3^2 - \mu_2^2\mu_3 - \mu_1\mu_3^2 + \mu_1^2\mu_3 + \mu_1\mu_2^2 - \mu_1^2\mu_2)} \\
&\quad \times \Big[e^{(\mu_1 - \frac{\alpha}{1-\alpha})t}(\mu_3 - \mu_2)\int \big(\alpha g_m(z) + (1-\alpha)g_m'(z)\big)e^{(\frac{\alpha}{1-\alpha} - \mu_1)z}\,dz \\
&\quad + e^{(\mu_2 - \frac{\alpha}{1-\alpha})t}(\mu_1 - \mu_3)\int (\alpha g_m(z) + (1-\alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_2)z}\,dz \\
&\quad + e^{(\mu_3 - \frac{\alpha}{1-\alpha})t}(\mu_2 - \mu_1)\int (\alpha g_m(z) + (1-\alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_3)z}\,dz\Big].
\end{aligned} \tag{4.3}$$

Case $\Delta_m < 0$. The characteristic equation has one real (μ_1) and two complex-conjugate roots $(\mu_2 = \mu_{21} \pm i\mu_{22})$. Therefore, $T_m(t)$ has the form

$$\begin{aligned}
T_m(t) &= C_1 e^{(\mu_1 - \frac{\alpha}{1-\alpha})t} + (C_2\cos\mu_{22}t + C_3\sin\mu_{22}t)e^{(\mu_{21} - \frac{\alpha}{1-\alpha})t} \\
&\quad + \frac{1}{\mu_{22}^2 - 3\mu_{21}^2 - 2\mu_1\mu_{21} + \mu_1^2}\Big[e^{(\mu_1 - \frac{\alpha}{1-\alpha})t}\int (\alpha g_m(z) \\
&\quad + (1-\alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_1)z}\,dz + \frac{1}{\mu_{22}}e^{(\mu_{21} - \frac{\alpha}{1-\alpha})t}\cos\mu_{22}t\int (\mu_1\sin\mu_{22}z \\
&\quad - \mu_{21}\sin\mu_{22}z - \mu_{22}\cos\mu_{22}z)\,(\alpha g_m(z) + (1-\alpha)g_m'(z))\,e^{(\frac{\alpha}{1-\alpha} - \mu_{21})z}\,dz \\
&\quad + \frac{1}{\mu_{22}}e^{(\mu_{21} - \frac{\alpha}{1-\alpha})t}\sin\mu_{22}t\int (\mu_{21}\cos\mu_{22}z - \mu_{22}\sin\mu_{22}z \\
&\quad - \mu_1\cos\mu_{22}z)\,(\alpha g_m(z) + (1-\alpha)g_m'(z))\,e^{(\frac{\alpha}{1-\alpha} - \mu_{21})z}\,dz\Big].
\end{aligned}$$

Case $\Delta_m = 0$. We have have 2 sub-cases:
(a) Three real roots, two of which are equal, third one is different $(\mu_1 = \mu_2,\ \mu_3)$:

$$\begin{aligned}
T_m(t) &= C_1 e^{(\mu_1 - \frac{\alpha}{1-\alpha})t} + C_2 t e^{(\mu_1 - \frac{\alpha}{1-\alpha})t} + C_3 e^{(\mu_3 - \frac{\alpha}{1-\alpha})t} \\
&\quad + \frac{1}{(\mu_1^2 - 2\mu_1\mu_3 + \mu_3)}\Big[e^{(\mu_1 - \frac{\alpha}{1-\alpha})t}\int (\mu_3 z - 1 - \mu_1 z)(\alpha g_m(z) \\
&\quad + (1-\alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_1)z}\,dz \\
&\quad + t e^{(\mu_1 - \frac{\alpha}{1-\alpha})t}(\mu_1 - \mu_3)\int (\alpha g_m(z) + (1-\alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_1)z}\,dz \\
&\quad + e^{(\mu_3 - \frac{\alpha}{1-\alpha})t}\int (\alpha g_m(z) + (1-\alpha)g_m'(z))e^{(\frac{\alpha}{1-\alpha} - \mu_3)z}\,dz\Big];
\end{aligned} \tag{4.4}$$

(b) all 3 real roots are the same ($\mu_1=\mu_2=\mu_3$):

$$\begin{aligned} T_m(t) &= C_1 e^{(\mu_1-\frac{\alpha}{1-\alpha})t} + C_2 t e^{(\mu_1-\frac{\alpha}{1-\alpha})t} + C_3 t^2 e^{(\mu_1-\frac{\alpha}{1-\alpha})t} \\ &\quad + \frac{1}{2} e^{(\mu_1-\frac{\alpha}{1-\alpha})t} \int z^2(\alpha g_m(z)+(1-\alpha)g_m'(z)) e^{(\frac{\alpha}{1-\alpha}-\mu_1)z}\,dz \\ &\quad - t e^{(\mu_1-\frac{\alpha}{1-\alpha})t} \int z(\alpha g_m(z)+(1-\alpha)g_m'(z)) e^{(\frac{\alpha}{1-\alpha}-\mu_1)z}\,dz \\ &\quad + \frac{1}{2} t^2 e^{(\mu_1-\frac{\alpha}{1-\alpha})t} \int (\alpha g_m(z)+(1-\alpha)g_m'(z)) e^{(\frac{\alpha}{1-\alpha}-\mu_1)z}\,dz. \end{aligned} \tag{4.5}$$

Here C_j ($j=\overline{1,3}$) are any constants, which will be defined using initial conditions.

4.2. **Convergence part.** We consider the case $\Delta_m>0$ in details. Found solution we satisfy to the initial conditions (3.5). Without losing generality, we assume that $\tilde{C}_i=0$ ($i=\overline{0,2}$). Regarding the C_j we will get the following algebraic system of equations

$$\begin{gathered} C_1+C_2+C_3=-d_1, \\ C_1(\mu_1-\frac{\alpha}{1-\alpha})+C_2(\mu_2-\frac{\alpha}{1-\alpha})+C_3(\mu_3-\frac{\alpha}{1-\alpha})=-d_2, \\ C_1(\mu_1-\frac{\alpha}{1-\alpha})^2+C_2(\mu_2-\frac{\alpha}{1-\alpha})^2+C_3(\mu_3-\frac{\alpha}{1-\alpha})^2=-d_3, \end{gathered}$$

where

$$\begin{aligned} &d_1 \\ &= \frac{1}{(\mu_2\mu_3^2-\mu_2^2\mu_3-\mu_1\mu_3^2+\mu_1^2\mu_3+\mu_1\mu_2^2-\mu_1^2\mu_2)}\Big[(\mu_3-\mu_2)\int^t \big(\alpha g_m(z) \\ &\quad + (1-\alpha)g_m'(z)\big)\,dz\big|_{t=0} + (\mu_1-\mu_3)\int^t \big(\alpha g_m(z)+(1-\alpha)g_m'(z)\big)\,dz\big|_{t=0} \\ &\quad + (\mu_2-\mu_1)\int^t \big(\alpha g_m(z)+(1-\alpha)g_m'(z)\big)\,dz\big|_{t=0}\Big], \end{aligned} \tag{4.6}$$

$$\begin{aligned} d_2 &= \frac{1}{(\mu_2\mu_3^2-\mu_2^2\mu_3-\mu_1\mu_3^2+\mu_1^2\mu_3+\mu_1\mu_2^2-\mu_1^2\mu_2)} \\ &\quad \times [(\mu_1-\frac{\alpha}{1-\alpha})(\mu_3-\mu_2)\int^t \big(\alpha g_m(z)+(1-\alpha)g_m'(z)\big)\,dz\big|_{t=0} \\ &\quad + (\mu_3-\mu_2)(\alpha g_m(0)+(1-\alpha)g_m'(0)) \\ &\quad + (\mu_2-\frac{\alpha}{1-\alpha})(\mu_1-\mu_3)\int^t \big(\alpha g_m(z)+(1-\alpha)g_m'(z)\big)\,dz\big|_{t=0} \\ &\quad + (\mu_1-\mu_3)(\alpha g_m(0)+(1-\alpha)g_m'(0)) + (\mu_3-\frac{\alpha}{1-\alpha})(\mu_2-\mu_1) \\ &\quad \times \int^t \big(\alpha g_m(z)+(1-\alpha)g_m'(z)\big)\,dz\big|_{t=0} \\ &\quad + (\mu_2-\mu_1)(\alpha g_m(0)+(1-\alpha)g_m'(0))\Big], \end{aligned} \tag{4.7}$$

$$
\begin{aligned}
d_3 = {} & \frac{1}{(\mu_2\mu_3^2 - \mu_2^2\mu_3 - \mu_1\mu_3^2 + \mu_1^2\mu_3 + \mu_1\mu_2^2 - \mu_1^2\mu_2)} \\
& \times \Big[(\mu_1 - \frac{\alpha}{1-\alpha})^2(\mu_3 - \mu_2)\int^t \big(\alpha g_m(z) + (1-\alpha)g_m'(z)\big)\,dz\big|_{t=0} \\
& + (\mu_1 - \frac{\alpha}{1-\alpha})(\mu_3 - \mu_2)(\alpha g_m(0) + (1-\alpha)g_m'(0)) \\
& + (\mu_3 - \mu_2)(\alpha g_m'(0) + (1-\alpha)g_m''(0)) + (\mu_2 - \frac{\alpha}{1-\alpha})^2(\mu_1 - \mu_3) \\
& \times \int^t (\alpha g_m(z) + (1-\alpha)g_m'(z))\ dz\big|_{t=0} + (\mu_2 - \frac{\alpha}{1-\alpha})(\mu_1 - \mu_3) \\
& \times (\alpha g_m(0) + (1-\alpha)g_m'(0)) + (\mu_1 - \mu_3)(\alpha g_m'(0) + (1-\alpha)g_m''(0)) \\
& + (\mu_3 - \frac{\alpha}{1-\alpha})^2(\mu_2 - \mu_1)\int^t (\alpha g_m(z) + (1-\alpha)g_m'(z))\ dz\big|_{t=0} \\
& + (\mu_3 - \frac{\alpha}{1-\alpha})(\mu_2 - \mu_1)(\alpha g_m(0) + (1-\alpha)g_m'(0)) \\
& + (\mu_2 - \mu_1)(\alpha g_m'(0) + (1-\alpha)g_m''(0))\Big].
\end{aligned}
\tag{4.8}
$$

Solving this system, we get

$$
\begin{aligned}
C_1 = {} & -d_1 - \frac{-d_2(\mu_1 - \frac{\alpha}{1-\alpha}) + d_3}{(\mu_2 - \frac{\alpha}{1-\alpha})(\mu_1 - \frac{\alpha}{1-\alpha}) - (\mu_2 - \frac{\alpha}{1-\alpha})^2} \\
& - \Big(- d_1(\mu_1 - \frac{\alpha}{1-\alpha})(\mu_2 - \frac{\alpha}{1-\alpha}) - d_2(\mu_2 - \frac{\alpha}{1-\alpha}) + d_2(\mu_1 - \frac{\alpha}{1-\alpha}) - d_3\Big) \\
& \div \Big(\big[\mu_1\mu_2 - \mu_2\mu_3 - \mu_3\mu_1 - \mu_3^2 - 2(\frac{\alpha}{1-\alpha})^2\big]\big[(\mu_2 - \frac{\alpha}{1-\alpha})(\mu_1 - \frac{\alpha}{1-\alpha}) \\
& - (\mu_2 - \frac{\alpha}{1-\alpha})^2\big]\Big) \\
& - \frac{-d_1(\mu_1 - \frac{\alpha}{1-\alpha})(\mu_2 - \frac{\alpha}{1-\alpha}) + d_2(\mu_2 - \frac{\alpha}{1-\alpha}) + d_2(\mu_1 - \frac{\alpha}{1-\alpha}) - d_3}{\mu_1\mu_2 - \mu_2\mu_3 - \mu_3\mu_1 - \mu_3^2 - 2(\frac{\alpha}{1-\alpha})^2}
\end{aligned}
$$

$$
\begin{aligned}
C_2 = {} & \frac{-d_2(\mu_1 - \frac{\alpha}{1-\alpha}) + d_3}{((\mu_2 - \frac{\alpha}{1-\alpha})(\mu_1 - \frac{\alpha}{1-\alpha}) - (\mu_2 - \frac{\alpha}{1-\alpha})^2)} \\
& - \Big((-d_1)(\mu_1 - \frac{\alpha}{1-\alpha})(\mu_2 - \frac{\alpha}{1-\alpha}) + d_2(\mu_2 - \frac{\alpha}{1-\alpha}) + d_2(\mu_1 - \frac{\alpha}{1-\alpha}) - d_3\Big) \\
& \div \Big(\big[\mu_1\mu_2 - \mu_2\mu_3 - \mu_3\mu_1 - \mu_3^2 - 2(\frac{\alpha}{1-\alpha})^2\big]\big[(\mu_2 - \frac{\alpha}{1-\alpha})(\mu_1 - \frac{\alpha}{1-\alpha}) \\
& - (\mu_2 - \frac{\alpha}{1-\alpha})^2\big]\Big)
\end{aligned}
$$

$$
C_3 = \frac{(-d_1)(\mu_1 - \frac{\alpha}{1-\alpha})(\mu_2 - \frac{\alpha}{1-\alpha}) + d_2(\mu_2 - \frac{\alpha}{1-\alpha}) + d_2(\mu_1 - \frac{\alpha}{1-\alpha}) - d_3}{\mu_1\mu_2 - \mu_2\mu_3 - \mu_3\mu_1 - \mu_3^2 - 2(\frac{\alpha}{1-\alpha})^2}
$$

In general, we can write

$$
|C_j| \le M_1|d_1| + M_2|d_2| + M_3|d_3|.
$$

Hence, we need the following estimations in order to provide convergence of used series:

$$\begin{aligned}|d_1| &\le \frac{1}{(m\pi)^4}\Big|M_4\int^t(\alpha f_{4,0}(z)+(1-\alpha)f_{4,1}(z))e^{(\frac{\alpha}{1-\alpha}-\mu_1)z}\,dz\big|_{t=0}\\ &\quad + M_5\int^t(\alpha f_{4,0}(z)+(1-\alpha)f_{4,1}(z))e^{(\frac{\alpha}{1-\alpha}-\mu_2)z}\,dz\big|_{t=0}\\ &\quad + M_6\int^t(\alpha f_{4,0}(z)+(1-\alpha)f_{4,1}(z))e^{(\frac{\alpha}{1-\alpha}-\mu_3)z}\,dz\big|_{t=0}\Big|\\ &\le \frac{1}{(m\pi)^4}M_7\end{aligned}$$

$$\begin{aligned}|d_2| &\le \frac{1}{(m\pi)^4}\Big|M_8e^{(\mu_1-\frac{\alpha}{1-\alpha})t}\int^t(\alpha f_{4,0}(z)+(1-\alpha)f_{4,1}(z))e^{(\frac{\alpha}{1-\alpha}-\mu_1)z}\,dz\big|_{t=0}\\ &\quad + M_9e^{(\mu_2-\frac{\alpha}{1-\alpha})t}\int^t(\alpha f_{4,0}(z)+(1-\alpha)f_{4,1}(z))e^{(\frac{\alpha}{1-\alpha}-\mu_2)z}\,dz\big|_{t=0}\\ &\quad + M_{10}e^{(\mu_3-\frac{\alpha}{1-\alpha})t}\int^t(\alpha f_{4,0}(z)+(1-\alpha)f_{4,1}(z))e^{(\frac{\alpha}{1-\alpha}-\mu_3)z}\,dz\big|_{t=0}\\ &\quad + M_{11}(\alpha f_{4,0}(z)+(1-\alpha)f_{4,1}(z))\Big|\\ &\le \frac{1}{(m\pi)^4}M_{12}\end{aligned}$$

$$\begin{aligned}|d_3| &\le \frac{1}{(m\pi)^4}\Big|M_{13}e^{(\mu_1-\frac{\alpha}{1-\alpha})t}\int^t(\alpha f_{4,0}(z)+(1-\alpha)f_{4,1}(z))e^{(\frac{\alpha}{1-\alpha}-\mu_1)z}\,dz\big|_{t=0}\\ &\quad + M_{14}e^{(\mu_2-\frac{\alpha}{1-\alpha})t}\int^t(\alpha f_{4,0}(z)+(1-\alpha)f_{4,1}(z))e^{(\frac{\alpha}{1-\alpha}-\mu_2)z}\,dz\big|_{t=0}\\ &\quad + M_{15}e^{(\mu_3-\frac{\alpha}{1-\alpha})t}\int^t(\alpha f_{4,0}(z)+(1-\alpha)f_{4,1}(z))e^{(\frac{\alpha}{1-\alpha}-\mu_3)z}\,dz\big|_{t=0}\\ &\quad + M_{16}(\alpha f_{4,0}(z)+(1-\alpha)f_{4,1}(z))+M_{17}(\alpha f_{4,1}(z)+(1-\alpha)f_{4,2}(z))\Big|\\ &\le \frac{1}{(m\pi)^4}M_{18}\end{aligned}$$

Here M_i ($i=\overline{1,18}$) are positive constants,

$$\begin{gathered}f_m(t)=\int_0^1 f(t,x)\sin m\pi x\,dx=\frac{1}{(m\pi)^4}f_{4,0}(t),\\ f_m'(t)=\frac{1}{(m\pi)^4}\int_0^1\frac{\partial}{\partial t}\big(\frac{\partial^4}{\partial x^4}f(t,x)\big)\sin m\pi x\,dx=\frac{1}{(m\pi)^4}f_{4,1}(t),\\ f_m''(t)=\frac{1}{(m\pi)^4}\int_0^1\frac{\partial}{\partial^2 t}\big(\frac{\partial^4}{\partial x^4}f(t,x)\big)\sin m\pi x\,dx=\frac{1}{(m\pi)^4}f_{4,2}(t),\\ f_{4,0}(t)=\int_0^1\frac{\partial^4 f(t,x)}{\partial x^4}\sin m\pi x\,dx,\\ f_{4,1}(t)=\int_0^1\frac{\partial^5 f(t,x)}{\partial t\partial x^4}\sin m\pi x\,dx,\end{gathered}$$

$$f_{4,2}(t) = \int_0^1 \frac{\partial^6 f(t,x)}{\partial t^2 \partial x^4} \sin m\pi x \, dx.$$

We note that for above-given estimations, we need to impose the following conditions to the given function $f(t,x)$:

$$\begin{gathered}
\frac{\partial f}{\partial t}|_{t=0} = 0, \quad \frac{\partial^2 f}{\partial t^2}|_{t=0} = 0, \quad \frac{\partial^3 f}{\partial t^3}|_{t=0} = 0, \\
f(t,1) = f(t,0) = 0, \quad \frac{\partial^2 f(t,1)}{\partial x^2} = \frac{\partial^2 f(t,0)}{\partial x^2} = 0.
\end{gathered} \tag{4.9}$$

Based on estimations (4.6)-(4.8), we obtain

$$|C_j| \le \frac{M_{19}}{(m\pi)^4}$$

and considering (4.3), finally we get

$$|T_m| \le \frac{M_{20}}{(m\pi)^4}.$$

Taking (3.4) into account, on can easily deduce that

$$|u(t,x)| \le \frac{M_{21}}{(m\pi)^4}, \quad |u_{xx}(t,x)| \le \frac{M_{22}}{(m\pi)^2}.$$

The required estimate

$$|{}_{CF}D_{0t}^{\alpha} u(t,x)| \le \frac{M_{23}}{(m\pi)^4}$$

can be deduced easily, as well.

Theorem 4.1. *If $f(t,x) \in C^2(\overline{\Omega})$, $\frac{\partial^3 f(t,x)}{\partial t^3} \in C(\Omega)$ and $\frac{\partial^3 f(t,x)}{\partial t^3}$ is continuous up to $t = 0$, $\frac{\partial^6 f(t,x)}{\partial t^2 \partial x^4} \in L^1(0,1)$ together with (4.9), then problem (3.1)-(3.3), when $k = 2$ has a unique solution represented by (3.4), where $T_m(t)$ are defined by (4.3)-(4.5) depending on the sign of Δ_m.*

Remark 4.2. Similar result can be obtained for general case, as well. For this, one needs to differentiate (3.9) once by t and write explicit form of $(k+1)$th order ordinary DE.

Remark 4.3. We note that used algorithm allows us to investigate fractional spectral problems such as

$$\begin{gathered}
{}_{CF}D_{0t}^{\alpha+1} T(t) + \mu T(t) = 0, \\
T(0) = 0, \quad T(1) = 0,
\end{gathered}$$

reducing it to the second order usual spectral problem.

Using above-given algorithm, we obtain the following second order spectral problem

$$\begin{gathered}
\bar{T}''(t) + (\mu - \frac{2\alpha}{1-\alpha})\bar{T}'(t) + (\frac{\alpha}{1-\alpha})^2 \bar{T}(t) = 0 \\
\bar{T}(0) = 0, \quad \bar{T}(1) = 0,
\end{gathered}$$

eigenvalues and corresponding eigenfunctions of which have a form

$$\mu_n = \frac{2\alpha}{1-\alpha} \pm 2\sqrt{(\frac{\alpha}{1-\alpha})^2 - (n\pi)^2}, \quad n \in \mathbb{N},$$

$$\bar{T}_n(t) = e^{\frac{\mu_n - \frac{2\alpha}{1-\alpha}}{2}} \sin n\pi x, \quad n \in \mathbb{N}.$$

Remark 4.4. We can apply this approach for studying the more general equation

$$\sum_{n=0}^{k} \lambda_n(t,x) \cdot {}_{CF}D_{0t}^{\alpha+n} u(t,x) - {}_{CF}D_{0t}^{\alpha+1} u(t,x) = f(t,x),$$

where $\lambda_n(t,x)$ might have singularity as well.

Remark 4.5. We are able to consider another kind of fractional derivative without singularity with αth order $(0 < \alpha < 1)$ such

$$D_{0t}^{\alpha} T(t) = G(\alpha) \int_0^t T'(s) K(t,s,\alpha) ds,$$

involving it in the fractional DE

$$\lambda_1 D_{0t}^{\alpha} T(t) + \lambda_2 T(t) = f(t).$$

In this case, our kernel should satisfy to the condition

$$\frac{\partial K(t,s,\alpha)}{\partial s} \frac{\partial K(s,t,\alpha)}{\partial t} = \big[\lambda_1 G(\alpha) K(s,s,\alpha) + \lambda_2\big]\big[\lambda_1 G(\alpha) K(t,t,\alpha) + \lambda_2\big].$$

Acknowledgments. The authors are very grateful to the anonymous referees for their helpful comments.

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