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EXISTENCE OF SOLUTIONS TO SUPERLINEAR P-LAPLACE EQUATIONS WITHOUT AMBROSETTI-RABINOWIZT CONDITION

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ABSTRACT. We study the existence of non-trivial weak solutions in $W_0^{1,p}(\Omega)$ of the super-linear Dirichlet problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where f satisfies the condition

$$|f(x,t)| \le |\omega(x)t|^{r-1} + b(x) \quad \forall (x,t) \in \Omega\times\mathbb{R},$$

where $r \in (p, \frac{Np}{N-p})$, $b \in L^{\frac{r}{r-1}}(\Omega)$ and $|\omega|^{r-1}$ may be non-integrable on Ω.

1. Introduction

Let N be an integer ≥ 3, Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, p be in $[1,N)$ and $p^* = \frac{Np}{N-p}$. Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space with the following norm

$$\|u\|_{1,p} = \Big\{ \int_\Omega |\nabla u|^p dx \Big\}^{1/p} \quad \forall u \in W_0^{1,p}(\Omega).$$

We consider the Dirichlet problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where f is a real Carathéodory function on $\Omega\times\mathbb{R}$ and satisfies the following conditions

- (A1) there exist $r \in (p,p^*)$, $\omega \in \mathcal{K}_{p,r}$ (see Definition 2.1) and $b \in L^{\frac{r}{r-1}}(\Omega)$ such that
$$|f(x,t)| \le |\omega(x)t|^{r-1} + b(x) \quad \forall (x,t) \in \Omega\times\mathbb{R},$$
- (A2) there exist $C \in [0,\infty)$ and $d \in L^1(\Omega)$ such that $|f(x,t)| \le d(x)$ for every x in Ω and $|t| \le C$,
- (A3) there is d_1 in $L^{\frac{N}{p}}(\Omega)$ such that $d_1(x) \le \frac{f(x,t)}{|t|^{p-2}t}$ for every $(x,t) \in \Omega\times\mathbb{R}$,
- (A4) $f(x,0) = 0$ for every x in Ω and $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$ a.e. in Ω, and

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(A5) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = \infty$ a.e. in Ω.

The integrability of $|\omega|^{r-1}$ is essential in [1, 5, 7, 11, 12, 13, 14, 16, 17], because these papers have used the differentiability of Nemytskii from $L^{q_1}(\Omega)$ into $L^{q_2}(\Omega)$ (see [5, 4]) and the Sobolev embedding from $W_0^p(\Omega)$ into $L^{q_1}(\Omega)$. In the present paper, using weighted Sobolev embeddings in [6, 10, 9, 15, 18] instead of classical one in [3], we can study the problem (1.1) with non-integrable functions $|\omega|^{r-1}$ in (A1).

In many applications, $\frac{f(x,t)}{|t|^{p-2}t}$ is non-negative for $t \neq 0$ and $|f(x,t)|$ is well-controlled when $|t|$ is sufficiently small. This observation is the motivation of (A2) and (A3). Here we consider the case, in which the positivity of $\frac{f(x,t)}{|t|^{p-2}t}$ can be disturbed by a function $d_1 \in L^{\frac{N}{p}}(\Omega)$.

In (A4) and (A5), we do not need the uniform convergence as in [1, 5, 7, 11, 12, 13, 14, 16, 17]. We study the problem (1.1) without Ambrosetti-Rabinowizt condition. Our main result is the following theorems under the assumption

(A6) $\frac{f(x,t)}{|t|^{p-2}t}$ is increasing in $t \geq C$ and decreasing in $t \leq -C$ for every x in Ω.

Theorem 1.1. *Assume f satisfies* (A1)-(A6). *Then there is a non-trivial weak solution in $W_0^{1,p}(\Omega)$ of the problem* (1.1).

Remark 1.2. If f is continuous on $\overline{\Omega} \times \mathbb{R}$ and satisfies the following conditions

(A1') There exist $r \in (p, p^* - 1)$ and a positive real number α such that
$$|f(x,t)| \leq \alpha(1 + |t|^{r-1}) \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

(A4') $f(x,0) = 0$ for every x in Ω and $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$ uniformly in Ω.

(A5') $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = \infty$ uniformly in Ω.

Then f satisfies (A1)–(A5). Therefore our theorem improves the corresponding results in [14, 16].

We study a method for constructing weight functions in weighted Sobolev embeddings and the Nemytskii operator from Sobolev spaces into Lebesgue spaces (see Theorems 2.8 and 2.9). We apply these results to prove the existence of non-trivial solutions of a class of super-linear p-Laplacian problem in the last section.

2. Nemytskii operators

Definition 2.1. Let σ be a measurable function on Ω. We put
$$T_\sigma u = \sigma u \quad h\forall u \in W_0^{1,p}(\Omega).$$
We say that

(i) σ is of class $\mathcal{C}_{p,s}$, if T_σ is a continuous mapping from $W_0^{1,p}(\Omega)$ into $L^s(\Omega)$,
(ii) σ is of class $\mathcal{K}_{p,s}$, if T_σ is a compact mapping from $W_0^{1,p}(\Omega)$ into $L^s(\Omega)$.

We have following results.

Theorem 2.2. *Let α_1 and α_2 be in $[1,\infty)$ such that $\alpha_1 < \alpha_2$. Let $\omega_1 \in \mathcal{C}_{p,\alpha_1}$, $\omega_2 \in \mathcal{C}_{p,\alpha_2}$ such that ω_1 and ω_2 are non-negative. Let $\beta \in (\alpha_1, \alpha_2)$ and $\omega = \omega_1^{\frac{\alpha_1(\alpha_2-\beta)}{\beta(\alpha_2-\alpha_1)}} \omega_2^{\frac{\alpha_2(\beta-\alpha_1)}{\beta(\alpha_2-\alpha_1)}}$. Then $w \in \mathcal{C}_{p,\beta}$.*

Proof. There is a positive real number C_1 such that

$$\Big\{\int_\Omega \omega_i^{\alpha_i}|u|^{\alpha_i}dx\Big\}^{1/\alpha_i} \le C_1\|u\|_{1,p} \quad h\forall u \in W_0^{1,p}(\Omega),\ i=1,2. \tag{2.1}$$

Since $\beta = \frac{\alpha_2-\beta}{\alpha_2-\alpha_1}\alpha_1 + \frac{\beta-\alpha_1}{\alpha_2-\alpha_1}\alpha_2$, by Hölder's inequality and (2.1), we get

$$\begin{aligned}
&\Big\{\int_\Omega \omega^\beta|u|^\beta dx\Big\}^{1/\beta}\\
&= \Big\{\int_\Omega \omega_1^{\frac{\alpha_2-\beta}{\alpha_2-\alpha_1}\alpha_1}|u|^{\frac{\alpha_2-\beta}{\alpha_2-\alpha_1}\alpha_1}\omega_2^{\frac{\beta-\alpha_1}{\alpha_2-\alpha_1}\alpha_2}|u|^{\frac{\beta-\alpha_1}{\alpha_2-\alpha_1}\alpha_2}dx\Big\}^{1/\beta}\\
&\le \Big\{\Big\{\int_\Omega \omega_1^{\alpha_1}|u|^{\alpha_1}dx\Big\}^{\frac{\alpha_2-\beta}{\alpha_2-\alpha_2}}\Big\{\int_\Omega \omega_2^{\alpha_2}|u|^{\alpha_2}dx\Big\}^{\frac{\beta-\alpha_1}{\alpha_2-\alpha_1}}\Big\}^{1/\beta}\\
&\le \Big\{\Big\{\int_\Omega \omega_1^{\alpha_1}|u|^{\alpha_1}dx\Big\}^{\frac{1}{\alpha_1}\frac{\alpha_2-\beta}{\alpha_2-\alpha_2}\alpha_1}\Big\{\int_\Omega \omega_2^{\alpha_2}|u|^{\alpha_2}dx\Big\}^{\frac{1}{\alpha_2}\frac{\beta-\alpha_1}{\alpha_2-\alpha_1}\alpha_2}\Big\}^{1/\beta}\\
&\le C_1\|u\|_{1,p} \quad \forall u \in W_0^{1,p}(\Omega).
\end{aligned}$$

□

Theorem 2.3. *Let s be in $[1,\frac{Np}{N-p})$, α be in $(0,1)$, $\omega \in \mathcal{C}_{p,s}$ and θ be measurable functions on Ω such that $\omega \ge 0$ and $|\theta| \le \omega^\alpha$. Then θ is of class $\mathcal{K}_{p,s}$.*

Proof. Since T_ω is in $\mathcal{C}_{p,s}$, T_ω is continuous from $W_0^{1,p}(\Omega)$ into $L^s(\Omega)$ and there is a positive real number C_2 such that

$$\Big\{\int_\Omega |u|^s\omega^s dx\Big\}^{1/s} \le C_2\|u\|_{1,p} \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.2}$$

Since $\omega^\alpha(x) \le 1+\omega(x)$ for every x in Ω and 1 and ω are in $\mathcal{C}_{p,s}$, ω^α belongs to $\mathcal{C}_{p,s}$. Thus T_θ is in $\mathcal{C}_{p,s}$. Let M be a positive real number and $\{u_n\}$ be a sequence in $W_0^{1,p}(\Omega)$, such that $\|u_n\|_{1,p} \le M$ for any n. By Rellich-Kondrachov's theorem [3, Theorem 9.16], $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ converging to u in $L^s(\Omega)$ and $\{u_{n_k}\}$ converging weakly to u in $W_0^{1,p}(\Omega)$, therefore $\|u\|_{1,p} \le \liminf_{k\to\infty}\|u_{n_k}\|_{1,p} \le M$. We shall prove $\{T_\theta(u_{n_k})\}$ converges to $T_\theta(u)$ in $L^s(\Omega)$.

Let ε be a positive real number. Choose a positive real number δ such that

$$(2C_2M)^s\delta^{(\alpha-1)s} < \frac{\varepsilon^s}{2}. \tag{2.3}$$

Put $\Omega' = \{x \in \Omega : \omega(x) > \delta\}$. By (2.2) and (2.3), we have

$$\begin{aligned}
&\int_\Omega |\theta(u_{n_k}-u)|^s dx\\
&= \int_\Omega |u_{n_k}-u|^s|\theta|^s dx\\
&\le \int_{\Omega'} |u_{n_k}-u|^s\omega^{\alpha s}dx + \int_{\Omega\setminus\Omega'} |u_{n_k}-u|^s\omega^{\alpha s}dx\\
&\le \delta^{(\alpha-1)s}\int_{\Omega'} |u_{n_k}-u|^s\omega^s dx + \delta^{\alpha s}\int_{\Omega\setminus\Omega'} |u_{n_k}-u|^s dx\\
&\le \delta^{(\alpha-1)s}\int_\Omega |u_{n_k}-u|^s\omega^s dx + \delta^{\alpha s}\int_\Omega |u_{n_k}-u|^s dx
\end{aligned}$$

$$
\begin{aligned}
&\le \delta^{(\alpha-1)s}\left(C_2\|u_{n_k}-u\|_{1,p}\right)^s+\delta^{\alpha s}\int_\Omega |u_{n_k}-u|^s dx\\
&\le \delta^{(\alpha-1)s}(2C_2M)^s+\delta^{\alpha s}\int_\Omega |u_{n_k}-u|^s dx\\
&\le \frac{\varepsilon^s}{2}+\delta^{\alpha s}\int_\Omega |u_{n_k}-u|^s dx. \qquad (2.4)
\end{aligned}
$$

Since $\{u_{n_k}\}$ converges in $L^s(\Omega)$, there is an integer k_0 such that

$$\int_\Omega |u_{n_k}-u|^s dx \le \delta^{-\alpha s}\frac{\varepsilon^s}{2} \quad h\forall k\ge k_0. \qquad (2.5)$$

Combining (2.4) and (2.5), we complete the proof. □

Corollary 2.4. *Let $p\in[1,N)$, $s\in(1,\frac{Np}{N-p})$, $\eta\in(\frac{sNp}{Np-s(N-p)},\infty)$ and $\theta\in L^\eta(\Omega)$. Then θ is in $\mathcal{K}_{p,s}$.*

Proof. Let $\beta\in(0,1)$ such that $\beta\eta=\frac{sNp}{Np-s(N-p)}$ and $\omega=|\theta|^{1/\beta}$. Then ω is in $L^{\frac{sNp}{Np-s(N-p)}}(\Omega)$. Since $\frac{Np-s(N-p)}{Np}+\frac{s(N-p)}{Np}=1$, by Hölder's inequality, we have

$$\int_\Omega |\omega u|^s dx \le \int_\Omega \left(|\omega|^{\frac{sNp}{Np-s(N-p)}}\right)^{\frac{Np-s(N-p)}{Np}}\Big(\int_\Omega |u|^{\frac{Np}{N-p}}\Big)^{\frac{s(N-p)}{Np}} \quad \forall u\in W_0^{1,p}(\Omega),$$

which implies that T_ω is continuous at 0 in $W_0^{1,p}(\Omega)$. Thus T_ω is a linear continuous map from $W_0^{1,p}(\Omega)$ into $L^s(\Omega)$. By Theorem 2.3, is of class $\mathcal{K}_{p,r}$. □

Example 2.5. Let $N=5$, $p=3$, $s=4$ and $\Omega=\{x\in\mathbb{R}^5:|x|<1\}$. Then $\frac{sNp}{Np-s(N-p)}=\frac{4.5.3}{5.3-4(5-3)}=\frac{60}{7}<10$. Put $\omega_0=|x|^{-\frac{1}{30}}\cos(16|x|)$, then ω_0 is in $L^{10}(\Omega)$. Thus by Corollary 2.4, ω_0 is of class $\mathcal{K}_{p,s}$.

Corollary 2.6. *Let $p\in[1,N)$, $s\in(1,\frac{Np}{N-p})$, α be in $(0,1)$ and $\eta\in\mathcal{C}_{p,p}$. Then $\theta=\eta^{\alpha\frac{p(p^*-s)}{s(p^*-p)}}$ is of class $\mathcal{K}_{p,s}$.*

Proof. Put $\omega_1=\eta$, $\omega_2=1$, $\alpha_1=p$, $\alpha_2=p^*$, $\beta=s$. By the Embedding theorem of Sobolev, $\omega_2\in\mathcal{C}_{p,p^*}$. By Theorem 2.2, we see that $\eta^{\frac{p(p^*-s)}{s(p^*-p)}}\in\mathcal{C}_{p,s}$. Thus by Theorem 2.3, $\eta^{\alpha\frac{p(p^*-s)}{s(p^*-p)}}$ is of class $\mathcal{K}_{p,s}$. □

Example 2.7. Let $\Omega=\{x\in\mathbb{R}^5:\|x\|<1\}$, $p=3$, $s=4$, $\alpha=\frac34$ and $\eta(x)=(1-\|x\|^2)^{-1}$ for every x in Ω. By [9, Theorem 8.4], $\eta\in\mathcal{C}_{p,p}$. Note that $p^*=\frac{Np}{N-p}=\frac{15}{2}$ and

$$\alpha\frac{p(p^*-s)}{s(p^*-p)}=\frac34\frac34\frac79=\frac{7}{16}.$$

Put $\theta(x)=(1-\|x\|^2)^{-\frac{7}{16}}$ for every x in Ω. Then $\theta\in\mathcal{K}_{3,4}$.

Theorem 2.8. *Let s be in $(1,p^*)$, ω be in $\mathcal{K}_{p,s}$, b be in $L^{\frac{s}{s-1}}(\Omega)$ and g be a Caratheodory function from $\Omega\times\mathbb{R}$ into $\mathbb{R}$. Assume*

$$|g(x,z)|\le|\omega(x)|^{s-1}|z|^{s-1}+b(x) \quad h\forall(x,z)\in\Omega\times\mathbb{R}. \qquad (2.6)$$

Put $N_g(v)(x)=g(x,v(x))$ for $v\in W_0^{1,p}(\Omega)$, $x\in\Omega$. We have

(i) *N_g is a continuous mapping from $W_0^{1,p}(\Omega)$ into $L^{\frac{s}{s-1}}(\Omega)$.*
(ii) *If A is a bounded subset in $W_0^{1,p}(\Omega)$, then $\overline{N_g(A)}$ is compact in $L^{\frac{s}{s-1}}(\Omega)$.*

Proof. (i) Put $\mu = s$, $q = s/(s-1)$ and

$$g_1(x,\zeta) = g(x,\omega(x)^{-1}\zeta) \quad \forall (x,\zeta) \in \Omega \times \mathbb{R},$$

By (2.6), we have

$$|g_1(x,\zeta)| \le |\zeta|^{s-1} + b(x) \quad h\forall (x,\zeta) \in \Omega \times \mathbb{R}.$$

On the other hand

$$N_g(v) = N_{g_1} \circ T_{|\omega|}(v) \quad \forall v \in W_0^{1,p}(\Omega).$$

Since $w \in \mathcal{K}_{p,s}$, applying [4, Theorem 2.3], we complete the proof. $\square$

Theorem 2.9. *Let $s \in (1,p^*)$, ω be in $\mathcal{K}_{p,s}$, a function $b \in L^{\frac{s}{s-1}}(\Omega)$ and g be a Caratheodory function from $\Omega \times \mathbb{R}$ into $\mathbb{R}$. Assume*

$$|g(x,z)| \le |\omega(x)|^{s-1}|z|^{s-1} + b(x) \quad \forall (x,z) \in \Omega \times \mathbb{R}.$$

Put

$$G(x,t) = \int_0^t g(x,\xi)d\xi \quad \forall (x,t) \in \Omega,$$
$$\Psi_g(u) = \int_\Omega G(x,t)dx \quad h\forall u \in W_0^{1,p}(\Omega).$$

We have

(i) *$\{N_G(w_n)\}$ converges to $N_G(w)$ in $L^1(\Omega)$ when $\{w_n\}$ weakly converges to w in $W_0^{1,p}(\Omega)$.*

(ii) *Ψ_g is continuously Fréchet differentiable mapping from $W_0^{1,p}(\Omega)$ into $\mathbb{R}$ and*

$$D\Psi_g(u)(\phi) = \int_\Omega g(x,\xi)\phi dx \quad h\forall u,\phi \in W_0^{1,p}(\Omega).$$

(iii) *If A is a bounded subset in $W_0^{1,p}(\Omega)$, then there is a positive real number M such that*

$$|\Psi_g(v)| + \|D\Psi_g(v)\| \le M \quad h\forall v \in A.$$

Proof. Let $\mu = s$, $q = \frac{s}{s-1}$ and g_1 be as in the proof of Theorem 2.8. Put

$$G_1(x,t) = \int_0^t g(x,\xi)d\xi \quad \forall (x,t) \in \Omega,$$
$$\Psi_{g_1}(u) = \int_\Omega \int_0^{u(x)} g_1(x,\xi)d\xi dx \quad \forall u \in L^p(\Omega).$$

By [4, Theorem 2.8], N_{G_1} is continuous from $L^{\frac{s}{s-1}}(\Omega)$ to $L^1(\Omega)$ and Ψ_{g_1} is continuously Fréchet differentiable mapping from $L^{\frac{s}{s-1}}(\Omega)$ to $\mathbb{R}$. We see that $N_G = N_{G_1} \circ T_\omega$ and $\Psi_g = \Psi_{g_1} \circ T_\omega$. By Theorem 2.3, we complete the proof. $\square$

For $\omega = 1$, Theorems 2.8 and 2.9 have been proved in [2, 4, 8].

Example 2.10. Let $\Omega = \{x \in \mathbb{R}^5 : \|x\| < 1\}$, $p = 3$, $s = 4$, $\alpha = \frac{3}{4}$ and $\rho(x) = (\frac{1}{2} - \|x\|^2)^2(1-\|x\|^2)^{-\frac{7}{16}}$ for every x in Ω. By Example 2.7, $\rho \in \mathcal{K}_{3,4}$. Put $a(x) = \rho(x)^{s-1} = (\frac{1}{2} - \|x\|^2)^6(1-\|x\|^2)^{-\frac{21}{16}}$ for every x in Ω. Thus a is not integrable on Ω and Theorem 2.9 improves corresponding results in [2, 4, 8].

3. Proof of main theorems

Put

$$J(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega F(x,u)dx \quad \forall u \in W_0^{1,p}(\Omega). \tag{3.1}$$

By [5, Theorem 9], Theorem 2.9 and (A1), J is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$ and

$$DJ(u)(v) = \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v dx - \int_\Omega f(x,u).v dx \ \forall u,v \in W_0^{1,p}(\Omega) \tag{3.2}$$

To prove the theorems, we need following lemmas.

Lemma 3.1. *Under conditions* (A3) *and* (A4)*, there exists positive numbers ρ and η such that $J(u) \geq \eta$ for all u in $W_0^{1,p}(\Omega)$ with $\|u\| = \rho$.*

Proof. Suppose by contradiction that

$$\inf\{J(u) : u \in W_0^{1,p}(\Omega), \|u\|_{1,p} = \frac{1}{n}\} \leq 0 \quad \forall n \in \mathbb{N}.$$

Then there is a sequence $\{u_n\}$ in $W_0^{1,p}(\Omega)$ such that $\|u_n\|_{1,p} = \frac{1}{n}$ and $J(u_n) < \frac{1}{n^{p+1}}$. By replacing $\{u_n\}$ by its subsequence, by [3, Theorem 4.9], we can suppose that $\lim_{n\to\infty} u_n(x) = 0$ for every x in Ω, $\{\frac{u_n}{\|u_n\|_{1,p}}\}$ strongly (resp. pointwisely) converges to w in $L^p(\Omega)$ (resp. on Ω) and

$$\begin{aligned}
\frac{1}{n} &> \frac{J(u_n)}{\|u_n\|_{1,p}^p} \\
&= \frac{1}{p} - \int_\Omega \frac{F(x,u_n(x))}{\|u_n\|_{1,p}^p}dx \\
&= \frac{1}{p} - \int_\Omega \int_0^1 f(x,su_n(x))\frac{u_n(x)}{\|u_n\|_{1,p}^p}\,ds\,dx \\
&= \frac{1}{p} - \int_\Omega \int_0^1 \frac{f(x,su_n(x))}{(su_n(x))^{p-2}su_n(x)}s^{p-1}\frac{|u_n(x)|^p}{\|u_n\|_{1,p}^p}\,ds\,dx.
\end{aligned}$$

Hence by the generalized Fatou Lemma, (A3) and (A4)

$$\begin{aligned}
0 &= \liminf_{n\to\infty} \frac{1}{n} \\
&= \frac{1}{p} - \limsup_{n\to\infty} \int_\Omega \int_0^1 \frac{f(x,su_n(x))}{(su_n(x))^{p-2}su_n(x)}s^{p-1}\frac{|u_n(x)|^p}{\|u_n\|_{1,p}^p}\,ds\,dx \\
&\geq \frac{1}{p} - \int_\Omega \int_0^1 \limsup_{n\to\infty}[\frac{f(x,su_n(x))}{(su_n(x))^{p-2}su_n(x)}s^{p-1}\frac{|u_n(x)|^p}{\|u_n\|_{1,p}^p}]\,ds\,dx = \frac{1}{p}.
\end{aligned}$$

This contradiction completes the proof. □

Lemma 3.2. *Let ρ be as in Lemma 3.1. Under conditions* (A3) *and* (A5)*, there is e in $W_0^{1,p}(\Omega) \setminus B(0,\rho)$ such that $J(e) < 0$.*

Proof. Let $u \in W_0^{1,p}(\Omega)$ such that $\|u\|_{1,p} = 1$ and $u > 0$ on Ω. By (3.1), we have

$$J(nu) = \frac{n^p}{p} - \int_\Omega \int_0^{nu(x)} f(x,s)\,ds\,dx$$

$$
\begin{aligned}
&= \frac{n^p}{p} - \int_\Omega \int_0^1 f(x, \xi n u(x)) n u(x) d\xi dx \\
&= \frac{n^p}{p}[1 - p \int_\Omega \int_0^1 \frac{f(x, \xi n u(x))}{(\xi n u(x))^{p-1}} \xi^{p-1} u(x)^p d\xi dx].
\end{aligned}
$$

By Sobolev's embedding theorem, u belongs to $L^{\frac{Np}{N-p}}(\Omega)$. By (A3), $d|u|^p$ is integrable and $\frac{f(x,\xi nu(x))}{|\xi nu(x)|^{p-2}\xi nu(x)}|u(x)|^p \geq d(x)|u(x)|^p$ for every integer n, $x \in \Omega$ and $\xi \in (0,1)$. Hence, by the generalized Fatou lemma, (A3) and (A5), we have

$$
\begin{aligned}
&\limsup_{n\to\infty}[1 - p \int_\Omega \int_0^1 \frac{f(x, \xi n u(x))}{|\xi n u(x)|^{p-2} \xi n u(x)} \xi^{p-1} |u(x)|^p d\xi dx] \\
&= 1 - \liminf_{n\to\infty}[p \int_\Omega \int_0^1 \frac{f(x, \xi n u(x))}{|\xi n u(x)|^{p-2} \xi n u(x)} \xi^{p-1} |u(x)|^p d\xi dx] \\
&\leq 1 - p \int_\Omega \int_0^1 \liminf_{n\to\infty}[\frac{f(x, \xi n u(x))}{|\xi n u(x)|^{p-2} \xi n u(x)} \xi^{p-1} |u(x)|^p] d\xi dx = -\infty,
\end{aligned}
$$

which implies $\lim_{n\to\infty} J(nu) = -\infty$. $\square$

Lemma 3.3. *Under conditions* (A2) *and* (A6)*, there is a positive real number* C_1 *such that*

$$
f(x,s)s - pF(x,s) \leq f(x,t)t - pF(x,t) + C_1 d(x) \quad \forall x \in \Omega, |s| \leq |t|.
$$

Proof. By the proof of [14, Lemma 2.3], (A2) and (A6), we have

$$
f(x,s)s - pF(x,s) \leq f(x,t)t - pF(x,t) \quad \forall x \in \Omega, C \leq s \leq t.
$$

Let $x \in \Omega$ and $\xi \in [-C, C]$. By (A2), we have

$$
|f(x,\xi)\xi| \leq Cd(x), \quad |F(x,\xi)| \leq \int_0^\xi d(x) dy \leq Cd(x).
$$

Hence

$$
\begin{aligned}
f(x,s)s - pF(x,s) &\leq f(x,t)t - pF(x,t) + 2(1+p)Cd(x) \quad \forall x \in \Omega,\ 0 \leq s \leq t \leq C, \\
f(x,s)s - pF(x,s) &\leq f(x,C)C - pF(x,C) + 2(1+p)Cd(x) \\
&\leq f(x,t)t - pF(x,t) + 2(1+p)Cd(x) \quad \forall x \in \Omega,\ 0 \leq s \leq C \leq t.
\end{aligned}
$$

Thus we get the lemma when $0 \leq s \leq t$. Similarly we obtain it if $t \leq s \leq 0$. $\square$

Lemma 3.4. *Assume* (A1)–(A3), (A5), (A6) *hold. Let* $\{u_n\}$ *be a sequence in* $W_0^{1,p}(\Omega)$ *such that* $\{J(u_n)\}$ *is bounded and* $\lim_{n\to\infty}(1 + \|u_n\|_{1,p})\|DJ(u_n)\| = 0$. *Then* $\{u_n\}$ *has a subsequence converging in* $W_0^{1,p}(\Omega)$.

Proof. We shall use the technique in [14, 16]. If $\{u_n\}$ is unbounded, up to a subsequence we may assume that for some c in $\mathbb{R}$ such that $\lim_n \|u\|_{1,p} = \infty$, $\lim_n J(u) = c$ and $\lim_{n\to\infty} \|u_n\|_{1,p} \|DJ(u_n)\| = 0$. Thus

$$
\begin{aligned}
&\lim_{n\to\infty} \int_\Omega (\frac{1}{p} f(x, u_n(x)) u_n(x) - F(x, u_n(x))) dx \\
&= \lim_{n\to\infty} (J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle) = c.
\end{aligned}
\tag{3.3}
$$

Put $w_n = \|u_n\|_{1,p}^{-1} u_n$ for every n in $\mathbb{N}$. Since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, by replacing $\{u_n\}$ by its subsequence, we can suppose $\{w_n\}$ converges weakly to w in $W_0^{1,p}(\Omega)$ (resp. strongly in $L^p(\Omega)$, pointwisely in Ω).

Consider the case $w = 0$. By the continuity of J, there is t_n in $[0, 1]$ such that $J(t_n u_n) = \max\{J(su_n) : s \in [0, 1]\}$ for every positive integer n. Fix a positive integer m and put $v_n = (2pm)^{1/p} w_n$ for every positive integer n. Then $\{v_n\}$ converges weakly to 0 in $W_0^{1,p}(\Omega)$ (resp. strongly in $L^p(\Omega)$, pointwisely in Ω). Therefore, by Theorem 2.9, $\{N_F((v_n)\}$ converges to $N_F(0) = 0$ in $L^1(\Omega)$. Thus

$$\lim_{n\to\infty} \int_\Omega F(x, v_n(x))dx = 0.$$

Since $\lim_{n\to\infty}(2pm)^{1/p}\|u_n\|_{1,p}^{-1} = 0$, there is an integer N_m such that $t_n \in [0, 1]$ and

$$J(t_n u_n) \geq J(v_m) = 2m - \int_\Omega F(x, v_m) \geq m \quad \forall n \geq N_m,$$

that is, $\lim_{n\to\infty} J(t_n u_n) = \infty$. Since $J(0) = 0$ and $\lim_{n\to\infty} J(u_n) = c$, it implies $t_n \in (0, 1)$ for any sufficiently large n and

$$\begin{aligned}\int_\Omega |\nabla(t_n u_n)|^p - \int_\Omega f(x, t_n u_n) t_n u_n dx &= \langle J'(t_n u_n), t_n u_n\rangle \\ &= t_n \frac{d}{dt}|_{t=t_n} J(t, u_n) = 0.\end{aligned}$$

Therefore, by Lemma 3.3, we get

$$\begin{aligned}&\int_\Omega (\frac{1}{p} f(x, u_n(x))u_n(x) - F(x, u_n(x)))dx \\ &\geq \int_\Omega (\frac{1}{p} f(x, t_n u_n(x)) t_n u_n(x) - F(x, t_n u_n(x)))dx - C_1\|d\|_{L^1(\Omega)} \\ &= \int_\Omega (\frac{1}{p} |\nabla t_n u_n(x)|^p - F(x, t_n u_n(x)))dx - C_1\|d\|_{L^1(\Omega)} \\ &= J(t_n u_n) - C_1\|d\|_{L^1(\Omega)} \to \infty,\end{aligned}$$

which contradicts (3.3).

If $w \neq 0$, the Lebesgue measure of the set $\Theta = \{x \in \Omega : w(x) \neq 0\}$ is positive. We have $\lim_{n\to\infty} |u_n(x)| = \infty$ for every x in Θ. Thus, By the generalized Fatou lemma, (A3) and (A5), we have

$$\begin{aligned}0 &= \liminf_{n\to\infty} [\frac{1}{p} - \frac{J(u_n)}{\|u_n\|_{1,p}}] = \liminf_{n\to\infty} \int_\Omega \frac{F(x, u_n(x))}{\|u_n\|_{1,p}^p} dx \\ &\geq \liminf_{n\to\infty} \Big[\int_\Theta \int_0^1 \frac{f(x, \xi u_n(x))}{|\xi u_n(x)|^{p-2} \xi u_n(x)} |\xi w_n(x)|^p \, d\xi \, dx \\ &\quad + \int_{\Omega\setminus\Theta} \int_0^1 d_1 |\xi w_n(x)|^p d\xi dx \Big] \\ &\geq \int_\Theta \int_0^1 \liminf_{n\to\infty} \frac{f(x, \xi u_n(x))}{(|\xi u_n(x)|^{p-2} \xi u_n(x)} |\xi w_n(x)|^p \, d\xi \, dx \\ &\quad - \|d_1\|_{L^{\frac{n}{p}}(\Omega)} \|w_n\|_{L^{\frac{Np}{N-p}}(\Omega)} = \infty,\end{aligned}$$

which is impossible. In any case, we obtain a contradiction. Therefore $\{u_n\}$ is bounded. By Theorem 2.8, there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$

weakly converges to u in $W_0^{1,p}(\Omega)$ and $\{N_f(u_{n_k})\}$ weakly converges to $N_f(u)$ in $L^{\frac{p}{p-1}}(\Omega)$. Arguing as in the proof of [4, Lemma 6.2], we see that $\{u_{n_k}\}$ converges to u in $W_0^{1,p}(\Omega)$. □

Proof of theorem 1.1. Using the Mountain-pas theorem with the Cerami condition, by Lemmas 3.1, 3.2 and 3.4, we obtain a non-trivial weak solution for the problem (1.1). □

Example 3.5. Let $N = 5$, $p = 3$, $r = 4$, $\alpha > 0$, $\Omega = \{x \in \mathbb{R}^5 : \|x\| < 1\}$,

$$\omega_0(x) = |x|^{-1/30}\cos(16|x|) \quad \forall x \in \Omega,$$

$$\omega_1(x) = (\frac{1}{2} - \|x\|^2)^2(1 - \|x\|^2)^{-7/6} \quad \forall x \in \Omega,$$

$$\varphi_0(t) = \begin{cases} |t|^{r-2}t(1-|t|) & \text{if } |t| \le 1, \\ 0 & \text{if } |t| \in \mathbb{R}\setminus[-1,1], \end{cases}$$

$$\varphi_1(t) = |t|^{p-2}t\varphi_1(t)\log(1+|t|) \quad h\forall t \in \mathbb{R},$$

$$f(x,t) = \omega_0(x)^{r-1}\varphi_0(t) + \omega_1(x)^{r-1}\varphi_1(t) \quad \forall (x,t) \in \Omega\times\mathbb{R}.$$

Let $\omega = |\omega_0| + \omega_1$, $C = 1$, $d(x) = |x|^{-\frac{1}{30}}$, $d_1(x) = -d(x)$ and $d_2(x) = |x|^{-\frac{1}{30}}$ for every x in Ω. We see that $d \in L^1(\Omega)$, $d_1 \in L^{\frac{N}{p}}(\Omega)$ and $d_2 \in L^1(\Omega)$. By Examples 2.5 and 2.7, ω is in $\mathcal{K}_{p,r}$. Thus f satisfies conditions (A1)–(A5). Since $\lim_{|x|\to 0}\omega_0(x) = \infty$ and $\lim_{|x|\to\frac{1}{2}}\omega_1(x) = 0$, the convergence in (A4) and (A5) are not uniform on Ω.

We have $\frac{f(x,t)}{|t|^{p-2}t} = \omega_1(x)(|t|-1)\log(1+|t|)$ for every $t \in [-2,2]\setminus[-1,1]$ and $\frac{f(x,t)}{|t|^{p-2}t} = \omega_1(x)\log(1+|t|)$ for every $t \in \mathbb{R}\setminus[-2,2]$. Thus f satisfies (A6). Therefore we can apply Theorem 1.1 to f with $C = 1$ respectively. Since $\omega^{r-1}(x) \ge (1 - \|x\|^2)^{-\frac{21}{16}}$ for every x in Ω, ω^{r-1} is not integrable on Ω. Therefore the results in [1, 5, 7, 11, 12, 13, 14, 16, 17] can not be applied to solve (1.1) in these cases.

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