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INVERSE PROBLEMS FOR STURM-LIOUVILLE OPERATORS WITH BOUNDARY CONDITIONS DEPENDING ON A SPECTRAL PARAMETER

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ABSTRACT. In this article, we study the inverse problem for Sturm-Liouville operators with boundary conditions dependent on the spectral parameter. We show that the potential $q(x)$ and coefficient $\frac{a_1\lambda+b_1}{c_1\lambda+d_1}$ functions can be uniquely determined from the particular set of eigenvalues.

1. INTRODUCTION

The theory of inverse problem for differential operators takes an important position in the trend development of the spectral theory of linear operators. Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics [1, 7, 14, 16, 18, 21]. Such problems often come along in mathematical physics, mechanics, electronics, geophysics and other branches of natural sciences. The inverse problem of a regular Sturm-Liouville operator was studied firstly by Ambarzumyan in 1929 [2] and secondly by Borg in 1945 [7]. From then on, Borg's result has been extended to various versions.

McLaughlin and Rundell in 1986 [19], established a new uniqueness theorem for the inverse Sturm-Liouville problem. They showed that the measurement of a particular set of eigenvalues was sufficient to define the obscure potential functions. They considered the eigenvalue problem

$$\begin{gathered} y'' + (\lambda - q(x))y = \lambda y, \quad 0 < x < 1, \\ y(0,\lambda) = 0, \quad y'(\pi,\lambda) + H_k y(\pi,\lambda) = 0. \end{gathered}$$

They indicated that the spectral knowledge, for a constant index n $(n = 0, 1, 2, \dots)$, $\{\lambda_n(q, H_k)\}_{k=1}^{+\infty}$ is equivalent to two spectra of boundary value problems with the equation and the first initial situation (one common boundary situation at $x = 0$) and the second boundary situation (two different boundary conditions at $x = \pi$). In [19] the spectral data was handled by the Hochstadt and Lieberman method [17]. Wang [22, 23] discussed the inverse problem for uncertain Sturm-Liouville operators on the finite interval $[a, b]$ and diffusion operators. Here, we consider inverse spectral problems for Sturm Liouville operators with boundary conditions

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dependent on the spectral parameter with the above spectral knowledge. As far as we know, inverse spectral problems for Sturm Liouville operators with boundary conditions depending on the spectral parameter have not been studied with the spectral data before.

Eigenvalue dependent boundary conditions have been studied extensively. References [3, 4, 5, 6, 8, 10, 12, 13] are well known examples for problems with boundary conditions depending linearly on the eigenvalue parameter. Recently inverse problems according to various spectral knowledge for eigenparameter linearly dependent Sturm-Liouville operator have been studied in [9, 11, 15, 20, 24, 25, 26, 27].

We consider the Sturm-Liouville operator $L := L(q, H_k)$ defined by

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.1}$$

with boundary conditions dependent on the spectral parameter

$$(a_1\lambda + b_1)y(0,\lambda) - (c_1\lambda + d_1)y'(0,\lambda) = 0, \tag{1.2}$$

$$y'(\pi,\lambda) + H_k y(\pi,\lambda) = 0. \tag{1.3}$$

Also we consider the Sturm-Liouville operator $\widetilde{L} := \widetilde{L}(\widetilde{q}, H_k)$ defined by

$$\widetilde{L}y \equiv -\widetilde{y}'' + \widetilde{q}(x)\widetilde{y} = \lambda\widetilde{y}, \ (0 \le x \le \pi), \tag{1.4}$$

with boundary conditions depending on the spectral parameter

$$(\widetilde{a}_1\lambda + \widetilde{b}_1)\widetilde{y}(0,\lambda) - (\widetilde{c}_1\lambda + \widetilde{d}_1)\widetilde{y}'(0,\lambda) = 0, \tag{1.5}$$

$$\widetilde{y}'(\pi,\lambda) + H_k\widetilde{y}(\pi,\lambda) = 0, \tag{1.6}$$

where $a_1, b_1, c_1, d_1, \widetilde{a}_1, \widetilde{b}_1, \widetilde{c}_1, \widetilde{d}_1, H_k \in \mathbb{R}$, such that $\delta_1 = a_1d_1 - b_1c_1 < 0$, $\widetilde{\delta}_1 = \widetilde{a}_1\widetilde{d}_1 - \widetilde{b}_1\widetilde{c}_1 < 0$, $0 < H_1 < H_2 < \dots < H_k < H_{k+1} < \dots < H_0$, the potentials $q(x)$ and $\widetilde{q}(x)$ are real valued functions, $q(x), \widetilde{q}(x) \in L^1[0,\pi]$ and λ is a spectral parameter.

For the boundary-value problem (1.1)-(1.3) with coefficient $H = -\frac{(a_2\lambda+b_2)}{(c_2\lambda+d_2)}$ where $c_2 \neq 0$ and $d_2 \neq 0$, instead of H_k describes the actual background of Sturm Liouville operators with boundary conditions dependent on a spectral parameter; see [12].

In this article, we construct a uniqueness theorem for Sturm-Liouville operators with boundary conditions depending on the spectral parameter on the finite interval $[0,\pi]$. i.e. for a constant index $n \in \mathbb{N}$, we demonstrate that if the spectral set $\{\lambda_n(q, H_k)\}_{k=1}^{+\infty}$ for different H_k can be restrained, then the spectral set $\{\lambda_n(q, H_k)\}_{k=1}^{+\infty}$ is sufficient to define the potential $q(x)$ and coefficient $\frac{a_1\lambda+b_1}{c_1\lambda+d_1}$ of the boundary condition. The techniques used here will be adopted from [17, 19, 26].

Lemma 1.1 ([4, 26]). *Eigenvalues λ_n $(n \neq 0)$ of the boundary-value problem* (1.1)-(1.3) *for coefficient $H = H_k = -\frac{(a_2\lambda+b_2)}{(c_2\lambda+d_2)}$ in* (1.3) *are roots of* (1.3) *and satisfy the asymptotic formula*

$$\sqrt{\lambda_n} = n + [1 + O(\frac{1}{n})]. \tag{1.7}$$

Lemma 1.2 ([26]). *The solution to the* (1.1) *with the initial conditions $y(0,\lambda) = (c_1\lambda + d_1)$ and $y'(0,\lambda) = (a_1\lambda + b_1)$ is*

$$\begin{aligned} y(x,\lambda) &= (c_1\lambda + d_1)\Big[\cos\sqrt{\lambda}x + \int_0^x A(x,t)\cos\sqrt{\lambda}t\,dt\Big] \\ &\quad + (a_1\lambda + b_1)\Big[\sin\frac{\sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\int_0^x B(x,t)\sin\sqrt{\lambda}t\,dt\Big] \end{aligned} \tag{1.8}$$

where the kernel $A(x,t)$ satisfies

$$\frac{\partial^2 A(x,t)}{\partial x^2} - q(x)A(x,t) = \frac{\partial^2 A(x,t)}{\partial t^2},$$

where $q(x) = 2\frac{dA(x,x)}{dx}$, $A(0,0) = h$, $\frac{\partial A(x,t)}{\partial t}\big|_{t=0} = 0$*; and the kernel* $B(x,t)$ *satisfies*

$$\frac{\partial^2 B(x,t)}{\partial x^2} - q(x)B(x,t) = \frac{\partial^2 B(x,t)}{\partial t^2}$$

where $q(x) = 2\frac{dB(x,x)}{dx}$, $B(x,0) = 0$.

2. Main results and their proofs

Theorem 2.1. *Let $\sigma(L_{k_j}) := \{\lambda_n(q, H_{k_j})\}$ $(j = 1,2)$ be the spectrum of the boundary value problem* (1.1)-(1.3) *with coefficient H_{k_j}. If $H_{k_1} \neq H_{k_2}$, then*

$$\sigma(L_{k_1}) \cap \sigma(L_{k_2}) = \emptyset \tag{2.1}$$

where $k_j \in \mathbb{N}$, and $\emptyset$ denotes an empty set.

Lemma 2.2. *Let $\lambda_n(q, H_k)$ be the n-th eigenvalue of the boundary-value problem* (1.1)-(1.3)*. Then the spectral set $\{\lambda_n(q, H_k)\}_{k=1}^{+\infty}$ is a bounded infinite set, where $0 < H_1 < H_2 < \dots < H_k < H_{k+1} < \dots < H_0$.*

The above Lemma carries a significant part in the proof of the next theorem.

Theorem 2.3. *Let $\lambda_n(q, H_k)$ be the n-th eigenvalue of the boundary-value problem* (1.1)-(1.3) *and $\lambda_n(\widetilde{q}, H_k)$ be the n-th eigenvalue of the boundary-value problem* (1.4)-(1.6)*, for a constant index $n (n \in \mathbb{N})$. If $\lambda_n(q, H_k) = \lambda_n(\widetilde{q}, H_k)$ for all $k \in \mathbb{N}$, then*

$$q(x) = \widetilde{q}(x) a.e. \text{ on } [0,\pi],$$

$$\frac{\widetilde{a}_1\lambda + \widetilde{b}_1}{\widetilde{c}_1\lambda + \widetilde{d}_1} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1}, \quad \forall\lambda \in C.$$

Proof of Theorem 2.1. Suppose the argument of Theorem 2.1 is false. Then there exists $\lambda_{n_1}(H_{k_1}) = \lambda_{n_2}(H_{k_2}) \in \mathbb{R}$, where $\lambda_{n_j}(H_{k_j}) \in \sigma(L_{k_j})$ for $j = 1,2$ and $n_j \in \mathbb{N}$. Let $y_{k_j}(x, \lambda_{n_j}(H_{k_j}))$ be the solution of (1.1)-(1.3) with the eigenvalue $\lambda_{n_j}(H_{k_j})$ and satisfy the initial conditions $y_{k_j}(0, \lambda_{n_j}(H_{k_j})) = (c_1\lambda + d_1)$ and $y'_{k_j}(0, \lambda_{n_j}(H_{k_j})) = (a_1\lambda + b_1)$. For a fixed index n, we have

$$-y''_{k_1}(x, \lambda_{n_1}(H_{k_1})) + q(x)y_{k_1}(x, \lambda_{n_1}(H_{k_1})) = \lambda_{n_1}(H_{k_1})y_{k_1}(x, \lambda_{n_1}(H_{k_1})) \tag{2.2}$$

and

$$-y''_{k_2}(x, \lambda_{n_2}(H_{k_2})) + q(x)y_{k_2}(x, \lambda_{n_2}(H_{k_2})) = \lambda_{n_2}(H_{k_2})y_{k_2}(x, \lambda_{n_2}(H_{k_2})). \tag{2.3}$$

By multiplying (2.2) by $y_{k_2}(x, \lambda_{n_2}(H_{k_2}))$ and (2.3) by $y_{k_1}(x, \lambda_{n_1}(H_{k_1}))$ respectively and subtracting and integrating from 0 to π, we obtain

$$(y_{k_2}y'_{k_1} - y_{k_1}y'_{k_2})\big|_0^\pi = 0. \tag{2.4}$$

Using the initial conditions, we obtain

$$y_{k_2}(\pi, \lambda_{n_2}(H_{k_2}))y'_{k_1}(\pi, \lambda_{n_1}(H_{k_1})) - y_{k_1}(\pi, \lambda_{n_1}(H_{k_1}))y'_{k_2}(\pi, \lambda_{n_2}(H_{k_2})) = 0. \tag{2.5}$$

On the other hand, we have the equality

$$
\begin{aligned}
&y_{k_2}(\pi,\lambda_{n_2}(H_{k_2}))y'_{k_1}(\pi,\lambda_{n_1}(H_{k_1})) - y_{k_1}(\pi,\lambda_{n_1}(H_{k_1}))y'_{k_2}(\pi,\lambda_{n_2}(H_{k_2}))\\
&= y_{k_2}(\pi,\lambda_{n_2}(H_{k_2}))[y'_{k_1}(\pi,\lambda_{n_1}(H_{k_1})) + H_{k_1}y_{k_1}(\pi,\lambda_{n_1}(H_{k_1}))]\\
&\quad - y_{k_1}(\pi,\lambda_{n_1}(H_{k_1}))[y'_{k_2}(\pi,\lambda_{n_2}(H_{k_2})) + H_{k_2}y_{k_2}(\pi,\lambda_{n_2}(H_{k_2}))]\\
&\quad + (H_{k_2}-H_{k_1})y_{k_1}(\pi,\lambda_{n_1}(H_{k_1}))y_{k_2}(\pi,\lambda_{n_2}(H_{k_2}))\\
&= (H_{k_2}-H_{k_1})y_{k_1}(\pi,\lambda_{n_1}(H_{k_1}))y_{k_2}(\pi,\lambda_{n_2}(H_{k_2})).
\end{aligned} \tag{2.6}
$$

Since $H_{k_2} - H_{k_1} \neq 0$, if $y_{k_1}(\pi,\lambda_{n_1}(H_{k_1}))y_{k_2}(\pi,\lambda_{n_2}(H_{k_2})) = 0$, it follows that

$$
y_{k_1}(\pi,\lambda_{n_1}(H_{k_1})) = 0 \text{ or } y_{k_2}(\pi,\lambda_{n_2}(H_{k_2})). \tag{2.7}
$$

By virtue of (2.7) together with (1.3), this yields

$$
y_{k_1}(\pi,\lambda_{n_1}(H_{k_1})) = y'_{k_1}(\pi,\lambda_{n_1}(H_{k_1})) = 0 \tag{2.8}
$$

or

$$
y_{k_2}(\pi,\lambda_{n_2}(H_{k_2})) = y'_{k_2}(\pi,\lambda_{n_2}(H_{k_2})) = 0. \tag{2.9}
$$

By (2.8) and (2.9), this yields

$$
y_{k_1}(x,\lambda_{n_1}(H_{k_1})) = 0 \text{ or } y_{k_2}(x,\lambda_{n_2}(H_{k_2})) = 0 \quad \text{on } [0,\pi], \tag{2.10}
$$

This is impossible. Thus, we obtain

$$
y_{k_2}(\pi,\lambda_{n_2}(H_{k_2}))y'_{k_1}(\pi,\lambda_{n_1}(H_{k_1})) - y_{k_1}(\pi,\lambda_{n_1}(H_{k_1}))y'_{k_2}(\pi,\lambda_{n_2}(H_{k_2})) \neq 0. \tag{2.11}
$$

Clearly, this contradicts (2.5); therefore (2.1)) holds. The proof is complete. □

Proof of Lemma 2.2. We will show that the following formula holds

$$
\lambda_n(H_0) < \cdots < \lambda_n(H_{k+1}) < \lambda_n(H_k) < \cdots < \lambda_n(H_1). \tag{2.12}
$$

Let $y(x,\lambda_n(H))$ be the solution of the boundary value problem (1.1)-(1.3) of the eigenvalue $\lambda_n(H)$ and satisfies the initial conditions $y(0,\lambda_n(H)) = (c_1\lambda + d_1)$ and $y'(0,\lambda_n(H)) = (a_1\lambda + b_1)$. We have

$$
-y''(x,\lambda_n(H)) + q(x)y(x,\lambda_n(H)) = \lambda_n(H)y(x,\lambda_n(H)), \tag{2.13}
$$

$$
\begin{aligned}
&-y''(x,\lambda_n(H+\Delta H)) + q(x)y(x,\lambda_n(H+\Delta H))\\
&= \lambda_n(H+\Delta H)y(x,\lambda_n(H+\Delta H))
\end{aligned} \tag{2.14}
$$

where ΔH is the enhancement of H. Multiplying (2.13) by $y(x,\lambda_n(H+\Delta H))$ and multiplying (2.14) by $y(x,\lambda_n(H))$ and subtracting from each other and integrating from 0 to π, we obtain

$$
\begin{aligned}
&\Delta\lambda_n(H)\int_0^\pi y(x,\lambda_n(H))y(x,\lambda_n(H+\Delta H))dx\\
&= \Delta H y(\pi,\lambda_n(H)y(\pi,\lambda_n(H+\Delta H)),
\end{aligned} \tag{2.15}
$$

where $\Delta\lambda_n(H) = \lambda_n(H+\Delta H) - \lambda_n(H)$.

It is well understood that $y(x,\lambda_n(H))$ and $\lambda_n(H)$ are real and continuous with respect to H. Dividing (2.15) by ΔH, and letting $\Delta H \to 0$ in (2.15), we have

$$
\frac{\partial\lambda_n(H)}{\partial H}\int_0^\pi y^2(x,\lambda_n(H))dx = y^2(\pi,\lambda_n(H)). \tag{2.16}
$$

If $y(\pi,\lambda_n(H)) = 0$, then $y'(\pi,\lambda_n(H)) = 0$. By the uniqueness theorem, this yields

$$
y(x,\lambda_n(H)) \equiv 0.
$$

This contradicts the eigenfunction $y(x,\lambda_n(H) \neq 0$ corresponding to eigenvalue $\lambda_n(H)$. Hence $y^2(\pi,\lambda_n(H)) > 0$ and $\int_0^\pi y^2(x,\lambda_n(H))dx > 0$. From (2.16), we have

$$\frac{\partial \lambda_n(H)}{\partial H} > 0.$$

This implies that (2.12) holds. Therefore the spectral set $\{\lambda_n(q,H_k)\}_{k=1}^{+\infty}$ is a bounded infinite set. The proof is complete. □

Finally, using Theorem 2.1, Lemma 2.2 and the properties of entire functions, we show that Theorem 2.3 holds.

Proof of Theorem 2.3. According to Lemma 1.2, solutions to equation (1.1) with boundary condition (1.2) and the equation (1.4) with boundary condition (1.5) can be stated in the integral forms:

$$\begin{aligned} y(x,\lambda) &= (c_1\lambda + d_1)\Big[\cos\sqrt{\lambda}x + \int_0^x A(x,t)\cos\sqrt{\lambda}t\,dt\Big] \\ &\quad + (a_1\lambda + b_1)\Big[\sin\frac{\sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\int_0^x B(x,t)\sin\sqrt{\lambda}t\,dt\Big] \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \widetilde{y}(x,\lambda) &= (\widetilde{c}_1\lambda + \widetilde{d}_1)\Big[\cos\sqrt{\lambda}x + \int_0^x \widetilde{A}(x,t)\cos\sqrt{\lambda}t\,dt\Big] \\ &\quad + (\widetilde{a}_1\lambda + \widetilde{b}_1)\Big[\sin\frac{\sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}}\int_0^x \widetilde{B}(x,t)\sin\sqrt{\lambda}t\,dt\Big] \end{aligned} \tag{2.18}$$

respectively. Let $\lambda = s^2$. From (2.17), (2.18) and [26, proof of Theorem 2.1] we obtain

$$\begin{aligned} y\widetilde{y} &= \frac{(c_1s^2+d_1)(\widetilde{c}_1s^2+\widetilde{d}_1)}{2}\Big[1+\cos 2sx + \int_0^x k(x,\tau)\cos 2s\tau d\tau\Big] \\ &\quad + \frac{(a_1s^2+b_1)(\widetilde{a}_1s^2+\widetilde{b}_1)}{2s^2}\Big[1-\cos 2sx + \int_0^x h(x,\tau)\cos 2s\tau d\tau\Big] \\ &\quad + \frac{1}{2s}(c_1s^2+d_1)(\widetilde{a}_1s^2+\widetilde{b}_1)\Big[\sin 2sx + \int_0^x l(x,\tau)\sin 2s\tau d\tau\Big] \\ &\quad + \frac{1}{2s}(\widetilde{c}_1s^2+\widetilde{d}_1)(a_1s^2+b_1)\Big[\sin 2sx + \int_0^x m(x,\tau)\sin 2s\tau d\tau\Big], \end{aligned} \tag{2.19}$$

where the functions $k(x,\tau)$, $h(x,\tau)$, $l(x,\tau)$ and $m(x,\tau)$ are continuous functions.

We define the function

$$w(\lambda) = (a_2\lambda + b_2)y(\pi,\lambda) - (c_2\lambda + d_2)y'(\pi,\lambda).$$

From (2.17), we obtain the asymptotic forms

$$\begin{gathered} y(\pi,\lambda) = (c_1\lambda + d_1)\cos\sqrt{\lambda}\pi + O(\sqrt{\lambda}e^{|\operatorname{Im}\sqrt{\lambda}|\pi}), \\ y'(\pi,\lambda) = -(c_1\lambda + d_1)\sqrt{\lambda}\sin\sqrt{\lambda}\pi + O(\sqrt{\lambda}e^{|\operatorname{Im}\sqrt{\lambda}|\pi}). \end{gathered}$$

Hence

$$w(\lambda) = (c_1\lambda + d_1)(c_2\lambda + d_2)\sqrt{\lambda}\sin\sqrt{\lambda}\pi + O(|\lambda|^2 e^{|\operatorname{Im}\sqrt{\lambda}|\pi}). \tag{2.20}$$

Zeros of $w(\lambda)$ are the eigenvalues of the Sturm-Liouville problem (1.1)-(1.3) where $H_k = H = -\frac{(a_2\lambda+b_2)}{(c_2\lambda+d_2)}$. $w(\lambda)$ is an entire function of order $\frac{1}{2}$ of λ. Multiplying (1.4) by y, (1.1) by $\widetilde{y}$ and subtracting and integrating from 0 to π, we take

$$(\widetilde{y}y' - y\widetilde{y}')\Big|_0^\pi + \int_0^\pi (\widetilde{q} - q)y\widetilde{y}dx = 0.$$

Using $y(0,\lambda) = (c_1\lambda + d_1)$, $\widetilde{y}(0,\lambda) = (\widetilde{c}_1\lambda + \widetilde{d}_1)$, $y'(0,\lambda) = (a_1\lambda + b_1)$ and $\widetilde{y}'(0,\lambda) = (\widetilde{a}_1\lambda + \widetilde{b}_1)$, this yields

$$\begin{aligned} 0 &= [\widetilde{y}(\pi,\lambda)y'(\pi,\lambda) - y(\pi,\lambda)\widetilde{y}'(\pi,\lambda)] + (a_1\lambda + b_1)(\widetilde{c}_1\lambda + \widetilde{d}_1) \\ &\quad - (c_1\lambda + d_1)(\widetilde{a}_1\lambda + \widetilde{b}_1) + \int_0^\pi (\widetilde{q}(x) - q(x))y\widetilde{y}dx. \end{aligned} \tag{2.21}$$

Let $Q(x) = (\widetilde{q}(x) - q(x))$ and

$$\begin{aligned} K(\lambda) &= (a_1\widetilde{c}_1 - \widetilde{a}_1c_1)\lambda^2 + (a_1\widetilde{d}_1 + b_1\widetilde{c}_1 - \widetilde{a}_1d_1 - \widetilde{b}_1c_1)\lambda \\ &\quad + (b_1\widetilde{d}_1 - \widetilde{b}_1d_1) + \int_0^\pi Q(x)y\widetilde{y}dx. \end{aligned} \tag{2.22}$$

Clearly, the function $K(\lambda)$ is an entire function. Because the first term of equation (2.21) for $\lambda = \lambda_n(q, H_k)$ is zero, then

$$K(\lambda_n(q, H_k)) = 0.$$

From Lemmas 1.1 and 2.2, we see that the spectral set $\{\lambda_n(q, H_k)\}_{k=1}^{+\infty}$ is a bounded infinite set. Therefore, if consists of $\lambda_{n0}(q) \in \mathbb{R}$, such that $\lambda_{n0}(q)$ is a finite accumulation dot of the spectrum set $\{\lambda_n(q, H_k)\}_{k=1}^{+\infty}$. It is well understood that the set of zeros of every entire function which is not identically zero hasn't any finite accumulation dot.

Hence

$$K(\lambda) = 0, \ \forall\lambda \in \mathbb{C}. \tag{2.23}$$

From (2.22), (2.23) and [26, proof of Theorem 2.1], we have

$$\begin{gathered} Q(x) = \widetilde{q}(x) - q(x) = 0, \quad \text{a.e. on } [0,\pi], \\ \frac{\widetilde{a}_1\lambda + \widetilde{b}_1}{\widetilde{c}_1\lambda + \widetilde{d}_1} = \frac{a_1\lambda + b_1}{c_1\lambda + d_1}, \quad \forall\lambda \in C. \end{gathered}$$

The proof is complete. □

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