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EXISTENCE OF SOLUTIONS TO A BOUNDARY-VALUE PROBLEM FOR AN INFINITE SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract. Using techniques associated with measures of noncompactness we prove an existence of solutions for a boundary-value problem for an infinite system of ordinary differential equations of second order. Our approach depends on transforming of the original boundary-value problem into an infinite system of integral equations of Fredholm type. The settings for this article are in the classical Banach sequence space l_p with $p \geq 1$.

1. Introduction and preliminaries concerning measures of noncompactness

To formulate a fixed point theorem Darbo used the so-called Kuratowski measure of noncompactness α, introduced in 1930 by Kuratowski [12]. Subsequently, in the literature have appeared a lot of functions being measures of noncompactness (cf. [1, 2, 3, 5]). Nevertheless, it turned out that among all classical realizations of measures of noncompactness the Hausdorff measure seems to be the most convenient and useful in applications.

The Hausdorff measure of noncompactness was introduced in 1965 [11] (see also [10]) by the formula

$$\chi(X) = \inf \big\{ \varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E \big\},$$

where X is a nonempty and bounded subset of the Banach space E.

The function χ has a lot of useful properties being essential in applications. For example, we have that $\chi(X) = 0$ if and only if X is a relatively compact subset of E. To recall other properties of χ let us introduce first some auxiliary notation. Namely, by the symbol $B(x, r)$ we denote the closed ball centered at x and with radius r. The symbol B_r will denote the ball $B(\theta, r)$, where θ stands for the zero element of E. If X is a subset of E then $\overline{X}$ and $\operatorname{conv} X$ denote the closure and the closed convex hull of X, respectively. Moreover, we use the standard notation $X + Y$, λX to denote the algebraic operations on subsets of E.

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Now, let us recall a few properties of the Hausdorff measure of noncompactness χ defined above. Namely, if X, Y are arbitrary nonempty bounded subsets of E and $\lambda \in \mathbb{R}$, then:

(1) $X \subset Y$ implies $\chi(X) \le \chi(Y)$.
(2) $\chi(\overline{X}) = \chi(\operatorname{conv} X) = \chi(X)$.
(3) $\chi(X+Y) \le \chi(X) + \chi(Y)$.
(4) $\chi(\lambda X) = |\lambda|\chi(X)$.

For other properties of the Hausdorff measure χ we refer to [3, 5], for example. In what follows we recall the fixed point theorem of Darbo type (cf. [1, 3]) which will be utilized in our considerations.

Theorem 1.1. *Let Ω be a nonempty, bounded, closed and convex subset of the space E and let $F : \Omega \to \Omega$ be a continuous operator such that $\chi(FX) \le k\chi(X)$ for any nonempty subset X of Ω, where $k \in [0, 1)$ is a constant. Then F has at least one fixed point in the set Ω.*

It is worthwhile mentioning that to apply efficiently Theorem 1.1 in a concrete Banach space E we have to know a formula expressing the Hausdorff measure of noncompactness χ in E in a convenient way, connected with the structure of the underlying Banach space E. It turns out that such formulas are known only in a few spaces [3, 5]. For our purposes we recall such a formula for the sequence space l_p.

To this end let us fix a number p, $p \ge 1$, and denote by l_p the classical Banach sequence space with the norm

$$\|x\|_{l_p} = \|(x_n)\|_{l_p} = \Big(\sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p}$$

for $x = (x_n) \in l_p$. In some considerations we will drop the index l_p if it does not lead to misunderstanding.

If X is a nonempty and bounded subset of l_p then

$$\chi(X) = \lim_{n\to\infty} \Big\{ \sup_{(x_k)\in X} \Big(\sum_{k=n}^{\infty} |x_k|^p\Big)^{1/p} \Big\} \tag{1.1}$$

(cf. [3]). The above formula will be utilized in the sequel of the paper.

Finally, let us mention that results of the present paper generalize those obtained in [14], where the considerations were conducted in the classical Banach sequence spaces c_0 and l_1.

2. Main result

Infinite systems of ordinary differential equations are closely related to several important problems appearing naturally in applications (cf. [5, 7, 8]). For example, let us mention that some methods of solving of partial differential equations based on the application of the so-called semidiscretization and numerical analysis lead to infinite systems of differential equations. Apart from this one can encounter other significant problems in engineering, mechanics, in the theory of branching processes and so on, which are associated with the theory of infinite systems of differential equations (see [4, 5, 7, 13]). On the other hand infinite systems of differential equations can be treated as ordinary differential equations in some Banach sequence spaces [4, 5, 7]. The above mentioned facts justify the interest in the theory of infinite systems of differential equations.

In this paper we study the infinite systems of differential equations of second order having the form

$$u_i''(t) = -f_i(t, u_1, u_2, \dots), \tag{2.1}$$

where $t \in I = [0, T]$ and $i = 1, 2, \dots$. The above system will be studied together with the boundary problem

$$u_i(0) = u_i(T) = 0, \tag{2.2}$$

for each $i = 1, 2, \dots$. In our study of problem (2.1)-(2.2) we will apply a technique associated with the Hausdorff measure of noncompactness χ, and the fixed point theorem of Darbo type presented in Theorem 1.1.

An essential tool applied in our approach to the study of problem (2.1)-(2.2) depends on converting (2.1)-(2.2) into the infinite system of integral equations of Fredholm type of the form

$$u_i(t) = \int_0^T G(t, s) f_i(s, u(s)) ds, \tag{2.3}$$

where $G(t, s)$ is the Green function corresponding to problem (2.1)-(2.2) on the interval $I = [0, T]$ and defined on the square I^2 in the following way (cf. [9])

$$G(t, s) = \begin{cases} \frac{t}{T}(T - s) & \text{for } 0 \le t \le s \le T, \\ \frac{s}{T}(T - t) & \text{for } 0 \le s \le t \le T\,. \end{cases} \tag{2.4}$$

Using standard methods it is easy to show the estimate

$$G(t, s) \le \frac{T}{4} \tag{2.5}$$

for all $(t, s) \in I^2$.

In what follows we write $f_i(t, u)$ instead of $f_i(t, u_1, u_2, \dots)$ ($i = 1, 2, \dots$). To introduce further auxiliary facts let us assume that E is a given Banach space with the norm $\| \cdot \|_E$. Denote by $C(I, E)$ the Banach space consisting of all functions $u = u(t)$ acting continuously from the interval I into the space E and endowed by the classical supremum norm

$$\|u\|_C = \sup\{\|u(t)\|_E : t \in I\}.$$

Remark 2.1. Assume that χ_E is the Hausdorff measure of noncompactness in the Banach space E. Next, let us take an arbitrary bounded subset X of the space $C(I, E)$ which is equicontinuous on the interval I. Then, it can be shown [3] that we have the following formula expressing the Hausdorff measure of noncompactness of the set X:

$$\chi(X) = \sup\{\chi_E(X(t)) : t \in I\}. \tag{2.6}$$

As a special case of the above discussed space $C(I, E)$ we will consider the space $C(I, \mathbb{R})$, which will be denoted by $C(I)$. The supremum norm in the space $C(I)$ will be denoted by $\| \cdot \|_\infty$. Moreover, the symbol $C^2(I)$ denotes the space of real functions defined and twice continuously differentiable on I with the standard norm $\| \cdot \|_{C^2}$ defined as follows

$$\|u\|_{C^2} = \sum_{i=0}^{2} \|u^{(i)}\|_\infty.$$

Now, let us observe that problem (2.1)-(2.2) has a solution $u = (u_i)$ belonging to the space $C^2(I)$ if and only if the infinite system of integral equations (2.3) has a solution $u \in C(I)$. To prove this fact, for arbitrarily fixed i and $t \in I$, let us write (cf. (2.4)):

$$u_i(t) = \int_0^t \frac{s}{T}(T-t) f_i(s, u(s)) ds + \int_t^T \frac{t}{T}(T-s) f_i(s, u(s)) ds.$$

Hence, differentiating the above equality we subsequently obtain

$$u_i'(t) = -\frac{1}{T}\int_0^t s f_i(s, u(s)) ds + \frac{1}{T}\int_t^T (T-s) f_i(s, u(s)) ds,$$
$$u_i''(t) = -f_i(t, u(t)),$$

which proves our assertion.

We will investigate the infinite system of integral equations (2.3) imposing the following assumptions.

(i) The function f_i is defined on the set $I \times \mathbb{R}^\infty$ and takes real values for $i = 1, 2, \dots$.

(ii) The operator f defined on the space $I \times l_p$ by the formula
$$(fu)(t) = (f_i(t, u)) = (f_1(t, u), f_2(t, u), \dots)$$
transforms the space $I \times l_p$ into l_p and is such that the family of functions $\{(fu)(t)\}_{t \in I}$ is equicontinuous at each point of the space l_p i.e., for each arbitrarily fixed $u \in l_p$ and for a given $\varepsilon > 0$ there exists $\delta > 0$ such that
$$\|(fv)(t) - (fu)(t)\|_{l_p} < \varepsilon$$
for each $t \in I$ and for any $v \in l_p$ such that $\|v - u\|_{l_p} < \delta$.

(iii) For any natural i there exist functions $g_i,\ h_i : I \to \mathbb{R}_+$ such that the inequality
$$|f_i(t, u)|^p \le g_i(t) + h_i(t)|u_i|^p$$
is satisfied for $t \in I$, $u = (u_i) \in l_p$ and $i = 1, 2, \dots$.
Moreover, we assume that the function g_i is continuous on I ($i = 1, 2, \dots$) and the function series $\sum_{i=1}^\infty g_i(t)$ is uniformly convergent, while the function sequence $(h_i(t))$ is equibounded on I.

Observe that in view of assumption (iii) the constant
$$H := \sup\{h_i(t) : t \in I,\ i = 1, 2, \dots\}$$
is finite. Moreover, we can define the function $g = g(t)$ on the interval I by putting
$$g(t) = \sum_{i=1}^\infty g_i(t).$$
Obviously the function $g(t)$ is continuous on I. Therefore, we can define the finite constant
$$G = \max\{g(t) : t \in I\}.$$

The concept of the equicontinuity utilized in assumption (iii) was introduced in the book [15]. Now, we can formulate our main result.

Theorem 2.2. *Under assumptions* (i)-(iii)*, if additionally* $(H/T)^{1/p} T^2 < 4$ *and* $T \le 1$*, the infinite system of integral equations* (2.3) *has at least one solution* $u(t) = (u_i(t))$ *in the space* l_p *i.e.,* $(u_i(t)) \in l_p$ *for each* $t \in I$.

Proof. At the beginning let us consider the space $C(I, l_p)$ of all functions continuous on the interval $I = [0, T]$ with values in the space l_p and furnished with the classical supremum norm
$$\|u\| = \sup\{\|u(t)\|_{l_p} : t \in I\}.$$
This space is a particular case of the space $C(I, E)$ introduced previously. Next, let us denote by F the operator defined on the space $C(I, l_p)$ by the formula
$$\begin{aligned}(Fu)(t) = ((Fu)_i(t)) &= \Big(\int_0^T G(t,s) f_i(s, u(s)) ds\Big) \\ &= \Big(\int_0^T G(t,s) f_1(s, u(s)) ds, \int_0^T G(t,s) f_2(s, u(s)) ds, \dots\Big).\end{aligned}$$
At first, let us notice that the operator F maps the space $C(I, l_p)$ into itself. Indeed, for a fixed $u = u(t) = (u_i(t)) \in C(I, l_p)$ and for an arbitrary $t \in I$, using the imposed assumptions and the Hölder inequality, we obtain
$$\begin{aligned}\|(Fu)(t)\|_{l_p}^p &= \sum_{i=1}^{\infty} \Big| \int_0^T G(t,s) f_i(s, u(s)) ds \Big|^p \\ &\le \sum_{i=1}^{\infty} \Big(\int_0^T |G(t,s)|^p |f_i(s, u(s))|^p ds \Big) \Big(\int_0^T ds \Big)^{p/q} \\ &\le T^{p/q} \sum_{i=1}^{\infty} \int_0^T |G(t,s)|^p [g_i(s) + h_i(s)|u_i(s)|^p] ds \\ &\le T^{p/q} \sum_{i=1}^{\infty} \Big[\int_0^T |G(t,s)|^p g_i(s) ds + \int_0^T |G(t,s)|^p h_i(s) |u_i(s)|^p ds \Big],\end{aligned}$$
where $q > 1$ is a number such that $1/p + 1/q = 1$.

Further, applying the Lebesgue dominated convergence theorem, from the above estimate we obtain
$$\begin{aligned}\|(Fu)(t)\|_{l_p}^p &\le T^{p/q} \int_0^T |G(t,s)|^p g(s) ds \\ &\quad + T^{p/q} H \int_0^T |G(t,s)|^p \{|u_1(s)|^p + |u_2(s)|^p + \dots\} ds.\end{aligned}$$
Hence, in view of (2.5), we derive the estimate
$$\begin{aligned}\|(Fu)(t)\|_{l_p}^p &\le T^{p/q} \int_0^T (T/4)^p g(s) ds + T^{p/q} H \int_0^T (T/4)^p \Big(\sum_{i=1}^{\infty} |u_i(s)|^p \Big) ds \\ &\le (T^{2p-1}/4^p) \int_0^T g(s) ds + (HT^{2p-1}/4^p) \int_0^T \Big(\sum_{i=1}^{\infty} |u_i(s)|^p \Big) ds \\ &\le (T^{2p}/4^p) G + H (T^{2p-1}/4^p) \int_0^T \|u\|^p ds \\ &= (T^{2p}/4^p)[G + H\|u\|^p].\end{aligned} \tag{2.7}$$
Hence we deduce that Fu is bounded on the interval I. This implies that the operator F transforms the space $C(I, l_p)$ into itself. Apart from this, from estimate

(2.7) we derive the following inequality

$$\|Fu\| \le \frac{T^2}{4}(G + H\|u\|^p)^{1/p}. \tag{2.8}$$

Further, taking into account assumption (iii) we deduce that the positive number

$$r_0 = \frac{T^2\, G^{1/p}}{(4^p - HT^{2p})^{1/p}}$$

is the optimal solution of the inequality

$$\frac{T^2}{4}(G + Hr^p)^{1/p} \le r.$$

Thus, in view of (2.8) we infer that the operator F transforms the ball B_{r_0} in the space $C(I, l_p)$ into itself.

Next we show that the operator F is continuous on the ball B_{r_0}. To this end let us fix arbitrarily a number $\varepsilon > 0$ and a function $u \in B_{r_0}$. Then, for an arbitrary $v \in B_{r_0}$ such that $\|v - u\| \le \varepsilon$ and for a fixed number $t \in I = [0, T]$, in view of the imposed assumptions we obtain

$$\begin{aligned}
&\|(Fv)(t) - (Fu)(t)\|_{l_p}^p \\
&= \sum_{i=1}^{\infty} \Big| \int_0^T G(t,s)[f_i(s, v(s)) - f_i(s, u(s))]ds \Big|^p \\
&\le \sum_{i=1}^{\infty} \Big(\int_0^T |G(t,s)|^p |f_i(s, v(s)) - f_i(s, u(s))|^p ds \Big) \Big(\int_0^T ds \Big)^{p/q} \\
&\le \sum_{i=1}^{\infty} T^{p/q} \int_0^T |G(t,s)|^p |f_i(s, v(s)) - f_i(s, u(s))|^p ds.
\end{aligned}$$

Hence, using estimate (2.5) and applying assumption (ii) concerning the equicontinuity of the family of functions $\{(fu)(t)\}_{t \in I}$, we obtain

$$\begin{aligned}
&\|(Fv)(t) - (Fu)(t)\|_{l_p}^p \\
&\le T^{p/q}(T/4)^p \sum_{i=1}^{\infty} \int_0^T |f_i(s, v(s)) - f_i(s, u(s))|^p ds \\
&= (T^{2p-1}/4^p) \lim_{k \to \infty} \sum_{i=1}^{k} \int_0^T |f_i(s, v(s)) - f_i(s, u(s))|^p ds \\
&= (T^{2p-1}/4^p) \lim_{k \to \infty} \int_0^T \Big(\sum_{i=1}^{k} |f_i(s, v(s)) - f_i(s, u(s))|^p \Big) ds.
\end{aligned} \tag{2.9}$$

Further on, let us observe that keeping in mind assumption (ii) on the equicontinuity of the family $\{(fu)(t)\}_{t \in I}$ at every point $u \in l_p$, we conclude that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, where $\delta = \delta(\varepsilon)$ is the function defined by the equality

$$\delta(\varepsilon) = \sup \big\{ |f_i(t, v) - f_i(t, u)| : u, v \in l_p,\ \|v - u\|_{l_p} \le \varepsilon,\ t \in I,\ i = 1, 2, \dots \big\}.$$

Combining the above fact with (2.9) and applying the Lebesgue dominated convergence theorem we arrive at the estimate

$$\|(Fv)(t) - (Fu)(t)\|_{l_p}^p \le \frac{T^{2p-1}}{4^p} \int_0^T (\delta(\varepsilon))^p ds = (T/2)^{2p} (\delta(\varepsilon))^p.$$

The above estimate allows us to infer that the operator F is continuous on the ball B_{r_0}.

Further, let us notice that the function $G(t,s)$ defined by (2.4) is uniformly continuous on the square I^2. Combining this fact with the definition of the operator F, it is easily seen that the set FB_{r_0} is equicontinuous on the interval I. Now, let us consider the set $B^1_{r_0} = \operatorname{conv} FB_{r_0}$. Obviously $B^1_{r_0} \subset B_{r_0}$ and the functions from the set $B^1_{r_0}$ are equicontinuous on I.

Next, let us take a nonempty subset X of the set $B^1_{r_0}$. From the above facts we conclude that X is equicontinuous on the interval I. Choose a function $u \in X$ and fix an arbitrary natural number n. Then, for arbitrarily fixed $t \in I$, in view of assumption (iii) we obtain

$$
\begin{aligned}
\sum_{i=n}^{\infty} |(Fu)_i(t)|^p &= \sum_{i=n}^{\infty} | \int_0^T G(t,s) f_i(s,u(s)) ds|^p \\
&\le \sum_{i=n}^{\infty} (\int_0^T |G(t,s)||f_i(s,u(s))| ds)^p.
\end{aligned}
$$

Hence, using the Hölder inequality, we obtain

$$
\begin{aligned}
\sum_{i=n}^{\infty} |(Fu)_i(t)|^p &\le \sum_{i=n}^{\infty} \Big[\Big(\int_0^T |G(t,s)|^p |f_i(s,u(s))|^p ds \Big)^{1/p} \Big(\int_0^T ds \Big)^{1/q} \Big]^p \\
&\le T^{p/q} \sum_{i=n}^{\infty} \int_0^T |G(t,s)|^p |f_i(s,u(s))|^p ds \\
&\le T^{p/q} (T/4)^p \sum_{i=n}^{\infty} \int_0^T |f_i(s,u(s))|^p ds.
\end{aligned}
$$

Now, applying the earlier conducted reasoning depending on the use of the Lebesgue dominated convergence theorem, from the above estimate we derive the inequality

$$
\begin{aligned}
&\sum_{i=n}^{\infty} |(Fu)_i(t)|^p \\
&\le (T^{2p-1}/4^p) \int_0^T \Big(\sum_{i=n}^{\infty} |f_i(s,u(s))|^p \Big) ds \\
&\le (T^{2p-1}/4^p) \int_0^T \Big(\sum_{i=n}^{\infty} [g_i(s) + h_i(s)|u_i(s)|^p] \Big) ds \\
&\le (T^{2p-1}/4^p) \Big\{ \int_0^T \Big(\sum_{i=n}^{\infty} g_i(s) \Big) ds + \int_0^T (\sum_{i=n}^{\infty} h_i(s) |u_i(s)|^p) ds \Big\} \\
&\le (T^{2p-1}/4^p) \Big\{ \int_0^T \Big(\sum_{i=n}^{\infty} g_i(s) \Big) ds + H \int_0^T \Big(\sum_{i=n}^{\infty} |u_i(s)|^p \Big) ds \Big\}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\sup_{u\in X}\sum_{i=n}^{\infty}|(Fu)_i(t)|^p &\le (T^{2p-1}/4^p)\int_0^T\Big(\sum_{i=n}^{\infty}g_i(s)\Big)ds \\
&\quad + (HT^{2p-1}/4^p)\sup_{u\in X}\int_0^T\Big(\sum_{i=n}^{\infty}|u_i(s)|^p\Big)ds.
\end{aligned}
\tag{2.10}
$$

Now, keeping in mind assumption (iii) and formula (1.1) expressing the Hausdorff measure of noncompactness χ in the space l_p and taking into account the fact that the set X consists of functions equicontinuous on the interval I on the base of Remark 2.1 and estimate (2.10) we infer the following inequality

$$\big(\chi(FX)\big)^p \le (HT^{2p-1}/4^p)(\chi(X))^p.$$

Consequently, we obtain

$$\chi(FX) \le [(H/T)^{1/p}\,T^2/4]\chi(X).$$

The above obtained inequality in conjunction with Theorem 1.1 applied to the operator F on the set $B^1_{r_0}$ completes the proof. □

Further, let us notice that in view of the equivalence of the infinite system of integral equations (2.3) and the boundary-value problem (2.1)-(2.2), we obtain the following reformulation of Theorem 2.2.

Theorem 2.3. *Under assumptions of Theorem 2.2 the infinite system of differential equations of the second order* (2.1) *satisfying boundary conditions* (2.2)*, has at least one solution $u(t) = (u_1(t), u_2(t), \dots)$ such that $u_i \in C^2(I)$ for $i = 1, 2, \dots$ and $u(t) \in l_p$ for any $t \in I$.*

Remark 2.4. Observe that reasoning conducted in the proof of Theorem 2.2 require to impose the additional assumption that $p > 1$. The case $p = 1$ was considered in [14].

Next we provide an example illustrating our considerations and results covered by Theorems 2.2 and 2.3.

Example 2.5. Let us consider the infinite system of differential equations of the second order of the form

$$u_i'' = -\frac{\sqrt{t}\,e^{-it}}{i} - \sum_{k=i}^{\infty}\frac{\ln(1+t)}{k(1+i)}\cdot\frac{u_k(t)}{1+(k-i)^2\,u_k^2(t)}, \tag{2.11}$$

for $t \in I = [0, T]$ and $i = 1, 2, \dots$. The system (2.11) is considered together with the boundary conditions

$$u_i(0) = u_i(T) = 0 \tag{2.12}$$

for $i = 1, 2, \dots$.

In our further considerations we will apply the fact that for any fixed positive β we have the equality

$$\max_{a>0}\frac{a}{1+\beta^2a^2} = \frac{1}{2\beta}. \tag{2.13}$$

Further, let us observe that problem (2.11)-(2.12) is a particular case of problem (2.1)-(2.2) if we put

$$f_i(t,u_1,u_2,\dots) = \frac{\sqrt{t}\,e^{-it}}{i} + \sum_{k=i}^{\infty} \frac{\ln(1+t)}{k(1+i)} \cdot \frac{u_k}{1+(k-i)^2\,u_k^2}$$

for an arbitrary $i = 1,2,\dots$.

Next fixing arbitrarily a natural number i, we obtain the estimates

$$\begin{aligned}
&|f_i(t,u_1,u_2,\dots)|^2 \\
&= \Big|\frac{\sqrt{t}\,e^{-it}}{i} + \sum_{k=i}^{\infty} \frac{\ln(1+t)}{k(1+i)} \cdot \frac{u_k}{1+(k-i)^2\,u_k^2}\Big|^2 \\
&\le 2\Big\{\frac{t\,e^{-2it}}{i^2} + \Big[\sum_{k=i}^{\infty} \frac{\ln(1+t)}{k(1+i)} \cdot \frac{u_k}{1+(k-i)^2\,u_k^2}\Big]^2\Big\} \\
&\le 2\Big\{\frac{t\,e^{-2it}}{i^2} + \Big[\Big(\sum_{k=i}^{\infty} \frac{\ln^2(1+t)}{k^2(1+i)^2}\Big)^{1/2}\Big(\sum_{k=i}^{\infty}(\frac{u_k}{1+(k-i)^2\,u_k^2})^2\Big)^{1/2}\Big]^2\Big\} \\
&\le \frac{2te^{-2it}}{i^2} + 2\Big(\sum_{k=i}^{\infty} \frac{\ln^2(1+t)}{k^2(1+i)^2}\Big) \cdot \sum_{k=i}^{\infty}\Big(\frac{u_k}{1+(k-i)^2\,u_k^2}\Big)^2.
\end{aligned}$$

Hence, applying (2.13), we obtain

$$\begin{aligned}
&|f_i(t,u_1,u_2,\dots)|^2 \\
&\le \frac{2te^{-2it}}{i^2} + \frac{2\ln^2(1+t)}{(1+i)^2} \cdot \frac{\pi^2}{6}\big(u_k^2 + \frac{1}{4} + \frac{1}{4\cdot 2^2} + \frac{1}{4\cdot 3^2} + \cdots\big) \\
&\le \frac{2te^{-2it}}{i^2} + \frac{2\ln^2(1+t)}{(1+i)^2} \cdot \frac{\pi^2}{24}\big(\frac{1}{1^2} + \frac{1}{2^2} + \cdots\big) + \frac{2\ln^2(1+t)}{(1+i)^2} \cdot \frac{\pi^2}{6} \cdot u_k^2 \\
&\le \frac{2te^{-2it}}{i^2} + \frac{\pi^4}{72} \cdot \frac{\ln^2(1+t)}{(1+i)^2} + \frac{\pi^2}{3} \cdot \frac{\ln^2(1+t)}{(1+i)^2} \cdot u_k^2.
\end{aligned} \tag{2.14}$$

Now, if we put

$$g_i(t) = \frac{2te^{-2it}}{i^2} + \frac{\pi^4}{72} \cdot \frac{\ln^2(1+t)}{(1+i)^2}, \quad h_i(t) = \frac{\pi^2}{3} \cdot \frac{\ln^2(1+t)}{(1+i)^2},$$

then the functions g_i, h_i $(i = 1,2,\dots)$ are continuous on the interval I. Moreover, let us notice that the series $\sum_{i=1}^{\infty} g_i(t)$ is uniformly convergent on I which is a simple consequence of the estimate

$$|g_i(t)| = g_i(t) \le \frac{2T}{i^2} + \frac{\pi^4}{72}\ln^2(1+T) \cdot \frac{1}{(i+1)^2} \le [2T + \frac{\pi^4}{72}\ln^2(1+T)]\frac{1}{i^2}$$

$(t \in I)$ and the classical Weierstrass test for uniform convergence of a function series. On the other hand we have the obvious inequality

$$|h_i(t)| = h_i(t) \le \frac{\pi^2\ln^2(1+T)}{12}$$

for $t \in I$ and for all $i = 1,2,\dots$. This means that the function sequence $(h_i(t))$ is equibounded on I.

Combining the above established facts with (2.14) we see that

$$|f_i(t,u_1,u_2,\dots)|^2 \le g_i(t) + h_i(t)|u_i|^2$$

for $t \in I$ and $i = 1, 2, \ldots$. This shows that assumption (iii) is satisfied. Moreover, we can accept the following constants G and H appearing in our considerations:

$$G = \frac{\pi^2}{6}[2T + \frac{\pi^4}{72}\ln^2(1+T)],$$
$$H = \frac{\pi^2}{12}\ln^2(1+T).$$

It is easily seen that the functions f_i satisfy assumption (i) for each $i = 1, 2, \ldots$.

To prove that assumption (ii) is satisfied let us fix arbitrarily $t \in I$ and $u = (u_i) = (u_1, u_2, \ldots) \in l_2$. Then, in view of the above estimates, we obtain

$$\sum_{i=1}^{\infty} |f_i(t, u_1, u_2, \ldots)|^2 \le \sum_{i=1}^{\infty} g_i(t) + \sum_{i=1}^{\infty} h_i(t)|u_i|^2 \le G + H\sum_{i=1}^{\infty}|u_i|^2.$$

Hence we deduce that the operator $f = (f_i)$ transforms the space $I \times l_2$ into l_2.

To show the remainder part of assumption (ii) let us fix arbitrarily a positive number $\varepsilon > 0$ and an arbitrary point $u = (u_i) \in l_2$. Then, taking a point $v = (v_i) \in l_2$ such that $\|v - u\|_{l_2} < \varepsilon$, we have

$$\begin{aligned}
&\|(fv)(t) - (fu)(t)\|_{l_2}\\
&= \sum_{i=1}^{\infty} |f_i(t, v_1, v_2, \ldots) - f_i(t, u_1, u_2, \ldots)|^2\\
&= \sum_{i=1}^{\infty} \Big| \sum_{k=i}^{\infty} \frac{\ln(1+t)}{k(1+i)} \cdot \frac{v_k(t)}{1+(k-i)^2\, v_k^2(t)} - \sum_{k=i}^{\infty} \frac{\ln(1+t)}{k(1+i)} \cdot \frac{u_k(t)}{1+(k-i)^2\, u_k^2(t)} \Big|^2\\
&\le \sum_{i=1}^{\infty} \Big| \sum_{k=i}^{\infty} \frac{\ln(1+t)}{k(1+i)} \Big[\frac{v_k(t)}{1+(k-i)^2\, v_k^2(t)} - \frac{u_k(t)}{1+(k-i)^2\, u_k^2(t)}\Big] \Big|^2\\
&\le \sum_{i=1}^{\infty} \Big[\frac{\ln(1+t)}{1+i} \sum_{k=i}^{\infty} \frac{1}{k}\Big|\frac{v_k(t)}{1+(k-i)^2\, v_k^2(t)} - \frac{u_k(t)}{1+(k-i)^2\, u_k^2(t)}\Big|\Big]^2\\
&\le \sum_{i=1}^{\infty} \frac{\ln^2(1+t)}{(1+i)^2} \Big[\sum_{k=i}^{\infty} \frac{1}{k}\Big|\frac{v_k(t)}{1+(k-i)^2\, v_k^2(t)} - \frac{u_k(t)}{1+(k-i)^2\, u_k^2(t)}\Big|\Big]^2.
\end{aligned}$$

Hence, applying the classical Cauchy-Schwarz inequality, we derive the estimates

$$\begin{aligned}
&\|(fv)(t) - (fu)(t)\|_{l_2}^2\\
&\le \sum_{i=1}^{\infty} \frac{\ln^2(1+t)}{(1+i)^2} \Big[\Big(\sum_{k=i}^{\infty} \frac{1}{k^2}\Big)^{1/2} \Big(\sum_{k=i}^{\infty} \Big|\frac{v_k(t)}{1+(k-i)^2 v_k^2(t)} - \frac{u_k(t)}{1+(k-i)^2 u_k^2(t)}\Big|^2\Big)^{1/2}\Big]^2\\
&\le \frac{\pi^2}{6}\ln^2(1+t) \sum_{i=1}^{\infty} \frac{1}{(1+i)^2} \Big(\sum_{k=i+1}^{\infty} \Big[\Big|\frac{v_k(t)}{1+(k-i)^2\, v_k^2(t)} - \frac{u_k(t)}{1+(k-i)^2\, u_k^2(t)}\Big|^2\\
&+ |v_i(t) - u_i(t)|^2\Big]\Big).
\end{aligned}$$

Hence, we obtain

$$\|(fv)(t) - (fu)(t)\|_{l_2}^2 \le \frac{\pi^2}{6}\ln^2(1+t) \sum_{i=1}^{\infty} \frac{1}{(1+i)^2} \Big(\sum_{k=i}^{\infty} |v_k - u_k|^2\Big).$$

Consequently, from the above inequality we derive the following one

$$\|(fv)(t)-(fu)(t)\|_{l_2}^2 \le \frac{\pi^2}{6}\ln^2(1+t)\sum_{i=1}^{\infty}\frac{1}{(1+i)^2}\varepsilon^2 \le \varepsilon^2\frac{\pi^4}{36}\ln^2(1+t).$$

Finally, we obtain

$$\|(fv)(t)-(fu)(t)\|_{l_2} \le \varepsilon\frac{\pi^2}{6}\ln(1+t) \le \varepsilon\frac{\pi^2}{6}\ln(1+T)$$

for any $t \in I$. This estimate proves the desired equicontinuity of the family of functions $\{(fu)(t)\}_{t\in I}$ and simultaneously shows that problem (2.11)-(2.12) has at least one solution $u(t) = (u_i(t))$ in the space $C(I, l_2)$ provided $(H/T)^{1/2}\, T^2 < 4$ and $T \le 1$. Keeping in mind our earlier obtained evaluation concerning the constant H, we conclude that in our considerations connected with problem (2.11)-(2.12) we can take any T such that $T\sqrt{T}\ln(1+T) < 8\sqrt{3}/\pi$. For example, the number $T = 1$ satisfies this inequality.

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