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A SIMPLIFIED APPROACH TO GRONWALL'S INEQUALITY ON TIME SCALES WITH APPLICATIONS TO NEW BOUNDS FOR SOLUTIONS TO LINEAR DYNAMIC EQUATIONS

CHRISTOPHER C. TISDELL, STEPHEN MEAGHER

Abstract. The purpose of this work is to advance and simplify our understanding of some of the basic theory of linear dynamic equations and dynamic inequalities on time scales.

Firstly, we revisit and simplify approaches to Gronwall's inequality on time scales. We provide new, simple and direct proofs that are accessible to those with only a basic understanding of calculus.

Secondly, we apply the ideas to second and higher order linear dynamic equations on time scales. Part of the novelty herein involves a strategic choice of metric, notably the taxicab metric, to produce *a priori* bounds on solutions. This choice of metric significantly simplifies usual approaches and extends ideas from the literature.

Thirdly, we examine mathematical applications of the aforementioned bounds. We form results concerning the non-multiplicity of solutions to linear problems; and error estimates on solutions to initial value problems when the initial conditions are imprecisely known.

1. Introduction

For hundreds of years, second and higher order differential equations of linear type have gained attention from mathematicians, engineers, scientists and educators due to their simplicity and accessibility [16]. These equations take the form of an initial value problem, namely

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_1(t)x' + a_0(t)x = f(t), \tag{1.1}$$

$$x^{(i)}(0) = b_i, \quad \text{for } i \in \{0, \dots, n-1\}. \tag{1.2}$$

Agnew makes the significance of (1.1), (1.2) clear via the now classic statement that they "are so important that many persons with few mathematical interests know enough about them to be able to use them in the solution of problems" Agnew [2, p.95].

As mathematical modelling has developed and matured, we have seen the rise of linear difference equations in the modelling of discrete phenomena and also as approximations to differential equations through numerical methods. These equations

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can take the classical form

$$\Delta^{(n)}x(t) + a_{n-1}(t)\Delta^{(n-1)}x(t) + \cdots + a_1(t)\Delta x(t) + a_0(t)x(t) = f(t), \tag{1.3}$$

$$\Delta^{(i)}x(0) = b_i, \quad \text{for } i \in \{0, \dots, n-1\}. \tag{1.4}$$

In the case of $q-$difference equations [5, p.1487], the "dynamic" equation with $n = 2$ looks like

$$\begin{gathered} D_h(D_h x)(t) + a_1(t)D_h x(t) + a_0(t)x(t) = f(t), \quad t \in h^{\mathbb{Z}}, \quad h > 1, \\ \text{where } D_h y(t) := \frac{y(ht) - y(t)}{ht - t}. \end{gathered} \tag{1.5}$$

In the past 20 years, or so, we have seen the birth and evolution of "dynamic equations on time scales" [7, 14]. The field of dynamic equations on time scales offers a mathematical framework that encompasses differential equations and difference equations simultaneously. Prototypical time scales are the set of real numbers (corresponding to differential equations) and the set of integers (corresponding to difference equations). This framework provides an opportunity to simultaneously model continuous, discrete and hybrid processes.

Let $\mathbb{T}$ be a time scale (precise definitions will be presented in Section 2). The general problem of solving an nth order linear "dynamic" equation, with initial values $b_i \in \mathbb{R}$, is to find an nth order delta differentiable function $x : \mathbb{T} \to \mathbb{R}$ satisfying

$$x^{\Delta^{(n)}} + a_{n-1}(t)x^{\Delta^{(n-1)}} + \cdots + a_1(t)x^{\Delta} + a_0(t)x = f(t), \tag{1.6}$$

$$x^{\Delta^{(i)}}(0) = b_i, \quad \text{for } i \in \{0, \dots, n-1\}. \tag{1.7}$$

on some suitable interval. Above, the $a_i : \mathbb{T}^{\kappa^i} \to \mathbb{R}$ and $f : \mathbb{T}^{\kappa^i} \to \mathbb{R}$ are arbitrary functions, and $0 \in \mathbb{T}$.

Equations (1.6) and (1.7) simultaneously encompass: (1.1), (1.2); and (1.3), (1.4); plus many more "in-between" and hybrid cases such as (1.5).

The purpose of this work is to advance and simplify our understanding of some of the basic theory of linear dynamic equations and dynamic inequalities on time scales, with Agnew's famous aforementioned quote taking on even more important meaning for (1.6), (1.7) given its wide-ranging and flexible characteristics.

Much work has been done generalising the basic inequalities found in Chapter 6 of Bohner and Peterson [7] (see [1] and the introduction of [13] for a recent overview). There have also been various generalisations to multi-variable situations (see e.g. [3, 4]), and to situations involving delay equations (see [9] and the references therein for a recent overview). However, unlike present article, none of these works provide such a simple and direct approach as we do herein; nor do they prove an inequality where the bounds depend on the classical real analysis exponential function *alone*, and are therefore independent of the time scale. The inequalities and methods that we show are striking in their simplicity and independence from the time scale.

Our work is organised as follows:

Section 2 briefly recalls some of the basic notation and concepts from the field of time scales to keep this work reasonably self contained.

In Section 3, we revisit and simplify approaches to Gronwall's inequality on time scales. This fundamental inequality has opened up many new directions for scientific investigation and mathematical research into nonlinear problems, and continues to be a fruitful resource within the area of time scales. Several of our results out important and novel and complement existing theorems and, in particular, provide new, simple and direct proofs that are accessible to those with only a basic understanding of calculus. Unlike more well–known approaches, the bounds that we obtain do not rely on the exponential function on times scales, rather they involve the exponential function from classical real analysis. This means the bounds are independent of the time scale itself and thus are easily calculable. Our results are also timely in view of the upcoming centenary of Gronwall's original results from 1919 [10] for differential inequalities.

In Section 4 we analyse second and higher order linear dynamic equations on time scales. The novelty herein involves a strategic choice of metric, notably the taxicab metric [16], to produce *a priori* bounds on solutions. This choice of metric significantly simplifies usual approaches and extends ideas from the literature in the second and higher order cases. Once again, these bounds are in terms of the classical exponential function and so are easily accessible and computable by a wide audience.

Finally, in Section 5, we look at mathematical applications of the aforementioned bounds. We form results concerning the non-multiplicity of solutions to second and higher order problems; and error estimates on solutions to initial value problems when the initial conditions are imprecisely known. Once again, the methods involved are direct and accessible, and differ from the existing literature by not relying on an understanding of matrix theory.

The present article is motivated by the recent works [15] and [16], where new Gronwall-type results were derived in the fractional integral operator setting; and the taxicab metric was applied to obtain *a priori* bounds on linear, ordinary differential equations.

2. Review of time scales

We briefly recall some of the basic notation and concepts from the field of time scales so that this work is reasonably self contained. For more details we refer the reader to the seminal work of Bohner and Peterson [7].

A time scale $\mathbb{T}$ is a closed (and nonempty) subset of $\mathbb{R}$. For each $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{R}$ is defined by

$$\sigma(t) := \begin{cases} \inf\{s \in \mathbb{T} \mid s > t\}, & \text{if } t \text{is not the maximum of } \mathbb{T}; \\ t, & \text{if } t \text{ is the maximum of } \mathbb{T}. \end{cases}$$

E.g. if $\mathbb{T} = \mathbb{R}$ then $\sigma(t) = t$, while if $\mathbb{T} = \mathbb{Z}$ then $\sigma(t) = t + 1$.

We define the set $\mathbb{T}^\kappa$ to be $\mathbb{T}$ if $\mathbb{T}$ does not have a discrete maximum,[1] otherwise $\mathbb{T}^\kappa$ is $\mathbb{T}$ with its discrete maximum removed. Note that $\mathbb{T}^\kappa$ is itself a time scale.

A function $x : \mathbb{T} \to \mathbb{R}$ is delta differentiable if there is a function $x^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ such that for each $t \in \mathbb{T}^\kappa$ and for each $\epsilon > 0$ there exists a $\delta > 0$ such that for any $s \in \mathbb{T}$ satisfying

$$|t - s| < \delta$$

we have

$$|x(\sigma(t)) - x(s) - x^\Delta(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|.$$

[1] In the time scale literature this is called a left-scattered maximum, see below for a definition of left-scattered.

For example if $\mathbb{T} = \mathbb{R}$ then this just the ordinary derivative of x. If $\mathbb{T} = \mathbb{Z}$ then

$$x^{\Delta}(t) = x(t+1) - x(t).$$

Note that $\mathbb{T}^{\kappa}$ is needed to ensure uniqueness of $x^{\Delta}(t)$: for if t_1 is a discrete maximum of $\mathbb{T}$, then for ϵ sufficiently small, $s = t_1$ and therefore $\sigma(t_1) = s$ which would mean $x^{\Delta}(t_1)$ could take any value.

The higher delta derivatives are defined recursively by

$$x^{\Delta^{(n)}}(t) = (x^{\Delta^{(n-1)}})^{\Delta}(t)$$

for $t \in \mathbb{T}^{\kappa^n}$ where $\mathbb{T}^{\kappa^n} = (\mathbb{T}^{\kappa^{n-1}})^{\kappa}$.

The anti-derivative X of x is a function such that $X^{\Delta} = x$, and the delta integral is given by

$$\int_{t_0}^{t} x(s)\Delta s = X(t) - X(t_0).$$

From this definition it is easy to see that delta integrals are linear operators in x.

To state existence results for anti-derivatives, we call on the notion of an rd-continuous function. It turns out that all rd-continuous functions have anti-derivatives. This necessitates defining the backward jump operator $\rho : \mathbb{T} \to \mathbb{R}$

$$\rho(t) := \begin{cases} \sup\{s \in \mathbb{T} \mid s < t\} & \text{if } t \text{ is not the minimum of } \mathbb{T}; \\ t & \text{if } t \text{ is the minimum of } \mathbb{T}. \end{cases}$$

A point t is called right-dense if $\sigma(t) = t$ and left-dense if $\rho(t) = t$, it is called right-scattered if $\sigma(t) > t$ and left-scattered if $\rho(t) < t$.

A function $x : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points and left-continuous at left-dense points.

If a function is delta differentiable it is rd-continuous, and if a function is continuous it is rd-continuous.

We will use the fact that a function of the form

$$|x(t)| + |x^{\Delta}(t)| + \dots + |x^{\Delta^{(n-1)}}(t)|$$

is rd-continuous if x has nth order delta derivatives.

If $I \subset \mathbb{R}$ is an interval we denote $I \cap \mathbb{T}$ by $I_{\mathbb{T}}$. If I is compact and $x : \mathbb{T} \to \mathbb{R}$ is rd-continuous, then x is bounded on $I_{\mathbb{T}}$, and x attains its maximum on $\mathbb{T}$, i.e. there exists a $t_1 \in I_{\mathbb{T}}$ such that $x(t_1) = \sup\{x(t) : t \in I_{\mathbb{T}}\}$ [7, Theorems 1.60 and 1.65, pp 22-23].

We will use the following facts regarding delta integrals:

If $x(s) \le y(s)$ for all $s \in [t_0, t_1]_{\mathbb{T}}$ then

$$\int_{t_0}^{t} x(s)\,\Delta s \le \int_{t_0}^{t} y(s)\,\Delta s, \quad \text{for all } t \in [t_0, t_1]_{\mathbb{T}} \tag{2.1}$$

(see, e.g. [7, Theorem 1.77, p29]).

In particular if $M > 0$ is a constant and $x \le M$ then

$$\int_{t_0}^{t} x(s)\,\Delta s \le M(t - t_0). \tag{2.2}$$

as the anti-derivative of a constant M is Ms (see [7, Example 1.13(ii)]).

If $h : [t_0, t] \to \mathbb{R}$ is continuous and non-decreasing then

$$\int_{t_0}^{t} h(s)\,\Delta s \le \int_{t_0}^{t} h(s)\,ds \tag{2.3}$$

(see e.g. [11, Theorem 2.3] or [6, Lemma 2.1]).

3. Gronwall-type results for dynamic equations on time scales

In this section, we present some Gronwall–type results on time scales. Gronwall's original results [10] are nearly 100 years old and they have had a profound effect on the study of differential and integral equations. For example, for recent results in this area, see [15].

There are two important distinctions between our approach and the results already in the literature [7, Chapter 6] regarding Gronwall's results on time scales. Firstly, we provide two methods of proof for the result that simplify existing approaches. Secondly, our bounds are in terms of the classical exponential function from real analysis. This means the bounds are independent of the time scale which means that the bounds are easier to calculate than traditional bounds that use the time-scale exponential function.

Theorem 3.1. *Let $a > 0$ be a constant and let $\rho : [0, a]_{\mathbb{T}} \to [0, \infty)$ be rd-continuous. If there are non–negative constants A and B such that*

$$\rho(t) \le B + \int_0^t A\rho(s)\,\Delta s, \quad \text{for all } t \in [0, a]_{\mathbb{T}} \tag{3.1}$$

then

$$\rho(t) \le Be^{At}, \quad \text{for all } t \in [0, a]_{\mathbb{T}}. \tag{3.2}$$

In the interest of diversity, we present two different styles of proof. They offer very simple approaches and each only requires a basic understanding of functions and time scales calculus. The style of first proof is motivated by [17, p82-83] with appropriate modifications for time scales.

Proof 1: The case $A = 0$ is trivial, so let $A > 0$. Since ρ is non–negative and rd-continuous on $[0, a]_{\mathbb{T}}$, there is a constant $M > 0$ such that

$$0 \le \rho(t) \le M, \quad \text{for all } t \in [0, a]_{\mathbb{T}}. \tag{3.3}$$

Inserting (3.3) into the right–hand side of (3.1) and using (2.2) we obtain, for all $t \in [0, a]_{\mathbb{T}}$:

$$\rho(t) \le B + \int_0^t AM\,\Delta s = B + MAt. \tag{3.4}$$

Now, in a similar fashion, inserting (3.4) into (3.1) and then applying (2.1) and (2.3) with $h(s) = B + MAs$, we obtain:

$$\begin{aligned}\rho(t) &\le B + \int_0^t A[B + MAs]\,\Delta s\\ &\le B + \int_0^t A[B + MAs]\,ds\\ &= B + BAt + \frac{MA^2t^2}{2!}.\end{aligned}$$

Continuing with this process, we see that the n-th iteration is

$$\rho(t) \le B\sum_{k=0}^{n-1}\frac{(At)^k}{k!} + \frac{M(At)^n}{n!}. \tag{3.5}$$

Taking limits as $n \to \infty$ in (3.5) we obtain (3.2). $\square$

Proof 2: The case $A = 0$ is trivial, so let $A > 0$. For $t \in [0, a]_{\mathbb{T}}$, define

$$g(t) := \frac{\rho(t)}{e^{At}}. \tag{3.6}$$

Since g is rd-continuous on a compact interval, it must attain its maximum value at some point $t_1 \in [0, a]_{\mathbb{T}}$. Let

$$m := \max_{t \in [0,a]_{\mathbb{T}}} g(t) = g(t_1).$$

Thus, from (3.6) we see that

$$\rho(t_1) = me^{At_1}. \tag{3.7}$$

Using (3.7), (3.6) and (3.1) we have

$$\begin{aligned}
me^{At_1} &= \rho(t_1) \\
&\leq B + \int_0^{t_1} A\rho(s)\,\Delta s \\
&= B + \int_0^{t_1} Ae^{As} g(s)\,\Delta s \\
&\leq B + \int_0^{t_1} Ae^{As} m\,\Delta s \\
&\leq B + \int_0^{t_1} Ae^{As} m\,ds \\
&= B + m[e^{At_1} - 1]
\end{aligned}$$

where, in the second last line we applied the fundamental inequality (2.3).

Thus, we have

$$me^{At_1} \leq B + m[e^{At_1} - 1]$$

from which we can eliminate the exponential function and simplify to

$$m \leq B. \tag{3.8}$$

Thus, from (3.6) and (3.8), for each $t \in [0, a]_{\mathbb{T}}$ we have

$$\rho(t) = g(t)e^{At} \leq me^{At} \leq Be^{At}. \tag{3.9}$$

□

Remark 3.2. We make no claim that inequality (3.2) is "sharp" (i.e., the least upper bound) for all time scales. Indeed, it can be considered as a rather "rough" estimate. There is a natural trade-off between our simple methods of proof and the degree of sharpness of the conclusion of Theorem 3.1. The significance, interest and distinction from existing literature is in the method of proof.

While inequality (3.2) could be classed as a "rough" estimate, this does not affect its applications in the remainder of this paper. Indeed, the value and importance of rough inequalities like (3.2) has been confirmed by well–known mathematicians such as Nirenberg and Friedrichs, who "often stressed the applicability of rough inequalities to various problems!" [12, p483].

The following generalisation of Theorem 3.1 is now presented.

Theorem 3.3. *Let A be a non-negative constant; let $B : [0,a]_{\mathbb{T}} \to [0,\infty)$ be rd-continuous and nondecreasing; and let $\rho : [0,a]_{\mathbb{T}} \to [0,\infty)$ be rd-continuous. If*

$$\rho(t) \le B(t) + \int_0^t A\rho(s)\,\Delta s, \quad \text{for all } t \in [0,a]_{\mathbb{T}} \tag{3.10}$$

then

$$\rho(t) \le B(t)e^{At}, \quad \text{for all } t \in [0,a]_{\mathbb{T}}. \tag{3.11}$$

Proof. If (3.10) holds then, for each $t_1 \in \mathbb{T}$ with $0 \le t \le t_1 \le a$ we have $B(t) \le B(t_1)$. Therefore

$$\rho(t) \le B(t_1) + \int_0^t A\rho(s)\,\Delta s, \quad t \in [0,t_1]_{\mathbb{T}}$$

where t_1 is now regarded as a constant. The conditions of Theorem 3.1 hold and the conclusion (3.2) can then be applied, so that we have

$$\rho(t) \le B(t_1)e^{At}. \tag{3.12}$$

Thus replacing t with t_1 in (3.12) we obtain

$$\rho(t_1) \le B(t_1)e^{At_1}, \quad \text{for all } \ t_1 \in [0,a]_{\mathbb{T}}.$$

so that (3.11) holds. □

4. *A priori* bounds via a taxicab approach

In this section we present our results concerning *a priori* bounds for the general homogeneous problem associated with (1.6), (1.7), namely

$$x^{\Delta^{(n)}} + a_{n-1}(t)x^{\Delta^{(n-1)}} + \dots + a_1(t)x^{\Delta} + a_0(t)x = 0, \tag{4.1}$$

$$x^{\Delta^{(i)}}(0) = b_i, \quad \text{for } i \in \{0,\dots,n-1\}. \tag{4.2}$$

Our methodology involves the taxicab size of a solution to homogeneous problems combined with applications of our earlier Gronwall inequalities from the previous section.

In [5] the *a priori* bounds on solutions to the basic second order ($n = 2$) form of (1.6), (1.7) with constant coefficients were obtained via an approach that used the Euclidean size of a solution, namely

$$d_1(t) := \sqrt{(x(t))^2 + (x'(t))^2}.$$

While the Euclidean approach to *a priori* bounds on solutions is somewhat manageable in the proofs concerning second–order, linear problems with constant coefficients, we believe it is not optimal. Moreover, the Euclidean method becomes unwieldy in the proofs involving higher-order cases, for example, when attempting to apply

$$d_{n-1}(t) := \sqrt{(x(t))^2 + (x'(t))^2 + \dots + (x^{(n-1)}(t))^2}$$

to nth order problems.

The purpose of this section is to propose a simpler approach that establishes *a priori* bounds on solutions by considering a different way of measuring the size of a solution to linear dynamic equations. We shall refer to this as the taxicab (or Manhattan) size, namely

$$\rho(t) := |x(t)| + |x^{\Delta}(t)| + \dots + |x^{\Delta^{(n-1)}}(t)| \tag{4.3}$$

for each t in an interval.

Taxicab geometry (in $\mathbb{R}^n$) dates back to mathematician Hermann Minkowski in the 19th century where the distance between points is the sum of the absolute difference of the Cartesian coordinates, as opposed to the straight line Euclidean distance.

The taxicab form (4.3) of the size of a solution to linear differential equations enables a simplification and extension of the mathematical literature such as [5], to higher order equations. For instance, there is no need to apply the AM–GM inequality ad nauseam in the proofs; and the product rule for delta differentiation is not required. The ideas are widely accessible to to those who have an understanding of the Fundamental Theorem of Calculus and the classic exponential function.

Theorem 4.1. *Consider the homogeneous IVP* (4.1)*,* (4.2) *where each function* $a_i : [0,a]_{\mathbb{T}}^{\kappa^i} \to \mathbb{R}$ *and* a_i *is rd-continuous. If* $x = x(t)$ *is a solution to* (4.1)*,* (4.2) *on* $[0,a]_{\mathbb{T}}$ *then*

$$|x^{\Delta^{(i)}}(t)| \le Be^{At}, \quad \text{for } i = 0,1,\dots,n-1 \text{ for each } t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}} \tag{4.4}$$

where

$$\begin{gathered} |a_i| \le A_i, \quad \text{on } [0,a]_{\mathbb{T}}^{\kappa^{n-1}},\ i = 0,1\dots,n-1; \\ A := \max\{A_0, A_1, \dots, A_{n-1}\} + (n-1); \\ B := |b_0| + |b_1| + \dots + |b_{n-1}|. \end{gathered}$$

The proofs of Theorems 4.1 and 4.2 are motivated by [8, Theorem B, p284] (which applies only to $[0,\infty)$), except that we make the constants explicit.

Proof. The constants A_i defined as if each a_i is rd-continuous on the compact set $[0,a]_{\mathbb{T}}^{\kappa^i}$ then they are uniformly bounded on $[0,a]_{\mathbb{T}}^{\kappa^i}$.

Let $x = x(t)$ be a solution to (4.1) on $[0,a]_{\mathbb{T}}$. We have for each $t \in [0,a]_{\mathbb{T}}^{\kappa^i}$ and each $i = 0,1,\dots,n-2$

$$\begin{aligned} |x^{\Delta^{(i)}}(t)| &= \big|b_i + \int_0^t x^{\Delta^{(i+1)}}(s)\,\Delta s\big| \\ &\le |b_i| + \big|\int_0^t |x^{\Delta^{(i+1)}}(s)|\,\Delta s\big| \\ &\le |b_i| + \big|\int_0^t |x(s)| + |x^{\Delta}(s)| + \dots + |x^{\Delta^{(n-1)}}(s)|\,\Delta s\big| \end{aligned} \tag{4.5}$$

In addition, using the dynamic equation (4.1) we have for each $t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}}$

$$\begin{aligned} &|x^{\Delta^{(n-1)}}(t)| \\ &\le |b_{n-1}| + \Big|\int_0^t |x^{\Delta^{(n)}}(s)|\,\Delta s\Big| \\ &= |b_{n-1}| + \Big|\int_0^t \big| -\big[a_{n-1}(s)x^{\Delta^{(n-1)}}(s) + \dots + a_1(s)x^{\Delta}(s) + a_0(s)x(s)\big]\big|\,\Delta s\Big| \\ &\le |b_{n-1}| + \Big|\int_0^t \big[|a_{n-1}(s)|\,|x^{\Delta^{(n-1)}}(s)| + \dots + |a_1(s)|\,|x^{\Delta}(s)| + |a_0(s)|\,|x(s)|\big]\,\Delta s\Big| \\ &\le |b_{n-1}| + \Big|\int_0^t (A-(n-1))\big[|x^{\Delta^{(n-1)}}(s)| + \dots + |x^{\Delta}(s)| + |x(s)|\big]\,\Delta s\Big|. \end{aligned}$$

$$\leq |b_{n-1}| + (A-(n-1))\Big|\int_0^t \big[|x^{\Delta^{(n-1)}}(s)| + \dots + |x^{\Delta}(s)| + |x(s)|\big]\,\Delta s\Big|. \tag{4.6}$$

Summing the inequalities in (4.5) with (4.6), for all $t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}}$, we obtain

$$\begin{aligned}
&|x(t)| + |x^{\Delta}(t)| + \dots + |x^{\Delta^{(n-1)}}(t)| \\
&\leq |b_0| + |b_1| + \dots + |b_{n-1}| \\
&\quad + (n-1)\Big|\int_0^t |x(s)| + |x^{\Delta}(s)| + \dots + |x^{\Delta^{(n-1)}}(s)|\,\Delta s\Big| \\
&\quad + \Big|\int_0^t (A-(n-1))\big[|x^{\Delta^{(n-1)}}(s)| + \dots + |x^{\Delta}(s)| + |x(s)|\big]\,\Delta s\Big| \\
&= B + \Big|\int_0^t A\big[|x^{\Delta^{(n-1)}}(s)| + \dots + |x^{\Delta}(s)| + |x(s)|\big]\,\Delta s\Big|.
\end{aligned} \tag{4.7}$$

For each $t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}}$, define ρ via

$$\rho(t) := |x(t)| + |x^{\Delta}(t)| + \dots + |x^{\Delta^{(n-1)}}(t)|$$

so that (4.7) now simplifies to

$$\rho(t) \leq B + \int_0^t A\rho(s)\,\Delta s, \quad \text{for all } t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}}.$$

Note that ρ is rd-continuous and non-negative. Thus, applying Theorem 3.1, we obtain

$$\rho(t) \leq Be^{At}, \quad \text{for all } t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}}$$

which, in turn, implies (4.4). □

We now examine the concept of exponential boundedness of solutions to the inhomogeneous problem (1.6), (1.7). We say that a function $\rho : I_{\mathbb{T}} \to \mathbb{R}$ is exponentially bounded on $I_{\mathbb{T}}$ if there exist non-negative constants M and L such that for each $t \in I_{\mathbb{T}}$ we have

$$|\rho(t)| \leq Me^{Lt}, \quad \text{for all } t \in I_{\mathbb{T}}.$$

Theorem 4.2. *Let each $a_i : [0,a]_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous and let f be exponentially bounded on $[0,a]_{\mathbb{T}}$. If x is a solution of (1.6), (1.7) on $[0,a]_{\mathbb{T}}$ then $x^{\Delta^{(i)}}$ is also exponentially bounded for $i = 0,\dots,n$, and the bound is independent of i. In particular, for all $t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}}$ we have*

$$|x^{\Delta^{(i)}}(t)| \leq \big(B + \frac{M}{L}\big)e^{(L+A)t}$$

where

$$\begin{gathered}
|a_i| \leq A_i, \quad \text{on } [0,a]_{\mathbb{T}}^{\kappa^{n-1}}, \quad i = 0,1\dots,n-1; \\
A := \max\{A_0,\ A_1,\dots,A_{n-1}\} + (n-1); \\
B := |b_0| + |b_1| + \dots + |b_{n-1}|; \\
|f(t)| \leq Me^{Lt} \quad \text{for all } t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}},
\end{gathered}$$

where M and L are non-negative constants independent of t.

Proof. The argument is very similar to that of Theorem 4.1 except that the inequality (4.6) is modified as follows. For all $t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}}$ we have

$$
\begin{aligned}
|x^{\Delta^{(n-1)}}(t)| &\le |b_{n-1}| + \Big| \int_0^t |x^{\Delta^{(n)}}(s)|\,\Delta s \Big| \\
&= |b_{n-1}| + \Big| \int_0^t \big| f(s) - \big[a_{n-1}(s) x^{\Delta^{(n-1)}}(s) + \dots + a_1(s) x^{\Delta}(s) \\
&\quad + a_0(s) x(s) \big] \big|\,\Delta s \Big| \\
&\le |b_{n-1}| + \Big| \int_0^t \big[|f(s)| + |a_{n-1}(s)|\,|x^{\Delta^{(n-1)}}(s)| + \dots \\
&\quad + |a_1(s)|\,|x^{\Delta}(s)| + |a_0(s)|\,|x(s)| \big]\,\Delta s \Big| \\
&\le |b_{n-1}| + \Big| \int_0^t M e^{Ls} + (A-(n-1)) \big[|x^{\Delta^{(n-1)}}(s)| + \dots \\
&\quad + |x^{\Delta}(s)| + |x(s)| \big]\,\Delta s \Big| \\
&\le |b_{n-1}| + \int_0^t M e^{Ls} \Delta s + (A-(n-1)) \Big| \int_0^t \big[|x^{\Delta^{(n-1)}}(s)| + \dots \\
&\quad + |x^{\Delta}(s)| + |x(s)| \big]\,\Delta s \Big|.
\end{aligned} \tag{4.8}
$$

Inequality (4.5) still holds and so putting

$$
\rho(t) := |x(t)| + |x^{\Delta}(t)| + \dots + |x^{\Delta^{(n-1)}}(t)|
$$

and using (4.5) and (4.8) we get

$$
\rho(t) \le B + \int_0^t M e^{Ls}\,\Delta s + \int_0^t A\rho(s)\,\Delta s \quad \text{for all } t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}}. \tag{4.9}
$$

Now using inequality (2.3) and (4.9) gives

$$
\begin{aligned}
\rho(t) &\le B + \int_0^t M e^{Ls}\,ds + \int_0^t A\rho(s)\,\Delta s \\
&= B + \frac{M}{L}(e^{Lt}-1) + \int_0^t A\rho(s)\,\Delta s. \\
&\le \big(B + \frac{M}{L}\big) e^{Lt} + \int_0^t A\rho(s)\,\Delta s
\end{aligned} \tag{4.10}
$$

Now we can apply Theorem 3.3 to (4.10) to obtain

$$
\rho(t) \le \big(B + \frac{M}{L}\big) e^{Lt} e^{At} = \big(B + \frac{M}{L}\big) e^{(L+A)t}
$$

for all $t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}}$ and the result follows. $\square$

Example 4.3. Consider the dynamic equation

$$
x^{\Delta^3}(t) + t x^{\Delta^2}(t) + t^2 x^{\Delta}(t) + t^3 x(t) = t,
$$

with initial conditions

$$
x(0) = 0, \quad x^{\Delta}(0) = 0, \quad x^{\Delta^2}(0) = 0.
$$

Within the context of Theorem 4.2 we have: $n = 3$; each $A_i = 1$; $A = 3$; and $B = 0$. Furthermore, we can choose $M = 1$ and $L = 1$.

By Theorem 4.2, we see that solutions $x(t)$ on the interval $[0, 1]_{\mathbb{T}}$ satisfy

$$|x(t)| \le e^{4t}.$$

5. Mathematical applications

In this section we apply the *a priori* bounds from earlier to obtain results regarding the nonmultiplicity of solutions to the inhomogeneous initial value problem (1.6), (1.7). We also explore error bounds on solutions to (1.6), (1.7) when the initial conditions are imprecisely known.

As previously assumed, throughout this section $\mathbb{T}$ will be a time scale which is unbounded above with $0 \in \mathbb{T}$.

Theorem 5.1. *If each $a_i : [0, \infty)_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous, then the inhomogeneous initial value problem* (1.6), (1.7) *has, at most, one solution on* $[0, \infty)_{\mathbb{T}}$.

Proof. Let $y = y(t)$ and $z = z(t)$ be two solutions to (1.6), (1.7) on $[0, \infty)_{\mathbb{T}}$. Define $r = r(t)$ on $[0, \infty)_{\mathbb{T}}$ via

$$r := y - z.$$

We show that $r \equiv 0$ on $[0, \infty)_{\mathbb{T}}$ and thus $y \equiv z$.

Due to the linearity of (1.6) we see that r satisfies the homogeneous problem

$$r^{\Delta^{(n)}} + a_{n-1}(t)r^{\Delta^{(n-1)}} + \dots + a_1(t)r^{\Delta} + a_0(t)r = 0 \tag{5.1}$$

subject to the homogeneous initial conditions

$$r(0) = 0,\ r^{\Delta}(0) = 0,\ \dots,\ r^{\Delta^{(n-1)}}(0) = 0. \tag{5.2}$$

Let $t \neq 0$ be any point in $[0, \infty)_{\mathbb{T}}$. As $\mathbb{T}$ has no right maximum there are points $t_1, \dots, t_{n-1} \in \mathbb{T}$ such that $t < t_1 < t_2 < \dots < t_{n-1}$. Let $J := [0, t_{n-1}]$. Then $J_{\mathbb{T}} \subset [0, \infty)_{\mathbb{T}}$ and $J_{\mathbb{T}}$ contains both 0 and t. Since we have assumed each a_i is rd-continuous, each a_i must be bounded on $J_{\mathbb{T}}$ (with the bound possibly depending on J). We can now apply Theorem 4.1 to (5.1), (5.2) on J. By construction $J_{\mathbb{T}}^{\kappa^{n-1}}$ contains $[0, t]_{\mathbb{T}}$.

Since the initial conditions (5.2) give $B = 0$, from Theorem 4.1, we see that r satisfies $|r| \le 0$ on $J_{\mathbb{T}}$, which means $r \equiv 0$ on $J_{\mathbb{T}}$. Hence $y \equiv z$ on $J_{\mathbb{T}}$. Now, since t was chosen to be any point in $[0, \infty)_{\mathbb{T}}$ with $t \neq 0$, we have in fact shown that $y(t) = z(t)$ for all $t \in [0, \infty)_{\mathbb{T}}$, that is, $y \equiv z$ on $[0, \infty)_{\mathbb{T}}$.

We conclude that the inhomogeneous initial value problem (1.6), (1.7) has, at most, one solution on $[0, \infty)_{\mathbb{T}}$. □

Example 5.2. Returning to Example 4.3 we see that the initial value problem

$$x^{\Delta^3}(t) + tx^{\Delta^2}(t) + t^2 x^{\Delta}(t) + t^3 x(t) = t,$$

with initial conditions

$$x(0) = 0, \quad x^{\Delta}(0) = 0, \quad x^{\Delta^2}(0) = 0$$

has, at most, one solution on $[0, \infty)_{\mathbb{T}}$.

Suppose we wish to solve (1.6), (1.7) for a solution $x = x(t)$ but the initial conditions (1.7) are imprecisely known. Let $y = y(t)$ be a solution to (1.6) subject to the initial conditions

$$y(0) = c_0,\ y^{\Delta}(0) = c_1,\ \dots,\ y^{\Delta^{(n-1)}}(0) = c_{n-1} \tag{5.3}$$

where the c_i are known constants (with each c_i ideally close to each b_i in (1.7)). The following result gives us an estimate on the error between x and y on $[0,a]_{\mathbb{T}}^{\kappa^{n-1}}$.

Theorem 5.3. *Let each $a_i : [0,a]_{\mathbb{T}}^{\kappa^i} \to \mathbb{R}$ be rd-continuous. If $x = x(t)$ solves* (1.6)*,* (1.7) *on $[0,a]_{\mathbb{T}}$ and $y = y(t)$ solves* (1.6)*,* (5.3) *on $[0,a]_{\mathbb{T}}$, then for each $t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}}$ we have*

$$|x^{(i)}(t) - y^{(i)}(t)| \le De^{At}, \quad \text{for } i = 0,1,\dots,n-1.$$

where

$$\begin{gathered} A_i := \max\{|a_i(t)| : t \in [0,a]_{\mathbb{T}}^{\kappa^{n-1}}\}; \\ A := \max\{A_0,\ A_1,\dots,A_{n-1}\} + (n-1); \\ D := |b_0 - c_0| + |b_1 - c_1| + \dots + |b_{n-1} - c_{n-1}|. \end{gathered}$$

Proof. In a similar way as in the proof of Theorem 5.1 we define $r = x - y$ and see that r satisfies (5.1) subject to the initial conditions

$$r(0) = b_0 - c_0,\ r^{\Delta}(0) = b_1 - c_1,\ \dots,\ r^{\Delta^{(n-1)}}(0) = b_{n-1} - c_{n-1}.$$

We can then apply Theorem 4.1 to obtain the conclusion. □

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Christopher C. Tisdell
School of Mathematics and Statistics, The University of New South Wales, UNSW, 2052, Australia.
YouTube: www.youtube.com/DrChrisTisdell
Facebook: www.facebook.com/DrChrisTisdell.Edu
Twitter: www.twitter.com/DrChrisTisdell ORCiD orcid.org/0000-0002-3387-2505
E-mail address: `cct@unsw.edu.au`

Stephen Meagher
School of Mathematics and Statistics, The University of New South Wales, UNSW, 2052, Australia.
ORCiD orcid.org/0000-0003-3543-6392
E-mail address: `s.meagher@unsw.edu.au`