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A PRIORI ERROR ESTIMATES OF FINITE VOLUME METHODS FOR GENERAL ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this article, we establish a priori error estimates for the finite volume approximation of general elliptic optimal control problems. We use finite volume methods to discretize the state and adjoint equation of the optimal control problems. For the variational inequality, we use the variational discretization methods to discretize the control. We show the existence and the uniqueness of the solution for discrete optimality conditions. Under some reasonable assumptions, we obtain some optimal order error estimates for the state, costate and control variables. On one hand, the convergence rate for the state, costate and control variables is $O(h^2)$ or $O(h^2\sqrt{|\log(\frac{1}{h})|})$ in the sense of L^2 norm or L^∞ norm. On the other hand, the convergence rate for the state and costate variables is $O(h)$ or $O(h|\log(\frac{1}{h})|)$ in the sense of H^1 norm or $W^{1,\infty}$ norm.

1. Introduction

In recent years, optimal control problems have attracted substantial interest due to their applications in aero-hydrodynamics, atmospheric, hydraulic pollution problems, combustion, exploration and extraction of oil and gas resources, and engineering. They must be solved successfully with efficient numerical methods. Finite element methods are an important numerical method for the problems of partial differential equations and widely used in the numerical solution of optimal control problems. There have been extensive studies in convergence of finite element approximation for optimal control problems. Let us mention two early papers devoted to linear optimal control problems by Falk [17] and Geveci [18]. A systematic introduction of finite element method for optimal control problems can be found in [6, 8, 9, 10, 26, 27, 28, 29, 30, 31], but there are very less published results on this topic for finite volume methods for optimal control problems. Recently, the adaptive finite element method has been investigated extensively and become one of the most popular methods in the scientific computation and numerical modeling. In [20], the authors studied a posteriori error estimates for adaptive finite element discretizations of boundary control problems. A posteriori error estimates

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and adaptive finite element approximation for parameter estimation problems have been obtained in [23, 25]. Some related works can also be found in [21, 22].

Finite volume methods have a long history as a class of important numerical tools for solving differential equations. Because of their local conservative property and other attractive properties such as the robustness with the unstructured meshes, the finite volume methods are widely used in computational fluid dynamics. In general, two different functional spaces are used in the finite volume methods, one for the trial space and one for the test space. Owing to the two different spaces, the numerical analysis of the finite volume methods is more difficult than that of the finite element methods and finite difference methods. So, the analysis of finite volume methods lags far behind that of finite element and finite difference methods. Early work for the finite volume methods can be found in [1, 2, 4, 11, 13, 16]. In [1], Bank and Rose obtain the result that the finite volume approximation is comparable with the finite element approximation in H^1 norm. The optimal L^2 error estimate is obtained in [11] under the assumption that $f \in H^1$. In [16], Ewing obtain the H^1 norm and maximum-norm error estimates. In [4], the author proposes a nonconforming finite volume element method and obtains the L^2 norm and H^1 norm error estimates. Chou and Ye propose a discontinuous finite volume element method. Unified error analysis for conforming, nonconforming and discontinuous finite volume method is presented in [14]. High order finite volume methods can be found in [5, 12]. For other recently development, we refer reader to see [3, 15, 24, 35].

For optimal control problems, the state and costate variables are discretized by continuous linear elements and the control variable by piecewise constant or piecewise linear polynomials in most references. The convergence rate of the control variable is $O(h)$ or $O(h^{3/2})$ in the sense of L^2 norm or L^∞ norm in [33]. In [19], Hinze proposes a variational discretization methods for optimal control problems with control constraints. With the variational discretization concept, the control variable is not discretized directly, but discretized by a projection of the discrete costate variable. The convergence rate of the control variable is $O(h^2)$. There are two approaches to find the approximate solution of the optimal control problems governed by partial differential equation. One is of the optimize-then-discretize type. One first applies the Lagrange multiplier methods to obtain an optimal system, at the continuous level, consisting of the state equation, an adjoint equation and an optimal condition. Then one use some numerical method to discretize the resulting system. The other is of the discretize-then-optimize type. One first discretizes the optimal control problems by some means and then applies the Lagrange multiplier rule to the resulting discrete optimization problem. The two discrete systems, determined by the two approaches, are the same when finite element method is used. In general, these discrete systems are not the same. In [36], the authors also use the optimize-then-discretize approach to solve the optimal control problem governed by convection dominated diffusion equation.

Recently, in [32], the authors discussed distributed optimal control problems governed by elliptic equations by using the finite volume element methods. The objective functional was $\frac{1}{2}||y - y_d||^2_{L^2(\Omega)} + \frac{1}{2}||u||^2_{L^2(\Omega)}$. They used finite volume methods to discretize the state and adjoint equation of the optimal control problems. Under some reasonable assumptions, they obtained some error estimates. In this paper, we will use the optimize-then-discretize methods to discretize general elliptic optimal control problems. We consider the elliptic optimal control with objective functional $g(y) + j(u)$. We show the existence and the uniqueness of the solution for discrete optimality conditions. Finally, we obtain some optimal order error estimates for the state, costate and control variables.

For $1 \le p < \infty$ and m a nonnegative integer let $W^{m,p}(\Omega) = \{v \in L^p(\Omega);\ D^\alpha v \in L^p(\Omega) \text{ if } |\alpha| \le m\}$ denote the Sobolev spaces endowed with the norm $\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p$, and the semi-norm $\mid v \mid_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v \mid_{\partial\Omega} = 0\}$. For p=2, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$.

We consider the general elliptic optimal control problems

$$\min_{u\in U}\{g(y) + j(u)\}, \tag{1.1}$$

$$-\operatorname{div}(A\nabla y) = f + u, \quad \text{in } \Omega, \tag{1.2}$$

$$y = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a convex bounded polygon with boundary $\partial\Omega$, g and j are convex functionals, $f \in H^1(\Omega)$, U is denoted by $U = \{u \in L^2(\Omega) : a \le u(x) \le b,\ a.e. \text{ in } \Omega,\ a, b \in \mathbb{R}\}$. Furthermore, we assume that the coefficient matrix $A(x) = (a_{i,j}(x))_{2\times 2} \in (W^{2,\infty}(\Omega))^{2\times 2}$ is a symmetric positive definite matrix and there is a constant $c > 0$ satisfying for any vector $\mathbf{X} \in \mathbb{R}^2$, $\mathbf{X}^t A\mathbf{X} \ge c\|\mathbf{X}\|_{\mathbb{R}^2}^2$.

This article is organized as follows. In next section, we describe the finite volume methods briefly and apply the piecewise linear finite volume elements to the optimal control problems (1.1)-(1.3). In Section 3, we prove the existence and the uniqueness of the solutions for discrete optimality conditions. And then the optimal order error estimates in L^2 norm are derived for the state, costate and control variables in Second 4. We estimate the error of the numerical solutions of control, state and costate in L^∞ norm. Finally we estimate $W^{1,\infty}$ and H^1 errors for the state and costate variables in Second 5.

2. Finite volume element methods

For the convex polygon Ω, we consider a quasi-uniform triangulation $\mathcal{T}_h$ consisting of closed triangle elements K such that $\bar{\Omega} = \cup_{K\in\mathcal{T}_h} K$. We use N_h to denote the set of all nodes or vertices of $\mathcal{T}_h$. To define the dual partition $\mathcal{T}_h^*$ of $\mathcal{T}_h$, we divide each $K \in \mathcal{T}_h$ into three quadrilaterals by connecting the barycenter C_K of K with line segments to the midpoints of edges of K as is shown in Figure 1.

The control volume V_i consists of the quadrilaterals sharing the same vertex z_i as is shown in Figure 2.

The dual partition $\mathcal{T}_h^*$ consists of the union of the control volume V_i. Let $h = \max\{h_K\}$, where h_K is the diameter of the triangle K. As is shown in [16], the dual partition $\mathcal{T}_h^*$ is also quasi-uniform, i.e., there exists a positive constant C such that

$$C^{-1}h^2 \le \operatorname{meas}(V_i) \le Ch^2, \quad \forall V_i \in \mathcal{T}_h^*.$$

We define the finite dimensional space V_h associated with $\mathcal{T}_h$ for the trial functions by

$$V_h = \{v \in C(\Omega) : v|_K \in P_1(K),\ \forall K \in \mathcal{T}_h,\ v|_{\partial\Omega} = 0\},$$

and define the finite dimensional space Q_h associated with the dual partition $\mathcal{T}_h^*$ for the test functions by

$$Q_h = \{q \in L^2(\Omega) : q|_V \in P_0(V),\ \forall V \in \mathcal{T}_h^*;\ q|_{V_z} = 0,\ z \in \partial\Omega\},$$

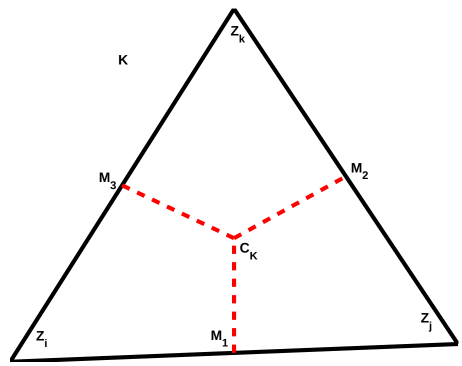

FIGURE 1. Dual partition of a triangular K.

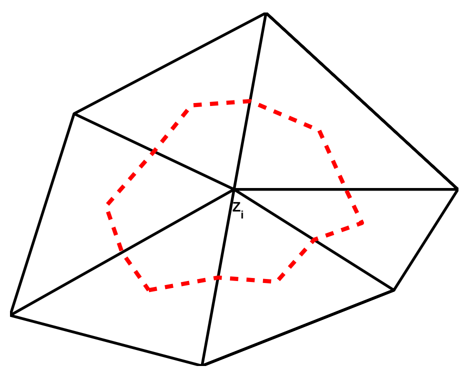

FIGURE 2. Control volume V_i sharing the same vertex z_i.

where $P_l(K)$ or $P_l(V)$ consists of all the polynomials with degree less than or equal to l defined on K or V.

To connect the trial space and test space, we define a transfer operator $I_h : V_h \to Q_h$ as follows:

$$I_h v_h = \sum_{z_i \in N_h} v_h(z_i)\chi_i, \quad I_h v_h|_{V_i} = v_h(z_i), \quad \forall V_i \in \mathcal{T}_h^*,$$

where χ_i is the characteristic function of V_i. For the operator I_h, it is well known that there exists a positive constant C such that for all $v \in V_h$,

$$\|v - I_h v\|_{0,\Omega} \le Ch\|v\|_{1,\Omega}. \tag{2.1}$$

To address the finite volume methods clearly, we consider the problem

$$-\operatorname{div}(A\nabla\varphi) = f, \quad \text{in } \Omega, \tag{2.2}$$

$$\varphi = 0, \quad \text{on } \partial\Omega, \tag{2.3}$$

where A, Ω, $\partial\Omega$ are the same as in (1.2)-(1.3), $f \in L^2(\Omega)$ or $H^1(\Omega)$.

The finite volume approximation φ_h of (2.2)-(2.3) is defined as the solution of the problem: find $\varphi_h \in V_h$ such that

$$a(\varphi_h, I_h v_h) = (f, I_h v_h), \quad \forall v_h \in V_h, \tag{2.4}$$

where the bilinear form $a(\varphi_h, I_h v_h)$ is defined by

$$a(\varphi, I_h v) = -\sum_{z_i \in N_h} v(z_i) \int_{\partial V_i} A\nabla\varphi \cdot \mathbf{n} ds, \quad \varphi, v \in H_0^1(\Omega),$$

where $\mathbf{n}$ is the unit outward normal vector to ∂V_i. The bilinear form $a(\cdot,\cdot)$ is not symmetric though the problem is self-adjoint. Then for all $w_h, v_h \in V_h$, there exist positive constants C and $h_0 \geq 0$ [13] such that for all $0 < h < h_0$,

$$|a(w_h, I_h v_h) - a(v_h, I_h w_h)| \leq Ch\|w_h\|_{1,\Omega} \ \|v_h\|_{1,\Omega}. \tag{2.5}$$

It is well known [27, 7] that the optimal control problems (1.1)-(1.3) have a solution (y, u), and that if a pair (y, u) is the solution of (1.1)-(1.3), then there is a co-state $p \in H_0^1(\Omega)$ such that the triplet $(y, p, u) \in H_0^1(\Omega) \times H_0^1(\Omega) \times U$ satisfies the optimality conditions:

$$(A\nabla y, \nabla w) = (f + u, w), \quad \forall w \in H_0^1(\Omega), \tag{2.6}$$

$$(A\nabla p, \nabla q) = (g'(y), q), \quad \forall q \in H_0^1(\Omega), \tag{2.7}$$

$$(j'(u) + p, v - u) \geq 0, \quad \forall v \in U. \tag{2.8}$$

If $y \in H_0^1(\Omega)\cap C^2(\Omega)$ and $p \in H_0^1(\Omega)\cap C^2(\Omega)$, then optimality conditions (2.6)-(2.8) can be written as

$$-\operatorname{div}(A\nabla y) = f + u, \quad \forall x \in \Omega, \tag{2.9}$$

$$y(x) = 0, \quad \forall x \in \partial\Omega, \tag{2.10}$$

$$-\operatorname{div}(A\nabla p) = g'(y), \quad \forall x \in \Omega, \tag{2.11}$$

$$p(x) = 0, \quad \forall x \in \partial\Omega, \tag{2.12}$$

$$(j'(u) + p, v - u) \geq 0, \quad \forall v \in U. \tag{2.13}$$

We use finite volume methods to discretize the state and costate equation directly. Then the optimality condition (2.9)-(2.13) can be approximated by: find $(y_h, p_h, u_h) \in V_h \times V_h \times U$ such that

$$a(y_h, I_h w_h) = (f + u_h, I_h w_h), \quad \forall w_h \in V_h, \tag{2.14}$$

$$a(p_h, I_h q_h) = (g'(y_h), I_h q_h), \quad \forall q_h \in V_h, \tag{2.15}$$

$$(j'(u_h) + p_h, v - u_h) \geq 0, \quad \forall v \in U. \tag{2.16}$$

For simplicity of notation, let $j(u) = \frac{1}{2}\|u\|^2_{L^2(\Omega)}$, then we derive $(j'(u), v - u) = (u, v - u)$ and $(j'(u_h), v - u_h) = (u_h, v - u_h)$. Then the variational inequality (2.13) can be restated as

$$(u + p, v - u) \geq 0, \quad \forall v \in U. \tag{2.17}$$

Similarly, the variational inequality (2.16) can be rewritten by

$$(u_h + p_h, v - u_h) \geq 0, \quad \forall v \in U. \tag{2.18}$$

Now, we introduce a projection [19]:

$$P_{[a,b]}(f(x)) = \max(a, \min(b, f(x))), \tag{2.19}$$

we can denote the variational inequality (2.17) by

$$u(x) = P_{[a,b]}(-p). \tag{2.20}$$

And the variational inequality (2.18) is equivalent to

$$u_h(x) = P_{[a,b]}(-p_h). \tag{2.21}$$

Then the discrete optimality conditions can be rewritten by: find $(y_h, p_h, u_h) \in V_h \times V_h \times U$ such that

$$a(y_h, I_h w_h) = (f + u_h, I_h w_h), \quad \forall w_h \in V_h, \tag{2.22}$$

$$a(p_h, I_h q_h) = (g'(y_h), I_h q_h), \quad \forall q_h \in V_h, \tag{2.23}$$

$$u_h(x) = P_{[a,b]}(-p_h). \tag{2.24}$$

For $\varphi \in W_h$, we shall write

$$g(\varphi) - g(\rho) = -\tilde{g}'(\varphi)(\rho - \varphi) = -g'(\rho)(\rho - \varphi) + \tilde{g}''(\varphi)(\rho - \varphi)^2, \tag{2.25}$$

where

$$\tilde{g}'(\varphi) = \int_0^1 g'(\varphi + s(\rho - \varphi))ds,$$

$$\tilde{g}''(\varphi) = \int_0^1 (1 - s)g''(\rho + s(\varphi - \rho))ds$$

are bounded functions in $\bar{\Omega}$ [34].

3. Existence and uniqueness

In this section, we show the existence and uniqueness of the solutions for discrete optimality conditions. We can easily see that the optimality conditions (2.22)-(2.24) are the finite volume approximation of (2.6)-(2.8). Now we show the existence and the uniqueness of the solution for (2.22)-(2.24). Let $y_h(u)$ be the solution of

$$a(y_h(u), I_h w_h) = (f + u, I_h w_h), \quad \forall w_h \in V_h, \tag{3.1}$$

and $p_h(y)$ be the solution of

$$a(p_h(y), I_h q_h) = (g'(y), I_h q_h), \quad \forall q_h \in V_h. \tag{3.2}$$

For $y_h(u)$ and $p_h(y)$, note that $y_h = y_h(u_h)$ and $p_h = p_h(y_h)$, we have the following results.

Lemma 3.1. *Assume that $y_h(u), p_h(u)$ are the solutions of* (3.1) *and* (3.2)*, respectively. Then*

$$\|p_h(y) - p_h\|_{1,\Omega} \le C\|y - y_h\|_{0,\Omega}, \quad \|y_h(u) - y_h\|_{1,\Omega} \le C\|u - u_h\|_{0,\Omega}. \tag{3.3}$$

Proof. Subtracting (2.15) from (3.2), and by using (2.25), we have

$$a(p_h(y) - p_h, I_h q_h) = (g'(y) - g'(y_h), I_h q_h) = (\tilde{g}''(y)(y - y_h), I_h q_h), \quad \forall q_h \in V_h. \tag{3.4}$$

Let $q_h = p_h(y) - p_h$, by using [16, Lemma 2.2] and the Cauchy-Schwarz's inequality, we can easily obtain that

$$\|p_h(y) - p_h\|_{1,\Omega} \le C\|y - y_h\|_{0,\Omega}. \tag{3.5}$$

Similarly, subtracting (2.14) from (3.1), we have

$$a(y_h(u) - y_h, I_h w_h) = (u - u_h, I_h w_h), \quad \forall w_h \in V_h, \tag{3.6}$$

let $w_h = y_h(y) - y_h$, we derive

$$\|y_h(u) - y_h\|_{1,\Omega} \le C\|u - u_h\|_{0,\Omega}. \tag{3.7}$$

This completes the proof. □

Lemma 3.2. *The optimality conditions* (2.14)-(2.16) *admit an unique solution for sufficiently small* h.

Proof. We first introduce a projection $P_k : L^2(\Omega) \to U$ defined by

$$\|z - P_k(z)\|_{0,\Omega} = \min_{z_h \in U} \|z - z_h\|_{0,\Omega}. \tag{3.8}$$

The projection P_k has the property that

$$\|P_k(z') - P_k(z'')\|_{0,\Omega} \le \|z' - z''\|_{0,\Omega}, \quad \forall z', z'' \in L^2(\Omega). \tag{3.9}$$

For a given $v_h \in L^2(\Omega)$, let $(y_h(v_h), p_h(v_h))$ be the solution of the following auxiliary problem: find $(y_h(v_h), p_h(v_h)) \in V_h \times\ V_h$ such that

$$a(y_h(v_h), I_h w_h) = (v_h + f, I_h w_h), \quad \forall w_h \in V_h, \tag{3.10}$$

$$a(p_h(v_h), I_h q_h) = (g'(y_h(v_h)), I_h q_h), \quad \forall q_h \in V_h. \tag{3.11}$$

Define a mapping $\Phi : L^2(\Omega) \to L^2(\Omega)$ by

$$\Phi(z_h) = z_h - \rho(z_h + p_h(z_h)), \quad \forall z_h \in L^2(\Omega),\ \rho > 0. \tag{3.12}$$

Let $T(z_h) = P_k\Phi(z_h)$, then the existence and uniqueness of (2.14)-(2.16) is to show that $T(z_h)$ is a contractive mapping. It follows from (3.9) that for all $z_h', z_h'' \in L^2(\Omega)$,

$$\begin{aligned}\|T(z_h') - T(z_h'')\|_{0,\Omega}^2 &= \|P_k(\Phi(z_h')) - P_k(\Phi(z_h''))\|_{0,\Omega}^2\\ &\le \|\Phi(z_h') - \Phi(z_h'')\|_{0,\Omega}^2 = (\Phi(z_h') - \Phi(z_h''), \Phi(z_h') - \Phi(z_h'')).\end{aligned}$$

Note that

$$\begin{aligned}&(\Phi(z_h') - \Phi(z_h''), \Phi(z_h') - \Phi(z_h''))\\ &= (1 - 2\rho)(z_h' - z_h'', z_h' - z_h'') - 2\rho(z_h' - z_h'', p_h(z_h') - p_h(z_h''))\\ &\quad + \rho^2\|z_h' - z_h'' + p_h(z_h') - p_h(z_h'')\|_{0,\Omega}^2.\end{aligned}$$

Then we have

$$\begin{aligned}&\|T(z_h') - T(z_h'')\|_{0,\Omega}^2\\ &=\le (1 - 2\rho)(z_h' - z_h'', z_h' - z_h'') - 2\rho(z_h' - z_h'', p_h(z_h') - p_h(z_h''))\\ &\quad + \rho^2\|z_h' - z_h'' + p_h(z_h') - p_h(z_h'')\|_{0,\Omega}^2.\end{aligned} \tag{3.13}$$

For $z_h', z_h'' \in L^2(\Omega)$, it follows from (3.10)-(3.11) and (2.25) that

$$\begin{aligned}a(y_h(z_h') - y_h(z_h''), I_h w_h) &= (z_h' - z_h'', I_h w_h), \quad \forall w_h \in V_h,\\ a(p_h(z_h') - p_h(z_h''), I_h q_h) &= (\tilde{g}''(y_h(z_h'))(y_h(z_h') - y_h(z_h'')), I_h q_h), \quad \forall q_h \in V_h.\end{aligned}$$

Let $w_h = p_h(z_h') - p_h(z_h'')$ and $q_h = y_h(z_h') - y_h(z_h'')$, we have

$$\begin{aligned}&(z_h' - z_h'', p_h(z_h') - p_h(z_h''))\\ &= (\tilde{g}''(y_h(z_h'))(y_h(z_h') - y_h(z_h'')), I_h(y_h(z_h') - y_h(z_h'')))\\ &\quad + a(y_h(z_h') - y_h(z_h''), I_h(p_h(z_h') - p_h(z_h'')))\\ &\quad - a(p_h(z_h') - p_h(z_h''), I_h(y_h(z_h') - y_h(z_h'')))\\ &\quad + (z_h' - z_h'', (p_h(z_h') - p_h(z_h'')) - I_h(p_h(z_h') - p_h(z_h'')))\end{aligned}$$

$$
\begin{aligned}
&\geq a(y_h(z_h') - y_h(z_h''), I_h(p_h(z_h') - p_h(z_h''))) \\
&\quad - a(p_h(z_h') - p_h(z_h''), I_h(y_h(z_h') - y_h(z_h''))) \\
&\quad + (z_h' - z_h'', (p_h(z_h') - p_h(z_h'')) - I_h(p_h(z_h') - p_h(z_h''))),
\end{aligned}
$$

where we have used the fact that $(v_h, I_h v_h) \geq 0$. Using [13, Lemma 2.4] and Lemma 3.1, we have

$$
\begin{aligned}
&a\big(y_h(z_h') - y_h(z_h''), I_h(p_h(z_h') - p_h(z_h''))\big) \\
&- a\big(p_h(z_h') - p_h(z_h''), I_h(y_h(z_h') - y_h(z_h''))\big) \\
&\geq -c_0 h \|p_h(z_h') - p_h(z_h'')\|_{1,\Omega} \cdot \|y_h(z_h') - y_h(z_h'')\|_{1,\Omega} \\
&\geq -c_0 c_1 h \|z_h' - z_h''\|_{0,\Omega}^2.
\end{aligned} \tag{3.14}
$$

Note that by (2.1) and Lemma 3.1, we have

$$
\begin{aligned}
&(z_h' - z_h'', (p_h(z_h') - p_h(z_h'')) - I_h(p_h(z_h') - p_h(z_h''))) \\
&\geq -c_2 h \|p_h(z_h') - p_h(z_h'')\|_{1,\Omega} \cdot \|z_h' - z_h''\|_{0,\Omega} \\
&\geq -c_2 c_3 h \|z_h' - z_h''\|_{0,\Omega}^2.
\end{aligned} \tag{3.15}
$$

Combining (3.14) and (3.15), we deduce that

$$
(z_h' - z_h'', p_h(z_h') - p_h(z_h'')) \geq -(c_0 c_1 + c_2 c_3) h \|z_h' - z_h''\|_{0,\Omega}^2. \tag{3.16}
$$

Now, it is easy to see that

$$
\|z_h' - z_h'' + p_h(z_h') - p_h(z_h'')\|_{0,\Omega}^2 \leq c_4 \|z_h' - z_h''\|_{0,\Omega}^2. \tag{3.17}
$$

Then it follows from (3.13), (3.16), and (3.17) that

$$
\|T(z_h') - T(z_h'')\|_{0,\Omega}^2 \leq C \|z_h' - z_h''\|_{0,\Omega}^2. \tag{3.18}
$$

For sufficiently small h we can ensure $0 < C < 1$. Therefore $T(z_h)$ is a contractive mapping and hence the optimality conditions (2.14)-(2.16) admit an unique solution. □

4. Optimal-order L^2 error estimates

In this section, we derive an optimal-order L^2 error estimates for the finite volume methods with the minimal regularity assumption for the exact solution u. Owing to the property of the variational inequality, we first estimate the error of the approximate control in L^2 norm. Using the properties of the control, we then estimate the errors of the numerical solutions for the state and the costate.

Theorem 4.1. *Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U$ and $(y_h, p_h, u_h) \in V_h \times V_h \times U$ be the solutions of (2.6)-(2.8) and (2.14)-(2.16), respectively. Assume that $u \in H^1(\Omega)$. Then there exists an $h_0 > 0$ such that for all $0 < h \leq h_0$,*

$$
\|u - u_h\| \leq C h^2 (\|y\|_{2,\Omega} + \|p\|_{2,\Omega}). \tag{4.1}
$$

Proof. Let $v = u$ in (2.16) and $v = u_h$ in the variational inequality of (2.13), by using (3.6), (2.17), and (2.18), then we have

$$
\begin{aligned}
(u-u_h, u-u_h) \leq & (p-p_h, u_h-u) \\
= & (p-p_h(y), u_h-u) + (p_h(y)-p_h, u_h-u) \\
= & (p-p_h(y), u_h-u) + (I_h(p_h(y)-p_h), u_h-u) \\
& + ((p_h(y)-p_h) - I_h(p_h(y)-p_h), u_h-u) \\
= & (p-p_h(y), u_h-u) + a(y_h-y_h(u), I_h(p_h(y)-p_h)) \\
& + ((p_h(y)-p_h) - I_h(p_h(y)-p_h), u_h-u).
\end{aligned} \tag{4.2}
$$

By using (2.25) and (3.4), the second term on the right hand side of (4.2) can be written by

$$
\begin{aligned}
& a(y_h-y_h(u), I_h(p_h(y)-p_h)) \\
& = a(y_h-y_h(u), I_h(p_h(y)-p_h)) - a(p_h(y)-p_h, I_h(y_h-y_h(u))) \\
& \quad + a(p_h(y)-p_h, I_h(y_h-y_h(u))) \\
& = a(y_h-y_h(u), I_h(p_h(y)-p_h)) - a(p_h(y)-p_h, I_h(y_h-y_h(u))) \\
& \quad + (\tilde{g}''(y)(y-y_h), I_h(y_h-y_h(u))) \\
& = a(y_h-y_h(u), I_h(p_h(y)-p_h)) - a(p_h(y)-p_h, I_h(y_h-y_h(u))) \\
& \quad + (\tilde{g}''(y)(y-y_h(u)), I_h(y_h-y_h(u))) \\
& \quad - (\tilde{g}''(y)(y_h-y_h(u)), I_h(y_h-y_h(u))) \\
& \leq (\tilde{g}''(y)(y-y_h(u)), I_h(y_h-y_h(u))) + a(y_h-y_h(u), I_h(p_h(y)-p_h)) \\
& \quad - a(p_h(y)-p_h, I_h(y_h-y_h(u))),
\end{aligned} \tag{4.3}
$$

where we have used that $(\tilde{g}''(y)(y_h-y_h(u)), I_h(y_h-y_h(u))) \geq 0$. Connecting (4.2) and (4.3), we obtain

$$
\begin{aligned}
& \alpha(u-u_h, u-u_h) \\
& \leq (\tilde{g}''(y)(y-y_h(u)), I_h(y_h-y_h(u))) + (p-p_h(y), u_h-u) \\
& \quad + ((p_h(y)-p_h) - I_h(p_h(y)-p_h), u_h-u) \\
& \quad + a(y_h-y_h(u), I_h(p_h(y)-p_h)) - a(p_h(y)-p_h, I_h(y_h-y_h(u))) \\
& \equiv E_1 + E_2 + E_3 + E_4.
\end{aligned} \tag{4.4}
$$

Note that

$$
(\tilde{g}''(y)(y-y_h(u)), I_h(y_h-y_h(u))) = (\tilde{g}''(y)(y-y_h(u)), y_h-y_h(u)). \tag{4.5}
$$

By using Lemma 3.1 and (4.5), we have

$$
\begin{aligned}
E_1 &= (\tilde{g}''(y)(y-y_h(u)), I_h(y_h-y_h(u))) \\
&\leq \|y-y_h(u)\|_{0,\Omega} \cdot \|y_h-y_h(u)\|_{0,\Omega} \\
&\leq \|y-y_h(u)\|_{0,\Omega} \cdot \|u_h-u\|_{0,\Omega} \leq Ch^2\|y\|_{2,\Omega} \cdot \|u_h-u\|_{0,\Omega}.
\end{aligned} \tag{4.6}
$$

Now, we can easily obtain

$$
\begin{aligned}
E_2 &= (p-p_h(y), u_h-u) \\
&\leq \|p-p_h(y)\|_{0,\Omega} \cdot \|u_h-u\|_{0,\Omega} \\
&\leq \|p-p_h(y)\|_{0,\Omega} \cdot \|u_h-u\|_{0,\Omega} \\
&\leq Ch^2\|p\|_{2,\Omega} \cdot \|u_h-u\|_{0,\Omega},
\end{aligned} \tag{4.7}
$$

where we have used the estimate in [16, Theorem 3.5]. Furthermore, by using Lemma 3.1, (2.1), and the triangle inequality, we derive

$$\begin{aligned}
E_3 &= ((p_h(y) - p_h) - I_h(p_h(y) - p_h), u_h - u) \\
&\le Ch\|p_h(y) - p_h\|_{1,\Omega} \cdot \|u_h - u\|_{0,\Omega} \\
&\le Ch\|y - y_h\|_{0,\Omega} \cdot \|u_h - u\|_{0,\Omega} \\
&\le Ch\|y - y_h\|_{1,\Omega} \cdot \|u_h - u\|_{0,\Omega} \\
&\le Ch(\|y - y_h(u)\|_{1,\Omega} + \|y_h(u) - y_h\|_{1,\Omega}) \cdot \|u_h - u\|_{0,\Omega} \\
&\le Ch(Ch\|y\|_{2,\Omega} + \|u_h - u\|_{0,\Omega})\|u_h - u\|_{0,\Omega} \\
&\le Ch\|u_h - u\|_{0,\Omega}^2.
\end{aligned} \tag{4.8}$$

Using (2.5) and Lemma 3.1, we have

$$\begin{aligned}
E_4 &= (a(y_h - y_h(u), I_h(p_h(y) - p_h)) - a(p_h(y) - p_h, I_h(y_h - y_h(u))) \\
&\le Ch\|y_h - y_h(u)\|_{1,\Omega} \cdot \|p_h(y) - p_h\|_{1,\Omega} \\
&\le Ch\|u_h - u\|_{0,\Omega} \cdot \|y - y_h\|_{0,\Omega} \\
&\le Ch\|y - y_h\|_{1,\Omega} \cdot \|u_h - u\|_{0,\Omega} \\
&\le Ch(\|y - y_h(u)\|_{1,\Omega} + \|y_h(u) - y_h\|_{1,\Omega}) \cdot \|u_h - u\|_{0,\Omega} \\
&\le Ch(Ch\|y\|_{2,\Omega} + \|u_h - u\|_{0,\Omega})\ \|u_h - u\|_{0,\Omega} \\
&\le Ch\|u_h - u\|_{0,\Omega}^2.
\end{aligned} \tag{4.9}$$

Hence, the estimate (4.1) follows from (4.4) and (4.6)-(4.9). □

Theorem 4.2. *Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U$ and $(y_h, p_h, u_h) \in V_h \times V_h \times U$ be the solutions of (2.6)-(2.8) and (2.14)-(2.16), respectively. Assume that $u \in L^2(\Omega)$. Then there exists an $h_0 > 0$ such that for all $0 < h \le h_0$,*

$$\|y - y_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \le Ch^2(\|y\|_{2,\Omega} + \|p\|_{2,\Omega}). \tag{4.10}$$

Proof. Using the triangle inequality, we have

$$\begin{aligned}
\|y - y_h\|_{0,\Omega} &\le \|y - y_h(u)\|_{0,\Omega} + \|y_h(u) - y_h\|_{0,\Omega}, \\
\|p - p_h\|_{0,\Omega} &\le \|p - p_h(y)\|_{0,\Omega} + \|p_h(y) - p_h\|_{0,\Omega}.
\end{aligned}$$

Lemma 3.1 implies that

$$\begin{aligned}
\|y - y_h\|_{0,\Omega} &\le \|y - y_h(u)\|_{0,\Omega} + C\|y_h(u) - y_h\|_{1,\Omega} \\
&\le \|y - y_h(u)\|_{0,\Omega} + C\|u - u_h\|_{0,\Omega},
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
\|p - p_h\|_{0,\Omega} &\le \|p - p_h(y)\|_{0,\Omega} + C\|p_h(y) - p_h\|_{1,\Omega} \\
&\le \|p - p_h(y)\|_{0,\Omega} + C\|y - y_h\|_{0,\Omega}.
\end{aligned} \tag{4.12}$$

By using [16, Theorem 3.5], we can easily obtain

$$\|y - y_h(u)\|_{0,\Omega} \le Ch^2\|y\|_{2,\Omega}. \tag{4.13}$$

From (4.11), (4.13), and Theorem 4.1, we derive

$$\|y - y_h\|_{0,\Omega} \le Ch^2\|y\|_{2,\Omega}. \tag{4.14}$$

Connecting (4.12), (4.14), and $\|p - p_h(y)\|_{0,\Omega} \le Ch^2\|p\|_{2,\Omega}$, we have

$$\|p - p_h\|_{0,\Omega} \le Ch^2\|p\|_{2,\Omega}. \tag{4.15}$$

From (4.14)-(4.15) we can immediately obtain (4.10). □

5. Optimal-order maximum-norm and H^1 error estimates

In this section, we first estimate the errors of the numerical solutions of control, state and costate in L^∞ norm. Then we estimate $W^{1,\infty}$ errors for the state and costate variables.

Theorem 5.1. *Let $(y,p,u) \in (H^2(\Omega)\cap H_0^1(\Omega)) \times (H^2(\Omega)\cap H_0^1(\Omega)) \times U$ and $(y_h,p_h,u_h) \in V_h \times V_h \times U$ be the solutions of (2.6)-(2.8) and (2.14)-(2.16), respectively. Assume that $u \in H^1(\Omega)$. Then there exists an $h_0 > 0$ such that for all $0 < h \le h_0$,*

$$\|u-u_h\|_{0,\infty} + \|y-y_h\|_{0,\infty} + \|p-p_h\|_{0,\infty} \le Ch^2\sqrt{|\log(\frac{1}{h})|}. \tag{5.1}$$

Proof. Using the definition of $P_{[a,b]}(\cdot)$ and (2.20)-(2.21), we have

$$\begin{aligned}
\|u-u_h\|_{0,\infty} &\le C\|p-p_h\|_{0,\infty} \\
&\le C(\|p-p_h(y)\|_{0,\infty} + \|p_h(y)-p_h\|_{0,\infty}) \\
&\le C\|p-p_h(y)\|_{0,\infty} + C\sqrt{|\log(\frac{1}{h})|}\|p_h(y)-p_h\|_{1,\Omega} \\
&\le C\|p-p_h(y)\|_{0,\infty} + C\sqrt{|\log(\frac{1}{h})|}\|y-y_h\|_{0,\Omega} \\
&\le Ch^2\sqrt{|\log(\frac{1}{h})|},
\end{aligned} \tag{5.2}$$

where we have used the inverse inequality, Lemma 3.1, [16, Theorem 3.11], and Theorem 4.1. Similarly, we obtain

$$\begin{aligned}
\|y-y_h\|_{0,\infty} &\le \|y-y_h(u)\|_{0,\infty} + \|y_h(u)-y_h\|_{0,\infty} \\
&\le \|y-y_h(u)\|_{0,\infty} + C\sqrt{|\log(\frac{1}{h})|}\|y_h(u)-y_h\|_{1,\Omega} \\
&\le \|y-y_h(u)\|_{0,\infty} + C\sqrt{|\log(\frac{1}{h})|}\|u-u_h\|_{0,\Omega} \\
&\le Ch^2\sqrt{|\log(\frac{1}{h})|}.
\end{aligned} \tag{5.3}$$

Then we complete the proof of (5.1). □

Theorem 5.2. *Let $(y,p,u) \in (H^2(\Omega)\cap H_0^1(\Omega)) \times (H^2(\Omega)\cap H_0^1(\Omega)) \times U$ and $(y_h,p_h,u_h) \in V_h \times V_h \times U$ be the solutions of (2.6)-(2.8) and (2.14)-(2.16), respectively. Assume that $u \in H^1(\Omega)$. Then there exists an $h_0 > 0$ such that for all $0 < h \le h_0$,*

$$\|p-p_h\|_{1,\infty} + \|y-y_h\|_{1,\infty} \le Ch|\log(\frac{1}{h})|. \tag{5.4}$$

Proof. Using the inverse inequality, Lemma 3.1, and [16, Theorem 3.10], we have

$$\begin{aligned}
\|\nabla(p-p_h)\|_{0,\infty} &\le \|\nabla(p-p_h(y))\|_{0,\infty} + \|\nabla(p_h(y)-p_h)\|_{0,\infty} \\
&\le \|\nabla(p-p_h(y))\|_{0,\infty} + Ch^{-1}\|\nabla(p_h(y)-p_h)\|_{0,\Omega} \\
&\le \|\nabla(p-p_h(y))\|_{0,\infty} + Ch^{-1}\|y-y_h\|_{0,\Omega} \\
&\le Ch\ |\log(\frac{1}{h})| + Ch \le Ch\ |\log(\frac{1}{h})|.
\end{aligned} \tag{5.5}$$

Similarly, we obtain

$$\begin{aligned}
\|\nabla(y-y_h)\|_{0,\infty} &\le \|\nabla(y-y_h(u))\|_{0,\infty} + \|\nabla(y_h(u)-y_h)\|_{0,\infty} \\
&\le \|\nabla(y-y_h(u))\|_{0,\infty} + Ch^{-1}\|y_h(u)-y_h\|_{0,\Omega} \\
&\le \|\nabla(y-y_h(u))\|_{0,\infty} + Ch^{-1}\|u-u_h\|_{0,\Omega} \\
&\le Ch\ |\log(\frac{1}{h})| + Ch \le Ch\ |\log(\frac{1}{h})|.
\end{aligned} \tag{5.6}$$

Then we complete the proof of (5.4). □

Now, we consider the errors of the state and costate in H^1 norm.

Theorem 5.3. *Let* $(y,p,u) \in (H^2(\Omega)\cap H_0^1(\Omega)) \times (H^2(\Omega)\cap H_0^1(\Omega)) \times U$ *and* $(y_h,p_h,u_h) \in V_h \times\ V_h \times U$ *are the solutions of* (2.6)-(2.8) *and* (2.14)-(2.16), *respectively. Then there exists an* $h_0 > 0$ *such that for all* $0 < h \le h_0$,

$$\|y-y_h\|_{1,\Omega} + \|p-p_h\|_{1,\Omega} \le Ch(\|y\|_{2,\Omega} + \|p\|_{2,\Omega}). \tag{5.7}$$

Proof. Using the triangle inequality, we have

$$\begin{gathered}
\|y-y_h\|_{1,\Omega} \le \|y-y_h(u)\|_{1,\Omega} + \|y_h(u)-y_h\|_{1,\Omega}, \\
\|p-p_h\|_{1,\Omega} \le \|p-p_h(y)\|_{1,\Omega} + \|p_h(y)-p_h\|_{1,\Omega}.
\end{gathered}$$

Lemma 3.1 implies

$$\|y-y_h\|_{1,\Omega} \le \|y-y_h(u)\|_{1,\Omega} + C\|u-u_h\|_{0,\Omega}, \tag{5.8}$$

$$\|p-p_h\|_{1,\Omega} \le \|p-p_h(y)\|_{1,\Omega} + C\|y-y_h\|_{0,\Omega}. \tag{5.9}$$

By using [16, Theorem 3.3], we obtain

$$\|y-y_h(u)\|_{1,\Omega} \le Ch\|y\|_{2,\Omega}, \quad \|p-p_h(y)\|_{1,\Omega} \le Ch\|p\|_{2,\Omega}. \tag{5.10}$$

From Theorem 4.2 and (5.8)-(5.10) we can easily obtain (5.7). □

6. Conclusion and future works

In this article, we presented the finite volume approximation of general elliptic optimal control problems. We prove the existence and the uniqueness of the solution for discrete optimality conditions. Under some reasonable assumptions, we obtain some optimal order error estimates for the state, costate and control variables. The convergence rate for the state, costate and control variables is $O(h^2)$ or $O(h^2\sqrt{|\log(\frac{1}{h})|})$ in the sense of L^2 norm or L^∞ norm. The convergence rate for the state and costate variables is $O(h)$ or $O(h|\log(\frac{1}{h})|)$ in the sense of H^1 norm or $W^{1,\infty}$ norm.

We presented a priori error estimates for the finite volume approximation of general elliptic optimal control problems. To our best knowledge in the context of optimal control problems, these priori error estimates for the general elliptic optimal control problems are new.

In the future, we shall consider the finite volume approximation of parabolic optimal control problems. Furthermore, we shall consider a posteriori error estimates and super-convergence of the finite volume solutions for parabolic optimal control problems.

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