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HARMONIC MEASURES AND POISSON KERNELS ON KLEIN SURFACES

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ABSTRACT. We introduce harmonic measure on a Klein surface and obtain a formula for the solution of the Dirichlet problem on a Klein surface, which is an analogue for the Poisson integral. We rewrite the Radon-Nikodym derivative of harmonic measure against the corresponding arc length.

1. Introduction

We study the Dirichlet problem for harmonic functions on Klein surfaces, through their double covers by symmetric Riemann surfaces in the sense of Klein, that is, Riemann surfaces endowed with fixed point free antianalytic involutions. We prove that symmetric conditions on the boundary determine symmetric solutions which lead to solutions for the similar problems for Klein surfaces. Thus, it is possible to solve boundary value problems on a Klein surface, once the harmonic measure on the symmetric Riemann surface is known. The idea of using the double cover has been successfully used to study objects on a Klein surface by Alling and Greenleaf [2], Andreian Cazacu [3], Bârză and Ghişa [5, 6].

In this paper, the methods introduced in [2, 4, 9] are used to extend the use of the Green's function to the study of harmonic measure for a Klein surface. Since dianalytic structures of Klein surfaces are related to symmetric conformal metrics on their double covers, we represent the harmonic measure in terms of such metrics, using the concept of normal derivative of the k-invariant Green's function introduced in [6]. The symmetric harmonic measure provides an explicit formula for the solution of the Dirichlet problem on a Klein surface, that is an analogue for the Poisson integral. Also, we rewrite the Radon-Nikodym derivative of harmonic measure against σ-arc length, the symmetric arc length.

2. Preliminaries

We present some definitions and basic results about the relationship between Klein surfaces and symmetric Riemann surfaces.

Let O_2 be a region in the complex plane, bounded by a finite number of analytic Jordan curves. Then $\overline{O_2} = O_2 \cup \partial O_2$ can be conceived as a bordered Riemann

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surface, see [1]. A symmetric Riemann surface is a pair (O_2, k), where O_2 is a Riemann surface and k is an antianalytic involution without fixed points.

If (X, A) is a a compact Klein surface, then there exists a symmetric Riemann surface (O_2, k) such that X is dianalytically equivalent with O_2/H, where H is the group generated by k, with respect to the usual composition of functions.Conversely, if (O_2, k) is a symmetric Riemann surface, then on the orbit space O_2/H there exists a dianalytic atlas A, such that $(O_2/H, A)$ is a Klein surface, see [2].

Remark 2.1. In this paper, we identify X with the orbit space O_2/H. The canonical projection of O_2 onto O_2/H is denoted by π. Also, we denote with $\widetilde{z}$ the H - orbit of $z \in O_2$, namely $\widetilde{z} = \widetilde{k(z)} = \pi(z) = \pi(k(z)) = \{z, k(z)\}$.

Let $F : X \to \mathbb{R}$ be a function on X. Its lifting f to O_2 is given by

$$F(\widetilde{z}) = f(z) = f(k(z)), z \in O_2, \widetilde{z} = \pi(z). \tag{2.1}$$

A function f on O_2 with the property (2.1) is called a symmetric function.

Conversely, if $g : O_2 \to \mathbb{R}$ is a function on O_2, then the function $f = g + g \circ k$ is a symmetric function on O_2. Thus, relation (2.1) defines a function F on X.

We consider the symmetric metric on O_2, defined by $d\sigma = \frac{1}{2}(|dz| + |dw|)$, where $w = k(z)$, $z \in O_2$. Then

$$d\Sigma(\widetilde{z}) = d\sigma(z) = d\sigma(k(z)), \quad z \in O_2,$$

is a metric on X. The metric $d\Sigma$ is invariant with respect to the group of conformal or anticonformal transition functions of X.

We denote by $\mathcal{B}(O_2)$ and $\mathcal{B}(X)$ the σ-algebra of Borel sets on O_2, respectively, on X. The σ-algebra of symmetric Borel sets of O_2 is denoted by $\mathcal{B}_s(O_2)$ and $\mathcal{B}_s(O_2) = \{U \cup k(U) : U \in \mathcal{B}(O_2)\}$, see [5].

Let $\widetilde{\gamma}$ be a piecewise smooth Jordan curve on X. Then $\widetilde{\gamma}$ has exactly two liftings γ and $k \circ \gamma$ on O_2 and by definition

$$\int_{\widetilde{\gamma}} F d\Sigma = \int_{\gamma} f d\sigma = \int_{k\circ\gamma} f d\sigma.$$

For more details about measure and integration on Klein surfaces, see [4].

Let u be a C^1- function defined in a neighborhood of the σ-rectifiable Jordan curve γ, parameterized in terms of the arc σ-length. Therefore, $\gamma : z = z(s) = x(s) + iy(s)$, $s \in [0, l]$, where l is the σ-length of γ. The normal derivative of u on γ with respect to $d\sigma$, denoted by $\frac{\partial u}{\partial n_\sigma}$, is the directional derivative of u in the direction of the unit normal vector $n_\sigma = (\frac{dy}{d\sigma}, -\frac{dx}{d\sigma})$.

Given Ω a relatively compact region of X, bounded by a finite number of σ-rectifiable Jordan curves, then $\pi^{-1}(\Omega) = D$ is a symmetric subset of O_2, since k is an antianalytic involution, without fixed points and $\pi \circ k = \pi$. For details about Green's identities for the symmetric region D in terms of $d\sigma$, see [6].

Let F be a continuous real-valued function on $\partial\Omega$. The Dirichlet problem on X for the region Ω, consists in finding a harmonic function U in Ω with prescribed values F on $\partial\Omega$. We define $f = F \circ \pi$ on ∂D. Then $f = f \circ k$ on ∂D, thus f is a symmetric, continuous real-valued functions on ∂D. The Dirichlet problem on X,

$$\begin{gathered} \Delta U = 0, \quad \text{in } \Omega \\ U = F, \quad \text{on } \partial\Omega \end{gathered} \tag{2.2}$$

is equivalent with the Dirichlet problem on O_2

$$\begin{aligned} \Delta u &= 0, \quad \text{in } D \\ u &= f, \quad \text{on } \partial D, \end{aligned} \tag{2.3}$$

see [5, 9].

The Dirichlet problem for the region D and the boundary function f has a unique solution, provided that ∂D has only regular points, see [10]. The symmetric conditions on the boundary imply symmetric solutions for the problem (2.3), for details see [5] and the original source [10].

Proposition 2.2. *The solution u of (2.3) is a symmetric function in D.*

3. Symmetric harmonic measure

First we recall some notions and results about harmonic measure. An extensive study of the harmonic measure is developed in [7].

Let D be a symmetric region in the complex plane and $\mathcal{B}_s(\partial D)$ the σ-algebra of symmetric Borel sets of ∂D. The harmonic measure for D is known (see [8]) to be a function $\omega_D : D \times \mathcal{B}_s(\partial D) \to [0,1]$ such that:

(1) for each $\zeta \in D$, the map $B \to \omega_D(\zeta, B)$ is a Borel probability measure on ∂D;
(2) if $f : \partial D \to \mathbb{R}$ is a continuous function, then the solution of the Dirichlet problem, for D and the boundary function f, is the generalized Poisson integral of f on D, $P_D f(z)$, given by

$$P_D f(\zeta) = \int_{\partial D} f(z) d\omega_D(\zeta, z), \zeta \in D. \tag{3.1}$$

Remark 3.1. The uniqueness of ω_D is a consequence of the Riesz representation theorem.

A method of determining the harmonic measure is given by the following characterization (see [8]).

Proposition 3.2. *The function $\omega_D(\cdot, B)$, is the solution of the generalized Dirichlet problem with boundary function $f = 1_B$.*

Remark 3.3. The function $\omega_D(\cdot, B)$ is well defined on a compact Riemann surface.

The harmonic measure for D is related to another conformal invariant, the Green's function for the symmetric region D. We are using the following integral Barza's representation [6]:

Theorem 3.4. *Let D be a symmetric region, whose boundary ∂D consists of a finite number of pairwise disjoint σ-rectifiable Jordan curves . If $u \in C(\overline{D})$ is harmonic on D, then for all ζ in D,*

$$u(\zeta) = \frac{1}{2\pi} \int_{\partial D} u(z) \frac{\partial g_D(z;\zeta)}{\partial n_\sigma} d\sigma(z). \tag{3.2}$$

Because of (3.2),

$$P_\zeta(z) = \frac{1}{2\pi} \frac{\partial g_D(z;\zeta)}{\partial n_\sigma}$$

is called the Poisson kernel for the region D.

Using the above theorem and the fact that Borel measures are determined by their actions on continuous functions, we obtain a representation of the harmonic measure in terms of the normal derivative of the Green's function with respect to $d\sigma$.

Proposition 3.5. *Let D be a symmetric region, whose boundary ∂D consists of a finite number of pairwise disjoint σ-rectifiable Jordan curves. If $\zeta \in D$, then*

$$d\omega_D(\zeta, z) = \frac{\partial g_D(z;\zeta)}{\partial n_\sigma} \cdot \frac{d\sigma(z)}{2\pi}, \quad z \in \partial D.$$

Thus, harmonic measure for $\zeta \in D$ is absolutely continuous to arc σ-length on ∂D and, the density is

$$\frac{d\omega_D}{d\sigma} = \frac{1}{2\pi}\frac{\partial g_D(z;\zeta)}{\partial n_\sigma} = P_\zeta(z), \quad \text{on } \partial D.$$

For ζ, a point inside D, let $g_D^{(k)}(z;\widetilde{\zeta})$ be the k-invariant Green's function for the region D, with singularities at ζ and $k(\zeta)$, defined by

$$g_D^{(k)}(z;\widetilde{\zeta}) = \frac{1}{2}[g_D(z;\zeta) + g_D(z;k(\zeta))] \quad \text{on } \overline{D}\backslash\{\zeta, k(\zeta)\}.$$

For additional information on this topic we refer to [6, 9]. One can also derive the following statement (see [6]).

Proposition 3.6. *For every symmetric region D, the function $g_D^{(k)}(\cdot;\widetilde{\zeta})$ is k-invariant on $\overline{D}$, i.e.*

$$g_D^{(k)}(z;\widetilde{\zeta}) = g_D^{(k)}(k(z);\widetilde{\zeta}), \quad \text{for every } z \in \overline{D}.$$

Let $\omega_D^{(k)} : D \times \mathcal{B}_s(\partial D) \to [0,1]$ be the function defined by

$$\omega_D^{(k)}(\widetilde{\zeta};B) = \frac{1}{2}[\omega_D(\zeta,B) + \omega_D(k(\zeta),B)], \quad \widetilde{\zeta} = \{\zeta, k(\zeta)\}, \quad \zeta \in D,\ B \in \mathcal{B}_s(\partial D).$$

Remark 3.7. The symmetry of the region D, implies that the function $\omega_D^{(k)}(\widetilde{\zeta};B)$ is symmetric with respect to B on $\mathcal{B}_s(\partial D)$, i.e. for every $B \in \mathcal{B}_s(\partial D)$:

$$\omega_D^{(k)}(\widetilde{\zeta};B) = \omega_D^{(k)}(\widetilde{\zeta};k(B)).$$

The function $\omega_D^{(k)}(\widetilde{\zeta};B)$ is called the symmetric harmonic measure for D. The function

$$P_{\widetilde{\zeta}}^{(k)}(z) = \frac{1}{2\pi}\frac{\partial g_D^{(k)}(z;\widetilde{\zeta})}{\partial n_\sigma}, \quad z \in D$$

is called the symmetric Poisson kernel for the region D.

4. Integral representation on the double cover

The next theorem yields a formula for the symmetric solution of problem (2.3).

Theorem 4.1. *Let D be a symmetric region bounded by a finite number of pairwise disjoint σ-rectifiable Jordan curves. Let f be a symmetric, continuous function on ∂D. There exists a unique symmetric function u on $\overline{D}$, which is harmonic on D, continuous on $\overline{D}$, such that $u = f$ on ∂D. For all ζ in D,*

$$u(\zeta) = \frac{1}{2}\int_{\partial D} f(z)[d\omega_D(\zeta,z) + d\omega_D(k(\zeta),z)]. \tag{4.1}$$

Proof. Since k is an involution of D, the function $\frac{u(\zeta)+u(k(\zeta))}{2}$ is a symmetric function on D. By (3.1),

$$u(\zeta) = \int_{\partial D} f(z) d\omega_D(\zeta, z), \zeta \in D. \tag{4.2}$$

Replacing ζ with $k(\zeta)$ in (4.2), we get

$$u(k(\zeta)) = \int_{\partial D} f(z) d\omega_D(k(\zeta), z), \zeta \in D. \tag{4.3}$$

Adding (4.2) to (4.3) and dividing by 2, it follows that

$$\frac{u(\zeta) + u(k(\zeta))}{2} = \frac{1}{2} \int_{\partial D} f(z)[d\omega(\zeta, z) + d\omega(k(\zeta), z)],\ \zeta \in D.$$

By Proposition 2.2, f is a symmetric function on D, then the left-hand side of the last equality is $u(\zeta)$ and we conclude that for all ζ in D,

$$u(\zeta) = \frac{1}{2} \int_{\partial D} f(z)[d\omega_D(\zeta, z) + d\omega_D(k(\zeta), z)].$$

The uniqueness of the solution of the Dirichlet problem for harmonic functions implies (4.1). □

By Theorem 4.1, we obtain the Radon-Nikodym derivative of symmetric harmonic measure for D against σ-arc length.

Proposition 4.2. *Let D be a symmetric region whose boundary ∂D consists of a finite number of pairwise disjoint σ- rectifiable Jordan curves. If $\zeta \in D$, then*

$$d\omega_D^{(k)}(\widetilde{\zeta}; z) = \frac{\partial g_D^{(k)}(z; \widetilde{\zeta})}{\partial n_\sigma} \cdot \frac{d\sigma(z)}{2\pi}, \quad z \in \partial D.$$

Thus, symmetric harmonic measure for D is absolutely continuous to arc σ-length on ∂D and, the density is

$$\frac{d\omega_D^{(k)}}{d\sigma} = \frac{1}{2\pi} \frac{\partial g_D^{(k)}(z; \widetilde{\zeta})}{\partial n_\sigma} = P_{\widetilde{\zeta}}^{(k)}(z), \quad \textit{on } \partial D.$$

5. Integral representations on a Klein surface

Let X be compact Klein surface and let Ω be a region of X bounded by a finite number of pairwise disjoint σ- rectifiable Jordan curves. Then there exists a symmetric Riemann surface (O_2, k) such that X is dianalytically equivalent with O_2/H, where H is the group generated by k, with respect to the usual composition of functions. Then, Ω is obtained from the symmetric region D by identifying the corresponding symmetric points.

The harmonic measure for Ω, $\omega_\Omega : \Omega \times \mathcal{B}(\partial\Omega) \to [0, 1]$, is defined by

$$\omega_\Omega(\widetilde{\zeta}, \widetilde{B}) = \omega_D^{(k)}(\widetilde{\zeta}, B) = \omega_D^{(k)}(\widetilde{\zeta}, k(B)), \widetilde{\zeta} \in \Omega, \widetilde{B} = \pi(B) \in \mathcal{B}(\partial\Omega)$$

The function

$$P_{\widetilde{\zeta}}(\widetilde{z}) = P_{\widetilde{\zeta}}^{(k)}(z) = P_{\widetilde{\zeta}}^{(k)}(k(z)), z \in D$$

is called the Poisson kernel for the region Ω.

Remark 5.1. From Remark 3.7, it follows that the function ω_Ω is well defined. From Proposition 3.6, it follows that the function $P_{\widetilde{\zeta}}$ is well defined.

By Theorem 4.1, we obtain the following representation of the solution of the problem (2.3) on a symmetric region D, in terms of the symmetric harmonic measure.

Theorem 5.2. *Let D be a symmetric region bounded by a finite number of disjoint σ- rectifiable Jordan curves. Let f be a symmetric, continuous function on ∂D. There exists a unique symmetric function u on $\overline{D}$, which is harmonic on D, continuous on $\overline{D}$, such that $u = f$ on ∂D. For all ζ in D we have*

$$u(\zeta) = \int_{\partial D} f(z) d\omega_D^{(k)}(\widetilde{\zeta}, z).$$

The symmetric solutions on O_2 determine the solutions of the similar problems on the Klein surface X.

We obtain the solution of the problem (2.2) on the region Ω, with respect to the harmonic measure for the region Ω.

Theorem 5.3. *Let F be a continuous real-valued function on the border $\partial\Omega$. The solution of the problem* (2.2) *with the boundary function F is the function U defined on $\overline{\Omega}$, by the relation $u = U \circ \pi$, where π is the canonical projection of O_2 on X and u is the solution* (4.1) *of the problem* (2.3) *on the symmetric region D, with the boundary function $f = F \circ \pi$.*

Proof. By definition, $\Delta U(\widetilde{\zeta}) = \Delta u(\zeta) = 0$, for all $\widetilde{\zeta} \in \Omega$, where $\widetilde{\zeta} = \pi(\zeta)$, thus U is a harmonic function. The symmetry of the function f on ∂D, implies

$$U(\widetilde{\zeta}) = u(\zeta) = f(\zeta) = f(k(\zeta)) = F(\widetilde{\zeta}), \quad \text{for all } \widetilde{\zeta} \in \partial\Omega.$$

Due to the uniqueness of the solution, the function U defined on $\overline{\Omega}$ by

$$U(\widetilde{\zeta}) = u(\zeta) = u(k(\zeta)),$$

for all $\widetilde{\zeta}$ in $\overline{\Omega}$, where $\widetilde{\zeta} = \pi(\zeta)$, is the solution of the problem (2.2) on Ω. □

By Proposition 4.2, we obtain the Radon-Nikodym derivative of harmonic measure for Ω against Σ-arc length.

Proposition 5.4. *Let Ω be a region bounded by a finite number of disjoint σ-rectifiable Jordan curves. If $\widetilde{\zeta} \in \Omega$, then*

$$d\omega_\Omega(\widetilde{\zeta}; \widetilde{z}) = d\omega_D^{(k)}(\widetilde{\zeta}, z) = d\omega_D^{(k)}(\widetilde{\zeta}, k(z)), \quad z \in \partial D.$$

Thus, harmonic measure for Ω is absolutely continuous to arc Σ-length on $\partial\Omega$ and, the density is

$$\frac{d\omega_\Omega}{d\sigma} = P_{\widetilde{\zeta}}(\widetilde{z}), \quad \text{on } \partial\Omega.$$

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