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MULTIPLE SOLUTIONS OF A FOURTH-ORDER NONHOMOGENEOUS EQUATION WITH CRITICAL GROWTH IN $\mathbb{R}^4$

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Abstract. In this article we study the existence of at least two positive weak solutions of an nonhomogeneous fourth-order Navier boundary-value problem involving critical exponential growth on a bounded domain in $\mathbb{R}^4$, with a parameter $\lambda > 0$. We establish upper and lower bounds for λ, which determine multiplicity and non-existence of solutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^4$ be a bounded domain with the boundary $\partial\Omega \in C^{2,\sigma}$ for some $0 < \sigma < 1$. In this context, we study the existence of multiple solutions in $W^{2,2}_{\mathcal{N}}(\Omega) = \{u \in W^{2,2}(\Omega) : u = 0 \text{ on } \partial\Omega\}$, for the following fourth-order Navier boundary value problem

$$\begin{gathered}
\Delta^2 u = \mu u|u|^p e^{u^2} + \lambda h(x) \quad \text{in } \Omega, \\
u, -\Delta u > 0 \quad \text{in } \Omega, \\
u = \Delta u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}$$

where $h \geq 0$ in Ω, $\|h\|_{L^2(\Omega)} = 1$, $\lambda > 0$, $\mu = 1$ if $p > 0$ and $\mu \in (0, \lambda_1(\Omega))$ if $p = 0$. Where $\lambda_1(\Omega)$ and ϕ_1 denote the first eigenvalue and the corresponding eigenfunction of Δ^2 on $W^{2,2}_{\mathcal{N}}(\Omega)$ respectively with respect to the Navier boundary condition. We note that $\lambda_1 > 0$ and ϕ_1 is strictly positive (see [4]). The existence of multiple solutions for analogous problems in higher dimensions with critical exponent, have been studied in [2, 8] for the Dirichlet boundary condition, and in [15] for Navier boundary condition. The existence of multiple solutions for the fourth-order nonhomogeneous quasilinear equation has been studied in [3]. The corresponding problem for second order elliptic equations have been studied in [11] for dimension two and in [12] for higher dimensions. We note that, the critical growth for the fourth-order equations is $u \mapsto |u|^{8/(N-4)}u$ for $N \geq 5$, from the point of view of the Sobolev imbedding theorem in $\mathbb{R}^n$. In 1971, Moser [10] proved the following theorem.

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Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded domain. There exists a constant $C_N > 0$ such that for any $u \in W_0^{1,N}(\Omega)$, $N \geq 2$ with $\|\nabla u\|_{L^N(\Omega)} \leq 1$, then*

$$\int_\Omega e^{\alpha|u|^p} \,\mathrm{d}x \leq C_N|\Omega|, \quad \forall \alpha \leq \alpha_N, \tag{1.2}$$

where

$$p = \frac{N}{N-1}, \quad \alpha_N := N w_{N-1}^{\frac{1}{N-1}},$$

and w_{N-1} is the surface measure of the unit sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$. Furthermore, the integral on the left hand side can be made arbitrarily large if $\alpha > \alpha_N$ by appropriate choice of u with $\|\nabla u\|_{L^N(\Omega)} \leq 1$. The embedding

$$W_0^{1,N}(\Omega) \ni u \mapsto e^{\alpha|u|^{\frac{N}{N-1}}} \in L^1(\Omega),$$

is compact for $\alpha < \alpha_N$ and it is not compact for $\alpha = \alpha_N$.

In 1988, Adams [1] extended the above result of Moser to higher order Sobolev spaces. To state the main theorem by Adams, we denote the m-th order derivatives of $u \in C^m(\Omega)$, by

$$\nabla^m u = \begin{cases} \Delta^{m/2} u, & \text{for } m \text{ even}, \\ \nabla \Delta^{(m-1)/2} u, & \text{for } m \text{ odd}. \end{cases}$$

Now denote by $W_0^{m,\frac{N}{m}}(\Omega)$ the completion of $C_0^\infty(\Omega)$, under the Sobolev norm

$$\|u\|_{W^{m,\frac{N}{m}}(\Omega)} = \Big(\|u\|_{N/m}^{N/m} + \sum_{|\alpha|=1}^{m} \|D^\alpha u\|_{N/m}^{N/m}\Big)^{m/N}. \tag{1.3}$$

Adams proved the following embedding.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. If m is a positive integer and $m \leq N$, then there exists a constant $C_0 = C_0(N,m) > 0$, such that for any $u \in W_0^{m,\frac{N}{m}}(\Omega)$ with $\|\nabla^m u\|_{L^{N/m}(\Omega)} \leq 1$, then*

$$\frac{1}{|\Omega|} \int_\Omega \exp\Big(\beta|u(x)|^{N/(N-m)}\Big) \,\mathrm{d}x \leq C_0, \tag{1.4}$$

for all $\beta \leq \beta_{N,m}$, where

$$\beta_{N,m} = \begin{cases} \frac{N}{w_{N-1}} \Big[\frac{\pi^{\frac{N}{2}} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{N-m+1}{2})}\Big]^{N/(N-m)}, & \text{when } m \text{ is odd}, \\ \frac{N}{w_{N-1}} \Big[\frac{\pi^{\frac{N}{2}} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{N-m}{2})}\Big]^{N/(N-m)}, & \text{when } m \text{ is even}. \end{cases}$$

Furthermore, for any $\beta > \beta_{N,m}$, the integral can be made as large as possible by appropriate choice of u with $\|\nabla^m u\|_{L^{N/m}(\Omega)} \leq 1$.

Now we define a subspace of $W^{m,\frac{N}{m}}(\Omega)$, by

$$W_{\mathcal{N}}^{m,\frac{N}{m}}(\Omega) := \big\{u \in W^{m,\frac{N}{m}}(\Omega) : \Delta^j u|_{\partial\Omega} = 0 \text{ for } 0 \leq j \leq \big[\frac{m-1}{2}\big]\big\}.$$

Note that, $W_0^{m,\frac{N}{m}}(\Omega)$ is strictly contained in $W_{\mathcal{N}}^{m,\frac{N}{m}}(\Omega)$. Therefore, one has

$$\sup_{u \in W_0^{m,\frac{N}{m}}(\Omega), \|\nabla^m u\|_{L^{N/m}(\Omega)} \leq 1} \int_\Omega e^{\beta_{N,m}|u|^{N/(N-m)}} \,\mathrm{d}x$$

$$\leq \sup_{u\in W_{\mathcal{N}}^{m,\frac{N}{m}}(\Omega),\|\nabla^m u\|_{L^{N/m}(\Omega)}\leq 1} \int_\Omega e^{\beta_{N,m}|u|^{N/(N-m)}}\,\mathrm{d}x.$$

In 2012, Tarsi [13] established that the Adams' inequality is also valid for the larger space $W_{\mathcal{N}}^{m,\frac{N}{m}}(\Omega)$. The key idea was to embed $W_{\mathcal{N}}^{m,\frac{N}{m}}(\Omega)$ into a Zygmund space. We state her embedding theorem below.

Theorem 1.3. *Let $N>2$, $m<N$ and $\Omega\subset\mathbb{R}^N$ be a bounded domain. Then, there is a constant $C'_N>0$, such that for all $u\in W_{\mathcal{N}}^{m,\frac{N}{m}}(\Omega)$ with $\|\nabla^m u\|_{L^{N/m}(\Omega)}\leq 1$, we have*

$$\int_\Omega e^{\beta|u|^{N/(N-m)}}\,\mathrm{d}x < C'_N|\Omega|,\quad \forall\beta\leq\beta_{N,m}, \tag{1.5}$$

and the constant $\beta_{N,m}$ appearing in (1.5) is sharp and $\beta_{N,m}$ is same as in Theorem 1.2.

Remark 1.4. When $N=4=2m$, we have $\beta_{4,2}=32\pi^2$.

Remark 1.5. We note that the bilinear form

$$(u,v)\mapsto\int_\Omega \nabla^m u\cdot\nabla^m v=\begin{cases}\int_\Omega \Delta^k u\Delta^k v, & \text{if } m=2k,\\ \int_\Omega \nabla(\Delta^k u)\cdot\nabla(\Delta^k v), & \text{if } m=2k+1,\end{cases} \tag{1.6}$$

defines a scalar product on both spaces $W_0^{m,2}(\Omega)$ and $W_{\mathcal{N}}^{m,2}(\Omega)$. Furthermore, if Ω is bounded, this scalar product induces a norm equivalent to (1.3).

Therefore, the above results imply that the nonlinearity of the problem (1.1) is of critical type. Now we state our main theorem regarding multiple solutions in this non-compact situation, given by problem (1.1).

Theorem 1.6. *There exist positive real numbers $\lambda_*\leq\lambda^*$, with λ_* independent of h, such that the problem (1.1) has at least two positive solutions for all $\lambda\in(0,\lambda_*)$ and no solution for all $\lambda>\lambda^*$.*

Though the Palais-Smale condition fails due to the presence of critical exponent, first we adapt the method of Tarantello (cf. [12]) to prove the existence of a first solution by a decomposition of Nehari manifold into three parts. Then, for the existence of second solution, we rely on a refined version of the Mountain-Pass Lemma, which was introduced by Ghoussoub and Preiss in [6].

2. Decomposition of the Nehari Manifold

Let $f(u)=\mu|u|^p ue^{u^2}$. The corresponding energy functional associated with problem (1.1), is

$$J(u)=\frac{1}{2}\int_\Omega|\Delta u|^2-\int_\Omega F(u)-\lambda\int_\Omega hu, \tag{2.1}$$

where $F(u)=\int_0^u f(s)ds$. As the energy functional is not bounded from below on $W_{\mathcal{N}}^{2,2}(\Omega)$, we need to study J on the Nehari manifold

$$\mathcal{M}:=\{u\in W_{\mathcal{N}}^{2,2}(\Omega):\langle J'(u),u\rangle=0\}, \tag{2.2}$$

where $J'(u)$ denotes the Fréchet derivative of J at u and $\langle\cdot,\cdot\rangle$ is the inner product. Here we note that, $\mathcal{M}$ contains every solution of the problem (1.1). For any $u\in$

$W_{\mathcal{N}}^{2,2}(\Omega)$, we note that

$$\langle J'(u), u\rangle = \int_\Omega |\Delta u|^2 - \int_\Omega f(u)u - \lambda \int_\Omega hu,$$
$$\langle J''(u)u, u\rangle = \int_\Omega |\Delta u|^2 - \int_\Omega f'(u)u^2.$$

Similar to the method used by Tarantello [12], we split $\mathcal{M}$ into three parts

$$\mathcal{M}^0 = \{u \in \mathcal{M} : \langle J''(u)u, u\rangle = 0\},$$
$$\mathcal{M}^+ = \{u \in \mathcal{M} : \langle J''(u)u, u\rangle > 0\},$$
$$\mathcal{M}^- = \{u \in \mathcal{M} : \langle J''(u)u, u\rangle < 0\}.$$

3. Topological Properties of $\mathcal{M}^0, \mathcal{M}^+, \mathcal{M}^-$

Our first aim is to show, $\mathcal{M}^0 = \{0\}$ for some small λ. For this, let $\zeta > 0$ if $p > 0$ and $\zeta < \frac{\lambda_1 - \mu}{\mu}$ if $p = 0$. Define, $\Lambda := \{u \in W_{\mathcal{N}}^{2,2}(\Omega) : \int_\Omega |\Delta u|^2 \le (1+\zeta)\int_\Omega f'(u)u^2\}$. Then, Lemma 3.3 implies that $\Lambda \ne \{0\}$. We now assume the following important hypotheses

$$\lambda > 0, \quad \|h\|_{L^2(\Omega)} = 1,$$
$$\inf_{u\in\Lambda\setminus\{0\}} \Big(\mu \int_\Omega (p+2u^2)|u|^{p+2}e^{u^2} - \lambda \int_\Omega hu\Big) > 0. \tag{3.1}$$

Condition (3.1) forces λ to be suitably small. Indeed, we can prove the following result.

Proposition 3.1. *Let*

$$\lambda < \mu C_0^{\frac{p+3}{p+4}} |\Omega|^{-(\frac{p+2}{2p+8})}, \tag{3.2}$$

where $C_0 = \inf_{u\in\Lambda\setminus\{0\}} \int_\Omega (p+2u^2)|u|^{p+2}e^{u^2} > 0$. *Then* (3.1) *holds.*

Proof. **Step 1:** $\inf_{u\in\Lambda\setminus\{0\}} \|u\|_{W_{\mathcal{N}}^{2,2}(\Omega)} > 0$. Toward a contradiction, suppose that, there exists a sequence $\{u_n\} \subset \Lambda \setminus \{0\}$, with $\|u_n\|_{W_{\mathcal{N}}^{2,2}(\Omega)} \to 0$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|_{W_{\mathcal{N}}^{2,2}(\Omega)}}$. Then $\|v_n\|_{W_{\mathcal{N}}^{2,2}(\Omega)} = 1$ and v_n satisfies

$$1 \le (1+\zeta) \int_\Omega f'(u_n)v_n^2, \quad \forall n. \tag{3.3}$$

Since $u_n \to 0$ in $W_{\mathcal{N}}^{2,2}(\Omega)$, by Adams' inequality for the higher order derivative in Theorem 1.3, we obtain $f'(u_n) \to f'(0)$ in $L^r(\Omega)$ for all $r \ge 1$. Since v_n is bounded in $W_{\mathcal{N}}^{2,2}(\Omega)$, v_n has a weak limit say v in $W_{\mathcal{N}}^{2,2}(\Omega)$. Certainly $\|v\|_{W_{\mathcal{N}}^{2,2}(\Omega)} \le 1$ and up to a subsequence which we still denote by v_n which converges strongly to v in $L^r(\Omega)$ for all $r \ge 1$. Hence from (3.3), we obtain

$$\int_\Omega |\Delta v|^2 \le 1 \le (1+\zeta)f'(0) \int_\Omega v^2. \tag{3.4}$$

This gives a contradiction for the case $p > 0$ since $f'(0) = 0$ in this case. For the case $p = 0$, by the assumption

$$\int_\Omega |\Delta v|^2 \ge \lambda_1 \int_\Omega v^2 > (1+\zeta)\mu \int_\Omega v^2,$$

which gives a contradiction to (3.3) since $f'(0) = \mu$. This completes Step 1. Since $\inf_{u\in\Lambda\setminus\{0\}} \|u\|_{W^{2,2}_{\mathcal{N}}(\Omega)} > 0$, from the definition of Λ, we obtain

$$0 < \inf_{u\in\Lambda\setminus\{0\}} \int_\Omega f'(u)u^2 = \inf_{u\in\Lambda\setminus\{0\}} \mu \int_\Omega (p+1+2u^2)e^{u^2}|u|^{p+2}. \tag{3.5}$$

Using (3.5), we can easily check that

$$C_0 = \inf_{u\in\Lambda\setminus\{0\}} \int_\Omega (p+2u^2)|u|^{p+2}e^{u^2} > 0. \tag{3.6}$$

Step 2: Finally, we have

$$\begin{aligned}
\lambda\Big|\int_\Omega hu\Big| &\le \lambda\|u\|_{L^2(\Omega)} \le \lambda|\Omega|^{\frac{p+2}{2p+8}}\Big(\int_\Omega |u|^{p+4}\Big)^{1/(p+4)} \\
&\le \frac{\lambda|\Omega|^{\frac{p+2}{2p+8}}}{(\mu\int_\Omega (p+2u^2)|u|^{p+2}e^{u^2})^{\frac{p+3}{p+4}}}\Big(\mu\int_\Omega (p+2u^2)|u|^{p+2}e^{u^2}\Big) \\
&\le \Big(\frac{\lambda|\Omega|^{\frac{p+2}{2p+8}}}{\mu C_0^{\frac{p+3}{p+4}}}\Big)\Big(\mu\int_\Omega (p+2u^2)|u|^{p+2}e^{u^2}\Big).
\end{aligned}$$

Hence, from the above inequality together with (3.2) and (3.6), we can complete the proof. $\square$

Lemma 3.2. *Suppose $\lambda > 0$ be such that (3.1) holds. Then $\mathcal{M}^0 = \{0\}$.*

Proof. For the sake of contradiction, suppose there exists $u \in \mathcal{M}^0$ and $u \not\equiv 0$. Then, we have

$$\int_\Omega |\Delta u|^2 = \int_\Omega f(u)u + \lambda\int_\Omega hu, \tag{3.7}$$

$$\int_\Omega |\Delta u|^2 = \int_\Omega f'(u)u^2. \tag{3.8}$$

It follows from (3.8), that

$$\int_\Omega |\Delta u|^2 = \int_\Omega f'(u)u^2 < (1+\zeta)\int_\Omega f'(u)u^2,$$

therefore, $u \in \Lambda\setminus\{0\}$. From (3.7) and (3.8), we obtain

$$\lambda\int_\Omega hu = \int_\Omega (f'(u)u - f(u))u = \mu\int_\Omega (p+2u^2)|u|^{p+2}e^{u^2},$$

which violates the condition (3.1). Therefore, $\mathcal{M}^0 = \{0\}$. $\square$

Now we discuss the topological properties of $\mathcal{M}^+$ and $\mathcal{M}^-$. The study of Nehari manifold is closely related to the behaviour of the map $s \mapsto J(su)$. This technique was first introduced in [5] by Drábek and Pohozaev. Given $u \in W^{2,2}_{\mathcal{N}}(\Omega)\setminus\{0\}$, we define a map, $\xi_u : \mathbb{R}^+ \to \mathbb{R}$ by

$$\xi_u(s) = s\int_\Omega |\Delta u|^2 - \int_\Omega f(su)u. \tag{3.9}$$

The choice of the above function is a consequence of the following expression

$$\langle J'(su), su\rangle = s\Big(s\int_\Omega |\Delta u|^2 - \int_\Omega f(su)u - \lambda\int_\Omega hu\Big).$$

So, $\xi_u(s) = \lambda \int_\Omega hu$ if and only if $su \in \mathcal{M}$ for $s > 0$.

Now we are ready to prove the following lemma.

Lemma 3.3. *For every $u \in W^{2,2}_{\mathcal{N}}(\Omega) \setminus \{0\}$, there exists unique $s_* = s_*(u) > 0$, such that $\xi_u(\cdot)$ has its maximum at s_* with $\xi_u(s_*) > 0$. Also there holds, $s_* u \in \Lambda \setminus \{0\}$.*

Proof. Differentiating (3.9), we have

$$\xi_u'(s) = \int_\Omega |\Delta u|^2 - \int_\Omega f'(su)u^2. \tag{3.10}$$

Therefore,

$$s^2 \xi_u'(s) = \int_\Omega |\Delta(su)|^2 - \int_\Omega f'(su)(su)^2 = \langle J''(su)su, su\rangle. \tag{3.11}$$

Now we observe that, $\xi_u(\cdot)$ is a concave function on $\mathbb{R}^+$ since

$$\xi_u''(s) = -\int_\Omega f''(su)u^3 < 0. \tag{3.12}$$

Also from the range of μ, we obtain

$$\lim_{s\to 0+} \xi_u'(s) > 0, \quad \lim_{s\to\infty} \xi_u(s) = -\infty.$$

Hence there exists a unique $s_* = s_*(u) > 0$, such that ξ_u is increasing on $(0, s_*)$, decreasing on (s_*, ∞) and $\xi_u'(s_*) = 0$. Now using (3.10) and $\xi_u'(s_*) = 0$, we deduce

$$\begin{aligned} \xi_u(s_*) &= s_* \int_\Omega f'(s_* u)u^2 - \int_\Omega f(s_* u)u \\ &= \frac{1}{s_*}\int_\Omega (f'(s_* u)s_* u - f(s_* u))s_* u \\ &= \frac{\mu}{s_*}\int_\Omega (p + 2(s_* u)^2)|s_* u|^{p+2} e^{(s_* u)^2} > 0. \end{aligned} \tag{3.13}$$

Here we note that, $f'(s)s - f(s) = \mu(p + 2s^2)|s|^p s e^{s^2}$. Finally

$$s_* \xi_u'(s_*) = \int_\Omega |\Delta(s_* u)|^2 - \int_\Omega f'(s_* u)(s_* u)^2 = 0,$$

which implies, $s_* u \in \Lambda \setminus \{0\}$. $\square$

Lemma 3.4. *Let λ be such that* (3.1) *holds. Then, for every $u \in W^{2,2}_{\mathcal{N}}(\Omega) \setminus \{0\}$, there exists a unique $s_- = s_-(u) > 0$ such that $s_- u \in \mathcal{M}^-$, $s_- > s_*$ and $J(s_- u) = \max_{s \ge s_*} J(su)$ $\forall s \in [s_*, \infty)$. Furthermore, if $\int_\Omega hu > 0$, then there exists a unique $s_+ = s_+(u) > 0$ such that $s_+ u \in \mathcal{M}^+$. In particular, $s_+ < s_*$ and*

$$J(s_+ u) \le J(su) \quad \text{for all } s \in [0, s_-]. \tag{3.14}$$

Proof. Define the functional, $\rho_u : [0,\infty) \to \mathbb{R}$ by $\rho_u(s) = J(su)$. Then it is easy to verify that $\rho_u \in C^2((0,\infty), \mathbb{R}) \cap C([0,\infty), \mathbb{R})$. Now we have

$$\rho_u'(s) = \xi_u(s) - \lambda \int_\Omega hu, \quad \rho_u''(s) = \xi_u'(s), \quad \forall s > 0.$$

Next from (3.1) and (3.13), we obtain

$$\xi_u(s_*) - \lambda \int_\Omega hu = \frac{1}{s_*}\Big\{\mu \int_\Omega (p + 2(s_* u)^2)|s_* u|^{p+2} e^{(s_* u)^2} - \lambda \int_\Omega h(s_* u)\Big\} > 0.$$

Since $\xi_u(\cdot)$ is strictly decreasing in (s_*,∞) and $\lim_{s\to\infty}\xi_u(s)=-\infty$, there exists a unique $s_-=s_-(u)>s_*$, such that $\xi_u(s_-)=\lambda\int_\Omega hu$. That is $s_-u\in\mathcal{M}$. One has $s_->s_*$ and $\rho_u'(s_)<0$, we obtain $s_-u\in\mathcal{M}^-$.

On the other hand, when $\int_\Omega hu>0$, we have $\lim_{s\to 0+}\xi_u(s)<0$, which implies, $\xi_u(s)-\lambda\int_\Omega hu<0$ for s near 0. Hence there exists a unique s_+, such that $\xi_u(s_+)=\lambda\int_\Omega hu$ which implies $s_+u\in\mathcal{M}$. From the graph, we see that $\xi_u(\cdot)$ is strictly increasing in $(0,s_*)$. Hence we have, $s_+u\in\mathcal{M}^+$.

And the remaining properties of s_-,s_+ can be proved by analyzing the identity $\rho_u(s)=\xi_u(s)-\lambda\int_\Omega hu$. □

Remark 3.5. If we define the positive cone $\mathcal{P}=\{u\in W^{2,2}_{\mathcal{N}}(\Omega):\int_\Omega hu>0\}$ in $W^{2,2}_{\mathcal{N}}(\Omega)$. Then, we obtain $\mathcal{M}^+\subset\mathcal{P}$.

The next corollary shows some topological properties of $\mathcal{M}^+,\mathcal{M}^-$.

Corollary 3.6. *Let $S_{W^{2,2}_{\mathcal{N}}(\Omega)}=\{u\in W^{2,2}_{\mathcal{N}}(\Omega):\|u\|_{W^{m,2}_{\mathcal{N}}(\Omega)}=1\}$. Then there exists a homeomorphism $S_-:S_{W^{2,2}_{\mathcal{N}}(\Omega)}\to\mathcal{M}^-$ defined by $S_-(u)=s_-(u)u$. Also $\mathcal{M}^+$ is homeomorphic to $S_{W^{2,2}_{\mathcal{N}}(\Omega)}\cap\mathcal{P}$.*

Proof. The function S_- is continuous, because s_- is continuous as an application of implicit function theorem applied to the map, $(s,u)\mapsto\xi_u(s)-\lambda\int_\Omega hu$. Also we deduce the continuity of $(S_-)^{-1}$ by the fact that $(S_-)^{-1}(w)=w/\|w\|$. In a similar manner, we can prove that $\mathcal{M}^+$ is homeomorphic to $S_{W^{m,2}_{\mathcal{N}}(\Omega)}\cap\mathcal{P}$. □

We set, $\theta_0=\inf\{J(u):u\in\mathcal{M}\}$. Relying on the embedding of $W^{2,2}_{\mathcal{N}}(\Omega)\hookrightarrow L^q(\Omega)$ for all $1\le q<\infty$ and using the estimate, $F(s)\le\frac{\mu|s|^p}{2}(e^{s^2}-1)$ for all $s\in\mathbb{R}$, we have the following two lemmas on the lower bound and upper bound of θ_0 in terms of λ, μ.

Lemma 3.7. *There exists $C_1=C_1(p)>0$, such that*
$$\theta_0\ge -C_1\lambda^{\frac{p+4}{p+3}}.$$

Proof. Let $u\in\mathcal{M}$. Then
$$\begin{aligned}J(u)&=\frac12\int_\Omega|\Delta u|^2-\int_\Omega F(u)-\lambda\int_\Omega hu\\&=\int_\Omega\big[\frac12 f(u)u-F(u)\big]-\frac{\lambda}{2}\int_\Omega hu.\end{aligned}$$
We note that a simple calculation gives
$$F(s)\le\frac{\mu|s|^p}{2}(e^{s^2}-1),\quad\forall s\in\mathbb{R}. \tag{3.15}$$
Using (3.15), we deduce
$$\begin{aligned}J(u)&\ge\frac{\mu}{2}\int_\Omega((u^2-1)e^{u^2}+1)|u|^p-\frac{\lambda}{2}\int_\Omega hu\\&\ge\frac{c\mu}{2}\int_\Omega|u|^{p+4}-\frac{\lambda}{2}\int_\Omega hu,\end{aligned}\tag{3.16}$$

since $(s^2-1)e^{s^2}+1 \geq cs^4$ for some $c>0$ and for all $s\in\mathbb{R}$. By applying Hölder inequality, we obtain

$$\int_\Omega hu \leq |\Omega|^{\frac{p+2}{2(p+4)}}\|u\|_{L^{p+4}(\Omega)}. \tag{3.17}$$

From (3.16) and (3.17), we obtain

$$J(u) \geq \frac{c\mu}{2}\|u\|^{p+4}_{L^{p+4}(\Omega)} - \Big(\frac{\lambda|\Omega|^{\frac{p+2}{2(p+4)}}}{2}\Big)\|u\|_{L^{p+4}(\Omega)}. \tag{3.18}$$

By considering the global minimum of the function

$$\omega(x) = (\frac{c\mu}{2})x^{p+4} - \Big(\frac{\lambda|\Omega|^{\frac{p+2}{2(p+4)}}}{2}\Big)x,$$

we deduce $\theta_0 \geq -C_1\lambda^{\frac{p+4}{p+3}}$. $\square$

Lemma 3.8. *There exists $C_2>0$, such that*

$$\theta_0 \leq -\frac{\mu(p+1)p}{2(p+2)}C_2.$$

Proof. Choose $v\in W^{2,2}_{\mathcal{N}}(\Omega)\setminus\{0\}$, such that $\int_\Omega hv>0$. Therefore, by Lemma 3.4, there exists $s_+=s_+(v)>0$, such that $s_+v\in\mathcal{M}^+$. Hence

$$\begin{aligned} J(s_+v) &= -\frac{s_+^2}{2}\int_\Omega|\Delta u|^2 + \int_\Omega[f(s_+v)s_+v - F(s_+v)] \\ &\leq \int_\Omega\Big[f(s_+v)s_+v - F(s_+v) - \frac{1}{2}f'(s_+v)(s_+v)^2\Big], \end{aligned} \tag{3.19}$$

since $s_+v\in\mathcal{M}$. Now we consider the function

$$\gamma(s) = f(s)s - F(s) - \frac{1}{2}f'(s)s^2.$$

We note that $\gamma'(s) = -\frac{1}{2}f''(s)s^2$. Since $\gamma(0)=0$, it follows that $\gamma(s)\leq 0$ for all $s\in\mathbb{R}$. Also we can verify the following limits

$$\begin{gathered} \lim_{s\to 0}\frac{\gamma(s)}{|s|^{p+2}} = -\frac{\mu p(p+1)}{2(p+2)} \quad \text{if } p>0, \\ \lim_{s\to 0}\frac{\gamma(s)}{s^4} = -\frac{3}{4}\mu \quad \text{if } p=0, \\ \lim_{s\to\infty}\frac{\gamma(s)}{|s|^{p+4}e^{s^2}} = -\mu \quad \forall p\geq 0. \end{gathered}$$

From these two estimates, we obtain

$$\gamma(s) \leq -\frac{\mu p(p+1)}{(p+2)}(p+2s^2)|s|^{2(p+2)}e^{s^2}, \quad \forall s\in\mathbb{R}. \tag{3.20}$$

Therefore, using (3.19) and (3.20) we obtain

$$\begin{aligned} J(s_+v) &\leq -\frac{\mu p(p+1)}{2(p+2)}\int_\Omega(p+2|s_+v|^2)|s_+v|^{p+2}e^{|s_+v|^2} \\ &\leq -\frac{\mu p(p+1)}{2(p+2)}\int_\Omega|s_+v|^{p+4}. \end{aligned} \tag{3.21}$$

Hence

$$\theta_0 \leq -\frac{\mu p(p+1)}{2(p+2)}C_2, \quad \text{where } C_2 = \int_\Omega|s_+v|^{p+4}.$$

□

As a consequence of Lemma 3.2, we have the following lemma.

Lemma 3.9. *Let λ and h satisfy (3.1). Given $u \in \mathcal{M} \setminus \{0\}$, there exists $\delta > 0$ and a differentiable function $s : \{w \in W^{2,2}_{\mathcal{N}}(\Omega) : \|w\|_{W^{2,2}_{\mathcal{N}}(\Omega)} < \delta\} \to \mathbb{R}$, with*

$$s(0) = 1, \quad s(w)(u - w) \in \mathcal{M}, \quad \forall \|w\|_{W^{2,2}_{\mathcal{N}}(\Omega)} < \delta$$

and

$$\langle s'(0), v\rangle = \frac{2\int_\Omega \Delta u \Delta v - \int_\Omega (f'(u)u + f(u))v - \lambda \int_\Omega hv}{\int_\Omega |\Delta u|^2 - \int_\Omega f'(u)u^2}. \tag{3.22}$$

Proof. Define a function $G : \mathbb{R} \times W^{2,2}_{\mathcal{N}}(\Omega) \to \mathbb{R}$ by

$$G(s, w) = s \int_\Omega |\Delta(u - w)|^2 - \int_\Omega f(s(u - w))(u - w) - \lambda \int_\Omega h(u - w).$$

Then $G \in C^1(\mathbb{R} \times W^{2,2}_{\mathcal{N}}(\Omega); \mathbb{R})$ and since $u \in \mathcal{M}$ it implies

$$G(1, 0) = \int_\Omega |\Delta u|^2 - \int_\Omega f(u)u - \lambda \int_\Omega hu = 0.$$

Also $G_s(1, 0) \neq 0$, indeed

$$G_s(1, 0) = \int_\Omega |\Delta u|^2 - \int_\Omega f'(u)u^2 \neq 0,$$

thanks to Lemma 3.2. Then by the Implicit Function Theorem, there exists $\delta > 0$ and a map $s : \{w \in W^{2,2}_{\mathcal{N}}(\Omega) : \|w\| < \delta\} \to \mathbb{R}$ of class C^1 that satisfies

$$\begin{gathered} G(s(w), w) = 0, \quad \text{if } \|w\|_{W^{2,2}_{\mathcal{N}}(\Omega)} < \delta, \\ s(0) = 1. \end{gathered}$$

Then

$$\begin{aligned} 0 &= s(w)G(s(w), w) \\ &= \int_\Omega (s(w)|\Delta(u - w)|)^2 - \int_\Omega f(s(w)(u - w))s(w)(u - w) \\ &\quad - \lambda \int_\Omega hs(w)(u - w), \end{aligned}$$

that is $s(w)(u-w) \in \mathcal{M}$ for all $w \in W^{2,2}_{\mathcal{N}}(\Omega)$ with $\|w\| < \delta$. Now if we differentiate the identity $G(s(w), w) = 0$ with respect to w, we obtain

$$0 = \langle G_s(s(w), w)s'(w) + G_w(s(w), w), v\rangle, \quad \forall v \in W^{2,2}_{\mathcal{N}}(\Omega).$$

Put $w = 0$ in the above identity and we obtain

$$0 = \langle G_s(1, 0)s'(0) + G_w(1, 0), v\rangle = G_s(1, 0)\langle s'(0), v\rangle + \langle G_w(1, 0), v\rangle.$$

Therefore,

$$\langle s'(0), v\rangle = -\frac{\langle G_w(1, 0), v\rangle}{G_s(1, 0)} = \frac{2\int_\Omega \Delta u \Delta v - \int_\Omega (f'(u)u + f(u))v - \lambda \int_\Omega hv}{\int_\Omega |\Delta u|^2 - \int_\Omega f'(u)u^2}.$$

□

4. Local minimum of J in $W^{2,2}_{\mathcal{N}}(\Omega)$

We are now in a position to prove the existence of a local minimizer for J, which ensures the existence of a first solution. Note that, $\mathcal{M} \subset W^{2,2}_{\mathcal{N}}(\Omega)$ is closed, hence a complete metric space. Now J is bounded below on $\mathcal{M}$. By the Ekeland's Variational Principle, there exists a sequence $\{u_n\} \subset \mathcal{M} \setminus \{0\}$, satisfying

$$J(u_n) < \theta_0 + \frac{1}{n}, \quad J(v) \geq J(u_n) - \frac{1}{n}\|v-u_n\|_{W^{2,2}_{\mathcal{N}}(\Omega)}, \quad \forall v \in \mathcal{M}. \tag{4.1}$$

Proposition 4.1. *Let λ and h satisfy* (3.1). *Then, we have*

$$\lim_{n\to\infty} \|J'(u_n)\|_{(W^{2,2}_{\mathcal{N}}(\Omega))^{-1}} = 0.$$

Proof. We proceed in three steps.

Claim 1: $\liminf_{n\to\infty} \|u_n\|_{W^{2,2}_{\mathcal{N}}(\Omega)} > 0$. Suppose this claim is false. Then, there exists a subsequence of $\{u_n\}$, which we still denote by $\{u_n\}$, such that $\|u_n\|_{W^{2,2}_{\mathcal{N}}(\Omega)} \to 0$ as $n \to \infty$. Therefore, $J(u_n) \to 0$ as $n \to \infty$ by continuity of the functional J. Which contradicts Lemma 3.8.

Claim 2: $\liminf_{n\to\infty} \int_\Omega (p+2u_n^2)|u_n|^{p+2}e^{u_n^2} > 0$. We argue by contradiction. Assume there exists a subsequence of $\{u_n\}$, which is still denoted by $\{u_n\}$, satisfying

$$\lim_{n\to\infty} \int_\Omega (p+2u_n^2)|u_n|^{p+2}e^{u_n^2} \to 0 \quad \text{as } n \to \infty. \tag{4.2}$$

Here we note that, $u_n \to 0$ in $L^q(\Omega)$ for all $q \in [1,\infty)$, by using (4.2), and for the case $p > 0$, we obtain

$$\int_\Omega f(u_n)u_n = \mu \int_\Omega |u_n|^{p+2}e^{u_n^2} \to 0 \quad \text{as } n \to \infty.$$

Therefore, we have $\int_\Omega f(u_n)u_n \to 0$ and $\int_\Omega hu_n \to 0$ as $n \to \infty$. Since $\{u_n\} \subset \mathcal{M}$, we deduce $\|u_n\|_{W^{2,2}_{\mathcal{N}}} \to 0$ as $n \to \infty$, hence a contradiction to Claim 1. Similar argument also holds for $p = 0$.

Claim 3: $\liminf_{n\to\infty} |\int_\Omega |\Delta u_n|^2 - \int_\Omega f'(u_n)u_n^2| > 0$. Suppose the claim does not hold. Then for a subsequence $\{u_n\}$, we have

$$\int_\Omega |\Delta u_n|^2 - \int_\Omega f'(u_n)u_n^2 = o_n(1). \tag{4.3}$$

From (4.3) and using Claim 1, we infer that

$$\liminf_{n\to\infty} f'(u_n)u_n^2 > 0.$$

Therefore, we have $u_n \in \Lambda \setminus \{0\}$ for large n. Since $\{u_n\} \subset \mathcal{M}$, we obtain

$$\begin{aligned} o_n(1) &= \lambda \int_\Omega hu_n + \int_\Omega (f(u_n) - f'(u_n)u_n)u_n \\ &= -\mu \int_\Omega (p+2u_n^2)|u_n|^{p+2}e^{u_n^2} + \lambda \int_\Omega hu_n, \end{aligned}$$

which contradicts (3.1). This completes the proof of the claim.

Now we complete the proof of the proposition. Suppose the statement of the proposition is false, i.e., $\|J'(u_n)\|_{(W^{2,2}_{\mathcal{N}}(\Omega))^{-1}} > 0$, for all large n (otherwise obvious). Now we set, $u = u_n \in \mathcal{M}$ and $w = \delta \frac{J'(u_n)}{\|J'(u_n)\|}$ for $\delta > 0$ small (by Riesz representation theorem, we identify $J'(u_n)$ as an element in $W^{2,2}_{\mathcal{N}}(\Omega)$). Applying Lemma 3.9, we obtain $s_n(\delta) := s[\delta \frac{J'(u_n)}{\|J'(u_n)\|}] > 0$, such that

$$w_\delta = s_n(\delta)\big[u_n - \delta \frac{J'(u_n)}{\|J'(u_n)\|}\big] \in \mathcal{M}.$$

Now from (4.1) and with the help of Taylor expansion, we have

$$\begin{aligned} \frac{1}{n}\|w_\delta - u_n\| &\geq J(u_n) - J(w_\delta) \\ &= (1 - s_n(\delta))\langle J'(w_\delta), u_n\rangle + \delta s_n(\delta)\big\langle J'(w_\delta), \frac{J'(u_n)}{\|J'(u_n)\|}\big\rangle + o(\delta). \end{aligned}$$

Dividing by $\delta > 0$ and taking limit as $\delta \to 0$, we obtain

$$\frac{1}{n}(1 + |s_n'(0)|\|u_n\|) \geq -s_n'(0)\langle J'(u_n), u_n\rangle + \|J'(u_n)\| = \|J'(u_n)\|.$$

Hence

$$\|J'(u_n)\| \leq \frac{1}{n}(1 + |s_n'(0)|\|u_n\|).$$

We complete the proof by noticing that, $|s_n'(0)|$ is uniformly bounded on n by (3.22) and using the Claim 2. □

Theorem 4.2. *Let λ and h satisfy* (3.1). *Then there exists a nonnegative function $u_0 \in \mathcal{M}^+$, such that $J(u_0) = \inf_{u\in\mathcal{M}\setminus\{0\}} J(u)$. Moreover, u_0 is a local minimum for J in $W^{2,2}_{\mathcal{N}}(\Omega)$.*

Proof. Let $\{u_n\}$ be a sequence which minimizes J on $\mathcal{M} \setminus \{0\}$ as in (4.1).

Step 1: $\liminf_{n\to\infty} \int_\Omega hu_n > 0$ and hence $u_n \in \mathcal{M}^+$. Indeed, $u_n \in \mathcal{M}$ and by using Lemma 3.8, there exists $C > 0$, such that

$$\begin{aligned} J(u_n) = \frac{p}{2(p+2)} &\int_\Omega |\Delta u_n|^2 + \int_\Omega \big(\frac{1}{p+2} f(u_n)u_n - F(u_n)\big) \\ &- \lambda \frac{p+1}{p+2}\int_\Omega hu_n < -C. \end{aligned} \tag{4.4}$$

Now we note that, $F(s) < \frac{1}{p+2} f(s)s$ for all $s \in \mathbb{R}$. Using (4.4), we conclude

$$\liminf_{n\to\infty} \int_\Omega hu_n > 0.$$

Step 2: $\limsup_{n\to\infty} \|u\|_{W^{2,2}_{\mathcal{N}}(\Omega)} < \infty$.

Case 1. If $p > 0$, then by using (4.4), we obtain directly

$$\int_\Omega |\Delta u_n|^2 \leq \lambda \int_\Omega hu_n,$$

and then with the help of Sobolev embedding we derive $\{u_n\}$ is bounded in $W^{2,2}_{\mathcal{N}}(\Omega)$.

Case 2. If $p = 0$, by using the fact that $\frac{1}{2} f(s)s - F(s) \geq Cs^4$ for all $s \in \mathbb{R}$ and for some $C > 0$, we deduce that $\{u_n\}$ is a bounded sequence in $L^2(\Omega)$. It implies that $\{F(u_n)\}$ is a bounded sequence in $L^1(\Omega)$ using (4.4) and hence $\{u_n\}$ is a bounded sequence in $W^{2,2}_{\mathcal{N}}(\Omega)$.

Step 3: Existence of $u_0 \in \mathcal{M}^+$. From the previous step up to a subsequence, $u_n \rightharpoonup u_0$ weakly in $W^{2,2}_{\mathcal{N}}(\Omega)$. Now from the Proposition 4.1, we note that $\{f(u_n)u_n\}$

is a bounded sequence in $L^1(\Omega)$. Therefore, by recalling Vitali convergence theorem (for details see Lemma 5.3), we obtain

$$\int_\Omega f(u_n)\phi \to \int_\Omega f(u_0)\phi, \quad \text{for all } \phi \in W^{2,2}_{\mathcal{N}}(\Omega).$$

Hence, u_0 will solve (1.1). It is obvious that $u_0 \neq 0$ as $h \neq 0$, that is $u_0 \in \mathcal{M}$. We see that $\theta_0 \leq J(u_0)$. From (4.1) we obtain by using Fatou's Lemma that $\theta_0 = \liminf_{n\to\infty} J(u_n) \geq J(u_0)$. Therefore, u_0 minimizes J on $\mathcal{M} \setminus \{0\}$. Now we have to show $u_0 \in \mathcal{M}^+$. Since u_0 satisfies $\int_\Omega hu_0 > 0$, by Lemma 3.14, there exists $s_+(u_0)$ such that $s_+(u_0)u_0 \in \mathcal{M}^+$. We claim $s_+(u_0) = 1$. Suppose $s_+(u_0) < 1$, then $s_-(u_0) = 1$ and hence $u_0 \in \mathcal{M}^-$. By using Lemma 3.14, we obtain

$$J(s_+(u_0)u_0) < J(u_0) = \theta_0,$$

which is impossible since $s_+(u_0)u_0 \in \mathcal{M} \setminus \{0\}$.

Step 4: u_0 is a local minimum for for J in $W^{2,2}_{\mathcal{N}}(\Omega)$. We see that $s_+(u_0) = 1$, since $u_0 \in \mathcal{M}^+$ (from Step 3). Also from (3.14), we have

$$s_+(u_0) = 1 < s_*(u_0).$$

Now by the continuity of $s_*(u_0)$, for sufficiently small $\delta > 0$, we have

$$1 < s_*(u_0 - w), \quad \forall \|w\|_{W^{2,2}_{\mathcal{N}}(\Omega)} < \delta. \tag{4.5}$$

By Lemma 3.9, for $\delta > 0$ small enough if necessary, there exists $s : \{w \in W^{2,2}_{\mathcal{N}}(\Omega) : \|w\| < \delta\} \to \mathbb{R}$, such that $s(w)(u_0 - w) \in \mathcal{M}$ and $s(0) = 1$. Whenever $s(w) \to 1$ as $\|w\| \to 0$, we have

$$s(w) < s_*(u_0 - w), \ \forall w \in W^{2,2}_{\mathcal{N}}(\Omega) \quad \text{with } \|w\| < \delta.$$

Hence, we obtain $s(w)(u_0 - w) \in \mathcal{M}^+$, using the above inequality and Lemma 3.14. Again by using the Lemma 3.14, we see that

$$J(u_0 - w) \geq J(s(w)(u_0 - w)) \geq J(u_0), \quad \forall s \in [0, s_*(u_0 - w)].$$

Therefore, from (4.5), we observe that $J(u_0 - w) \geq J(u_0)$ for every $\|w\|_{W^{2,2}_{\mathcal{N}}(\Omega)} < \delta$. Consequently, u_0 is a local minimizer.

Step 5: A positive local minimum for J. When $u_0, -\Delta u_0 > 0$, we are done. Otherwise, we obtain positive solution by the following procedure (cf. [3]). Since $-\Delta u_0 \in L^2(\Omega)$ (also we note that $-\Delta u_0 \not\equiv 0$, as $u_0 \in \mathcal{M}^+$), by standard elliptic PDE theory (see e.g [14, Theorem 9.1.4]), the boundary-value problem

$$\begin{gathered} -\Delta v = |-\Delta u_0| \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

has a strong solution in $W^{2,2}_{\mathcal{N}}(\Omega)$. Note that, $\|v\|_{W^{2,2}_{\mathcal{N}}} = \|u_0\|_{W^{2,2}_{\mathcal{N}}}$ and also by maximum principle, we obtain $v > |u_0|$ in Ω. Hence, we obtain

$$\|v\| = \|u_0\|, \quad |v|_{p+1} \geq |u_0|_{p+1}, \quad \int_\Omega hv > 0. \tag{4.6}$$

From (4.6), we have

$$1 = s_+(u_0) \leq s_+(v) \leq s_-(v) \leq s_-(u_0).$$

Then Lemma 3.4 implies

$$J(s_+(v)v) = \min_{s\in[0,s_-(v)]} J(sv) \leq J(v) \leq J(u_0), \tag{4.7}$$

where the first inequality is not strict if and only if $s_+(v) = 1$.

Since $J(s_+(v)v) \geq \theta_0$ and $J(u_0) = \theta_0$, we obtain $s_+(v) = 1$ from (4.7), which implies $v \in \mathcal{M}^+$ and $J(v) = J(u_0) = \theta_0$. Then v is also a local minimum for J. When u_0 does not satisfy $u_0, -\Delta u_0 > 0$ a.e in Ω, we replace u_0 by v. □

5. Existence of a second solution

The existence of a second solution for (1.1), depends on whether we can apply some version of Mountain Pass Lemma. We wish to look for a solution of the form $u_1 = v + u_0$, where u_0 is the local minimum for the functional (2.1). Then, we see that u_1 will solve (1.1), whenever v solves the equation

$$\begin{gathered} \Delta^2 v = f(v+u_0) - f(u_0) \quad \text{in } \Omega, \\ v, -\Delta v > 0 \quad \text{in } \Omega, \\ v = \Delta v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{5.1}$$

We can write the above problem as

$$\begin{gathered} \Delta^2 v = \tilde{f}(x,v) \quad \text{in } \Omega, \\ v, -\Delta v > 0 \quad \text{in } \Omega, \\ v = \Delta v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{5.2}$$

when we define the map $\tilde{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(x,s) = \begin{cases} f(s+u_0(x)) - f(u_0(x)) & \text{if } s \geq 0, \\ 0 & \text{if } s < 0. \end{cases}$$

The energy functional corresponding to (5.2) is $J_{u_0} : W^{2,2}_{\mathcal{N}}(\Omega) \to \mathbb{R}$, defined by

$$J_{u_0}(v) = \frac{1}{2}\int_\Omega |\Delta v|^2 - \int_\Omega \tilde{F}(x,v),$$

where $\tilde{F}(x,s) = \int_0^s \tilde{f}(x,t)dt$. Now onwards, we denote J_{u_0} by J_0. These type of functionals were studied in many articles, for example see [16, 2]. We now state the Generalized Mountain Pass Lemma that was introduced by Ghoussoub and Preiss [6].

Definition 5.1. Let H be a closed subset of the Banach Space $W^{2,2}_{\mathcal{N}}(\Omega)$. We say that a sequence $\{v_n\} \subset W^{2,2}_{\mathcal{N}}(\Omega)$ is a Palais-Smale sequence for J_0 at the level c around H, if

- (i) $\lim_{n\to\infty} \operatorname{dist}(v_n, H) = 0$,
- (ii) $\lim_{n\to\infty} J_0(v_n) = c$,
- (iii) $\lim_{n\to\infty} \|J_0'(v_n)\|_{(W^{2,2}_{\mathcal{N}}(\Omega))^{-1}} = 0$.

In short, we say such a sequence is a $(PS)_{H,c}$ sequence.

Remark 5.2. In case $H = W^{2,2}_{\mathcal{N}}(\Omega)$, the above definition coincides with the usual Palais-Smale sequence at the level c.

Lemma 5.3. *Let $H \subset W^{2,2}_{\mathcal{N}}(\Omega)$ be a closed set, $c \in \mathbb{R}$. Assume, $\{v_n\} \subset W^{2,2}_{\mathcal{N}}(\Omega)$ be a $(PS)_{H,c}$ sequence. Then (up to a subsequence), $v_n \rightharpoonup v_0$ weakly in $W^{2,2}_{\mathcal{N}}(\Omega)$, and*

$$\lim_{n\to\infty}\int_\Omega \tilde{f}(x,v_n) = \int_\Omega \tilde{f}(x,v_0), \quad \lim_{n\to\infty}\int_\Omega \tilde{F}(x,v_n) = \int_\Omega \tilde{F}(x,v_0). \tag{5.3}$$

Proof. From the fact that $\{v_n\}$ is a $(PS)_{H,c}$ sequence, we have

$$\frac{1}{2}\int_\Omega |\Delta v_n|^2 - \int_\Omega \tilde{F}(x,v_n) = c + o_n(1), \tag{5.4}$$

$$\Big|\int_\Omega \Delta v_n \Delta\phi - \int_\Omega \tilde{f}(x,v_n)\phi\Big| \le o_n(1)\|\phi\|_{W^{2,2}_{\mathcal{N}}(\Omega)}, \quad \forall \phi \in W^{2,2}_{\mathcal{N}}(\Omega). \tag{5.5}$$

Now we make the following claim.

Claim: $\sup_n \|v_n\|_{W^{2,2}_{\mathcal{N}}(\Omega)} < \infty$ and $\sup_n \int_\Omega \tilde{f}(x,v_n) < \infty$.

Given any $\epsilon > 0$, there exists $s_\epsilon > 0$, such that

$$\int_\Omega \tilde{F}(x,s) \le \epsilon \int_\Omega s\tilde{f}(x,s), \quad \forall |s| \ge s_\epsilon. \tag{5.6}$$

Using (5.4) and (5.6), we have

$$\begin{aligned}
\frac{1}{2}\int_\Omega |\Delta v_n|^2 &\le \int_{\Omega\cap\{|v_n|\le s_\epsilon\}} \tilde{F}(x,v_n) + \int_{\Omega\cap\{|v_n|\ge s_\epsilon\}} \tilde{F}(x,v_n) + c + o_n(1) \\
&\le \int_{\Omega\cap\{|v_n|\le s_\epsilon\}} \tilde{F}(x,v_n) + \epsilon\int_\Omega \tilde{f}(x,v_n)v_n + c + o_n(1) \\
&\le C_\epsilon + \epsilon\int_\Omega \tilde{f}(x,v_n)v_n.
\end{aligned} \tag{5.7}$$

Now from (5.7) and by substituting $\phi = v_n$ in (5.5), we obtain

$$\begin{aligned}
\int_\Omega \tilde{f}(x,v_n)v_n &\le \int_\Omega |\Delta v_n|^2 + o_n(1)\|v_n\|_{W^{2,2}_{\mathcal{N}}(\Omega)} \\
&\le 2C_\epsilon + 2\epsilon\int_\Omega \tilde{f}(x,v_n)v_n + o_n(1)\|v_n\|_{W^{2,2}_{\mathcal{N}}(\Omega)},
\end{aligned}$$

Hence, by choosing ϵ small enough if needed, we obtain

$$\int_\Omega \tilde{f}(x,v_n)v_n \le \frac{2C_\epsilon}{1-2\epsilon} + o_n(1)\|v_n\|_{W^{2,2}_{\mathcal{N}}(\Omega)}. \tag{5.8}$$

We conclude the claim by using (5.5) and (5.8). Also note that, $\sup_n \int_\Omega \tilde{f}(x,v_n)v_n < \infty$. Since $\{v_n\} \subset W^{2,2}_{\mathcal{N}}(\Omega)$ is bounded, up to a subsequence $v_n \rightharpoonup v_0$ weakly in $W^{2,2}_{\mathcal{N}}(\Omega)$, for some $v_0 \in W^{2,2}_{\mathcal{N}}(\Omega)$.

Let $|A|$ denote the Lebesgue measure of $A \subset \mathbb{R}^4$. Now we set

$$C := \sup_n \int_\Omega |\tilde{f}(x,v_n)v_n|,$$

and notice that $C < \infty$, from the above claim. Given $\epsilon > 0$, we define

$$\mu_\epsilon = \max_{x\in\bar{\Omega}, |s|\le \frac{2C}{\epsilon}} |\tilde{f}(x,s)|.$$

Then, for any $A \subset \Omega$ with $|A| \le \frac{\epsilon}{2\mu_\epsilon}$, we have

$$\begin{aligned}
\int_A |\tilde{f}(x,v_n)| &\le \int_{A\cap\{|v_n|\ge \frac{2C}{\epsilon}\}} \frac{|\tilde{f}(x,v_n)v_n|}{|v_n|} + \int_{A\cap\{|v_n|\le\frac{2C}{\epsilon}\}} |\tilde{f}(x,v_n)| \\
&\le \frac{\epsilon}{2} + \mu_\epsilon|A| \le \epsilon.
\end{aligned}$$

Therefore, $\{\tilde{f}(x,v_n)\}$ is an equi-integrable family in $L^1(\Omega)$ and so is $\{\tilde{F}(x,v_n)\}$ (we note that, $|\tilde{F}(x,t)| \le C_1|t||\tilde{f}(x,t)|$ for all $(x,t)\in\bar{\Omega}\times\mathbb{R}$ and for some $C_1>0$). By applying the Vitali convergence theorem, we complete the proof. □

Certainly, $J_0(0)=0$ and $v=0$ is a local minimum for J_0. Also, we have

$$\lim_{s\to\infty} J_0(sv) = -\infty, \quad \forall v \in W^{2,2}_{\mathcal{N}}(\Omega)\setminus\{0\}.$$

Hence, we can fix $e\in W^{2,2}_{\mathcal{N}}(\Omega)\setminus\{0\}$, such that $J_0(e)<0$. Now, we define the mountain pass level

$$c_0 = \inf_{\gamma\in\Gamma}\sup_{s\in[0,1]} J_0(\gamma(s)),$$

where $\Gamma=\{\gamma\in C([0,1],W^{2,2}_{\mathcal{N}}(\Omega)) : \gamma(0)=0,\gamma(1)=e\}$. Note that, from the definition, we have $c_0\ge 0$. Define, $R_0=\|e\|_{W^{2,2}_{\mathcal{N}}(\Omega)}$. We note that, $\inf\{J_0(v) : \|v\|_{W^{2,2}_{\mathcal{N}}(\Omega)}=R\}=0$ for all $R\in(0,R_0)$. We consider, $H=W^{2,2}_{\mathcal{N}}(\Omega)$ if $c_0>0$ and $H=\{\|v\|_{W^{2,2}_{\mathcal{N}}(\Omega)}=\frac{R_0}{2}\}$ if $c_0=0$. We now present an upper bound for c_0.

Lemma 5.4. *The upper bound c_0 for the Mountain Pass level satisfies*

$$c_0 < 16\pi^2. \tag{5.9}$$

Proof. Without loss of generality, we can assume that the unit ball $B_0(1)\subset\Omega$. For any $\epsilon>0$, we define

$$\tilde{\tau}_n(x) := \begin{cases} \sqrt{\frac{1}{16\pi^2}\log n} + \frac{1}{\sqrt{16\pi^2\log n}}(1-n|x|^2), & \text{if } |x|\in[0,\frac{1}{\sqrt{n}}),\\ -\sqrt{\frac{1}{4\pi^2\log n}}\log|x|, & \text{if } |x|\in[\frac{1}{\sqrt{n}},1),\\ \chi_n(x), & \text{if } |x|\in[1,\infty), \end{cases} \tag{5.10}$$

where

$$\chi_n\in\mathcal{C}^\infty_0(\Omega), \quad \chi_n\big|_{\partial B_1(0)} = \chi_n|_{\partial\Omega}=0.$$

Furthermore, $\Delta\chi_n|_{\partial\Omega}=0$ and $\chi_n,|\nabla\chi_n|,\Delta\chi_n$ are of $O(\frac{1}{\sqrt{2\log n}})$ as $n\to\infty$. Then, $\tilde{\tau}_n\in W^{2,2}_{\mathcal{N}}(\Omega)$. Now we normalize $\tilde{\tau}_n$, by setting

$$\tau_n := \frac{\tilde{\tau}_n}{\|\tilde{\tau}_n\|_{W^{2,2}_{\mathcal{N}}(\Omega)}} \in W^{2,2}_{\mathcal{N}}(\Omega).$$

Suppose (5.9) is not true. This implies for all n, there is $s_n>0$ (see [7]), such that

$$J_0(s_n\tau_n) = \sup_{s>0} J_0(s\tau_n) \ge 16\pi^2, \quad \forall n.$$

Hence

$$\frac{s_n^2}{2} - \int_\Omega \tilde{F}(x,s_n\tau_n) \ge 16\pi^2, \quad \forall n. \tag{5.11}$$

In particular

$$s_n^2 \ge 32\pi^2, \quad \forall n. \tag{5.12}$$

It follows that, $\frac{d}{ds}J_0(s\tau_n)=0$ at the point of maximum $s=s_n$ for J_0, we obtain

$$s_n^2 = \int_\Omega \tilde{f}(x,s_n\tau_n)(s_n\tau_n). \tag{5.13}$$

Now, from the definition of $\tilde{f}$, we have $\inf_{x\in\bar{\Omega}} \tilde{f}(x,s) \geq e^{s^2}$ for $|s|$ large. Then, from (5.12) for sufficiently large n, we obtain

$$\begin{aligned} s_n^2 &\geq \int_{\{|x|\leq \frac{1}{\sqrt{n}}\}} \tilde{f}(x, s_n\tau_n)(s_n\tau_n) \\ &\geq \int_{\{|x|\leq \frac{1}{\sqrt{n}}\}} e^{s_n^2\tau_n^2}(s_n\tau_n) \\ &\geq e^{s_n^2 \frac{\log n}{16\pi^2}} \frac{s_n}{\sqrt{16\pi^2}} \sqrt{\log n}\frac{\pi^2}{2}\frac{1}{n^2} \\ &= \frac{\pi}{8} e^{(\frac{s_n^2}{16\pi^2}-2)\log n} s_n(\log n)^{\frac{1}{2}}. \end{aligned} \tag{5.14}$$

Using (5.12) and (5.14), it follows that s_n is bounded and also $s_n^2 \to 32\pi^2$. Also from (5.14), we have $s_n \geq \frac{\pi}{8}(\log n)^{\frac{1}{2}}$ for all large n, which gives a contradiction. $\square$

We now prove the theorem regarding the existence of a second solution.

Theorem 5.5. *Given a local minimum u_0 of J in $W_{\mathcal{N}}^{2,2}(\Omega)$, there exists an element $v_0 \in W_{\mathcal{N}}^{2,2}(\Omega)$ with $v_0 > 0$ in Ω, such that $J_0'(v_0) = 0$.*

Proof. From Lemma 5.4, we have $c_0 \in [0, 16\pi^2)$. Consider $\{v_n\}$ be a Palais-Smale sequence for J_0 at the level c_0 around H (such a $(PS)_{H,c_0}$ sequence exists [6]). Then, up to a subsequence, $v_n \rightharpoonup v_0$ weakly in $W_{\mathcal{N}}^{2,2}(\Omega)$ for some $v_0 \in W_{\mathcal{N}}^{2,2}(\Omega)$ by Lemma 5.3 and (5.3) holds. We can easily check that, v_0 is a solution of (5.2) and therefore a critical point of J_0. It remains to show that v_0 is not a trivial solution. We prove this by contradiction.

Case I. $c_0 = 0$, $v_0 = 0$. We note that, $H = \{\|v\|_{W_{\mathcal{N}}^{2,2}(\Omega)} = \frac{R_0}{2}\}$ in this case. Also,

$$o_n(1) = J_0(v_n) = \frac{1}{2}\int_\Omega |\Delta v_n|^2 - \int_\Omega \tilde{F}(x, v_n) = \frac{1}{2}\int_\Omega |\Delta v_n|^2 + o_n(1),$$

which contradicts that $\operatorname{dist}(v_n, H) \to 0$ as $n \to \infty$.

Case II. $c_0 \in (0, 16\pi^2)$, $v_0 = 0$. Using the fact that $J_0(v_n) \to c_0$, we see that for given any $\epsilon > 0$, $\|v_n\|^2_{W_{\mathcal{N}}^{2,2}(\Omega)} \leq 32\pi^2 - \epsilon$ for all large n. Let, $0 < \delta < \frac{\epsilon}{32\pi^2}$ and $q = \frac{32\pi^2}{(1+\delta)(32\pi^2-\epsilon)} > 1$. We have

$$\int_\Omega |\tilde{f}(x, v_n)v_n|^q \leq C\int_\Omega e^{((1+\delta)q\|v_n\|^2)(\frac{v_n^2}{\|v_n\|^2})^2},$$

since, $\sup_{x\in\bar{\Omega}} |\tilde{f}(x,s)s| \leq Ce^{(1+\delta)s^2}$ for all $s \in \mathbb{R}$ and for some $C > 0$. Now using Tarsi's embedding (1.5), we obtain $\sup_{x\in\bar{\Omega}}\int_\Omega |\tilde{f}(x, v_n)v_n|^q < \infty$ since $(1+\delta)q\|v_n\|^2 \leq 32\pi^2$. Since $v_n \to 0$ pointwise almost everywhere in Ω, by recalling Vitali convergence theorem, one obtains $\int_\Omega \tilde{f}(x, v_n)v_n \to 0$ as $n \to \infty$. Therefore,

$$\begin{aligned} o_n(1)\|v_n\|_{W_{\mathcal{N}}^{2,2}(\Omega)} = \langle J_0'(v_n), v_n\rangle &= \frac{1}{2}\int_\Omega |\Delta v_n|^2 - \int_\Omega \tilde{f}(x, v_n)v_n \\ &= \frac{1}{2}\int_\Omega |\Delta v_n|^2 + o_n(1), \end{aligned}$$

which contradicts, $\frac{1}{2}\int_\Omega |\Delta v_n|^2 \to c_0$ as $n \to \infty$.

Therefore, $v_0 \not\equiv 0$ in Ω and the positivity of v_0 and $-\Delta v_0$ follows from the fact that, $\tilde{f}(x,s) \geq 0$ for all $(x,s) \in \Omega \times \mathbb{R}$ and using the maximum principle. $\square$

6. Proof of Theorem 1.6

Define, $\lambda_* := \mu C_0^{\frac{p+3}{p+4}} |\Omega|^{-\frac{p+2}{2p+8}}$ where C_0 is same as in the Proposition (3.1). Then, condition (3.1) is true whenever $0 < \lambda < \lambda_*$. From the Theorem 4.2 and 5.5, we show the existence of at least two positive solutions for (1.1). Also, we define

$$\lambda^* := p\mu^{-1/p}\Big(\frac{\lambda_1}{p+1}\Big)^{\frac{p+1}{p}}\Big(\frac{\int_\Omega \phi_1}{\int_\Omega h\phi_1}\Big).$$

We prove, there is no solution of (1.1) when $\lambda > \lambda^*$. Assume, u_λ be a solution of (1.1). Now multiply (1.1) by ϕ_1 and then integrating by parts over Ω, we obtain

$$\int_\Omega \phi_1(\Delta^2 u_\lambda) = \int_\Omega f(u_\lambda)\phi_1 + \lambda\int_\Omega h\phi_1,$$

which implies

$$\lambda\int_\Omega h\phi_1 = \int_\Omega (\lambda_1 u_\lambda - f(u_\lambda))\phi_1. \tag{6.1}$$

We see that, $\lambda_1 t - f(t) \le \lambda_1 t - \mu t^{p+1} = \Theta(t)$ for all $t > 0$. The global maximum for the function Θ is $p\mu^{-1/p}(\frac{\lambda_1}{p+1})^{\frac{p+1}{p}}$ on $(0,\infty)$. Then, from (6.1) and the definition of λ^*, we obtain $\lambda \le \lambda^*$. This completes the proof of Theorem 1.6.

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