EXISTENCE OF BOUNDED SOLUTIONS OF NEUMANN PROBLEM FOR A NONLINEAR DEGENERATE ELLIPTIC EQUATION

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Abstract. We prove the existence of bounded solutions of Neumann problem for nonlinear degenerate elliptic equations of second order in divergence form. We also study some properties as the Phragmén-Lindelöf property and the asymptotic behavior of the solutions of Dirichlet problem associated to our equation in an unbounded domain.

1. Introduction

We consider the equation

\[ \sum_{i=1}^{m} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) - c_0 |u|^{p-2} u = f(x, u, \nabla u) \quad \text{in } \Omega, \]

where \( \Omega \) is a bounded open set of \( \mathbb{R}^m, m \geq 2, \) \( c_0 \) is a positive constant, \( \nabla u \) is the gradient of unknown function \( u \) and \( f \) is a nonlinear function which has the growth of rate \( p, 1 < p < m, \) respect to gradient \( \nabla u. \) We assume that the following degenerate ellipticity condition is satisfied,

\[ \lambda(|u|) \sum_{i=1}^{m} a_i(x, u, \eta) \eta_i \geq \nu(x)|\eta|^p, \]

where \( \eta = (\eta_1, \eta_2, \ldots, \eta_m), \) \( |\eta| \) denotes the modulus of \( \eta, \) \( \nu \) and \( \lambda \) are positive functions with properties to be specified later on.

We study the nonlinear Neumann boundary problem for \( 1.1 \) with the boundary condition

\[ \sum_{i=1}^{m} a_i(x, u, \nabla u) \cos(\overrightarrow{n}, x_i) + c_2 |u|^{p-2} u + F(x, u) = 0 \quad (c_2 > 0, \ x \in \partial \Omega), \]

where \( \partial \Omega \) is locally Lipschitz boundary (see \( [1] \)) and \( \overrightarrow{n} = \overrightarrow{n}(x) \) is the outwardly directed (relative to \( \Omega \)) unit vector normal to \( \partial \Omega \) at every point \( x \in \partial \Omega. \)

Many results for linear and quasilinear elliptic equations of second order have been established starting with pioneering papers \( [13, 16], \) and arriving to the most
recent [2] [7] [20] [21] [22]. For example, in the very recent paper [21] the existence of positive solutions for p-Laplacian, with nonlinear Neumann boundary conditions, is proved by a priori estimates and topological methods.

The Dirichlet problem for the equation of the type (1.1) in nondegenerate case on bounded domains was studied by Boccardo, Murat and Puel in [3, 4], using the method of sub and supersolutions. Afterwards, Drabek and Nicolosi in [8], assuming condition (1.2), studied the weak solvability of general boundary value problem for equation (1.1), obtaining more general results than [3, 4]. Let us also mention, on the related topic and in degenerate-case, [5] [6] and [10] [11].

In this article the basic idea of [8] is used: the question of the existence of solutions is handled by priori estimates, in the energy space corresponding to the given problem and in \( L^\infty \), together with the theory of equations with pseudomonotone operators.

This article is organized as follows. In Section 2 we formulate the hypotheses, we state our problem and the main existence theorem. Section 3 consists of preliminary assertions which are sufficient in the proof of our main results. In Section 4 we prove the existence theorem and we give an example where all our assumptions are satisfied. In Section 5 we study asymptotic behavior of the solution of the Dirichlet problem associated to equation (1.1) in an unbounded domain. Finally, in Section 6 we shall show that a theorem, like the Phragmén-Lindelöf one, holds for Dirichlet problem, in the case of p-Laplacian, in a cylindrical unbounded domain of \( \mathbb{R}^m \); the analogous question for higher-order linear equations was first investigated by P.D. Lax in [14].

2. Hypotheses and formulation of the main results

We shall suppose that \( \mathbb{R}^m \) \((m \geq 2)\) is the \( m \)-dimensional Euclidean space with elements \( x = (x_1, x_2, \ldots, x_m) \). Let \( \Omega \) be an open bounded nonempty subset of \( \mathbb{R}^m \), \( \partial \Omega \) be locally lipschitzian. The symbols \( \text{meas}_m(\cdot) \) and \( \text{meas}(\cdot) \) will denote the \( m \)-dimensional Lebesgue measure and the \((m - 1)\)-dimensional Hausdorff measure, respectively.

We denote by \( L^q(\partial \Omega) \), \((1 \leq q < \infty)\) the Lebesgue space of \( q \)-summable functions on \( \partial \Omega \) with respect to the \((m - 1)\)-dimensional Hausdorff measure, with obvious modifications if \( q = \infty \).

Let \( p \) be a real number such that \( 1 < p < m \). We use, on the weight function \( \nu(x) \), the hypothesis

\( \text{(H1) } \nu : \Omega \to (0, +\infty) \) is a measurable function such that

\[
\nu(x) \in L^1_{\text{loc}}(\Omega), \quad \left( \frac{1}{\nu(x)} \right)^{\frac{1}{p-1}} \in L^1_{\text{loc}}(\Omega).
\]

We shall denote by \( W^{1,p}(\nu, \Omega) \) the set of all real functions \( u \in L^p(\Omega) \) having the weak derivative \( \frac{\partial u}{\partial x_i} \) with the property \( \nu | \frac{\partial u}{\partial x_i} |^p \in L^1(\Omega) \), for \( i = 1, \ldots, m \). \( W^{1,p}(\nu, \Omega) \) is a Banach space respect to the norm

\[
\| u \|_{1,p} = \left[ \int_\Omega (|u|^p + \nu |\nabla u|^p) \, dx \right]^{1/p}.
\]

The space \( \tilde{W}^{1,p}(\nu, \Omega) \) is the closure of \( C^\infty_0(\Omega) \) in \( W^{1,p}(\nu, \Omega) \). Put \( \tilde{W} = W^{1,p}(\nu, \Omega) \cap L^\infty(\Omega) \).
Remark 2.1. There exists a positive number $K_0$ such that for every $u \in W^{1,p}(\nu, \Omega)$ it is also $\min_{\Omega}(u, K) \in W^{1,p}(\nu, \Omega)$ for every $K \geq K_0$. Details concerning this assertion can be found in Nicolosi [19].

Remark 2.2. For every $u \in W$ and for every $\gamma > 0$ it is $u|\nu|^\gamma \in W$. Details concerning this assertion can be found in Guglielmino and Nicolosi [10].

We have also the following hypotheses

(H2) There exists $t > \frac{m}{p-1}$ such that
\[
\frac{1}{\nu(x)} \in L^t(\Omega).
\]

From (H1) and (H2) there is a continuous inclusion $\xi$ of $W^{1,p}(\nu, \Omega)$ in $W^{1,p^*}(\Omega)$, where $\tau = (1 + \frac{1}{t})^{-1}$. So, from Sobolev embedding, if we set
\[
p^* = \frac{mp}{m-p+m/t},
\]
then, we have $W^{1,p}(\nu, \Omega) \subset L^{p^*}(\Omega)$ and there exists $\hat{c} > 0$ depending only on $m, p, t, \Omega$ and $\|1/\nu\|_{L^1(\Omega)}$ such that for every $u \in W^{1,p}(\nu, \Omega)$
\[
\left( \int_\Omega |u|^{p^*} \, dx \right)^{1/p^*} \leq \hat{c}\|u\|_{1,p}.
\]

In this connection see, for instance, [11, 12] and [17, Theorem 3.1].

Next, by the theorem of trace for Sobolev spaces (see for instance [18, Cap. 2, pag.103]), we know that for any $u \in W^{1,p^*}(\Omega)$, there exists a unique element $\gamma_0 u \in L^p(\partial\Omega)$ where
\[
\hat{p} = p\tau(m-1)(m-p\tau)^{-1} = \frac{(m-1)p}{m-p+m/t}
\]
and, the mapping $\gamma_0$ is continuous linear from $W^{1,p^*}(\Omega)$ to $L^\hat{p}(\partial\Omega)$. Obviously, $\gamma_0 \circ \xi$ is a continuous linear map of $W^{1,p}(\nu, \Omega)$ to $L^\hat{p}(\partial\Omega)$ and for $u|_{\partial\Omega} = (\gamma_0 \circ \xi)(u)$, the trace of $u$ on $\partial\Omega$, the following inequality holds:
\[
\left( \int_{\partial\Omega} |u|_{\partial\Omega}^{\hat{p}} \, ds \right)^{1/\hat{p}} \leq c'\|u\|_{1,p}, \quad \text{for all } u \in W^{1,p}(\nu, \Omega),
\]
where $c'$ is a positive constant depending only on $m, p, t, \Omega$ and $\|1/\nu\|_{L^1(\Omega)}$.

When clear from the context, for $u \in W^{1,p}(\nu, \Omega)$, we shall write $u$ instead of $u|_{\partial\Omega}$.

Remark 2.3. Hypotheses (H1) and (H2) imply that $W^{1,p}(\nu, \Omega)$ is compactly embedded in $L^p(\Omega)$. The proof of this assertion is the same as that for $p = 2$ (see [11]). Furthermore, as the linear and continuous map $\gamma_0$ from $W^{1,p^*}(\Omega)$ in $L^\hat{p}(\partial\Omega)$ is compact for every $q$: $1 \leq q < \hat{p}$ (see [18, Cap. 2, pag.103]), then, it is also compact the embedding $\gamma_0 \circ \xi$ of $W^{1,p}(\nu, \Omega)$ in $L^q(\partial\Omega)$. It will be useful to note that $W^{1,p}(\nu, \Omega)$ is reflexive. For the proof of this fact it is possible to use the same procedure as in [11, pag.46].

We need the following structural hypotheses:

(H3) The functions $f(x, u, \eta)$, $a_i(x, u, \eta)$ ($i = 1, 2, \ldots, m$) are Caratheodory functions in $\Omega \times \mathbb{R} \times \mathbb{R}^m$, i.e. measurable with respect to $x$ for every $(u, \eta) \in \mathbb{R} \times \mathbb{R}^m$ and continuous with respect to $(u, \eta)$ for almost all $x \in \Omega$. 

(H4) The function \( F(x, u) \) is a Caratheodory function in \( \partial \Omega \times \mathbb{R} \), i.e. measurable with respect to \( x \) for every \( u \in \mathbb{R} \) and continuous with respect to \( u \) for almost all \( x \in \partial \Omega \).

(H5) There exist a number \( \sigma \) and a function \( f^*(x) \) such that

\[
\max \left( 0, \frac{2 - p}{2} \right) < \sigma < 1, \quad f^* \in L^1(\Omega),
\]

\[
|f(x, u, \eta)| \leq \lambda(|u|) \left[ f^*(x) + |u|^{p-1+\sigma} + \left( \nu^{1/p}(x)\eta \right)^{p-1+\sigma} + \nu(x)|\eta|^p \right] \tag{2.1}
\]

holds for almost all \( x \in \Omega \) and for all real numbers \( u, \eta_1, \eta_2, \ldots, \eta_m \).

(H6) There exists a function \( F^* \in L^\infty(\partial \Omega) \) such that

\[
|F(x, u)| \leq \lambda(|u|) + F^*(x) \tag{2.2}
\]

holds for almost all \( x \in \partial \Omega \) and for every \( u \in \mathbb{R} \).

(H7) There exists a function \( F_0 \in L^\infty(\partial \Omega) \) such that

\[
u |F(x, u)| + F_0(x) \geq 0 \tag{2.3}
\]

holds for almost all \( x \in \partial \Omega \) and for every \( u \in \mathbb{R} \).

(H8) There exist a nonnegative number \( c_1 < c_0 \) and a function \( f_0 \in L^\infty(\Omega) \) such that for almost all \( x \in \Omega \) and for all real numbers \( u, \eta_1, \eta_2, \ldots, \eta_m \),

\[
u u f(x, u, \eta) + c_1 |u|^p + \lambda(|u|) \nu(x)\eta^p + f_0(x) \geq 0. \tag{2.4}
\]

(H9) There exists a function \( a^* \in L^p/(p-1)(\Omega) \) such that for almost all \( x \in \Omega \) and for all real numbers \( u, \eta_1, \eta_2, \ldots, \eta_m \),

\[
\frac{|a_i(x, u, \eta)|}{\nu^{1/p}(x)} \leq \lambda(|u|) \left[ a^*(x) + |u|^{p-1} + \nu^{p-1}(x)\eta|^{p-1} \right]. \tag{2.5}
\]

(H10) Condition \( (1.2) \) is satisfied for almost all \( x \in \Omega \) and for all real numbers \( u, \eta_1, \eta_2, \ldots, \eta_m \); the function \( \lambda : [0, +\infty) \rightarrow [1, +\infty) \) is monotone and nondecreasing.

(H11) For almost all \( x \in \Omega \) and all real numbers \( u, \eta_1, \eta_2, \ldots, \eta_m, \tau_1, \tau_2, \ldots, \tau_m \), the inequality

\[
\sum_{i=1}^{m} \left[ a_i(x, u, \eta) - a_i(x, u, \tau) \right] (\eta_i - \tau_i) \geq 0 \tag{2.6}
\]

holds while the inequality holds if and only if \( \eta \neq \tau \).

In this article we study the problem of finding a function \( u \in W \) such that

\[
\int_{\Omega} \left\{ \sum_{i=1}^{m} a_i(x, u, \nabla u) \frac{\partial w}{\partial x_i} + c_0 |u|^{p-2} uw + f(x, u, \nabla u) w \right\} \, dx
\]

\[
+ \int_{\partial \Omega} \left\{ c_2 |u|^{p-2} uw + F(x, u) w \right\} \, ds = 0 \tag{2.7}
\]

holds for every \( w \in W \). Hypotheses (H1)–(H6)and (H10) provide the correctness for this problem. We shall prove the following result:

**Theorem 2.4.** Let (H1)–(H11) be satisfied. Then (2.7) has at least one solution.
3. Auxiliary results

The first result of this section is an a priori estimate in $L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ for every solution of (2.7).

**Lemma 3.1.** Let (H1)–(H10) be satisfied and let $u$ be a solution of (2.7). Then

$$
\|u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\partial\Omega)} \leq K
$$

(3.1)

where

$$
K = 2 \left( \frac{2}{c_3} \left( \|f_0\|_{L^\infty(\Omega)} + \|F_0\|_{L^\infty(\partial\Omega)} \right) \right)^{1/p}, \quad c_3 = \min(c_2, c_0 - c_1).
$$

**Proof.** Let us take $w = u|u|^\gamma$ as a test function in (2.7) (see Remark 2.2), where $\gamma$ is a positive number. We deduce that

$$
\int_{\Omega} |u|^\gamma \left\{ (\gamma + 1) \sum_{i=1}^{m} a_i(x, u, \nabla u) \frac{\partial u}{\partial x_i} + c_0|u|^p + f(x, u, \nabla u) \right\} \, dx
$$

$$
+ \int_{\partial\Omega} \left\{ c_2|u|^\gamma + F(x, u)|u|^\gamma \right\} \, ds = 0.
$$

By using (H7), (H8) and (H10) we obtain

$$
\int_{\Omega} |u|^\gamma \left\{ \frac{\gamma + 1}{\lambda(\|u\|_{L^\infty(\Omega)})} - \lambda(\|u\|_{L^\infty(\Omega)}) \right\} |\nabla u|^p + (c_0 - c_1)|u|^p - f_0 \right\} \, dx
$$

$$
+ \int_{\partial\Omega} \{c_2|u|^\gamma + F_0|u|^\gamma \} \, ds \leq 0.
$$

Set $\gamma$ such that $\gamma > [\lambda(\|u\|_{L^\infty(\Omega)})]^2 - 1$, from the above inequality it follows that

$$
c_3 \left[ \int_{\Omega} |u|^\gamma \, dx + \int_{\partial\Omega} |u|^\gamma \, ds \right] \leq \int_{\Omega} |f_0| |u|^\gamma \, dx + \int_{\partial\Omega} |F_0| |u|^\gamma \, ds.
$$

Then, by Hölder’s inequality

$$
c_3 \left[ \int_{\Omega} |u|^\gamma \, dx + \int_{\partial\Omega} |u|^\gamma \, ds \right]
$$

$$
\leq \left[ \left( \int_{\Omega} |u|^\gamma \, dx \right)^{\frac{1}{\gamma+p}} + \left( \int_{\partial\Omega} |u|^\gamma \, ds \right)^{\frac{1}{\gamma+p}} \right]
$$

$$
\times \left[ \left( \int_{\Omega} |f_0|^{(\gamma+p)/p} \, dx \right)^{\frac{p}{\gamma+p}} + \left( \int_{\partial\Omega} |F_0|^{(\gamma+p)/p} \, ds \right)^{\frac{p}{\gamma+p}} \right].
$$

The above inequality implies

$$
\left( \int_{\Omega} |u|^\gamma \, dx \right)^{\frac{1}{\gamma+p}} + \left( \int_{\partial\Omega} |u|^\gamma \, ds \right)^{\frac{1}{\gamma+p}}
$$

$$
\leq \frac{2^{\frac{p}{\gamma+p} + 1}}{c_3} \left\{ \|f_0\|_{L^p(\Omega)} (\text{meas}_m \Omega)^{\frac{1}{\gamma+p}} + \|F_0\|_{L^p(\partial\Omega)} (\text{meas} \partial\Omega)^{\frac{1}{\gamma+p}} \right\}
$$

Letting $\gamma \to +\infty$ we obtain (3.1). The proof is complete. \qed

The second result of this Section is an a priori estimate for every solution $u$ of (2.7), in the norm of $W^{1,p}(\nu, \Omega)$. 

Lemma 3.2. Let (H1)–(H10) be satisfied and let $u$ be a solution of (2.7). Then there exists a constant $M > 0$ such that

$$
\|u\|_{1,p} \leq M,
$$

where $M$ depends only on $c_0$, $c_1$, $c_2$, $\sigma$, $p$, $\|f_0\|_{L^\infty(\Omega)}$, $\|f^*\|_{L^1(\Omega)}$, $\lambda(s)$, $\text{meas}_m \Omega$, $\text{meas} \partial \Omega$ and $\|F_0\|_{L^\infty(\partial \Omega)}$.

Proof. We have (see the proof of the Lemma 3.1):

$$
\int_\Omega |u|^\gamma \left\{ \frac{\gamma + 1}{\lambda \|u\|_{L^\infty(\Omega)}} - \lambda(\|u\|_{L^\infty(\Omega)}) \right\} |\nabla u|^p + (c_0 - c_1)|u|^p \, dx \\
+ \int_{\partial \Omega} c_2 |u|^{\gamma + p} \, ds \\
\leq \int_\Omega |F_0| |u|^\gamma \, dx + \int_\Omega |f_0| |u|^\gamma \, dx.
$$

Set $\gamma$ such that $\gamma > \lambda(K)[1 + \lambda(K)] - 1$, where $K$ is the constant defined in previous Lemma. Then, from the last inequality we obtain

$$
\int_\Omega |u|^\gamma |\nabla u|^p + (c_0 - c_1)|u|^p \, dx \leq K \gamma \left( \int_\Omega |f_0| \, dx + \int_{\partial \Omega} |F_0| \, ds \right).
$$

(3.2)

On the other hand if we take $w(x) = u(x)$ as a test function in relation (2.7), we have

$$
\int_\Omega \left\{ \sum_{i=1}^m a_i(x, u, \nabla u) \frac{\partial u}{\partial x_i} + c_0 |u|^p + f(x, u, \nabla u) \right\} \, dx + \int_{\partial \Omega} F(x, u) u \, ds \leq 0.
$$

Applying inequalities (1.2), (2.1), (2.3) and Lemma 3.1 we obtain

$$
\min \left( \frac{1}{\lambda(K)}, c_0 \right) \|u\|_{1,p}^p \\
\leq \lambda(K) \int_\Omega [f^* |u| + |u|^{p+\sigma} + |u|^{(\nu^{1/p}) \right\} |\nabla u|^{p-1+\sigma} + |u| \nu |\nabla u|^\gamma \, dx + \int_{\partial \Omega} |F_0| \, ds.
$$

Then, there exists a constant $K_1$, depending only on $c_0$, $c_1$, $c_2$, $\sigma$, $\lambda(s)$, $\|f_0\|_{L^\infty(\Omega)}$ and $\|F_0\|_{L^\infty(\partial \Omega)}$, such that

$$
\|u\|_{1,p}^p \leq K_1 \int_\Omega [f^* |u| + |u|^{p+\sigma} + |u|^{\tau' |\nabla u|^\gamma} \, dx + \|F_0\|_{L^\infty(\partial \Omega)} \text{meas} \partial \Omega, 
$$

(3.3)

where $\tau' = \frac{\sigma}{\nu^{1/p} + \sigma}$ (see also (3.4)).

We use (3.1), (3.2) to estimate the first term on the right-hand side of previous inequality:

$$
\int_\Omega f^* |u| \, dx \leq \|u\|_{L^\infty(\Omega)} \|f^*\|_{L^1(\Omega)} \leq K \|f^*\|_{L^1(\Omega)}
$$

$$
\int_\Omega |u|^{p+\sigma} \, dx \leq \|u\|_{L^\infty(\Omega)}^{p+\sigma} \text{meas}_m \Omega \leq K^{p+\sigma} \text{meas}_m \Omega,
$$

$$
\int_\Omega |u|^{\tau' |\nabla u|^\gamma} \, dx \leq K \tau' \left( \int_\Omega |f_0| \, dx + \int_{\partial \Omega} |F_0| \, ds \right) \text{ if } \tau' > \lambda(K)[1 + \lambda(K)] - 1.
$$

In the case $\tau' \leq \lambda(K)[1 + \lambda(K)] - 1$, we first apply Young’s inequality to obtain

$$
|u|^{\tau'} \leq \epsilon + C(\epsilon, \tau', \gamma) |u|^\gamma, \quad \gamma > \lambda(K)[1 + \lambda(K)] - 1;
$$

where
hence,
\[
\int_{\Omega} |u|^\gamma \nu |\nabla u|^p \, dx \leq \epsilon \|u\|_{L^p}^p + C(\epsilon, \gamma, \gamma) K^p \left( \int_{\Omega} |f_0| \, dx + \int_{\partial\Omega} |F_0| \, ds \right).
\]

The above inequalities and (3.3) give \(\|u\|_{1,p} \leq M\), where \(M\) depends only on \(c_0, c_1, c_2, p, \sigma, \|f_0\|_{L^\infty(\Omega)}, \|F_0\|_{L^\infty(\partial\Omega)}\), \(\text{meas}_m \Omega, \text{meas} \partial\Omega, \|f^*\|_{L^1(\Omega)}\), \(\lambda(s)\). The proof is complete. \(\square\)

We want to emphasize that the constants \(K\) and \(M\) in previous Lemmas do not depend on \(u\). Moreover, Hypothesis (H2) in such Lemmas is only used for defining the trace of \(u\) on \(\partial\Omega\).

The following lemma will be useful in verifying the assumptions of the Leray-Lions Theorem in the proof of Lemma 3.4.

**Lemma 3.3.** Let (H1), (H3), (H9)–(H11) be satisfied. Let \(u \in W^{1,p}(\nu, \Omega)\) and \(\{u_n\}\) be a sequence in \(W^{1,p}(\nu, \Omega)\) such that there exists a constant \(\Lambda > 0\) for which \(\|u_n\|_{1,p} \leq \Lambda\) and \(\lambda(|u_n(x)|) \leq \Lambda\) for almost all \(x \in \Omega\) and for every \(n = 1, 2, \ldots\). Moreover, let us suppose \(\lim_{n \to \infty} \|u_n - u\|_{L^p(\Omega)} = 0\) and
\[
\lim_{n \to \infty} \int_{\Omega} \sum_{i=1}^{m} a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u) \frac{\partial(u_n - u)}{\partial x_i} \, dx = 0. \tag{3.4}
\]
Then
\[
\lim_{n \to \infty} \int_{\Omega} \nu |\nabla u_n - \nabla u|^p \, dx = 0.
\]

The proof of the above lemma is an easy modification of the proof of [8, Lemma 3.3]. The following Lemma is a direct application of the Leray-Lions Theorem.

**Lemma 3.4.** Assume that \(\lambda(s) \equiv \lambda\), with \(\lambda\) a positive constant. Let us suppose that (H1)–(H4), (H9)–(H11) are satisfied. Let us suppose moreover that for every \(u \in \mathbb{R}\), \((\eta_1, \ldots, \eta_m) \in \mathbb{R}^m\) and for almost all \(x \in \Omega\), it holds
\[
|f(x, u, \eta)| \leq \lambda,
\]
and for almost all \(x \in \partial\Omega\) and for all \(u \in \mathbb{R}\),
\[
|F(x, u)| \leq \lambda.
\]

Then (2.7) has at least one solution.

**Proof.** Let us consider the operator
\[
A(u, v) : W^{1,p}(\nu, \Omega) \times W^{1,p}(\nu, \Omega) \to (W^{1,p}(\nu, \Omega))^*,
\]
defined by
\[
\langle A(u, v), w \rangle = \int_{\Omega} \left\{ \sum_{i=1}^{m} a_i(x, u, \nabla v) \frac{\partial w}{\partial x_i} + c_0 |u|^{p-2} uv + f(x, u, \nabla v) w \right\} \, dx
\]
\[
\quad + \int_{\partial\Omega} [c_2 |u|^{p-2} uv + F(x, u) w] \, ds
\]
for every \(w \in W^{1,p}(\nu, \Omega)\), and the operator \(T : W^{1,p}(\nu, \Omega) \to (W^{1,p}(\nu, \Omega))^*\) defined by
\[
T(u) = A(u, u), \quad u \in W^{1,p}(\nu, \Omega).
\]
Moreover, it is easy to check that the operator $A(u,v)$ is a bounded operator. Moreover, we observe that
\[ \langle A(v,v),v \rangle \geq \min \left( \frac{1}{\lambda}, c_0 \right) \|v\|^p_{1,p} - \lambda \|v\|_{1,p} \left[ (\text{meas}_{\partial \Omega})^{(p-1)/p} + c'(\text{meas} \partial \Omega)^{(p-1)/p} \right]. \]
Hence
\[ \lim_{\|v\|_{1,p} \to +\infty} \frac{\langle T(v),v \rangle}{\|v\|_{1,p}} = +\infty. \]

Now, we shall verify that the operator $A(u,v)$ satisfies the other assumptions of the Leray-Lions Theorem (see [15, Theorem 1]; see, also, [9]):
(i) Continuity and monotony in $v$: from (H11),
\[ (A(u,u) - A(u,v), u-v) \geq 0. \]
Moreover, we observe that
\[ \lim_{n \to +\infty} \langle A(u_n,v_n),w \rangle = \langle A(u,v),w \rangle \quad \text{for every } w \in W^{1,p}(\nu, \Omega), \]
if
\[ (u_n,v_n) \to (u,v) \quad \text{in } W^{1,p}(\nu, \Omega) \times W^{1,p}(\nu, \Omega). \]
For example, we prove that
\[ \lim_{n \to +\infty} \int_{\partial \Omega} |v_n|^{p-2} v_n w \, ds = \int_{\partial \Omega} |v|^{p-2} v w \, ds. \tag{3.5} \]
Now, Hypothesis (H2) implies
\[ \|v_n - v\|_{L^p(\partial \Omega)} \leq c'(\text{meas} \partial \Omega)^{(p-1)/p} \|v_n - v\|_{1,p} \]
then $v_n \to v$ in $L^p(\partial \Omega)$. Let $E$ be an arbitrary measurable subset of $\partial \Omega$. It results
\[ \int_E |v_n|^{p-1} |w| \, ds \leq \int_E |v_n|^p \, ds + \int_E |w|^p \, ds. \]
The strong convergence of $v_n$ to $v$ in $L^p(\partial \Omega)$ implies that $\{ |v_n|^p \}$ are equiintegrable. Then the above inequality together with Hypothesis (H2) imply that $\{ |v_n|^{p-1} |w| \}$ is also an equiintegrable sequence of functions. Hence (3.5) follows from Vitali’s theorem.
(ii) Continuity of $A(u,v)$ with respect to $v$: let $u_n \to u$ in $W^{1,p}(\nu, \Omega)$ and
\[ \lim_{n \to -\infty} \langle A(u_n,u_n) - A(u_n,u), u_n - u \rangle = 0, \]
then, by Lemma [3.3], $u_n \to u$ in $W^{1,p}(\nu, \Omega)$; hence, by previous observation, we have that $A(u_n,v) \to A(u,v)$ in $(W^{1,p}(\nu, \Omega))^*$, for every $v \in W^{1,p}(\nu, \Omega)$.
(iii) Continuity of $\langle A(u,v),w \rangle$ in $u$: we observe that if $v \in W^{1,p}(\nu, \Omega)$, $u_n \to u$ in $W^{1,p}(\nu, \Omega)$ and $A(u_n,v) \to v'$ in $(W^{1,p}(\nu, \Omega))^*$, then $u_n \to u$ in $L^p(\Omega)$, $u_n \to u$ in $L^p(\partial \Omega)$, hence
\[ \lim_{n \to -\infty} \langle A(u_n,v), u_n - u \rangle = 0 \]
and, therefore, $\langle A(u_n,v), u_n \rangle \to \langle v', u \rangle$ (see, also, [11, note (15)], where the special case $p = 2$ is treated, but for Dirichlet problem, and, Remark 2.3).

Thus, all the assumptions of the Leray-Lions theorem (Hypothesis II) are satisfied. Hence the equation $Tu = 0$ has at least one solution $u \in W^{1,p}(\nu, \Omega)$.

We shall prove that $u \in L^\infty(\Omega) \cap L^\infty(\partial \Omega)$. We set:
\[ \Omega_k = \{ x \in \Omega : u > k \}, \quad \partial \Omega_k = \{ x \in \partial \Omega : u > k \}. \]
From (2.7), choosing \( w = u - \min(u, k) \), \( k > K_0 \) (for \( K_0 \) see Remark 2.1), we have

\[
\int_{\Omega_k} \left\{ \sum_{i=1}^m a_i(x, w + k, \nabla w) \frac{\partial w}{\partial x_i} + c_0|w + k|^{p-1}w + f(x, w + k, \nabla w) \right\} \, dx
\]

\[
+ \int_{\partial\Omega_k} \{c_2|w + k|^{p-1}w + F(x, w + k)\} \, ds = 0.
\]

Applying condition (1.2) we obtain

\[
\min \left( \frac{1}{\lambda}, c_0 \right) \|u\|_{1,p}^p \leq \lambda \int_{\Omega} w \, dx + \lambda \int_{\partial\Omega_k} w \, ds.
\]

The above inequality and (H4) imply

\[
\|u\|_{1,p}^p \leq \frac{\lambda(\text{meas}_m \Omega)|\theta - \rho|/\rho \dot{\theta} + c'| \left( \text{meas}_m \Omega_k \right)^{(\dot{\theta} - 1)/\rho} + \left( \text{meas} \partial\Omega_k \right)^{(\dot{\theta} - 1)/\rho}}{\min \left( \frac{1}{\lambda}, c_0 \right)}.
\]

For \( h > k \) we have

\[
\left( \int_{\Omega} |w|^{\tilde{p}} \, dx \right)^{\frac{\tilde{p}-1}{\tilde{p}}} + \left( \int_{\partial\Omega} |w|^{\tilde{p}} \, ds \right)^{\frac{\tilde{p}-1}{\tilde{p}}}
\]

\[
\geq (h - k)^{p-1} \left( \text{meas}_m \Omega_k \right)^{(\dot{\theta} - 1)/\rho} + \left( \text{meas} \partial\Omega_k \right)^{(\dot{\theta} - 1)/\rho}.
\]

For \( h > 0 \), denote

\[
\varphi(h) = \{\text{meas}_m \Omega_h + \text{meas} \partial\Omega_h\}.
\]

We have

\[
\varphi(h) \leq \frac{\alpha}{(h - k)^{\dot{\theta}} \varphi(k)^{\frac{1}{\rho-1}}}, \text{ if } h > K_0.
\]

where the positive constant \( \alpha \) depends only on \( \hat{c}, \hat{c}', c_0, \lambda, m, p, t, \Omega \).

Note that \( \frac{\tilde{p}-1}{\rho} > 1 \), then it follows from a lemma of Stampacchia [17, Lemma 3.11] that \( \text{ess sup}_\Omega u + \text{ess sup}_{\partial\Omega} u < +\infty \). By this way also \( \text{ess sup}_\Omega (-u) + \text{ess sup}_{\partial\Omega} (-u) < +\infty \). Hence \( u \in L^\infty(\Omega) \cap L^\infty(\partial\Omega) \).

\section*{4. Proof of Theorem 2.4}

\textbf{Proof.} Let \( K \) be the constant defined in Lemma 3.1. We define

\[
A_i(x, u, \eta) = \begin{cases}
  a_i(x, -K, \eta) & \text{if } u < -K \\
  a_i(x, u, \eta) & \text{if } |u| \leq K \\
  a_i(x, K, \eta) & \text{if } u > K,
\end{cases}
\]

in \( \Omega \times \mathbb{R} \times \mathbb{R}^m \). For every positive integer \( n \) we define:

\[
f_n(x, u, \eta) = \begin{cases}
  f(x, u, \eta) & \text{if } |f| \leq n \\
  \frac{f(x, u, \eta)}{|f(x, u, \eta)|} & \text{if } |f| > n
\end{cases}
\]

in \( \Omega \times \mathbb{R} \times \mathbb{R}^m \),

\[
F_n(x, u) = \begin{cases}
  F(x, u) & \text{if } |F| \leq n \\
  \frac{F(x, u)}{|F(x, u)|} & \text{if } |F| > n
\end{cases}
\]

in \( \partial\Omega \times \mathbb{R} \).

The functions \( A_i(x, u, \eta), f_n(x, u, \eta), F_n(x, u) \), satisfy (H3)–(H11). It is sufficient to note, for example, that in \( \Omega \times \mathbb{R} \times \mathbb{R}^m \),

\[
|f_n(x, u, \eta)| \leq |f(x, u, \eta)|,
\]
and, that (H8) holds with \(|f_0(x)|\) instead of \(f_0(x)\). Analogous considerations verify the others assumptions. On the other hand, for every \(u \in \mathbb{R}, \ (\eta_1, \ldots, \eta_n) \in \mathbb{R}^m\) and for almost all \(x \in \Omega\) it holds that
\[
|f_n(x,u,\eta)| \leq n,
\]
and for almost all \(x \in \partial\Omega\) and for all \(u \in \mathbb{R},\)
\[
|F_n(x,u)| \leq n.
\]
Then, it follows from Lemma 3.4 that, for every \(n \in \mathbb{N}\), there exists \(u_n \in W\) such that
\[
\int_\Omega \left[ \sum_{i=1}^m a_i(x,u_n,\nabla u_n) \frac{\partial w}{\partial x_i} + c_0|u_n|^{p-2}u_n w + f_n(x,u_n,\nabla u_n)w \right] dx
\]
\[
+ \int_{\partial\Omega} [c_2|u_n|^{p-2}u_n w + F_n(x,u_n)w] ds = 0
\]
for every \(w \in W\). An a priori estimate of Lemma 3.1 yields
\[
\|u_n\|_{L^\infty(\Omega)} + \|u_n\|_{L^\infty(\partial\Omega)} \leq K, \quad \text{for every } n \in \mathbb{N},
\]
and hence (4.1) can be written in the equivalent form
\[
\int_\Omega \left[ \sum_{i=1}^m a_i(x,u_n,\nabla u_n) \frac{\partial w}{\partial x_i} + c_0|u_n|^{p-2}u_n w + f_n(x,u_n,\nabla u_n)w \right] dx
\]
\[
+ \int_{\partial\Omega} [c_2|u_n|^{p-2}u_n w + F_n(x,u_n)w] ds = 0.
\]
It follows from Lemma 3.2 that for every \(n \in \mathbb{N}\),
\[
\|u_n\|_{L^p} \leq M.
\]
On the basis of (4.2) and (4.4) there exists a subsequence of \(\{u_n\}\) (denoted again by \(\{u_n\}\)) such that \(\{u_n\}\) converges weakly to \(u\) in \(W^{1,p}(\nu,\Omega)\) and \(\{u_n\}\) converges weakly* in \(L^\infty(\Omega)\) and in \(L^\infty(\partial\Omega)\) where \(u \in W\) and \(\|u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\partial\Omega)} \leq K\).

We shall prove that \(u \in W\) is the solution of (2.7).

To pass to the limit in (4.3) for \(n \to +\infty\) we have to prove that
\[
\lim_{n \to +\infty} \int_\Omega \nu|\nabla u_n - \nabla u|^p dx = 0.
\]
Now, the compact embedding of \(W^{1,p}(\nu,\Omega)\) in \(L^p(\Omega)\) implies the strong convergence of \(u_n\) to \(u\) in \(L^p(\Omega)\) and hence also almost everywhere in \(\partial\Omega\) (see Remark 2.3). Then, taking into account Lemma 3.3 to get (4.5) it will be sufficient to prove that (3.4) it holds.

Let us take \(w = |u_n - u|^{\gamma}(u_n - u)\) as a test function in (4.3) where \(\gamma\) is a positive number. We deduce
\[
\int_\Omega \left\{ \sum_{i=1}^m a_i(x,u_n,\nabla u_n)(\gamma + 1)|u_n - u|^{\gamma} \frac{\partial(u_n - u)}{\partial x_i} + c_0|u_n|^{p-2}u_n |u_n - u|^\gamma(u_n - u) + f_n(x,u_n,\nabla u_n)|u_n - u|^\gamma(u_n - u) \right\} dx
\]
\[
+ \int_{\partial\Omega} \left\{ c_2|u_n|^{p-2}u_n |u_n - u|^\gamma(u_n - u) + F_n(x,u_n)|u_n - u|^\gamma(u_n - u) \right\} ds
\]
\[
= 0.
\]
From the above inequality, taking into account (1.2), (2.1), (2.2), (4.2), we obtain
\[
\int_\Omega |u_n - u|^\gamma |\nabla u_n|^{p \nu} \, dx
\leq \int_\Omega \sum_{i=1}^m a_i(x, u_n, \nabla u_n)(\gamma + 1)|u_n - u|^\gamma \frac{\partial u}{\partial x_i} dx
\]
\[
+ c_0 K^{p-1} \int_\Omega |u_n - u|^{\gamma + 1} dx + 2K\lambda(K) \int_\Omega [|f^*| + K^{p-1+\sigma} + 1]|u_n - u|^{\gamma} dx
\]
\[
+ c_2 \int_{\partial \Omega} |u_n|^{p-1} |u_n - u|^{\gamma + 1} ds + \int_{\partial \Omega} [|F^*| + \lambda(K)]|u_n - u|^{\gamma + 1} ds,
\]
where \( \gamma \) is such that \( \frac{\gamma + 1}{\lambda(K)} - 4K\lambda(K) > 1 \).

By Lebesgue theorem, the first three addends in the right hand side of previous inequality go to 0 as \( n \to +\infty \) (see, [8] Lemma 3.4, pp. 229-230). We prove, for example, that
\[
\lim_{n \to +\infty} \int_{\partial \Omega} [|F^*| + \lambda(K)]|u_n - u|^{\gamma + 1} ds = 0,
\]
this integral is absent in [8]. It results that a.e. \( x \in \partial \Omega, \)
\[
||F^* + \lambda(K)||u_n - u|^{\gamma + 1} \leq (2K)^{\gamma + 1} ||F^* + \lambda(K)|| \in L^1(\partial \Omega).
\]
As \( u_n \to u \) a.e. in \( \partial \Omega \), it will be enough to apply Lebesgue theorem again. Then, it follows
\[
\lim_{n \to +\infty} \int_\Omega |u_n - u|^{\gamma} |\nabla u_n|^{p \nu} \, dx = 0,
\]
and, so, applying Hölder inequality
\[
\lim_{n \to +\infty} \int_\Omega |u_n - u||\nabla u_n|^{p \nu} \, dx = 0. \tag{4.6}
\]
By (4.3) we obtain
\[
\int_\Omega \sum_{i=1}^m \left[ a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u) \right] \frac{\partial (u_n - u)}{\partial x_i} \, dx
\]
\[
= - \int_\Omega c_0 |u_n|^{p-2} u_n (u_n - u) \, dx - \int_\Omega f_n(x, u_n, \nabla u_n)(u_n - u) \, dx
\]
\[
- \int_\Omega \sum_{i=1}^m \left[ a_i(x, u, \nabla u) - a_i(x, u_n, \nabla u) \right] \frac{\partial (u_n - u)}{\partial x_i} \, dx
\]
\[
+ \int_\Omega \sum_{i=1}^m [a_i(x, u, \nabla u) - a_i(x, u_n, \nabla u)] \frac{\partial (u_n - u)}{\partial x_i} \, dx
\]
\[
- \int_{\partial \Omega} c_2 |u_n|^{p-2} u_n (u_n - u) \, ds - \int_{\partial \Omega} F_n(x, u_n)(u_n - u) \, ds.
\]
Now, all addends in the right-hand side of previous inequality go to 0 as \( n \to +\infty \).
For example, we shall estimate the second and the last addend. We have
\[
\int_\Omega |f_n(x, u_n, \nabla u_n)||u_n - u| \, dx
\]
\[
\leq \lambda(K) \int_\Omega [K^{p-1+\sigma} + 1 + |f^*||u_n - u| \, dx + 2\lambda(K) \int_\Omega |u_n - u||\nabla u_n|^{p \nu} \, dx.
\]
From the Lebesgue theorem and (4.6), the above inequality implies
\[
\lim_{n \to +\infty} \int_{\Omega} f_n(x, u_n, \nabla u_n) (u_n - u) \, dx = 0.
\]
Next
\[
\int_{\partial \Omega} |F_n(x, u_n)| |u_n - u| \, ds \leq [\lambda(K) + \|F^*\|_{L^\infty(\partial \Omega)}] \int_{\partial \Omega} |u_n - u| \, ds.
\]
Taking into account that the imbedding of \(W^{1,p}(\Omega)\) in \(L^1(\partial \Omega)\) is compact (see Remark 2.3), the above inequality implies
\[
\lim_{n \to +\infty} \int_{\partial \Omega} F_n(x, u_n) (u_n - u) \, dx = 0.
\]
For details concerning others passage to the limit see [8, pag. 228]. Consequently
\[
\int \sum_{i=1}^m (a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u)) \frac{\partial(u_n - u)}{\partial x_i} \, dx
\]
tends to zero as \(n \to +\infty\). So, \(u_n \to u\) in \(W^{1,p}(\nu, \Omega)\).

Now, to prove that the function \(u \in W\) is the solution of (2.7) it is sufficient to pass to the limit as \(n \to \infty\). For example, we prove that
\[
\lim_{n \to +\infty} \int_{\partial \Omega} F_n(x, u_n) w \, ds = \int_{\partial \Omega} F(x, u) w \, ds \tag{4.7}
\]
for every \(w \in W\).

We fix \(\epsilon > 0\) and a point \(x_0 \in \partial \Omega\) such that \(u_n(x_0) \to u(x_0)\) as \(n \to +\infty\) and the function \(F(x_0, u)\) is continuous with respect \(u\). Then there is a number \(n_\epsilon \in \mathbb{N}\) such that for any \(n > n_\epsilon\),
\[
-n < F(x_0, u(x_0)) - \epsilon < F(x_0, u_n(x_0)) < \epsilon + F(x_0, u(x_0)) < n.
\]
These inequalities and the definition of the function \(F_n(x, u)\) imply that for any \(n > n_\epsilon\), \(F_n(x_0, u_n(x_0)) = F(x_0, u_n(x_0))\) and
\[
|F_n(x_0, u_n(x_0)) - F(x_0, u(x_0))| < \epsilon.
\]
In this way \(F_n(x, u_n(x)) \to F(x, u(x))\) a.e. on \(\partial \Omega\). Next, from definition of \(F_n(x, u)\) and (2.2) we have
\[
|F_n(x, u_n(x)) w(x)| \leq [\lambda(K) + \|F^*\|_{L^\infty(\partial \Omega)}] |w(x)|
\]
a.e. \(x \in \partial \Omega\). Now, a new application of the Lebesgue theorem gives (4.7). The proof is complete.

Now, we show an example where all assumptions are satisfied. Let \(\Omega\) be a bounded open set of \(\mathbb{R}^m\) such that \(0 \in \partial \Omega\). Put
\[
\nu(x) = |x|^\gamma \quad \text{for} \ 0 < \gamma < p - 1.
\]
Then the function \(\nu\) satisfies Hypotheses (H1) and (H2) with \(t\) such that
\[
\frac{m}{p-1} < t < \frac{m}{\gamma}.
\]
Consider the boundary-value problem
\[
-\nabla \left( \frac{|x|^\gamma}{1 + |u|^p} |\nabla u|^{p-2} \nabla u \right) + e^u - |u|^p + |x|^\gamma |\nabla u|^p = g(x) \quad \text{in} \ \Omega, \tag{4.8}
\]
\[
\frac{|x|^\gamma}{1 + |u|^p} |\nabla u|^{p-2} \sum_{i=1}^{m} \partial u \cos(x_i, x_i) + \frac{1}{e} u |u|^{p-2} + \frac{e^{u-1}}{2} = 0 \quad \text{on } \partial \Omega,
\]

where \(g(x) \in L^\infty(\Omega)\). In this case we have:

\[
a_i(x, u, \nabla u) = \frac{|x|^\gamma}{1 + |u|^p} |\nabla u|^{p-2} \partial_i u, \quad i = 1, 2, \ldots, m;
\]

\[
f(x, u, \nabla u) = e^u - |u|^p - u |u|^{p-2} + |x|^\gamma |\nabla u|^p - g(x), \quad c_0 = 1;
\]

\[
F(x, u) = \frac{1}{2e} u |u|^{p-2} + \frac{e^{u-1}}{2}; \quad c_2 = \frac{1}{2e}.
\]

If we put \(\lambda(\|u\|) = e^{\|u\|^p}\), it is possible to verify all the Hypotheses (H3)–(H11). To verify (H3), for example, it will be sufficient to note that the function \((|u|^p + e^{u^p})\) has minimum \((\leq 0)\) in \((-\infty, +\infty)\).

Hence, BVP (4.8), (4.9) has at least one weak solution in the sense (2.7), i.e. there exists at least one \(u \in W\) such that

\[
\int_\Omega \frac{|x|^\gamma}{1 + |u|^p} |\nabla u|^{p-2} \nabla u \nabla w \, dx + \int_\Omega \left[ e^u - |u|^p + |x|^\gamma |\nabla u|^p \right] w \, dx
\]

\[
+ \int_{\partial \Omega} \left\{ \frac{1}{e} u |u|^{p-2} + \frac{e^{u-1}}{2} \right\} w \, ds
\]

\[
= \int_\Omega g w \, dx
\]

holds for every \(w \in W\).

Examples concerning the Dirichlet problem related to (1.1) can be found in \(\S\) Section 6).

5. Asymptotic behavior near infinity of solutions to the Dirichlet problem for (1.1)

Let \(\Omega = \{ x \in \mathbb{R}^m : |x| > r \}, \ r \) be a positive constant. For \(n \in \mathbb{N}\), we denote

\[
\Omega_n = \Omega \cap \{ x \in \mathbb{R}^m : |x| < n \}.
\]

We introduce the hypothesis

(H12) The function \(\nu = \nu(x) : \Omega \to (0, +\infty)\) is a measurable function such that

\(\nu \in L^\infty(\Omega)\). For every \(n \in \mathbb{N}\), there exists a real number \(\delta_n > \max(\frac{m}{p}, \frac{1}{p-1})\)

such that \(1/\nu \in L^{1n}(\Omega_n)\).

We set

\[
L^1(\Omega) + L^{p/(p-1)}(\Omega) = \{ f_1(x) + f_2(x) : f_1 \in L^1(\Omega), f_2 \in L^{p/(p-1)}(\Omega) \}.
\]

Let (H3), (H5), (H8)–(H12) be satisfied with \(f_0 \in L^1(\Omega) \cap L^\infty(\Omega), \ f^* \in L^1(\Omega) + L^{p/(p-1)}(\Omega)\) and let \(u \in W^{1,p}(\nu, \Omega) \cap L^\infty(\Omega)\) such that

\[
\int_\Omega \left\{ \sum_{i=1}^{m} a_i(x, u, \nabla u) \frac{\partial w}{\partial x_i} + c_0 u |u|^{p-2} u w + f(x, u, \nabla u) w \right\} \, dx = 0
\]

for every \(w \in W^{1,p}(\nu, \Omega) \cap L^\infty(\Omega)\). The function \(u\) exists because of \(\S\) Theorem 2.2.

Moreover, let us suppose that real numbers $L$, $\tau$, $\gamma$ and $\delta_1$ positive constants. Let us consider $u \in \dot{W}^1^p (\nu, \Omega) \cap L^\infty (\Omega)$ that satisfies (5.1) for every $w \in \dot{W}^1^p (\nu, \Omega) \cap L^\infty (\Omega)$. Then

$$\int_{|x| \geq \lambda} |u|^p dx \leq C e^{-\delta_1 \lambda}$$

(5.3)

for every $\lambda \geq r$, where $\delta_3$ and $C$ are positive constants depending on known parameters.

**Proof.** Let us define in $\mathbb{R}^m$ a Lipschitzian function $\theta(x)$, $0 \leq \theta(x) \leq 1$, such that $\theta(x) = 0$ if $0 < |x| < r + 1$, $\theta(x) = 1$ if $|x| > r + 2$. Define in $\mathbb{R}^m$ the function $\vartheta_R(x)$, $0 \leq \vartheta_R(x) \leq 1$, such that $\vartheta_R(x) = 1$ if $|x| < R$, $\vartheta_R(x) = 0$ if $|x| > R + 1$, and let $\vartheta_R(x)$ be a Lipschitzian function.

Take in (5.1) as a test function $w = u |\gamma e^\gamma x| \vartheta_R$ where $\gamma(x) = \beta |x|$ if $|x| < L$, $\gamma(x) = \beta L$ for $|x| > L$ and the positive constants $\gamma, \beta$ will be stated later on. Moreover, let us suppose that real numbers $L, R$ are such that $r + 2 < L < R$.

After easy computations, by (1.2) and (2.4), we obtain

$$\int_{\mathbb{R}^m} e^{\gamma \tau(x)} |u|^\gamma \vartheta_R \left\{ \frac{\gamma + 1}{\lambda (\|u\|_{L^\infty (\Omega)})} \lambda (\|u\|_{L^\infty (\Omega)}) \right\} \nu |\nabla u|^p dx$$

$$+ (c_0 - c_1) |u|^p \right\} dx$$

$$\leq \gamma \int_{\mathbb{R}^m} \sum_{i=1}^m |a_i(x, u, \nabla u) | \frac{\partial \tau(x)}{\partial x_i} |u|^\gamma e^{\gamma \tau(x)} \vartheta_R dx$$

$$+ \int_{\mathbb{R}^m} \sum_{i=1}^m |a_i(x, u, \nabla u)| |u|^\gamma e^{\gamma \tau(x)} |\nabla \vartheta_R| dx$$

(5.4)

$$+ \int_{\mathbb{R}^m} \sum_{i=1}^m |a_i(x, u, \nabla u)| |u|^\gamma e^{\gamma \tau(x)} |\vartheta_R| dx$$

$$+ \int_{\mathbb{R}^m} e^{\gamma \tau(x)} |f_0||u|^\gamma \vartheta_R dx.$$}

Now, we choose $\gamma$ in such that

$$\frac{\gamma + 1}{\lambda (\|u\|_{L^\infty (\Omega)})} \lambda (\|u\|_{L^\infty (\Omega)}) = 2.$$ 

Then, we can estimate from below the left-hand side of (5.4) by

$$2 \int_{r + 2 < |x| < L} e^{\gamma |x|} |u|^\gamma |\nabla u|^p dx + (c_0 - c_1) \int_{r + 2 < |x| < L} e^{\gamma |x|} |u|^p dx. \ (5.5)$$
Next, we shall estimate every addend of right hand side of (5.4). By (2.5), (5.2) and the definitions of \( \tau(x), \theta(x), \theta_R(x) \), it results that

\[
\gamma \int_{\mathbb{R}^m} \sum_{i=1}^{m} |a_i(x, u, \nabla u)| \left| \frac{\partial \tau(x)}{\partial x_i} \right| |u|^{\gamma+1} e^{\gamma \tau(x)} \theta \theta_R \, dx \\
\leq \gamma \beta \int_{|x|<L} e^{\gamma \beta |x|} \sum_{i=1}^{m} |a_i(x, u, \nabla u)||u|^{\gamma+1} \, dx \\
\leq \gamma \beta d_1 \left[ \int_{\mathbb{R}^m} |a^*(x)| e^{\gamma \beta |x|} \, dx + e^{\gamma \beta (r+2)} \right] \\
+ 2m \gamma \beta \lambda(\|u\|_{L^\infty(\Omega)}) \|\nu\|_{L^\infty(\Omega)}^{1/p} \int_{r+2<|x|<L} e^{\gamma \beta |x|} |u|^{\gamma+p} \, dx \\
+ m \gamma \beta \lambda(\|u\|_{L^\infty(\Omega)}) \|\nu\|_{L^\infty(\Omega)}^{1/p} \int_{r+2<|x|<L} e^{\gamma \beta |x|} |u|^{\gamma+p} \, dx;
\]

\[
\int_{\mathbb{R}^m} \sum_{i=1}^{m} |a_i(x, u, \nabla u)||u|^{\gamma+1} e^{\gamma \tau(x)} |\nabla \theta \theta_R \, dx \\
\leq e^{\gamma \beta (r+2)} \int_{|x|<r+2} \sum_{i=1}^{m} |a_i(x, u, \nabla u)||u|^{\gamma+1} |\nabla \theta| \, dx \\
\leq d_2 e^{\gamma \beta (r+2)} \lambda(\|u\|_{L^\infty(\Omega)}) \|\nu\|_{L^\infty(\Omega)}^{1/p} \int_{r+2<|x|<L} |u|^{\gamma+1} \, dx \\
\times \int_{|x|<r+2} [a^*(x) \nu^{1/p} + |u|^{p-1} \nu^{1/p} + \nu |\nabla u|^{p-1}] \, dx \\
\leq d_3 e^{\gamma \beta (r+2)} ;
\]

\[
\int_{\mathbb{R}^m} \sum_{i=1}^{m} |a_i(x, u, \nabla u)||u|^{\gamma+1} e^{\gamma \tau(x)} |\nabla \theta \theta_R | \, dx \\
\leq e^{\gamma \beta L} \int_{0<|x|<R} \sum_{i=1}^{m} |a_i(x, u, \nabla u)||u|^{\gamma+1} |\nabla \theta \theta_R | \, dx \\
\leq d_4 e^{\gamma \beta L} \lambda(\|u\|_{L^\infty(\Omega)}) \|\nu\|_{L^\infty(\Omega)}^{1/p} \left( \|u\|_{L^\infty(\Omega)}^{\gamma+1} + 1 \right) \\
\times \left[ \int_{0<|x|<R} [a^*(x) \nu^{-1/p} + |u|^{p} + \nu |\nabla u|^{p}] \, dx \right] ;
\]

\[
\int_{\mathbb{R}^m} e^{\gamma \tau(x)} |f_0| \|u\|^{\gamma \theta \theta_R} \, dx \leq \tilde{c} \|u\|_{L^\infty(\Omega)}^{\gamma} \int_{\mathbb{R}^m} e^{(\gamma \beta - \delta_1)|x|} \, dx,
\]

where the constants \( d_i (i = 1, 2, 3, 4) \) are positive and depend only on \( m, p, \lambda(s), \|u\|_{L^\infty(\Omega)}, \|u\|_{L^1(\Omega)}, \|\nu\|_{L^\infty(\Omega)}, \|a^*\|_{L^1(\Omega)} \) and \( r \).

From (5.4), estimates (5.5)–(5.9), letting \( R \to +\infty \), we obtain

\[
2 \int_{r+2<|x|<L} e^{\gamma \beta |x|} |u|^{\gamma+p} \, dx + (c_0 - c_1) \int_{r+2<|x|<L} e^{\gamma \beta |x|} |u|^{\gamma+p} \, dx \\
\leq d_5 e^{\gamma \beta (r+2)} + \gamma \beta d_2 \left[ \tilde{c} \int_{\mathbb{R}^m} e^{(\gamma \beta - \delta_1)|x|} \, dx + e^{\gamma \beta (r+2)} \right] \\
+ 2m \gamma \beta \lambda(\|u\|_{L^\infty(\Omega)}) \|\nu\|_{L^\infty(\Omega)}^{1/p} \int_{r+2<|x|<L} e^{\gamma \beta |x|} |u|^{\gamma+p} \, dx
\]
where $M$ and for every real numbers $L > r + 2$, $\beta > 0$; where $\gamma$ is a fixed real number, $\gamma > 2$.

Fix $\beta$ such that

$$0 < \beta < \min\left(\frac{\delta_1}{\gamma}, \frac{c_0 - c_1}{2m\gamma\lambda(||u||_{L^\infty(\Omega)})\|\nu\|_{L^\infty(\Omega)}^{1/p}}, \frac{2}{m\gamma\lambda(||u||_{L^\infty(\Omega)})\|\nu\|_{L^\infty(\Omega)}^{1/p}}\right).$$

Then, for every $L > r + 2$, we obtain

$$\int_{r + 2 < |x| < L} e^{\gamma|\nu|} |u|^{\gamma + p} dx \leq M$$

where $M$ depends only on $m$, $p$, $r$, $\beta$, $\gamma$, $c_0$, $c_1$, $\tilde{c}$, $\lambda(s)$, $||u||_{L^\infty(\Omega)}$, $||u||_{1,p}$, $||\nu||_{L^\infty(\Omega)}$ and $\delta_1$. Letting $L \to +\infty$, the above inequality implies

$$\int_{|x| > r} e^{\gamma\delta_2} |u|^{\gamma + p} dx \leq M_1$$

where $\delta_2 = \gamma \beta$ and $M_1 = e^{\gamma\beta(r + 2)||u||_{L^\infty(\Omega)}}\text{meas}_m(r < |x| < r + 2) + M$. Hence (5.3) follows from (5.10). The proof is complete. \(\square\)

We give an example where Hypothesis (H12) is satisfied. Let $\Omega = \{x \in \mathbb{R}^m : |x| > 1\}$. We consider the function $\nu : \Omega \to (0, +\infty)$ defined by

$$\nu(x) = \left((|x| - 1)e^{-(|x| - 1)}\right)^\gamma, \quad \gamma \in (0, (p - 1)/m).$$

Then

$$\nu(x) \leq \left(\frac{1}{e}\right)^\gamma, \quad x \in \Omega.$$ 

For every integer $n \geq 2$, we set $\Omega_n = \{x \in \mathbb{R}^m : 1 < |x| < n\}$. Then, the function $1/\nu(x) \in L^{\infty}(\Omega_n)$ for every $\delta_n$ satisfying $m/(p - 1) < \delta_n < 1/\gamma$.

6. **Phragmén-Lindelöf theorem**

Now, we shall consider weak solutions of (1.1) for the Dirichlet problem, with $p$-Laplacian, in a cylindrical unbounded domain.

Let $0 \leq a < b \leq +\infty$ and define the set

$$\pi_{a,b} = \{x \in \mathbb{R}^m : x' \in \Omega', a < x_m < b\},$$

where $x' = (x_1, \ldots, x_{m-1})$, $\Omega'$ is a bounded domain in $\mathbb{R}^{m-1}$, $m \geq 3$, with a smooth boundary $\partial \Omega'$; $\pi_a = \pi_{a,\infty}$. Let $p$ be a real number such that $1 < p < m - 1$.

For the next theorem, we need the following hypotheses:

(H13) Let $\tilde{\nu} = \nu(x') : \Omega' \to (0, +\infty)$ be a measurable such that

$$\tilde{\nu} \in L^\infty(\Omega'), \quad \left(\frac{1}{t}\right) \in L^p(\Omega'),$$

with $t > \max\left(\frac{m}{p}, \frac{1}{p - 1}\right)$;
Theorem 6.1. Let \( f(x, u, \eta) \) be a Caratheodory function in \( \pi_0 \times \mathbb{R} \times \mathbb{R}^m \) such that for almost all \( x = (x', x_m) \in \pi_0 \) and for all \( (u, \eta) \in \mathbb{R} \times \mathbb{R}^m \),

\[
|f(x, u, \eta)| \leq \lambda(u)|f^*(x) + \hat{\nu}(x')|\eta|^p|,
\]

\[
f^* \in L^1(\pi_0) + L^{p/(p-1)}(\pi_0),
\]

\[
c_1|u|^p + uf(x, u, \eta) \geq -f_0(x),
\]

where \( \lambda : [0, +\infty) \to [1, +\infty) \) is a monotone nondecreasing function and \( c_1 \) is a positive constant.

Theorem 6.1. Let (H13), (H14) be satisfied. Let \( \bar{\lambda} : [0, +\infty) \to [1, +\infty) \) be a nondecreasing function such that \( \bar{\lambda}(s) \leq \lambda(s) \) for all \( s \geq 0 \). Let \( c_0 \) be a positive constant such that \( c_0 > c_1 \). Let \( u \in W^{1,p}(\hat{\nu}, \pi_0) \cap L^\infty(\pi_0) \) satisfy

\[
\int_{\pi_0} \left\{ \frac{\hat{\nu}}{\lambda(|u|)} \sum_{i=1}^m \left| \frac{\partial u}{\partial x_i} \right|^p - \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_i} + c_0|u|^{p-2}uw + f(x, u, \nabla u)w \right\} dx = 0 \quad (6.1)
\]

for an arbitrary function \( w \in W^{1,p}(\hat{\nu}, \pi_0) \cap L^\infty(\pi_0) \) (the function \( u \) exists by [9, Theorem 2.2]). Let us assume that for some \( a \geq 0 \),

\[
c_1|u|^p + uf(x, u, \eta) \geq 0
\]

for almost all \( x \in \pi_a \) and for all \( (u, \eta) \in \mathbb{R} \times \mathbb{R}^m \).

Then there exists a positive constant \( \alpha \), depending on \( m, p, t, \Omega', \|u\|_{L^\infty(\pi_0)}, \|u\|_{1,p}, \lambda, \lambda(s), \|\hat{\nu}\|_{L^\infty(\Omega')}, \|1/\hat{\nu}\|_{L^1(\Omega')}, \) such that

\[
\int_{\pi_0} e^{\alpha x - \hat{\nu}} \sum_{i=1}^m \left| \frac{\partial u}{\partial x_i} \right|^p dx \leq D,
\]

where \( D \) is a positive number depending only on known parameters.

Proof. For the sake of simplicity, we will assume throughout that

\[
c_1|u|^p + uf(x, u, \eta) \geq 0 \quad (6.2)
\]

for almost all \( x \in \pi_0 \) and for all \( (u, \eta) \in \mathbb{R} \times \mathbb{R}^m \). Let \( \theta(x) \in C^1(\mathbb{R}) \) be a function such that \( \theta(x) = 1 \) if \( x < \frac{1}{2}, \theta(x) = 0 \) if \( x > 1, 0 \leq \theta(x) \leq 1, |\theta'(x)| \leq \beta \).

For every \( b \geq 0 \), we consider \( \theta_b(x_m) = \theta(x_m - b) \). It results \( 0 \leq \theta_b(x_m) \leq 1 \) and \( |\theta_b'(x_m)| \leq \beta \) for all \( b \geq 0 \). Let \( b \) be a real number, \( b > 0 \). Let us prove that

\[
\int_{\pi_0} \frac{\hat{\nu}}{\lambda(|u|)} \sum_{i=1}^m \left| \frac{\partial u}{\partial x_i} \right|^p dx + \int_{\pi_0} \{c_0|u|^p + f(x, u, \nabla u)u\} dx \quad (6.3)
\]

\[
= \int_{\pi_0} \frac{\hat{\nu}}{\lambda(|u|)} \sum_{i=1}^m \left| \frac{\partial u}{\partial x_i} \right|^p \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i} (\theta_b u) dx + \int_{\pi_0} \{c_0|u|^p + f(x, u, \nabla u)\} \theta_b dx.
\]

The function \( w = (\theta_c(x_m) - \theta_b(x_m))u \in W^{1,p}(\hat{\nu}, \pi_0) \cap L^\infty(\pi_0), c > b > 0, \) so by [6.1], we have

\[
\int_{\pi_0} \frac{\hat{\nu}}{\lambda(|u|)} \sum_{i=1}^m \left| \frac{\partial u}{\partial x_i} \right|^p \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_i} [(\theta_c - \theta_b)u] + c_0|u|^p(\theta_c - \theta_b) + f(x, u, \nabla u)(\theta_c - \theta_b) u dx = 0,
\]
hence, in (6.3) the right hand side does not depend on $b$. It results
\[
\int_{\pi_0} \frac{\dot{\nu}}{\lambda(|u|)} \sum_{i=1}^{m} \left| \frac{\partial u}{\partial x_i} \right|^p + \left| \frac{\partial \theta_b u}{\partial x_i} \right|^p dx + \int_{\pi_0} c_0 |u|^p \theta_b dx + \int_{\pi_0} f(x, u, \nabla u) \theta_b dx \\
= \int_{\pi_0} \frac{\dot{\nu}}{\lambda(|u|)} \sum_{i=1}^{m} \left| \frac{\partial u}{\partial x_i} \right|^p \theta_b dx + \int_{\pi_0} \frac{\dot{\nu}}{\lambda(|u|)} \left| \frac{\partial u}{\partial x_m} \right|^p \left( \theta_b \frac{\partial u}{\partial x_m} \right) \theta_b' dx + \int_{\pi_0} c_0 |u|^p \theta_b dx + \int_{\pi_0} f(x, u, \nabla u) \theta_b dx.
\]
(6.4)

By (H13) and (6.2), Hölder’s inequality and the definition of function $\theta_b$ it follows that
\[
\left| \int_{\pi_0} \frac{\dot{\nu}}{\lambda(|u|)} \left| \frac{\partial u}{\partial x_m} \right|^p u \theta_b' dx \right| \\
\leq \beta (sup \nu) \int_{\Omega'} \left| \frac{\dot{\nu}}{\lambda(|u|)} \right|^p \sum_{i=1}^{m} \left| \frac{\partial u}{\partial x_i} \right|^p \left( \theta_b \frac{\partial u}{\partial x_i} \right) \theta_b' dx \]
(6.5)

Next, from the weighted Friedrichs inequality (see, [17, Corollary 3.3]), we have
\[
\int_{\Omega'} |u|^p dx' \leq \alpha_1 \int_{\Omega'} \nu(x') \sum_{i=1}^{m} \left| \frac{\partial u}{\partial x_i} \right|^p dx',
\]
(6.6)

where the positive constant $\alpha_1$ depends only on $m$, $p$, $\Omega'$ and $1/\nu \| L^1(\Omega')$.

From (6.5) and (6.6) we obtain
\[
\left| \int_{\pi_0} \frac{\dot{\nu}}{\lambda(|u|)} \left| \frac{\partial u}{\partial x_m} \right|^p u \theta_b' dx \right| \\
\leq \beta (sup \nu) \int_{\Omega'} \left| \frac{\dot{\nu}}{\lambda(|u|)} \right|^p \sum_{i=1}^{m} \left| \frac{\partial u}{\partial x_i} \right|^p \left( \theta_b \frac{\partial u}{\partial x_i} \right) \theta_b' dx,
\]
(6.7)

where the positive constant $\alpha_2$ depends only on $m$, $p$, $\beta$, $\Omega'$ and $1/\nu \| L^1(\Omega')$. Hence
\[
\lim_{b \to +\infty} \int_{\pi_0} \frac{\dot{\nu}}{\lambda(|u|)} \left| \frac{\partial u}{\partial x_m} \right|^p u \theta_b' dx = 0.
\]
(6.8)

From (6.4), letting $b \to +\infty$, taking into account that the left hand side does not depend on $b$, by Lebesgue theorem and (6.8) we obtain (6.3).

Next, by (6.2), (6.3), $c_0 > c_1$, an easy computation gives
\[
\int_{\pi_0} \frac{\dot{\nu}}{\lambda(|u|)} \sum_{i=1}^{m} \left| \frac{\partial u}{\partial x_i} \right|^p dx \leq \int_{\pi_0} \frac{\dot{\nu}}{\lambda(|u|)} \sum_{i=1}^{m} \left| \frac{\partial u}{\partial x_i} \right|^p \left( \theta_b \frac{\partial u}{\partial x_i} \right) dx,
\]
(6.9)

for every $b > 0$.

From (6.9) and (6.7) we obtain
\[
\int_{\pi_{b+1/2}} \left| \frac{\partial u}{\partial x_i} \right|^p dx \leq \lambda(\|u\|_{L^\infty(\pi_0)}) [\alpha_2 (sup \nu)^{1/p} + 1] \int_{\pi_{b+1/2}} \left| \frac{\partial u}{\partial x_i} \right|^p dx
\]
\[= (\alpha_3 + 1) \int_{\pi_{b+1/2}} \left| \frac{\partial u}{\partial x_i} \right|^p dx,
\]
for every $b > 0$. 

From (6.9) and (6.7) we obtain
\[
\int_{\pi_{b+1/2}} \left| \frac{\partial u}{\partial x_i} \right|^p dx \leq \lambda(\|u\|_{L^\infty(\pi_0)}) [\alpha_2 (sup \nu)^{1/p} + 1] \int_{\pi_{b+1/2}} \left| \frac{\partial u}{\partial x_i} \right|^p dx
\]
\[= (\alpha_3 + 1) \int_{\pi_{b+1/2}} \left| \frac{\partial u}{\partial x_i} \right|^p dx,
\]
for every $b > 0$, where the positive constant $\alpha_3$ depends on $m$, $p$, $\beta$, $\Omega'$, $\|\tilde{\nu}\|_{L^\infty(\Omega')}$, $\|u\|_{L^\infty(\pi_0)}$, $\lambda(s)$ and $\|1/\tilde{\nu}\|_{L^1(\Omega')}$. Consequently,

$$I_{b+1}(u) \leq \frac{\alpha_3}{\alpha_3 + 1} I_b(u), \quad \forall b > 0,$$

where, for every $a \geq 0$,

$$I_a(u) = \int_{\pi_a} \tilde{\nu} \sum_{i=1}^m \left| \frac{\partial u}{\partial x_i} \right|^p dx, \quad A = I_0(u) < \infty.$$

This formula, by induction, gives

$$I_{b+n}(u) \leq s^n I_b(u) \leq A s^n,$$

for $n \in \mathbb{N}$, $b > 0$ and $s = \frac{\alpha_3}{\alpha_3 + 1}$. We can write last relation in this way

$$I_{b+n}(u) \leq A e^{n \log s}, \quad \text{for every } b > 0, \ n \in \mathbb{N} \cup \{0\}.$$

It is simple to verify that above inequality gives

$$I_\lambda(u) \leq \alpha_4 e^{-\lambda \tilde{\alpha}}, \quad \text{for all } \lambda > 0,$$

where $\alpha_4 = A e^\tilde{\alpha}$ and $\tilde{\alpha} = -\log s > 0$.

Now, fixing $\alpha$ such that $0 < \alpha < \tilde{\alpha}$, we have

$$\int_{\pi_0} e^{\alpha x_m} \tilde{\nu} \sum_{i=1}^m \left| \frac{\partial u}{\partial x_i} \right|^p dx = \sum_{j=0}^{+\infty} \int_{\pi_{j+1}} e^{\alpha x_m} \tilde{\nu} \sum_{i=1}^m \left| \frac{\partial u}{\partial x_i} \right|^p dx
\leq \sum_{j=0}^{+\infty} e^{\alpha(j+1)} \int_{\pi_{j+1}} \tilde{\nu} \sum_{i=1}^m \left| \frac{\partial u}{\partial x_i} \right|^p dx
\leq \sum_{j=0}^{+\infty} e^{\alpha(j+1)} I_j(u)
\leq \alpha_4 \sum_{j=0}^{+\infty} e^{\alpha(j+1)} e^{-j \tilde{\alpha}} < +\infty.$$

The proof is complete. \hfill \Box

As in Section 4, we will show an example where all assumptions are fulfilled. Let $\Omega' = \{x' = (x_1, x_2, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}: |x'| < 1\}$. Put

$$\tilde{\nu}(x') = [d(x', \partial \Omega')]^p = (1 - |x'|)^p$$

for $\rho : 0 < \rho < \min \left( \frac{p}{m}, (p-1) \right)$. Then the function $\tilde{\nu}$ satisfies (H13) with $t$ arbitrarily taken as follows:

$$\max \left( \frac{m}{p}, \frac{1}{p-1} \right) < t < \frac{1}{\rho}.$$

Let us define in $\pi_0 \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ the function $f(x, u, \eta)$ by

$$f(x, u, \eta) = u e^u (1 - |x'|)^p |\eta|^p - g_1(x),$$

where $g_1(x) \in L^\infty(\pi_0)$ has compact support. It is possible to verify (H14) by setting $\lambda(|u|) = e^{2|u|}$, and, taking into account that

$$\frac{1}{2} |u|^p + uf(x, u, \eta) \geq -2^{\frac{1}{p-1}} |g_1(x)|^{\frac{p}{p-1}}$$

(6.10)
for almost all \( x \in \pi \) and for all \((u, \eta) \in \mathbb{R} \times \mathbb{R}^m \). Then, from \([8\text{, Theorem 2.2}]\), there exists a function \( u \in \tilde{W}^{1,p}(\hat{\nu}, \pi) \cap L^\infty(\pi) \) such that

\[
\int_{\pi} \left\{ \frac{(1 - |x'|)^p}{e^{2u}} \sum_{i=1}^{m} \frac{\partial u}{\partial x_i} |p - 2\frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_i}| + |u|^{p-2}uw + ue^{u(1 - |x'|)^p} |\nabla u|^p \right\} dx = \int_{\pi} g_1 w \, dx
\]

for every arbitrary function \( w \in \tilde{W}^{1,p}(\hat{\nu}, \pi) \cap L^\infty(\pi) \). In this case \( c_0 = 1 \).

From (6.10) because of the support of \( g_1 \), there exists a positive number \( \alpha \) such that

\[
\frac{1}{2}|u|^p + uf(x, u, \eta) \geq 0
\]

for almost all \( x \in \pi \) and for all \((u, \eta) \in \mathbb{R} \times \mathbb{R}^m \). So, it is possible to apply Theorem 6.1 to the function \( u \).

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**References**


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