Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 275, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

MULTIPLE POSITIVE SOLUTIONS FOR A NONLOCAL PROBLEM INVOLVING CRITICAL EXPONENT

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Communicated by Paul H. Rabinowitz

Abstract. This article concerns the nonlocal problem

$$
\begin{gathered}
-\Big(a-b\int_{\mathbb{R}^4}|\nabla u|^2\,dx\Big)\Delta u=|u|^2u+\mu f(x),\quad \text{in } \mathbb{R}^4,\\
u\in\mathcal{D}^{1,2}(\mathbb{R}^4),
\end{gathered}
$$

where a,b are positive constants, μ is a non-negative parameter, $f(x)\in L^{4/3}(\mathbb{R}^4)$ is a non-negative function. By using the variational method, the existence of multiple positive solutions are obtained.

1. Introduction and main results

In this article, we focus on multiple positive solutions to the nonlocal problem

$$
\begin{gathered}
-\Big(a-b\int_{\mathbb{R}^4}|\nabla u|^2\,dx\Big)\Delta u=|u|^2u+\mu f(x),\quad \text{in } \mathbb{R}^4,\\
u\in\mathcal{D}^{1,2}(\mathbb{R}^4),
\end{gathered}
\tag{1.1}
$$

here a,b are positive constants, μ is a parameter, $f(x)\in L^{4/3}(\mathbb{R}^4)$ is a non-negative function. The problem (1.1) is related to the stationary problem

$$
\varrho h\frac{\partial^2u}{\partial t^2}+\delta\frac{\partial u}{\partial t}+f_1\big(\frac{\partial u}{\partial t}\big)=\Big(p_0+\frac{Eh}{2L}\int_0^L\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\Big)\frac{\partial^2u}{\partial x^2}+f_2(x,u)
\tag{1.2}
$$

with $0<x<L$ and $t\geq 0$. Where $u=u(x,t)$ is the lateral displacement, ϱ the mass density, E the Young modulus, h the cross-section area, L the length, δ the resistance modulus, p_0 the initial axial tension, f_1 and f_2 the external forces. More precisely, this problem as an extension of the classical d'Alembert's wave equation for free vibrations of elastic strings and first proposed by Kirchhoff [11] when $f_1=f_2=0$. The equation (1.2) with external forces is considered for analyzing phenomena in real world and it is studied by many researchers (see for instance [23, 28] and the references therein).

The distinguishing feature of (1.2) is that the equation contains a nonlocal coefficient $\big(p_0+\frac{Eh}{2L}\int_0^L\big|\frac{\partial u}{\partial x}\big|^2\,dx\big)$ which depends on the average $\frac{1}{2L}\int_0^L\big|\frac{\partial u}{\partial x}\big|^2\,dx$ of the Kinetic energy $\frac{1}{2L}\big|\frac{\partial u}{\partial x}\big|^2$ on $[0,L]$, (1.2) is no longer a pointwise identity and therefore

2010 *Mathematics Subject Classification.* 35A15, 35B09, 35B33.

Key words and phrases. Multiple positive solutions; nonlocal problem; critical exponent.

Submitted August 15, 2017. Published November 5, 2017.

it is often called nonlocal problem. Restating that (1.2) received much attention after the abstract functional analysis framework was proposed by Lions [18].

It is worth paying more concerns for Young's modulus, which is also known as the elastic modulus, is a measure of the sensitivity of the variable to the independent variable. It allows the elastic modulus to be sign-changing in others fields, because of elasticities may be change sign (e.g. the price elasticities of demand [8]). Young's modulus can also be used in computing tension, where the atoms are pulled apart instead of squeezed together. In those cases, the strain is negative because the atoms are stretched instead of compressed, this leads to minus Young's modulus. Indeed, for example, an elastic meta-material which exhibits simultaneously negative effective mass density and bulk modulus with a single unit structure made of solid materials was presented in [21], authors of [29, 30] got the Young's modulus of the nanoplate exhibits a negative temperature coefficient, the meta-material model that possess simultaneously negative effective mass density and negative effective Young's modulus were proposed in [9, 26]. Therefore, problem (1.2) with $E < 0$ is still an interesting model.

Recently, the Kirchhoff type problem

$$-\Big(a + b\textstyle\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u), \quad \text{in } \Omega$$

with $a, b \geq 0, a + b > 0$, $\Omega = \mathbb{R}^N$ or Ω is a smooth bounded domain in $\mathbb{R}^N$ has been studied by many researchers; we refer the reader to [3, 5, 6, 15, 22, 27, 39] with sub-critical growth, and [10, 12, 16, 17, 19, 20, 24, 31, 32, 33, 35, 36, 37, 38, 40] with critical cases. Particularly, [10, 16, 17, 37] $N = 4$ and some show interesting results. Only a few authors mentioned problem of the form

$$-\Big(a - b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u). \tag{1.3}$$

Yin and Liu [34] researched problem (1.3) when $f(x,u) = |u|^{p-2}u$ (where $2 < p < 2^* = \frac{2N}{N-2}$ as $N \geq 3$ and $2^* = +\infty$ as $N = 1, 2$) and they got (1.3) has at least a nontrivial non-negative solution and a nontrivial non-positive solution with Dirichlet's boundary condition. Lei et al [13] studied (1.3) assuming $f(x,u) = f_\lambda(x)|u|^{q-2}u$ $(1 < q < 2)$ with $N \geq 3$, with the assumption $f_\lambda(x) \in L^\infty(\Omega)$, they concluded that (1.3) has at least two positive solutions. Also Lei et al [14] obtained many solutions for $f(x,u) = u^{-\gamma}$ with $1 < q < 2$ and $0 < \gamma < 1$.

To the best of our knowledge, there is no result for equation (1.1). From [2, pp.7], $\mathcal{D}^{1,2}(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$ continuously but this embedding is never compact. Motivated by [13, 14, 34], since the typical difficulty is the lack of compactness of the embedding $D^{1,2}(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$, we overcome the difficulty by using the methods from [10, 16, 17, 37]. Our main results can be stated as follows:

Theorem 1.1. *Problem* (1.1) *has infinitely many positive solutions when* $\mu = 0$.

Theorem 1.2. *Assume that* $f(x) \in L^{4/3}(\mathbb{R}^4)$ *is a positive function, then there exists* $\mu_* > 0$ *such that problem* (1.1) *has at least two positive solutions when* $\mu \in (0, \mu_*]$.

2. Preliminaries

In this section, we give some notation and definitions. All results are based on $\mathcal{D}^{1,2}(\mathbb{R}^4) = \big\{u \in L^4(\mathbb{R}^4) \big| \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^4), i = 1, \dots, 4\big\}$. For $u, v \in \mathcal{D}^{1,2}(\mathbb{R}^4)$, the inner product is $\langle u,v\rangle = \int_{\mathbb{R}^4} \nabla u \nabla v\,dx$ and the norm is

$$\|u\| = \langle u,u\rangle^{1/2} = \Big(\int_{\mathbb{R}^4} |\nabla u|^2\,dx\Big)^{1/2}.$$

We recall that a function $u \in \mathcal{D}^{1,2}(\mathbb{R}^4)$ is called a solution of problem (1.1) if

$$\big(a - b\|u\|^2\big)\int_{\mathbb{R}^4} \nabla u \nabla v\,dx = \int_{\mathbb{R}^4} |u|^2 uv\,dx + \mu\int_{\mathbb{R}^4} fv\,dx$$

hold for all $v \in \mathcal{D}^{1,2}(\mathbb{R}^4)$. Throughout this paper, we denote by $\|\cdot\|_s$ the usual L^s-norm and $\to$ (resp. $\rightharpoonup$) the strong (resp. weak) convergence. Set

$$S = \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^4)\setminus\{0\}} \frac{\|u\|^2}{\big(\int_{\mathbb{R}^4} u^4\,dx\big)^{1/2}}. \tag{2.1}$$

It is well known, for any $\varepsilon > 0$ and $y \in \mathbb{R}^4$, all positive solutions for the problem

$$\begin{gathered} -\Delta u = u^3, \quad x \in \mathbb{R}^4, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^4) \end{gathered}$$

can be expressed as

$$u_{\varepsilon,y} := \frac{2\sqrt{2}\varepsilon}{\varepsilon^2 + |x-y|^2}, \tag{2.2}$$

as a consequence, S can be archive by (2.2) and $\|u_{\varepsilon,y}\|^2 = \|u_{\varepsilon,y}\|_4^4 = S^2$.

Because of that we are looking for positive solution, for equation (1.1), set the energy $I : \mathcal{D}^{1,2}(\mathbb{R}^4) \mapsto \mathbb{R}$ be the functional defined by

$$I(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{4}\int_{\mathbb{R}^4} (u^+)^4\,dx - \mu\int_{\mathbb{R}^4} fu\,dx, \tag{2.3}$$

here $u^+ = \max\{0,u\}$. It is able to verify $I(u) \in C^1(\mathcal{D}^{1,2}(\mathbb{R}^4),\mathbb{R})$, and for all $v \in \mathcal{D}^{1,2}(\mathbb{R}^4)$, I has the Gâteaux derivative given by

$$\langle I'(u), v\rangle = (a - b\|u\|^2)\int_{\mathbb{R}^4} \nabla u\nabla v\,dx - \int_{\mathbb{R}^4} (u^+)^3 v\,dx - \mu\int_{\mathbb{R}^4} fv\,dx. \tag{2.4}$$

3. Main Lemmas

Lemma 3.1. *Assume that $f(x) \in L^{4/3}(\mathbb{R}^4)$ is a positive function, then, there exist $r, \rho, \mu_1 > 0$ such that, for any $\mu \in (0, \mu_1]$, one has*

(i) *$I(u) \ge \rho$ with $\|u\| = r$;*
(ii) *$\inf I(u) < 0$ with $\|u\| < r$;*
(iii) *There exists $e \in \mathcal{D}^{1,2}(\mathbb{R}^4)$ which satisfies $I(e) < 0$ with $\|e\| > r$.*

Proof. (i) From (2.3) and (2.1), we obtain

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{4}\int_{\mathbb{R}^4} (u^+)^4\,dx - \mu\int_{\mathbb{R}^4} fu\,dx \\ &\ge \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{4S^2}\|u\|^4 - \frac{\mu}{\sqrt{S}}\|f\|_{4/3}\|u\| \\ &= \|u\|\Big(\frac{a}{2}\|u\| - \frac{bS^2+1}{4S^2}\|u\|^3 - \frac{\mu}{\sqrt{S}}\|f\|_{4/3}\Big). \end{aligned}$$

Set $g_1(t,\mu) := \frac{a}{2}t - \frac{bS^2+1}{4S^2}t^3 - \frac{\mu}{\sqrt{S}}\|f\|_{4/3}$ for all $t \geq 0$, we can see that there exist constants $r = \sqrt{\frac{2aS^2}{3(bS^2+1)}} > 0$ and $\mu_1 = \frac{aS}{4\|f\|_{4/3}}\sqrt{\frac{aS}{bS^2+1}} > 0$ such that

$$\begin{aligned}\max_{t>0} g_1(t,\mu) = g_1(r,\mu) &= \frac{aS}{3}\sqrt{\frac{2a}{3(bS^2+1)}} - \frac{\mu}{\sqrt{S}}\|f\|_{4/3} \\ &\geq \frac{aS}{3}\sqrt{\frac{2a}{3(bS^2+1)}} - \frac{\mu_1}{\sqrt{S}}\|f\|_{4/3}\end{aligned}$$

for any $\mu \in (0,\mu_1]$. Particularly, we have $I(u) \geq rg(r,\mu_1)$ when $\|u\| = r$. Thus

$$I(u) \geq \sqrt{\frac{2aS^2}{3(bS^2+1)}}\Big(\frac{aS}{3}\sqrt{\frac{2a}{3(bS^2+1)}} - \frac{\mu_1}{\sqrt{S}}\|f\|_{4/3}\Big) > \frac{a^2S^2}{72(bS^2+1)} := \rho.$$

Therefore, there exist $r, \rho, \mu_1 > 0$ such that $I(u) \geq \rho > 0$.

(ii) For $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^4)$ with $\|u_0\| = r$ such that $\int_{\mathbb{R}^4} fu_0\,dx > 0$, then

$$\lim_{t\to 0^+}\frac{I(tu_0)}{t} = -\mu\int_{\mathbb{R}^4} fu_0\,dx < 0.$$

Hence, there exists some $u \in \mathcal{D}^{1,2}(\mathbb{R}^4)$ such that $I(u) < 0$ when $\|u\|$ enough small. Therefore, $c_1 := \inf_{\|u\|<r} I(u) < 0$ is well defined.

(iii) For any $t \in \mathbb{R}$ and a $u_0 \in \mathcal{D}^{1,2}(\mathbb{R}^4)$ is fixed with $\|u_0\| = r$, we have

$$\lim_{t\to\infty}\frac{I(tu_0)}{t^4} = -\frac{b}{4}\|u_0\|^4 - \frac{1}{4}\int_{\mathbb{R}^4}(u_0^+)^4\,dx \leq -\frac{br^4}{4} < 0,$$

so, there is $t_e > 1$ satisfies $I(t_eu_0) < 0$. Let $e := t_eu_0 \in \mathcal{D}^{1,2}(\mathbb{R}^4)$, then $I(e) < 0$ and $\|e\| = t_er > r$. For example, take $e \in \mathcal{D}^{1,2}(\mathbb{R}^4)$ with $\|e\|^2 = \frac{4a}{b} + 4(\frac{\mu}{b\sqrt{S}}\|f\|_{4/3})^{2/3}$, we can verify $\|e\| > r$ and $I(e) < 0$. The proof is complete. $\square$

Lemma 3.2. *Assume that $\mu > 0$ and $f(x) \in L^{4/3}(\mathbb{R}^4)$ is a positive function, then I satisfies the $(PS)_c$ condition with*

$$c < \frac{a^2S^2}{4(bS^2+1)} - \Lambda\mu^{4/3}, \quad \Lambda = \Big(\frac{4bS^2}{bS^2+1}\Big)^{-1/3}\|f\|_{4/3}^{4/3}.$$

Proof. Let $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^4)$ is a $(PS)_c$ sequence such that $I(u_n) \to c$, $I'(u_n) \to 0$ as $n \to \infty$. So by the Hölder and Sobolev inequalities, for n large enough, one has

$$c + o(\|u_n\|) \geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \geq \frac{a}{4}\|u_n\|^2 - \frac{3\mu}{4\sqrt{S}}\|f\|_{4/3}\|u_n\|.$$

This means that $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^4)$. That is, there exist a subsequence (still denoted by $\{u_n\}$) and u_0 in $\mathcal{D}^{1,2}(\mathbb{R}^4)$ such that $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^4)$, $u_n \to u_0$ in $L^p_{\rm loc}$ $(1 \leq p < 4)$, $u_n(x) \to u_0(x)$ in $\mathbb{R}^4$ as $n \to \infty$.

Set $\omega_n := u_n - u_0$, then $\|\omega_n\| \to 0$ as $n \to \infty$. Otherwise $\|\omega_n\| \not\to 0$. Through of contradiction, we can assume there is a subsequence (still denoted by $\{\omega_n\}$) such that $\lim_{n\to\infty}\|\omega_n\| = l > 0$. For $v \in \mathcal{D}^{1,2}(\mathbb{R}^4)$, it holds that

$$\big(a - b\|u_n\|^2\big)\int_{\mathbb{R}^4}\nabla u_n\nabla v\,dx - \int_{\mathbb{R}^4}(u_n^+)^3v\,dx - \mu\int_{\mathbb{R}^4} fv\,dx = o(1)$$

as $n \to \infty$. Lebesgue's dominated convergence theorem (see [25, pp.27]) leads to

$$\int_{\mathbb{R}^4} fu_n\,dx = \int_{\mathbb{R}^4} fu_0\,dx + o(1). \tag{3.1}$$

Using the Brézis-Lieb lemma (see [4, Theorem 1]) and (3.1), it satisfies

$$\Big(a-(bl^2+b\|u_0\|^2)\Big)\int_{\mathbb{R}^4}\nabla u_0\nabla v\,dx-\int_{\mathbb{R}^4}(u_0^+)^3v\,dx-\mu\int_{\mathbb{R}^4}fv\,dx=0. \tag{3.2}$$

Particularly, take $v=u_0$ in (3.2), there is

$$\big(a-bl^2-b\|u_0\|^2\big)\|u_0\|^2-\int_{\mathbb{R}^4}(u_0^+)^4\,dx-\mu\int_{\mathbb{R}^4}fu_0\,dx=0. \tag{3.3}$$

Furthermore, as $n\to\infty$, it holds

$$\langle I'(u_n),u_n\rangle=a\|u_n\|^2-b\|u_n\|^4-\int_{\mathbb{R}^4}(u_n^+)^4\,dx-\mu\int_{\mathbb{R}^4}fu_n\,dx=o(1).$$

Using the Brézis-Lieb lemma again, we get

$$\begin{aligned}o(1)&=a\|\omega_n\|^2+a\|u_0\|^2-2b\|\omega_n\|^2\|u_0\|^2-b\|u_0\|^4-b\|\omega_n\|^4\\&\quad-\int_{\mathbb{R}^4}(u_0^+)^4\,dx-\int_{\mathbb{R}^4}(\omega_n^+)^4\,dx-\mu\int_{\mathbb{R}^4}fu_0\,dx\end{aligned} \tag{3.4}$$

Cutting (3.3) out of (3.4), we have

$$a\|\omega_n\|^2-b\|\omega_n\|^4-b\|\omega_n\|^2\|u_0\|^2=\int_{\mathbb{R}^4}(\omega_n^+)^4\,dx+o(1). \tag{3.5}$$

Noting that $\int_{\mathbb{R}^4}(\omega_n^+)^4\,dx\le\int_{\mathbb{R}^4}\omega_n^4\,dx$, we obtain $0\le l^2(a-b\|u_0\|^2-bl^2)\le\frac{l^4}{S^2}$, $l>0$. So that

$$l^2\ge\frac{S^2(a-b\|u_0\|^2)}{bS^2+1}>0. \tag{3.6}$$

On the one hand, applying (3.5)–(3.6), it holds that

$$\begin{aligned}I(u_0)&=\frac a2\|u_0\|^2-\frac b4\|u_0\|^4-\frac14\int_{\mathbb{R}^4}(u_0^+)^4\,dx-\mu\int_{\mathbb{R}^4}fu_0\,dx\\&=c-\frac a2\|\omega_n\|^2+\frac b4\|\omega_n\|^4+\frac b2\|\omega_n\|^2\|u_0\|^2+\frac14\int_{\mathbb{R}^4}(\omega_n^+)^4\,dx+o(1)\\&=c-\frac a2l^2+\frac b4l^4+\frac b2l^2\|u_0\|^2+\frac14\left(al^2-bl^4-bl^2\|u_0\|^2\right)\\&=c-\frac{a-b\|u_0\|^2}{4}l^2\\&\le c-\frac{a^2S^2}{4(bS^2+1)}+\frac{abS^2}{2(bS^2+1)}\|u_0\|^2-\frac{b^2S^2}{4(bS^2+1)}\|u_0\|^4\\&<-\Lambda\mu^{4/3}+\frac{abS^2}{2(bS^2+1)}\|u_0\|^2-\frac{b^2S^2}{4(bS^2+1)}\|u_0\|^4.\end{aligned} \tag{3.7}$$

On the other hand, from (3.3) it follows that

$$a\|u_0\|^2=b\|u_0\|^4+bl^2\|u_0\|^2+\int_{\mathbb{R}^4}(u_0^+)^4\,dx+\mu\int_{\mathbb{R}^4}fu_0\,dx. \tag{3.8}$$

Moreover, Hölder and Yang's inequalities lead to $\frac\mu2\int_{\mathbb{R}^4}fu_0\,dx\le\frac{\mu}{2\sqrt S}\|f\|_{4/3}\|u_0\|$ and

$$\|f\|_{4/3}\|u_0\|\le\frac{\sqrt Sb}{2\mu(bS^2+1)}\|u_0\|^4+\Big(\frac{\sqrt Sb}{2\mu(bS^2+1)}\Big)^{-1/3}\|f\|_{4/3}^{4/3}. \tag{3.9}$$

Therefore, from (2.3), (3.8) and (3.9), we get

$$\begin{aligned} I(u_0) &= \frac{a}{2}\|u_0\|^2 - \frac{b}{4}\|u_0\|^4 - \frac{1}{4}\int_{\mathbb{R}^4}(u_0^+)^4\,dx - \mu\int_{\mathbb{R}^4} fu_0\,dx \\ &= \frac{bl^2}{2}\|u_0\|^2 + \frac{b}{4}\|u_0\|^4 + \frac{1}{4}\int_{\mathbb{R}^4}(u_0^+)^4\,dx - \frac{\mu}{2}\int_{\mathbb{R}^4} fu_0\,dx \\ &\geq \frac{b}{2}\|u_0\|^2\cdot\frac{S^2(a-b\|u_0\|^2)}{bS^2+1} + \frac{b}{4}\|u_0\|^4 - \frac{\mu}{2}\int_{\mathbb{R}^4} fu_0\,dx \\ &\geq \frac{abS^2}{2(bS^2+1)}\|u_0\|^2 - \frac{b^2S^2}{4(bS^2+1)}\|u_0\|^4 - \Big(\frac{4bS^2}{bS^2+1}\Big)^{-1/3}\|f\|_{4/3}^{4/3}\mu^{4/3}. \\ &= \frac{abS^2}{2(bS^2+1)}\|u_0\|^2 - \frac{b^2S^2}{4(bS^2+1)}\|u_0\|^4 - \Lambda\mu^{4/3}. \end{aligned} \tag{3.10}$$

Which is a contradiction by comparing the calculations from (3.7) with (3.10). Hence $l = 0$. As a consequence, we get $u_n \to u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^4)$. This proof is complete. □

By (2.2), we can obtain the following estimate for the mountain pass level.

Lemma 3.3. *There exists $\mu_* \in (0,\mu_1]$ such that $\sup_{t\geq 0} I(tu_{\varepsilon,y}) < \frac{a^2S^2}{4(bS^2+1)} - \Lambda\mu^{4/3}$ with $\mu \in (0,\mu_*]$ (μ_1 is defined in the Lemma 3.1).*

Proof. Set $g(t) = I(tu_{\varepsilon,y})$ and $h(t) = I(tu_{\varepsilon,y}) + \mu t\int_{\mathbb{R}^4} fu_{\varepsilon,y}\,dx$ with $t \geq 0$, then

$$\begin{aligned} g(t) &= \frac{a}{2}\|tu_{\varepsilon,y}\|^2 - \frac{b}{4}\|tu_{\varepsilon,y}\|^4 - \frac{1}{4}\int_{\mathbb{R}^4}(tu_{\varepsilon,y})^4\,dx - \mu\int_{\mathbb{R}^4} f\cdot tu_{\varepsilon,y}\,dx \\ &= \frac{aS^2}{2}t^2 - \frac{bS^4}{4}t^4 - \frac{S^2}{4}t^4 - \mu t\int_{\mathbb{R}^4} fu_{\varepsilon,y}\,dx \end{aligned}$$

and

$$h(t) = \frac{aS^2}{2}t^2 - \frac{bS^4}{4}t^4 - \frac{S^2}{4}t^4.$$

So, there exists $t_1 = \sqrt{\frac{a}{bS^2+1}}$ such that $\max_{t>0} h(t) = h(t_1) = \frac{a^2S^2}{4(bS^2+1)}$. For any $\mu \in (0,\mu_1)$ and $t \in (0,t_1)$, noticing $\mu_1 = \frac{aS}{4\|f\|_{4/3}}\sqrt{\frac{aS}{bS^2+1}}$, we can see that

$$\mu t\int_{\mathbb{R}^4} fu_{\varepsilon,y}\,dx < \mu_1 t_1\int_{\mathbb{R}^4} fu_{\varepsilon,y}\,dx \leq \frac{\mu_1 t_1}{\sqrt{S}}\|f\|_{4/3}\|u_{\varepsilon,y}\| = \frac{a^2S^2}{4(bS^2+1)} = \max_{t>0} h(t).$$

Therefore, $\max_{t>0} g(t) > 0$. Take $\mu_2 \in (0,\mu_1] \cap \Big(0, \frac{aS^2(a^2b)^{1/4}}{2(bS^2+1)\|f\|_{4/3}}\Big)$, then

$$\frac{a^2S^2}{4(bS^2+1)} - \Lambda\mu^{4/3} > \frac{a^2S^2}{4(bS^2+1)} - \Lambda\mu_2^{4/3} > 0 \tag{3.11}$$

for all $\mu \in (0,\mu_2)$. Thus there exists $t_2 \in (0,t_1)$ such that

$$\begin{aligned} \max_{0\leq t\leq t_2} g(t) &\leq \max_{0\leq t\leq t_2}\Big\{\frac{aS^2}{2}t^2 - \frac{bS^4}{4}t^4\Big\} \\ &\leq \frac{a^2S^2}{4(bS^2+1)} - \Lambda\mu_2^{4/3} \leq \frac{a^2S^2}{4(bS^2+1)} - \Lambda\mu^{4/3}. \end{aligned}$$

for all $\mu \in (0, \mu_2)$. Choose $\mu_* \in (0, \mu_2)$ such that, for any $\mu \in (0, \mu_*]$, it holds

$$\mu t_2 \int_{\mathbb{R}^4} f u_{1,0}\, dx > \Lambda \mu^{4/3}.$$

Hence for all $\mu \in (0, \mu_*]$, one has

$$\sup_{t \geq t_2} g(t) \leq \sup_{t \geq t_2} h(t) - \mu t_2 \int_{\mathbb{R}^4} f u_{1,0}\, dx < h(t_1) - \Lambda \mu^{4/3} = \frac{a^2 S^2}{4(bS^2+1)} - \Lambda \mu^{4/3}.$$

Consequently,

$$c^+ := \sup_{t \geq 0} I(t u_{1,0}) = \sup_{t \geq 0} g(t) < \frac{a^2 S^2}{4(bS^2+1)} - \Lambda \mu^{4/3}.$$

Thus the proof is complete. □

4. Proof Theorem 1.1

Proof. For any $\lambda > 0$, let $v_{\varepsilon,y} = \lambda^{1/2} u_{\varepsilon,y}$, then $u_{\varepsilon,y} = \lambda^{-1/2} v_{\varepsilon,y}$ by (2.2). So

$$-\lambda \Delta v_{\varepsilon,y} = -\lambda \Delta \lambda^{1/2} u_{\varepsilon,y} = -\lambda^{\frac{3}{2}} \Delta u_{\varepsilon,y} = \lambda^{\frac{3}{2}} (\lambda^{-1/2} v_{\varepsilon,y})^3 = v_{\varepsilon,y}^3. \tag{4.1}$$

Noting that there are infinitely many $u_{\varepsilon,y}$, we can verify all $v_{\varepsilon,y}$ are infinitely many and its are positive solutions of (4.1) for any $\lambda > 0$. Considering the equation

$$\lambda = a - b\|v_\lambda\|^2 = a - b\lambda S^2. \tag{4.2}$$

Obviously, the solution of equation (4.2) is $\lambda_0 = \frac{a}{bS^2+1}$. As a consequence, we have

$$v_{\lambda_0} = \lambda_0^{1/2} u_{\varepsilon,y} = \big(\frac{a}{bS^2+1}\big)^{1/2} u_{\varepsilon,y}. \tag{4.3}$$

Therefore, for equation

$$-\Big(a - b\int_{\mathbb{R}^4} |\nabla u|^2\, dx\Big)\Delta u = |u|^2 u, \quad \text{in } \mathbb{R}^4, \tag{4.4}$$

we can verify that all v_{λ_0} are positive solutions of (4.4) easily by (4.1)–(4.3). Thus equation (4.4) has infinitely positive solutions $\big(\frac{a}{bS^2+1}\big)^{1/2} u_{\varepsilon,y}$ when $\mu = 0$. □

5. Proof of Theorem 1.2

Existence of the first positive solution. Taking $B_r := \{u \in \mathcal{D}^{1,2}(\mathbb{R}^4) : \|u\| \leq r\}$ and μ_* from the Lemma 3.3, where $r = \sqrt{\frac{2aS^2}{3(bS^2+1)}}$. Reason by the Lemma 3.1, there exists $\mu_1 > 0$ such that $\inf I(B_r) < 0$ for any $\mu \in (0, \mu_*] \subset (0, \mu_1)$. By the Ekeland variational principle (see[7, Lemma 1.1]), there exists a sequence $\{u_n\} \subset B_r$ such that

$$I(u_n) \leq \inf I(B_r) + \frac{1}{n} \quad \text{and} \quad I(u) \geq I(u_n) - \frac{1}{n}\|u - u_n\| \tag{5.1}$$

for all $n \in \mathbb{N}$ and for any $u \in \overline{B_r}$. Therefore, we get $I(u_n) \to c_1$ and $I'(u_n) \to 0$ in dual space of $\mathcal{D}^{1,2}(\mathbb{R}^4)$ as $n \to \infty$. Noting that $c_1 < 0 < \frac{a^2S^2}{4(bS^2+1)} - \Lambda\mu^{4/3}$ (see Lemma 3.1 and inequality (3.11)), by Lemma 3.2, there exist a subsequence (still denoted by $\{u_n\}$) and $u_* \in B_r$ such that $u_n \to u_*$ as $n \to \infty$. Then, $I(u_*) = c_1 < 0$ and $I'(u_*) = 0$. Which implies that u_* is a local minimizer for c_1. Consequently, u_* is a solution of problem (1.1). Define $u_*^- = \max\{0, -u_*\}$, then $u_*^- = 0$ by both $\|u_*\| < \sqrt{\frac{2aS^2}{3(bS^2+1)}}$ and $\langle I'(u_*), u_*^- \rangle = 0$, which deduces $u_* \geq 0$. By the strong maximum principle, we obtain $u_* > 0$. The proof is complete. □

Existence of the second positive solution. We shall divide it into three steps.

Step 1. There exists a critical point u_{**} with $I(u_{**}) > 0$. By Lemma 3.1, the functional I has mountain pass geometry. Set

$$\Gamma^+ = \Big\{\tau(t) \in C^1\big([0,1], \mathcal{D}^{1,2}(\mathbb{R}^4)\big); \tau(0) = 0, \tau(1) = e\Big\}.$$

By (2.3)–(2.4), $I\big(\tau(t)\big)$ has continuity. Besides, $I\big(\tau(0)\big) = 0$, $I\big(\tau(1)\big) \le 0$ and $I\big(\tau(t)\big) > \rho$ with some $t \in (0,1)$. Moreover, by Lemma 3.3, there is a $\mu_* \in (0, \mu_1)$, such that

$$0 < \rho \le c_2 := \inf_{\tau \in \Gamma^+} \sup_{t \in [0,1]} I\big(\tau(1)\big) \le c^+ < \frac{a^2 S^2}{4(bS^2+1)} - \Lambda \mu^{4/3}.$$

hold for all $\mu \in (0, \mu_*]$. Via Lemma 3.2 and the mountain pass theorem (see [1, Theorem 2.1–2.4]) implies that for I, there exist u_{**} and a sequence $\{u_k\}$ in $\mathcal{D}^{1,2}(\mathbb{R}^4)$ such that $u_k \to u_{**}$ in $\mathcal{D}^{1,2}(\mathbb{R}^4)$, $I(u_k) \to c_2 = I(u_{**})$ and $I'(u_k) \to 0 = I'(u_{**})$ in dual space of $\mathcal{D}^{1,2}(\mathbb{R}^4)$. Hence u_{**} is a solution of problem (1.1) with $\|u_{**}\| \ge \sqrt{\frac{2aS^2}{3(bS^2+1)}}$. Because of $I(u_*) < 0 < I(u_{**})$, we get $u_{**} \ne u_*$.

Step 2. The critical point u_{**} satisfies $\|u_{**}\|^2 < a/b$. Note that u_{**} is a critical point of I. Relying on $\langle I'(u_{**}), u_{**}\rangle = 0$, one has

$$(a - b\|u_{**}\|^2)\|u_{**}\|^2 = \int_{\mathbb{R}^4} (u_{**}^+)^4\,dx + \mu \int_{\mathbb{R}^4} f u_{**}\,dx. \tag{5.2}$$

Obviously $\|u_{**}\|^2 < \frac{a}{b}$ if u_{**} is trivial one. Without loss of generality, we can suppose that there satisfies $\|u_{**}\|^2 \ge \frac{a}{b}$, then $(a - b\|u_{**}\|^2)\|u_{**}\|^2 \le 0$, which implies $\int_{\mathbb{R}^4} f u_{**}\,dx \le 0$ from (5.2). By $I(u_{**}) = c$ and $I'(u_{**}) = 0$, there is

$$\frac{a^2 S^2}{4(bS^2+1)} - \Lambda\mu^{4/3} > I(u_{**}) - \frac{1}{4} I'(u_{**}) = \frac{a}{4}\|u_{**}\|^2 - \frac{3\mu}{4}\int_{\mathbb{R}^4} f u_{**}\,dx \ge \frac{a^2}{4b}. \tag{5.3}$$

This is a contradiction. So $\|u_{**}\|^2 < \frac{a}{b}$.

Step 3. u_{**} is a positive critical point of I. Define $u^- = \max\{0, -u\}$, then $u = u^+ - u^-$. By $I'(u_{**}) = 0$, we have

$$\begin{aligned} 0 &= \langle I'(u_{**}), u_{**}^-\rangle \\ &= (a - b\|u_{**}\|^2)\int_{\mathbb{R}^4} \nabla u_{**} \nabla u_{**}^-\,dx - \int_{\mathbb{R}^4} (u_{**}^+)^3 u_{**}^-\,dx - \mu \int_{\mathbb{R}^4} f u_{**}^-\,dx \\ &= (a - b\|u_{**}\|^2)\|u_{**}^-\|^2 - \mu \int_{\mathbb{R}^4} f u_{**}^-\,dx \\ &\ge (a - b\|u_{**}\|^2)\|u_{**}^-\|^2, \end{aligned}$$

which implies $\|u_{**}^-\| = 0$. Hence u_{**} is non-negative. According to the Lemma 3.3 that $\|u_{**}\| \ge \sqrt{\frac{2aS^2}{3(bS^2+1)}}$, we have $u_{**} \not\equiv 0$. By the strong maximum principle, we obtain $u_{**} > 0$. Hence the problem (1.1) has a positive solution u_{**} which different with u_*. Therefore, the problem (1.1) has at least two positive solutions. The proof is complete. $\square$

Acknowledgments. The authors would like to thank the anonymous referees and the editors for their helpful comments and suggestions, which led to an improvement of the original manuscript. This work was supported by the National Natural Science Foundation of China (No. 11661021), by the Innovation Group Major Program of Guizhou Province (No. KY[2016]029) and by the Natural Science Foundation of Education of Guizhou Province (No. KY[2016]163, No. KY[2013]405).

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