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NONEXISTENCE OF GLOBAL SOLUTIONS FOR FRACTIONAL TEMPORAL SCHRÖDINGER EQUATIONS AND SYSTEMS

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Abstract. We, first, consider the nonlinear Schrödinger equation

$$i^\alpha {}_0^C D_t^\alpha u + \Delta u = \lambda |u|^p + \mu a(x)\cdot \nabla |u|^q, \quad t>0,\ x\in\mathbb{R}^N,$$

where $0<\alpha<1$, i^α is the principal value of i^α, ${}_0^C D_t^\alpha$ is the Caputo fractional derivative of order α, $\lambda\in\mathbb{C}\backslash\{0\}$, $\mu\in\mathbb{C}$, $p>q>1$, $u(t,x)$ is a complex-valued function, and $a:\mathbb{R}^N\to\mathbb{R}^N$ is a given vector function. We provide sufficient conditions for the nonexistence of global weak solution under suitable initial data. Next, we extend our study to the system of nonlinear coupled equations

$$i^\alpha {}_0^C D_t^\alpha u + \Delta u = \lambda |v|^p + \mu a(x)\cdot \nabla |v|^q, \quad t>0,\ x\in\mathbb{R}^N,$$
$$i^\beta {}_0^C D_t^\beta v + \Delta v = \lambda |u|^\kappa + \mu b(x)\cdot \nabla |u|^\sigma, \quad t>0,\ x\in\mathbb{R}^N,$$

where $0<\beta\le\alpha<1$, $\lambda\in\mathbb{C}\backslash\{0\}$, $\mu\in\mathbb{C}$, $p>q>1$, $\kappa>\sigma>1$, and $a,b:\mathbb{R}^N\to\mathbb{R}^N$ are two given vector functions. Our approach is based on the test function method.

1. Introduction

In recent years, fractional calculus received a great attention from many researchers in different disciplines. In fact, it was discovered that in many situations, physical problems can be modeled more adequately using fractional derivatives rather than ordinary derivatives. In particular, there have been different fractional generalizations of the Schrödinger equation in the literature: a spatial fractional Schrödinger equation which involves fractional order space derivatives (see [7, 8, 9]), a fractional temporal Schrödinger equation involving a fractional time derivative (see [11, 12]), and a spatio-temporal fractional Schrödinger equation with both time and space fractional derivatives (see [1, 13]).

This paper is concerned with the nonexistence of global solutions for fractional temporal Schrödinger equations and systems. We start by considering the nonlinear time fractional Schrödinger equation

$$\begin{gathered} i^\alpha {}_0^C D_t^\alpha u + \Delta u = \lambda |u|^p + \mu a(x)\cdot \nabla |u|^q,\ quadt>0,\ x\in\mathbb{R}^N,\\ u(0,x)=g(x),\quad x\in\mathbb{R}^N, \end{gathered} \tag{1.1}$$

where $u(t,x)$ is a complex-valued function, $0<\alpha<1$, i^α is the principal value of i^α, ${}_0^C D_t^\alpha$ is the Caputo fractional derivative of order α, $\lambda=\lambda_1+i\lambda_2$, $(\lambda_1,\lambda_2)\in$

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$\mathbb{R}^2\backslash\{(0,0)\}$, $\mu=\mu_1+i\mu_2$, $(\mu_1,\mu_2)\in\mathbb{R}^2$, $p>q>1$, the symbol ∇ denotes the gradient with respect to x, $a(x)=(A_1(x),A_2(x),\dots,A_N(x))\in\mathbb{R}^N$, $a(x)\cdot\nabla|u|^q$ is the scalar product of $a(x)$ and $\nabla|u|^q$, and $g(x)=g_1(x)+ig_2(x)$, $(g_1(x),g_2(x))\in\mathbb{R}^2$, $g\in L^1(\mathbb{R}^N)$. Sufficient conditions for the nonexistence of a global weak solution to (1.1) are derived. Next, we are concerned with the system of nonlinear coupled equations

$$
\begin{gathered}
i^{\alpha}\,{}_0^C D_t^{\alpha} u+\Delta u=\lambda|v|^p+\mu a(x)\cdot\nabla|v|^q, \quad t>0,\ x\in\mathbb{R}^N,\\
i^{\beta}\,{}_0^C D_t^{\beta} v+\Delta v=\lambda|u|^\kappa+\mu b(x)\cdot\nabla|u|^\sigma, \quad t>0,\ x\in\mathbb{R}^N,\\
u(0,x)=g(x), \quad v(0,x)=h(x), \quad x\in\mathbb{R}^N,
\end{gathered}
\tag{1.2}
$$

where $0<\beta\le\alpha<1$, $\lambda=\lambda_1+i\lambda_2$, $(\lambda_1,\lambda_2)\in\mathbb{R}^2\backslash\{(0,0)\}$, $\mu=\mu_1+i\mu_2$, $(\mu_1,\mu_2)\in\mathbb{R}^2$, $p>q>1$, $\kappa>\sigma>1$, $a(x)=(A_1(x),A_2(x),\dots,A_N(x))\in\mathbb{R}^N$, $b(x)=(B_1(x),B_2(x),\dots,B_N(x))\in\mathbb{R}^N$, $g(x)=g_1(x)+ig_2(x)$, $(g_1(x),g_2(x))\in\mathbb{R}^2$, $g\in L^1(\mathbb{R}^N)$, and $h(x)=h_1(x)+ih_2(x)$, $(h_1(x),h_2(x))\in\mathbb{R}^2$, $h\in L^1(\mathbb{R}^N)$. The used approach in this paper is based on the test function method [10].

Before we state and prove our results, let us dwell on some existing results on nonexistence of global solutions of nonlinear Schrödinger equations.

Ikeda and Wakasugi [4] studied the nonlinear Schrödinger equation

$$
i\partial_t u+\Delta u=\lambda|u|^p, \quad t>0,\ x\in\mathbb{R}^N,
$$

where $p>1$ and $\lambda\in\mathbb{C}\backslash\{0\}$. They proved that under a condition related to the sign of the initial data, a blow-up of the L^2-norm of solutions occurs if $1<p\le 1+\frac{2}{N}$. This exponent reveals the close relation between the Schrödinger equation and the heat equation as it is the critical exponent for the heat equation in $\mathbb{R}^N$.

Ikeda and Inui [3] derived the blow-up of solution for the semilinear Schrödinger equation with small data when $1<p<1+\frac{4}{N}$. Moreover, they obtained the critical exponent and provided an estimate of the upper bound of the life span.

Kirane and Nabti [6] studied the nonlocal in time nonlinear Schrödinger equation

$$
i\partial_t u+\Delta u=\lambda J_{0|t}^{\alpha}|u|^p, \quad t>0,\ x\in\mathbb{R}^N,
$$

where $p>1$, $\lambda\in\mathbb{C}\backslash\{0\}$, and $J_{0|t}^{\alpha}$ is the Riemann-Liouville fractional integral of order $0<\alpha<1$. Using the test function method, they derived a blow-up exponent. Moreover, they derived an estimate of the life span.

Fino et al. [2] studied the fractional Schrödinger equation

$$
i\partial_t u=(-\Delta)^{\alpha/2}u+\lambda|u|^p, \quad t>0,\ x\in\mathbb{R}^N,
$$

where $(-\Delta)^{\alpha/2}$ is the fractional Laplacian operator of order $\alpha/2$, $0<\alpha<2$, $\lambda\in\mathbb{C}\backslash\{0\}$, and $p>1$. They established a finite-time blow-up result under suitable assumptions on the initial data.

Zhang et al. [14] studied the particular case of (1.1), when $\mu=0$. Under suitable initial data, they proved that the problem admits no global weak solution when $1<p<1+\frac{2}{N}$. Moreover, under certain conditions, they proved that the problem has no global weak solution for all $p>1$.

Motivated by the above cited works, our aim in this paper is to study the nonexistence of global weak solutions to (1.1) and (1.2). This article is organized as follows. In Section 2, we recall some preliminaries on fractional calculus and we fix some notations. In Section 3, we state and prove our results.

2. Preliminaries and notation

In this section, we recall some basic concepts on fractional calculus and present some properties that will be used later. For more details on fractional calculus, we refer the reader to [5]. Next, we fix some notations that will be used through this paper.

Let $f \in L^1(0,T)$, $T>0$, be a given function. The Riemann-Liouville left-sided fractional integral ${}_0I_t^\alpha f$ of order $\alpha>0$ is defined by

$$({}_0I_t^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds,\quad \text{for a.e. } t\in[0,T],$$

where Γ is the Gamma function. The Riemann-Liouville right-sided fractional integral ${}_tI_T^\alpha f$ of order $\alpha>0$ is defined by

$$({}_tI_T^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}\int_t^T (s-t)^{\alpha-1}f(s)\,ds,\ quad\text{for a.e. } t\in[0,T].$$

Let $0<\alpha<1$ and $f\in AC^1[0,T]$, $T>0$. The Caputo left-sided and right-sided fractional derivatives of order α of f are defined, respectively, by

$$({}_0^CD_t^\alpha f)(t)={}_0I_t^{1-\alpha}f'(t),\quad \text{for a.e. } t\in[0,T]$$

and

$$({}_t^CD_T^\alpha f)(t)=-{}_tI_T^{1-\alpha}f'(t),\quad \text{for a.e. } t\in[0,T].$$

The following fractional integration by parts will be used later to define the weak solutions to (1.1) and (1.2).

Lemma 2.1. *Let $0<\alpha<1$. If $f\in C[0,T]$, ${}_0^CD_t^\alpha f\in L^1(0,T)$, $g\in C^1[0,T]$ and $g(T)=0$, then*

$$\int_0^T ({}_0^CD_t^\alpha f)(t)g(t)\,dt=\int_0^T (f(t)-f(0))({}_t^CD_T^\alpha g)(t)\,dt.$$

The following result will be useful later.

Lemma 2.2. *Let $T>0$, $r\geq 1$ and $f:[0,T]\to\mathbb{R}$ be the function given by*

$$f(t)=\Big(1-\frac{t}{T}\Big)^r,\quad 0\leq t\leq T.$$

Then, for any $0<\alpha<1$, we have

$$({}_t^CD_T^\alpha f)(t)=\frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)}T^{-r}(T-t)^{r-\alpha},\quad 0\leq t\leq T.$$

Given a complex number $z\in\mathbb{C}$, we denote by $\operatorname{Re} z$ its real part, and by $\operatorname{Im} z$ its imaginary part. For $T>0$, Let

$$Q_T=(0,T)\times\mathbb{R}^N.$$

Given a function $w(x)$, $x\in\mathbb{R}^N$, $T>0$ and $r>1$, we define the functional space

$$\begin{aligned}&L^r_{\rm loc}(Q_T,w(x)dt\,dx)\\&=\big\{u:Q_T\to\mathbb{C}:\int_K |u|^r w(x)\,dt\,dx<\infty,\ \text{for any compact } K\subset Q_T\big\}.\end{aligned}$$

We define the functions $\mathcal{J},\mathcal{K}:(0,1)\times\mathbb{R}^2\to\mathbb{R}$ by

$$\mathcal{J}(\alpha,(a,b))=\cos\big(\frac{\alpha\pi}{2}\big)a-\sin\big(\frac{\alpha\pi}{2}\big)b,\quad (\alpha,(a,b))\in(0,1)\times\mathbb{R}^2,$$

$$\mathcal{K}(\alpha,(a,b)) = \cos\big(\frac{\alpha\pi}{2}\big)b + \sin\big(\frac{\alpha\pi}{2}\big)a, \quad (\alpha,(a,b)) \in (0,1)\times\mathbb{R}^2.$$

3. Results and proofs

3.1. **Nonexistence of global weak solution for** (1.1)**.** The vector function $a(x) = (A_1(x), A_2(x), \dots, A_N(x))$ is assumed to satisfy the following hypotheses:

- (H1) $\|a(R^{\alpha/2}y)\| = \|y\|^{\delta}O(R^{\tau})$ as $R \to +\infty$, for any $y \neq 0_{\mathbb{R}^N}$ that belongs to a bounded domain of $\mathbb{R}^N$, where $(\delta,\tau)\in\mathbb{R}^2$, and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^N$.
- (H2) $|\operatorname{div} a(R^{\alpha/2}y)| = \|y\|^{\gamma}O(R^{\nu})$ as $R \to +\infty$, for any $y \neq 0_{\mathbb{R}^N}$ that belongs to a bounded domain of $\mathbb{R}^N$, where $\gamma > -\frac{(p-q)}{p}$, $\nu\in\mathbb{R}$, and $\operatorname{div} a$ is the divergence of the vector function a defined by
$$\operatorname{div} a(x) = \sum_{j=1}^{N}\frac{\partial A_j(x)}{\partial x_j}, \quad x\in\mathbb{R}^N.$$

Using the fractional integration by parts given by Lemma 2.1, we define a weak solution to (1.1) as follows.

Definition 3.1. We say that u is a local weak solution to (1.1) if there exists some $0 < T < \infty$ such that

$$u \in L^1((0,T); L^p_{\rm loc}(\mathbb{R}^N)) \cap L^q_{\rm loc}(Q_T, A_j(x)dt\,dx) \cap L^q_{\rm loc}\Big(Q_T, \frac{\partial A_j}{\partial x_j}dt\,dx\Big),$$

for $j = 1,\dots,N$, and

$$\begin{aligned}
&\int_{Q_T} u\left(\Delta\varphi + i^{\alpha}\,{}^{C}_{t}D^{\alpha}_{T}\varphi\right)\,dt\,dx\\
&= \lambda\int_{Q_T}|u|^p\varphi\,dt\,dx - \mu\int_{Q_T}|u|^q a(x)\cdot\nabla\varphi\,dt\,dx\\
&\quad - \mu\int_{Q_T}|u|^q \mathrm{div\,a(x)}\varphi\,\mathrm{dt\,dx} + \mathrm{i}^{\alpha}\int_{\mathrm{Q_T}} \mathrm{g}^{\mathrm{C}}_{\mathrm{t}}\mathrm{D}^{\alpha}_{\mathrm{T}}\varphi\,\mathrm{dt\,dx},
\end{aligned}$$

for every test function $\varphi \in C^{1,2}_{t,x}([0,T]\times\mathbb{R}^N)$ with $\operatorname{supp}_x\varphi \subset\subset \mathbb{R}^N$ and $\varphi(T,\cdot)\equiv 0$. Moreover, if $T>0$ can be arbitrarily chosen, then u is said to be a global weak solution to (1.1).

We have the following result concerning the nonexistence of global solution for (1.1).

Theorem 3.2. *Let $p > q > 1$ and $g \in L^1(\mathbb{R}^N)$. Suppose that one of the following cases holds:*

(I)
$$\lambda_1\int_{\mathbb{R}^N}\mathcal{J}(\alpha,(g_1(x),g_2(x))\,dx > 0\,,$$
and $\mu_1 = 0$, $1 < p < 1+\frac{2}{N}$ or
$$\mu_1 \neq 0, \quad N < \min\Big\{\frac{2}{p-1}, \frac{2\alpha q - 2\tau p - \alpha p}{\alpha(p-q)}, \frac{2\alpha q - 2\nu p - \alpha p}{\alpha(p-q)}\Big\}.$$

(II)
$$\lambda_2 \int_{\mathbb{R}^N} \mathcal{K}(\alpha, (g_1(x), g_2(x))\, dx > 0$$
and $\mu_2 = 0$, $1 < p < 1 + \frac{2}{N}$ *or*
$$\mu_2 \neq 0, \quad N < \min\Big\{\frac{2}{p-1}, \frac{2\alpha q - 2\tau p - \alpha p}{\alpha(p-q)}, \frac{2\alpha q - 2\nu p - \alpha p}{\alpha(p-q)}\Big\}.$$

Then (1.1) *admits no global weak solution.*

Proof. Let $\Phi \in C_0^\infty(\mathbb{R}^N)$ be a function satisfying
$$0 \le \Phi(x) \le 1; \quad \Phi(x) = \begin{cases} 1 & \text{if } 0 \le \|x\| \le 1, \\ 0 & \text{if } \|x\| \ge 2. \end{cases} \tag{3.1}$$
For $T > 0$, we define the functions
$$\varphi_1(x) = \Big(\Phi\big(T^{\frac{-\alpha}{2}} x\big)\Big)^{\omega}, \quad x \in \mathbb{R}^N,$$
$$\varphi_2(t) = \big(1 - \frac{t}{T}\big)^m, \quad 0 \le t \le T, \tag{3.2}$$
$$\varphi(t,x) = \varphi_1(x)\varphi_2(t), \quad (t,x) \in Q_T, \tag{3.3}$$
where $\omega \ge \max\Big\{\frac{2p}{p-1}, \frac{p}{p-q}\Big\}$ and $m \ge \max\{1, \frac{\alpha p}{p-1}\}$. It can be easily seen that $\varphi \in C^{1,2}_{t,x}([0,T] \times \mathbb{R}^N)$ with $\operatorname{supp}_x \varphi \subset\subset \mathbb{R}^N$ and $\varphi(T,\cdot) \equiv 0$.

Suppose that u is a global weak solution to (1.1). First, we consider the case
$$\lambda_1 \int_{\mathbb{R}^N} \mathcal{J}(\alpha, (g_1(x), g_2(x))\, dx > 0. \tag{3.4}$$
By Definition 3.1, we have
$$\begin{aligned} &\operatorname{Re} \int_{Q_T} u \left(\Delta\varphi + i^{\alpha}\, {}^C_t D^{\alpha}_T \varphi\right)\, dt\, dx \\ &= \operatorname{Re}\Big(\lambda \int_{Q_T} |u|^p \varphi\, dt\, dx - \mu \int_{Q_T} |u|^q a(x) \cdot \nabla\varphi\, dt\, dx \\ &\quad - \mu \int_{Q_T} |u|^q \operatorname{div} a(x) \varphi\, dt\, dx + i^{\alpha} \int_{Q_T} g\, {}^C_t D^{\alpha}_T \varphi\, dt\, dx\Big), \end{aligned}$$
which implies
$$\begin{aligned} &\int_{Q_T} |u|^p \varphi\, dt\, dx + \frac{1}{\lambda_1} \int_{Q_T} \mathcal{J}(\alpha, (g_1(x), g_2(x))\, {}^C_t D^{\alpha}_T \varphi\, dt\, dx \\ &= \frac{1}{\lambda_1} \int_{Q_T} \Big[(\operatorname{Re} u)\Delta\varphi + \Big(\cos\big(\frac{\alpha\pi}{2}\big) \operatorname{Re} u - \sin\big(\frac{\alpha\pi}{2}\big) \operatorname{Im} u\Big)\, {}^C_t D^{\alpha}_T \varphi\Big]\, dt\, dx \\ &\quad + \frac{\mu_1}{\lambda_1}\Big(\int_{Q_T} |u|^q a(x) \cdot \nabla\varphi\, dt\, dx + \int_{Q_T} |u|^q \operatorname{div} a(x) \varphi\, dt\, dx\Big). \end{aligned} \tag{3.5}$$
Next, we shall estimate each term of the right-hand side of the above inequality. First, we have
$$\begin{aligned} &\frac{1}{\lambda_1} \int_{Q_T} \Big[(\operatorname{Re} u)\Delta\varphi + \Big(\cos\big(\frac{\alpha\pi}{2}\big) \operatorname{Re} u - \sin\big(\frac{\alpha\pi}{2}\big) \operatorname{Im} u\Big)\, {}^C_t D^{\alpha}_T \varphi\Big]\, dt\, dx \\ &\le \frac{1}{|\lambda_1|} \int_{Q_T} \big[|u||\Delta\varphi| + 2|u||{}^C_t D^{\alpha}_T \varphi|\big]\, dt\, dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|\lambda_1|}\int_{Q_T} |u| \left(|\Delta\varphi| + 2|{}_t^C D_T^\alpha \varphi|\right)\, dt\, dx \\
&= \frac{1}{|\lambda_1|}\int_{Q_T} |u|\varphi^{\frac{1}{p}} \left(|\Delta\varphi| + 2|{}_t^C D_T^\alpha \varphi|\right) \varphi^{-\frac{1}{p}}\, dt\, dx.
\end{aligned}
$$

Further, using the ε-Young inequality with parameters p and $\frac{p}{p-1}$, we obtain

$$
\begin{aligned}
&\frac{1}{\lambda_1}\int_{Q_T}\Big[(\operatorname{Re} u)\Delta\varphi + \Big(\cos\big(\frac{\alpha\pi}{2}\big)\operatorname{Re} u - \sin\big(\frac{\alpha\pi}{2}\big)\operatorname{Im} u\Big)\, {}_t^C D_T^\alpha \varphi\Big]\, dt\, dx \\
&\le \frac{1}{|\lambda_1|}\Big(\varepsilon\int_{Q_T} |u|^p\varphi\, dt\, dx + c_\varepsilon \int_{Q_T}\Big(|\Delta\varphi| + 2|{}_t^C D_T^\alpha\varphi|\Big)^{\frac{p}{p-1}}\varphi^{-\frac{1}{p-1}}\, dt\, dx\Big),
\end{aligned} \tag{3.6}
$$

where $\varepsilon > 0$ and $c_\varepsilon > 0$ is a constant. Using the ε-Young inequality with parameters $\frac{p}{q}$ and $\frac{p}{p-q}$, we obtain

$$
\begin{aligned}
&\frac{\mu_1}{\lambda_1}\Big(\int_{Q_T} |u|^q a(x)\cdot\nabla\varphi\, dt\, dx + \int_{Q_T} |u|^q \operatorname{div} a(x)\varphi\, dt\, dx\Big) \\
&\le \frac{|\mu_1|}{|\lambda_1|}\int_{Q_T} |u|^q \left(\|a(x)\|\|\nabla\varphi\| + |\operatorname{div} a(x)||\varphi|\right)\, dt\, dx \\
&\le \frac{|\mu_1|}{|\lambda_1|}\Big(\varepsilon\int_{Q_T}|u|^p\varphi\, dt\, dx + d_\varepsilon\int_{Q_T}\big[(\|a(x)\|\|\nabla\varphi\| \\
&\quad + |\operatorname{div} a(x)||\varphi|)\varphi^{\frac{-q}{p}}\big]^{\frac{p}{p-q}}\, dt\, dx\Big),
\end{aligned} \tag{3.7}
$$

where $\varepsilon > 0$ and $d_\varepsilon > 0$ is a constant. Combining (3.5), (3.6) and (3.7), we obtain

$$
\begin{aligned}
&\Big(1 - \frac{\varepsilon}{|\lambda_1|} - \frac{\varepsilon|\mu_1|}{|\lambda_1|}\Big)\int_{Q_T}|u|^p\varphi\, dt\, dx + \frac{1}{\lambda_1}\int_{Q_T}\mathcal{J}(\alpha,(g_1(x),g_2(x))_t^C D_T^\alpha\varphi\, dt\, dx \\
&\le \frac{c_\varepsilon}{|\lambda_1|}\int_{Q_T}\left(|\Delta\varphi| + 2|{}_t^C D_T^\alpha\varphi|\right)^{\frac{p}{p-1}}\varphi^{-\frac{1}{p-1}}\, dt\, dx \\
&\quad + \frac{d_\varepsilon|\mu_1|}{|\lambda_1|}\int_{Q_T}\Big[\left(\|a(x)\|\|\nabla\varphi\| + |\operatorname{div} a(x)||\varphi|\right)\varphi^{\frac{-q}{p}}\Big]^{\frac{p}{p-q}}\, dt\, dx \\
&:= I_1 + I_2.
\end{aligned} \tag{3.8}
$$

Estimate of I_1. Using the inequality

$$
(a+b)^{\frac{p}{p-1}} \le 2^{\frac{1}{p-1}}\left(a^{\frac{p}{p-1}} + b^{\frac{p}{p-1}}\right), \quad a \ge 0,\ b \ge 0,
$$

we obtain

$$
\frac{|\lambda_1|}{c_\varepsilon} I_1 \le C_p\Big(\int_{Q_T}|\Delta\varphi|^{\frac{p}{p-1}}\varphi^{\frac{-1}{p-1}}\, dt\, dx + \int_{Q_T}|{}_t^C D_T^\alpha\varphi|^{\frac{p}{p-1}}\varphi^{\frac{-1}{p-1}}\, dt\, dx\Big), \tag{3.9}
$$

where $C_p = 2^{\frac{p+1}{p-1}}$. On the other hand, by (3.3), we have

$$
\begin{aligned}
&\int_{Q_T}|\Delta\varphi|^{\frac{p}{p-1}}\varphi^{\frac{-1}{p-1}}\, dt\, dx \\
&= \Big(\int_0^T\Big(1 - \frac{t}{T}\Big)^m dt\Big)\Big(\int_{0\le\|x\|\le 2T^{\alpha/2}}\left|\Delta\big(\Phi(T^{\frac{-\alpha}{2}}x)\big)^\omega\right|^{\frac{p}{p-1}}\big(\Phi(T^{\frac{-\alpha}{2}}x)\big)^{\frac{-\omega}{p-1}}\, dx\Big).
\end{aligned} \tag{3.10}
$$

A simple computation yields

$$\int_0^T \Big(1-\frac{t}{T}\Big)^m\,dt = \frac{T}{m+1}. \tag{3.11}$$

Moreover, using the change of variable $y = T^{\frac{-\alpha}{2}}x$, we obtain

$$\begin{aligned}
&\int_{0\le\|x\|\le 2T^{\alpha/2}} |\Delta\Big(\Phi(T^{\frac{-\alpha}{2}}x)\Big)^{\omega}|^{\frac{p}{p-1}}\Big(\Phi(T^{\frac{-\alpha}{2}}x)\Big)^{\frac{-\omega}{p-1}}\,dx \\
&= T^{\frac{\alpha N}{2}-\frac{\alpha p}{p-1}}\int_{0\le\|y\|\le 2}|\Delta(\Phi(y))^{\omega}|^{\frac{p}{p-1}}(\Phi(y))^{\frac{-\omega}{p-1}}\,dy.
\end{aligned}\tag{3.12}$$

Note that because $\omega \ge \frac{2p}{p-1}$, we have

$$\int_{0\le\|y\|\le 2}|\Delta(\Phi(y))^{\omega}|^{\frac{p}{p-1}}(\Phi(y))^{\frac{-\omega}{p-1}}\,dy < \infty.$$

Combining (3.10), (3.11) and (3.12), we obtain

$$\begin{aligned}
&\int_{Q_T}|\Delta\varphi|^{\frac{p}{p-1}}\varphi^{\frac{-1}{p-1}}\,dt\,dx \\
&= \Big(\frac{1}{m+1}\int_{0\le\|y\|\le 2}|\Delta(\Phi(y))^{\omega}|^{\frac{p}{p-1}}(\Phi(y))^{\frac{-\omega}{p-1}}\,dy\Big)T^{\frac{\alpha N}{2}-\frac{\alpha p}{p-1}+1}.
\end{aligned}\tag{3.13}$$

Next, by (3.3), we have

$$\begin{aligned}
&\int_{Q_T}|{}_t^C D_T^{\alpha}\varphi|^{\frac{p}{p-1}}\varphi^{\frac{-1}{p-1}}\,dt\,dx \\
&= \Big(\int_{\mathbb{R}^N}\varphi_1(x)\,dx\Big)\Big(\int_0^T|{}_t^C D_T^{\alpha}\varphi_2(t)|^{\frac{p}{p-1}}(\varphi_2(t))^{\frac{-1}{p-1}}\,dt\Big).
\end{aligned}\tag{3.14}$$

On the other hand, it can be easily seen that

$$\int_{\mathbb{R}^N}\varphi_1(x)\,dx = T^{\frac{\alpha N}{2}}\int_{0\le\|y\|\le 2}\Phi(y)\,dy. \tag{3.15}$$

Further, by Lemma 2.2, we have

$${}_t^C D_T^{\alpha}\varphi_2(t) = \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}T^{-m}(T-t)^{m-\alpha}.$$

A simple computation yields

$$\begin{aligned}
&\int_0^T|{}_t^C D_T^{\alpha}\varphi_2(t)|^{\frac{p}{p-1}}(\varphi_2(t))^{\frac{-1}{p-1}}\,dt \\
&= \Big[\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}\Big]^{\frac{p}{p-1}}\frac{(p-1)T^{1-\frac{\alpha p}{p-1}}}{p(m-\alpha+1)-(m+1)}.
\end{aligned}\tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we obtain

$$\int_{Q_T}|{}_t^C D_T^{\alpha}\varphi|^{\frac{p}{p-1}}\varphi^{\frac{-1}{p-1}}\,dt\,dx = C(m,p)T^{1-\frac{\alpha p}{p-1}+\frac{\alpha N}{2}}, \tag{3.17}$$

where

$$C(m,p) = \Big(\int_{0\le\|y\|\le 2}\Phi(y)\,dy\Big)\Big[\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}\Big]^{\frac{p}{p-1}}\frac{(p-1)}{p(m-\alpha+1)-(m+1)}.$$

Hence, combining (3.9), (3.13) and (3.17), we obtain

$$I_1 \le C_1 T^{1-\frac{\alpha p}{p-1}+\frac{\alpha N}{2}}, \tag{3.18}$$

where $C_1 > 0$ is a certain constant (independent of T).

Estimate of I_2. Using the inequality

$$(a+b)^{\frac{p}{p-q}} \le 2^{\frac{q}{p-q}} \left(a^{\frac{p}{p-q}} + b^{\frac{p}{p-q}}\right), \quad a \ge 0,\, b \ge 0,$$

we obtain

$$\begin{aligned} &\frac{|\lambda_1|}{d_\varepsilon |\mu_1|} I_2 \\ &\le C_{p,q}\Big(\int_{Q_T} \|a(x)\|^{\frac{p}{p-q}} \|\nabla\varphi\|^{\frac{p}{p-q}} \varphi^{\frac{-q}{p-q}}\, dt\, dx + \int_{Q_T} |\operatorname{div} a(x)|^{\frac{p}{p-q}} \varphi\, dt\, dx\Big), \end{aligned} \tag{3.19}$$

where $C_{p,q} = 2^{\frac{q}{p-q}}$. On the other hand, by (3.3) and (3.11), we have

$$\begin{aligned} &\int_{Q_T} \|a(x)\|^{\frac{p}{p-q}} \|\nabla\varphi\|^{\frac{p}{p-q}} \varphi^{\frac{-q}{p-q}}\, dt\, dx \\ &= \frac{T}{m+1}\left(\int_{\mathbb{R}^N} \|a(x)\|^{\frac{p}{p-q}} \|\nabla\varphi_1\|^{\frac{p}{p-q}} \varphi_1^{\frac{-q}{p-q}}\, dx\right). \end{aligned} \tag{3.20}$$

Further, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \|a(x)\|^{\frac{p}{p-q}} \|\nabla\varphi_1\|^{\frac{p}{p-q}} \varphi_1^{\frac{-q}{p-q}}\, dx \\ &= \int_{T^{\alpha/2} \le \|x\| \le 2T^{\alpha/2}} \|a(x)\|^{\frac{p}{p-q}} \|\nabla \left(\Phi(T^{\frac{-\alpha}{2}}x)\right)^{\omega}\|^{\frac{p}{p-q}} \left(\Phi(T^{\frac{-\alpha}{2}}x)\right)^{\frac{-q\omega}{p-q}}\, dx. \end{aligned}$$

Using the change of variable $y = T^{\frac{-\alpha}{2}}x$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \|a(x)\|^{\frac{p}{p-q}} \|\nabla\varphi_1\|^{\frac{p}{p-q}} \varphi_1^{\frac{-q}{p-q}}\, dx \\ &= T^{\frac{-\alpha p}{2(p-q)}+\frac{\alpha N}{2}} \int_{1\le\|y\|\le 2} \|a(T^{\alpha/2}y)\|^{\frac{p}{p-q}} \|\nabla(\Phi(y))^{\omega}\|^{\frac{p}{p-q}} (\Phi(y))^{\frac{-q\omega}{p-q}}\, dy. \end{aligned}$$

Using (H1), for T large enough, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \|a(x)\|^{\frac{p}{p-q}} \|\nabla\varphi_1\|^{\frac{p}{p-q}} \varphi_1^{\frac{-q}{p-q}}\, dx \\ &\le C_a T^{\frac{-\alpha p}{2(p-q)}+\frac{\alpha N}{2}+\frac{\tau p}{p-q}} \int_{1\le\|y\|\le 2} \|\nabla(\Phi(y))^{\omega}\|^{\frac{p}{p-q}} (\Phi(y))^{\frac{-q\omega}{p-q}}\, dy, \end{aligned} \tag{3.21}$$

where $C_a > 0$ is a certain constant. Note that since $\omega \ge \frac{p}{p-q}$, we have

$$\int_{1\le\|y\|\le 2} \|\nabla(\Phi(y))^{\omega}\|^{\frac{p}{p-q}} (\Phi(y))^{\frac{-q\omega}{p-q}}\, dy < \infty.$$

Combining (3.20) and (3.21), we obtain

$$\begin{aligned} &\int_{Q_T} \|a(x)\|^{\frac{p}{p-q}} \|\nabla\varphi\|^{\frac{p}{p-q}} \varphi^{\frac{-q}{p-q}}\, dt\, dx \\ &\le \frac{C_a}{m+1} T^{\frac{-\alpha p}{2(p-q)}+\frac{\alpha N}{2}+\frac{\tau p}{p-q}+1} \int_{1\le\|y\|\le 2} \|\nabla(\Phi(y))^{\omega}\|^{\frac{p}{p-q}} (\Phi(y))^{\frac{-q\omega}{p-q}}\, dy. \end{aligned} \tag{3.22}$$

Again, by (3.3) and (3.11), we have

$$\int_{Q_T} |\operatorname{div} a(x)|^{\frac{p}{p-q}} \varphi\, dt\, dx = \frac{T}{m+1} \int_{0\le\|x\|\le 2T^{\alpha/2}} |\operatorname{div} a(x)|^{\frac{p}{p-q}} \Big(\Phi\big(T^{\frac{-\alpha}{2}} x\big)\Big)^{\omega}\, dx.$$

Using the same change of variable $y = T^{\frac{-\alpha}{2}} x$, we obtain

$$\int_{Q_T} |\operatorname{div} a(x)|^{\frac{p}{p-q}} \varphi\, dt\, dx = \frac{T^{1-\frac{\alpha p}{2(p-q)}+\frac{\alpha N}{2}}}{m+1} \int_{0\le\|y\|\le 2} |\operatorname{div} a(T^{\alpha/2} y)|^{\frac{p}{p-q}} (\Phi(y))^{\omega}\, dy.$$

Next, by (H2), for T large enough, we have

$$\begin{aligned} &\int_{Q_T} |\operatorname{div} a(x)|^{\frac{p}{p-q}} \varphi\, dt\, dx \\ &\le C_a' T^{1-\frac{\alpha p}{2(p-q)}+\frac{\alpha N}{2}+\frac{\nu p}{p-q}} \int_{0\le\|y\|\le 2} \|y\|^{\frac{\gamma p}{p-q}} (\Phi(y))^{\omega}\, dy, \end{aligned} \tag{3.23}$$

where $C_a' > 0$ is a certain constant. Note that since $\gamma > -\frac{(p-q)}{p}$, we have

$$\int_{0\le\|y\|\le 2} \|y\|^{\frac{\gamma p}{p-q}} (\Phi(y))^{\omega}\, dy < \infty.$$

Further, combining (3.19), (3.22) and (3.23), we obtain

$$I_2 \le C_2 |\mu_1| \Big(T^{1-\frac{\alpha p}{2(p-q)}+\frac{\alpha N}{2}+\frac{\tau p}{p-q}} + T^{1-\frac{\alpha p}{2(p-q)}+\frac{\alpha N}{2}+\frac{\nu p}{p-q}} \Big), \tag{3.24}$$

where $C_2 > 0$ is a certain constant. On the other hand, we have

$$\begin{aligned} &\frac{1}{\lambda_1} \int_{Q_T} \mathcal{J}(\alpha, (g_1(x), g_2(x)) {}_t^C D_T^{\alpha} \varphi\, dt\, dx \\ &= \frac{1}{\lambda_1} \frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)} T^{1-\alpha} \int_{\mathbb{R}^N} \mathcal{J}(\alpha, (g_1(x), g_2(x)) \Phi\big(T^{\frac{-\alpha}{2}} x\big)^{\omega}\, dx. \end{aligned} \tag{3.25}$$

Combining (3.8), 3.18 and (3.24), and taking

$$\varepsilon = \frac{|\lambda_1|}{2(1+|\mu_1|)},$$

we obtain

$$\begin{aligned} &\frac{1}{2} \int_{Q_T} |u|^p \varphi\, dt\, dx \\ &+ \frac{1}{\lambda_1} \frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)} T^{1-\alpha} \int_{\mathbb{R}^N} \mathcal{J}(\alpha, (g_1(x), g_2(x)) \Phi\big(T^{\frac{-\alpha}{2}} x\big)^{\omega}\, dx \\ &\le C \Big(T^{1-\frac{\alpha p}{p-1}+\frac{\alpha N}{2}} + |\mu_1| \Big(T^{1-\frac{\alpha p}{2(p-q)}+\frac{\alpha N}{2}+\frac{\tau p}{p-q}} + T^{1-\frac{\alpha p}{2(p-q)}+\frac{\alpha N}{2}+\frac{\nu p}{p-q}} \Big) \Big), \end{aligned}$$

where $C > 0$ is a certain constant. This implies that

$$\begin{aligned} &\frac{1}{\lambda_1} \frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)} \int_{\mathbb{R}^N} \mathcal{J}(\alpha, (g_1(x), g_2(x)) \Phi\big(T^{\frac{-\alpha}{2}} x\big)^{\omega}\, dx \\ &\le C \Big(T^{\alpha-\frac{\alpha p}{p-1}+\frac{\alpha N}{2}} + |\mu_1| \Big(T^{\alpha-\frac{\alpha p}{2(p-q)}+\frac{\alpha N}{2}+\frac{\tau p}{p-q}} + T^{\alpha-\frac{\alpha p}{2(p-q)}+\frac{\alpha N}{2}+\frac{\nu p}{p-q}} \Big) \Big). \end{aligned} \tag{3.26}$$

We discuss two cases.

Case 1: $\mu_1 = 0$, $1 < p < 1 + \frac{2}{N}$. In this case, passing to the limit as $T \to +\infty$ in (3.26), we obtain

$$\frac{1}{\lambda_1} \int_{\mathbb{R}^N} \mathcal{J}(\alpha, (g_1(x), g_2(x))\, dx \leq 0,$$

which contradicts (3.4).

Case 2: $\mu_1 \neq 0$ and

$$N < \min\big\{\frac{2}{p-1}, \frac{2\alpha q - 2\tau p - \alpha p}{\alpha(p-q)}, \frac{2\alpha q - 2\nu p - \alpha p}{\alpha(p-q)}\big\}.$$

Passing to the limit as $T \to +\infty$ in (3.26), we obtain

$$\frac{1}{\lambda_1} \int_{\mathbb{R}^N} \mathcal{J}(\alpha, (g_1(x), g_2(x))\, dx \leq 0,$$

which contradicts (3.4).

Next, we suppose that

$$\lambda_2 \int_{\mathbb{R}^N} \mathcal{K}(\alpha, (g_1(x), g_2(x))\, dx > 0. \tag{3.27}$$

Observe that

$$v(t,x) = \frac{u(t,x)}{i}, \quad t \geq 0,\ x \in \mathbb{R}^N$$

is a global weak solution to the problem

$$\begin{gathered} i^{\alpha}\,{}^C_0 D^{\alpha}_t v + \Delta v = \lambda' |v|^p + \mu' a(x) \cdot \nabla |v|^q, \quad t > 0,\, x \in \mathbb{R}^N, \\ v(0,x) = \widetilde{g(x)}, \quad x \in \mathbb{R}^N, \end{gathered}$$

where

$$\begin{gathered} \lambda' = \lambda_2 + (-\lambda_1) i := \lambda'_1 + i\lambda'_2, \\ \mu' = \mu_2 + (-\mu_1) i := \mu'_1 + i\mu'_2, \widetilde{g(x)} = g_2(x) + (-g_1(x)) i := \widetilde{g}_1(x) + i\widetilde{g}_2(x), \end{gathered}$$

for $x \in \mathbb{R}^N$. It can be easily seen that (3.27) is equivalent to

$$\lambda'_1 \int_{\mathbb{R}^N} \mathcal{J}(\alpha, (\widetilde{g}_1(x), \widetilde{g}_2(x))\, dx > 0.$$

Therefore, from the previous case, if

$$\mu_2 = 0,\ 1 < p < 1 + \frac{2}{N},$$

we obtain a contradiction with (3.27). Similarly, if

$$\mu_2 \neq 0, \quad N < \min\big\{\frac{2}{p-1}, \frac{2\alpha q - 2\tau p - \alpha p}{\alpha(p-q)}, \frac{2\alpha q - 2\nu p - \alpha p}{\alpha(p-q)}\big\},$$

we obtain a contradiction with (3.27). □

Remark 3.3. Taking $\mu = 0$ in Theorem 3.2, we obtain the result given by [14, Theorem 2.2].

3.2. **Nonexistence of global weak solution for System** (1.2)**.** The vector functions $a(x) = (A_1(x), A_2(x), \dots, A_N(x))$ and $b(x) = (B_1(x), B_2(x), \dots, B_N(x))$ are assumed to satisfy the following hypotheses:

(H1) $\|a(R^{\frac{\alpha+\beta}{4}}y)\| = \|y\|^{\delta}O(R^{\tau})$ as $R \to +\infty$, for any $y \neq 0_{\mathbb{R}^N}$ that belongs to a bounded domain of $\mathbb{R}^N$, where $(\delta, \tau) \in \mathbb{R}^2$.

(H2) $|\operatorname{div} a\big(R^{\frac{\alpha+\beta}{4}}y\big)| = \|y\|^{\gamma}O(R^{\nu})$ as $R \to +\infty$, for any $y \neq 0_{\mathbb{R}^N}$ that belongs to a bounded domain of $\mathbb{R}^N$, where $\gamma > -\frac{(p-q)}{p}$ and $\nu \in \mathbb{R}$.

(H3) $\|b(R^{\frac{\alpha+\beta}{4}}y)\| = \|y\|^{\xi}O\left(R^{\chi}\right)$ as $R \to +\infty$, for any $y \neq 0_{\mathbb{R}^N}$ that belongs to a bounded domain of $\mathbb{R}^N$, where $(\xi, \chi) \in \mathbb{R}^2$.

(H4) $|\operatorname{div} b\big(R^{\frac{\alpha+\beta}{4}}y\big)| = \|y\|^{\theta}O(R^{\ell})$ as $R \to +\infty$, for any $y \neq 0_{\mathbb{R}^N}$ that belongs to a bounded domain of $\mathbb{R}^N$, where $\theta > -\frac{(\kappa-\sigma)}{\kappa}$ and $\ell \in \mathbb{R}$.

We adopt the following definition for weak solutions to (1.2).

Definition 3.4. We say that (u, v) is a local weak solution to (1.2) if there exists some $0 < T < \infty$ such that

$$u \in L^1((0,T); L^{\kappa}_{\rm loc}(\mathbb{R}^N)) \cap L^{\sigma}_{\rm loc}(Q_T, B_j(x)dt\,dx) \cap L^{\sigma}_{\rm loc}\Big(Q_T, \frac{\partial B_j}{\partial x_j}dt\,dx\Big),$$

$$v \in L^1((0,T); L^{p}_{\rm loc}(\mathbb{R}^N)) \cap L^{q}_{\rm loc}(Q_T, A_j(x)dt\,dx) \cap L^{q}_{\rm loc}V = \Big(Q_T, \frac{\partial A_j}{\partial x_j}dt\,dx\Big),$$

for $j = 1, 2, \dots, N$, and

$$\begin{aligned} &\int_{Q_T} u\left(\Delta\varphi + i^{\alpha}{}^C_t D^{\alpha}_T\varphi\right)\,dt\,dx \\ &= \lambda\int_{Q_T} |v|^p\varphi\,dt\,dx - \mu\int_{Q_T} |v|^q a(x)\cdot\nabla\varphi\,dt\,dx \\ &\quad - \mu\int_{Q_T} |v|^q \mathrm{div\,a(x)}\varphi\,\mathrm{dt\,dx} + \mathrm{i}^{\alpha}\int_{\mathrm{Q_T}} \mathrm{g}^{\mathrm{C}}_{\mathrm{t}}\mathrm{D}^{\alpha}_{\mathrm{T}}\varphi\,\mathrm{dt\,dx}, \end{aligned} \tag{3.28}$$

$$\begin{aligned} &\int_{Q_T} v\left(\Delta\varphi + i^{\beta}{}^C_t D^{\beta}_T\varphi\right)\,dt\,dx \\ &= \lambda\int_{Q_T} |u|^{\kappa}\varphi\,dt\,dx - \mu\int_{Q_T} |u|^{\sigma} b(x)\cdot\nabla\varphi\,dt\,dx \\ &\quad - \mu\int_{Q_T} |u|^{\sigma} \mathrm{div\,b(x)}\varphi\,\mathrm{dt\,dx} + \mathrm{i}^{\beta}\int_{\mathrm{Q_T}} \mathrm{h}^{\mathrm{C}}_{\mathrm{t}}\mathrm{D}^{\beta}_{\mathrm{T}}\varphi\,\mathrm{dt\,dx}, \end{aligned} \tag{3.29}$$

for every test function $\varphi \in C^{1,2}_{t,x}([0,T] \times \mathbb{R}^N)$ with $\operatorname{supp}_x \varphi \subset\subset \mathbb{R}^N$ and $\varphi(T, \cdot) \equiv 0$. Moreover, if $T > 0$ can be arbitrarily chosen, then (u, v) is said to be a global weak solution to (1.2).

For any $(r, s) \in (0,1) \times (0,1)$, let us define the quantities:

$$\theta_1(r,s) = \frac{4}{r+s}\big(\frac{sp}{p-1} - r\big), \quad \theta_2(r,s) = \frac{4}{r+s}\big(\frac{(r+s)\kappa}{2(\kappa-1)} - r\big),$$

$$\theta_3(r,s) = \frac{4}{r+s}\big(\frac{(r+s)p}{4(p-q)} - \frac{\tau p}{p-q} - r\big), \quad \theta_4(r,s) = \frac{4}{r+s}\big(\frac{(r+s)p}{4(p-q)} - \frac{\nu p}{p-q} - r\big),$$

$$\theta_5(r,\beta) = \frac{4}{r+s}\Big(\frac{(r+s)\kappa}{4(\kappa-\sigma)} - \frac{\chi\kappa}{\kappa-\sigma} - r\Big),$$

$$\theta_6(r,s)=\frac{4}{r+s}\Big(\frac{(r+s)\kappa}{4(\kappa-\sigma)}-\frac{\ell\kappa}{\kappa-\sigma}-r\Big).$$

We have the following nonexistence result for (1.2).

Theorem 3.5. *Let $0<\beta\leq\alpha<1$, $p>q>1$, $\kappa>\sigma>1$, and $g,h\in L^1(\mathbb{R}^N)$. Suppose that one of the following cases holds:*

(I)

$$\alpha=\beta,\quad \lambda_1\int_{\mathbb{R}^N}\left(\mathcal{J}(\alpha,(g_1(x),g_2(x))+\mathcal{J}(\beta,(h_1(x),h_2(x))\right)\,dx>0$$

and

$$\mu_1=0,\ N<2\min\big\{\frac{1}{p-1},\frac{1}{\kappa-1}\big\}\quad or$$
$$\mu_1\neq 0,\quad N<\min_{j=1,\dots,6}\theta_j(\alpha,\alpha).$$

(II)

$$\beta<\alpha,\quad \lambda_1\int_{\mathbb{R}^N}\mathcal{J}(\beta,(h_1(x),h_2(x))\,dx>0$$

and

$$\mu_1=0,\quad N<\frac{4}{\alpha+\beta}\min\big\{\frac{\beta p}{p-1}-\alpha,\frac{(\alpha+\beta)\kappa}{2(\kappa-1)}-\alpha\big\}\quad or$$
$$\mu_1\neq 0,\quad N<\min_{j=1,\dots,6}\theta_j(\alpha,\beta).$$

(III)

$$\alpha=\beta,\quad \lambda_2\int_{\mathbb{R}^N}\left(\mathcal{K}(\alpha,(g_1(x),g_2(x))+\mathcal{K}(\beta,(h_1(x),h_2(x))\right)\,dx>0$$

and

$$\mu_2=0,\ N<2\min\big\{\frac{1}{p-1},\frac{1}{\kappa-1}\big\}\quad or$$
$$\mu_2\neq 0,\ N<\min_{j=1,\dots,6}\theta_j(\alpha,\alpha).$$

(IV)

$$\beta<\alpha,\quad \lambda_2\int_{\mathbb{R}^N}\mathcal{K}(\beta,(h_1(x),h_2(x))\,dx>0$$

and

$$\mu_2=0,\quad N<\frac{4}{\alpha+\beta}\min\big\{\frac{\beta p}{p-1}-\alpha,\frac{(\alpha+\beta)\kappa}{2(\kappa-1)}-\alpha\big\}\quad or$$
$$\mu_2\neq 0,\quad N<\min_{j=1,\dots,6}\theta_j(\alpha,\beta).$$

Then (1.2) *admits no global weak solution.*

Proof. Let φ be the test function defined by

$$\varphi(t,x)=\varphi_1(x)\varphi_2(t),\quad (t,x)\in Q_T,$$

where φ_2 is given by (3.2),

$$\varphi_1(x)=\Big(\Phi\Big(T^{\frac{-(\alpha+\beta)}{4}}x\Big)\Big)^{\omega},\quad x\in\mathbb{R}^N,$$

Φ is given by (3.1), and ω and m are supposed to be large enough.

Suppose that (u,v) is a global weak solution to (1.2). First, we consider the case $\lambda_1 \neq 0$. Therefore, by (3.28), we have

$$\begin{aligned}
&\operatorname{Re}\int_{Q_T} u\left(\Delta\varphi + i^{\alpha}\,{}_t^C D_T^{\alpha}\varphi\right)\,dt\,dx\\
&= \operatorname{Re}\Big(\lambda\int_{Q_T}|v|^p\varphi\,dt\,dx - \mu\int_{Q_T}|v|^q a(x)\cdot\nabla\varphi\,dt\,dx\\
&\quad - \mu\int_{Q_T}|v|^q \operatorname{div} a(x)\varphi\,dt\,dx + i^{\alpha}\int_{Q_T} g_t^C D_T^{\alpha}\varphi\,dt\,dx\Big),
\end{aligned}$$

which implies

$$\begin{aligned}
&\int_{Q_T}|v|^p\varphi\,dt\,dx + \frac{1}{\lambda_1}\int_{Q_T}\mathcal{J}(\alpha,(g_1(x),g_2(x)))_t^C D_T^{\alpha}\varphi\,dt\,dx\\
&= \frac{1}{\lambda_1}\int_{Q_T}\Big[(\operatorname{Re}u)\Delta\varphi + \Big(\cos\big(\frac{\alpha\pi}{2}\big)\operatorname{Re}u - \sin\big(\frac{\alpha\pi}{2}\big)\operatorname{Im}u\Big){}_t^C D_T^{\alpha}\varphi\Big]\,dt\,dx \qquad (3.30)\\
&\quad + \frac{\mu_1}{\lambda_1}\Big(\int_{Q_T}|v|^q a(x)\cdot\nabla\varphi\,dt\,dx + \int_{Q_T}|v|^q\operatorname{div} a(x)\varphi\,dt\,dx\Big).
\end{aligned}$$

Next, let us estimate each term of the right-hand side of the above inequality. First, we have

$$\begin{aligned}
&\frac{1}{\lambda_1}\int_{Q_T}\Big[(\operatorname{Re}u)\Delta\varphi + \Big(\cos\big(\frac{\alpha\pi}{2}\big)\operatorname{Re}u - \sin\big(\frac{\alpha\pi}{2}\big)\operatorname{Im}u\Big){}_t^C D_T^{\alpha}\varphi\Big]\,dt\,dx\\
&\le \frac{1}{|\lambda_1|}\int_{Q_T}\big[|u||\Delta\varphi| + 2|u||{}_t^C D_T^{\alpha}\varphi|\big]\,dt\,dx\\
&= \frac{1}{|\lambda_1|}\int_{Q_T}|u|\left(|\Delta\varphi| + 2|{}_t^C D_T^{\alpha}\varphi|\right)\,dt\,dx\\
&= \frac{1}{|\lambda_1|}\int_{Q_T}|u|\varphi^{\frac{1}{\kappa}}\left(|\Delta\varphi| + 2|{}_t^C D_T^{\alpha}\varphi|\right)\varphi^{-\frac{1}{\kappa}}\,dt\,dx.
\end{aligned}$$

Further, using the ε-Young inequality with parameters κ and $\frac{\kappa}{\kappa-1}$, we obtain

$$\begin{aligned}
&\frac{1}{\lambda_1}\int_{Q_T}\Big[(\operatorname{Re}u)\Delta\varphi + \Big(\cos\big(\frac{\alpha\pi}{2}\big)\operatorname{Re}u - \sin\big(\frac{\alpha\pi}{2}\big)\operatorname{Im}u\Big){}_t^C D_T^{\alpha}\varphi\Big]\,dt\,dx\\
&\le \frac{1}{|\lambda_1|}\Big(\varepsilon\int_{Q_T}|u|^{\kappa}\varphi\,dt\,dx + c_{\varepsilon}\int_{Q_T}\left(|\Delta\varphi| + 2|{}_t^C D_T^{\alpha}\varphi|\right)^{\frac{\kappa}{\kappa-1}}\varphi^{-\frac{1}{\kappa-1}}\,dt\,dx\Big),
\end{aligned} \qquad (3.31)$$

where $\varepsilon > 0$ and $c_{\varepsilon} > 0$ is a constant. Using the ε-Young inequality with parameters $\frac{p}{q}$ and $\frac{p}{p-q}$, we obtain

$$\begin{aligned}
&\frac{\mu_1}{\lambda_1}\Big(\int_{Q_T}|v|^q a(x)\cdot\nabla\varphi\,dt\,dx + \int_{Q_T}|v|^q\operatorname{div} a(x)\varphi\,dt\,dx\Big)\\
&\le \frac{|\mu_1|}{|\lambda_1|}\Big(\varepsilon\int_{Q_T}|v|^p\varphi\,dt\,dx + d_{\varepsilon}\int_{Q_T}\big[\big(\|a(x)\|\|\nabla\varphi\| \qquad (3.32)\\
&\quad + |\operatorname{div} a(x)||\varphi|\big)\varphi^{\frac{-q}{p}}\big]^{\frac{p}{p-q}}\,dt\,dx\Big),
\end{aligned}$$

where $\varepsilon > 0$ and $d_\varepsilon > 0$ is a constant. Combining (3.30), (3.31) and (3.32), we obtain

$$
\begin{aligned}
&\Big(1-\frac{\varepsilon|\mu_1|}{|\lambda_1|}\Big)\int_{Q_T}|v|^p\varphi\,dt\,dx+\frac{1}{\lambda_1}\int_{Q_T}\mathcal{J}(\alpha,(g_1(x),g_2(x))_t^C D_T^\alpha\varphi\,dt\,dx\\
&\le\frac{\varepsilon}{|\lambda_1|}\int_{Q_T}|u|^\kappa\varphi\,dt\,dx+\frac{c_\varepsilon}{|\lambda_1|}\int_{Q_T}\big(|\Delta\varphi|+2|_t^C D_T^\alpha\varphi|\big)^{\frac{\kappa}{\kappa-1}}\varphi^{-\frac{1}{\kappa-1}}\,dt\,dx\\
&\quad+\frac{d_\varepsilon|\mu_1|}{|\lambda_1|}\int_{Q_T}\Big[(\|a(x)\|\|\nabla\varphi\|+|\operatorname{div}a(x)||\varphi|)\,\varphi^{\frac{-q}{p}}\Big]^{\frac{p}{p-q}}\,dt\,dx\\
&:=\frac{\varepsilon}{|\lambda_1|}\int_{Q_T}|u|^\kappa\varphi\,dt\,dx+I_1+I_2.
\end{aligned}\tag{3.33}
$$

On the other hand, by (3.29), we have

$$
\begin{aligned}
&\operatorname{Re}\int_{Q_T}v\left(\Delta\varphi+i^\beta{}_t^C D_T^\beta\varphi\right)dt\,dx\\
&=\operatorname{Re}\Big(\lambda\int_{Q_T}|u|^\kappa\varphi\,dt\,dx-\mu\int_{Q_T}|u|^\sigma b(x)\cdot\nabla\varphi\,dt\,dx\\
&\quad-\mu\int_{Q_T}|u|^\sigma\operatorname{div}b(x)\varphi\,dt\,dx+i^\beta\int_{Q_T}h_t^C D_T^\beta\varphi\,dt\,dx\Big),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_{Q_T}|u|^\kappa\varphi\,dt\,dx+\frac{1}{\lambda_1}\int_{Q_T}\mathcal{J}(\beta,(h_1(x),h_2(x))_t^C D_T^\beta\varphi\,dt\,dx\\
&=\frac{1}{\lambda_1}\int_{Q_T}\Big[(\operatorname{Re}v)\Delta\varphi+\Big(\cos\big(\frac{\beta\pi}{2}\big)\operatorname{Re}v-\sin\big(\frac{\beta\pi}{2}\big)\operatorname{Im}v\Big){}_t^C D_T^\beta\varphi\Big]\,dt\,dx\\
&\quad+\frac{\mu_1}{\lambda_1}\Big(\int_{Q_T}|u|^\sigma b(x)\cdot\nabla\varphi\,dt\,dx+\int_{Q_T}|u|^\sigma\operatorname{div}b(x)\varphi\,dt\,dx\Big).
\end{aligned}
$$

As previously, using the ε-Young inequality, we obtain

$$
\begin{aligned}
&\Big(1-\frac{\varepsilon|\mu_1|}{|\lambda_1|}\Big)\int_{Q_T}|u|^\kappa\varphi\,dt\,dx+\frac{1}{\lambda_1}\int_{Q_T}\mathcal{J}(\beta,(h_1(x),h_2(x))_t^C D_T^\beta\varphi\,dt\,dx\\
&\le\frac{\varepsilon}{|\lambda_1|}\int_{Q_T}|v|^p\varphi\,dt\,dx+\frac{e_\varepsilon}{|\lambda_1|}\int_{Q_T}\Big(|\Delta\varphi|+2|_t^C D_T^\beta\varphi|\Big)^{\frac{p}{p-1}}\varphi^{-\frac{1}{p-1}}\,dt\,dx\\
&\quad+\frac{f_\varepsilon|\mu_1|}{|\lambda_1|}\int_{Q_T}\Big[(\|b(x)\|\|\nabla\varphi\|+|\operatorname{div}b(x)||\varphi|)\,\varphi^{\frac{-\sigma}{\kappa}}\Big]^{\frac{\kappa}{\kappa-\sigma}}\,dt\,dx\\
&:=\frac{\varepsilon}{|\lambda_1|}\int_{Q_T}|v|^p\varphi\,dt\,dx+J_1+J_2,
\end{aligned}\tag{3.34}
$$

where $e_\varepsilon > 0$ and $f_\varepsilon > 0$ are certain constants. Next, adding (3.33) to (3.34), we obtain

$$
\begin{aligned}
&\Big(1-\frac{\varepsilon|\mu_1|}{|\lambda_1|}\Big)\int_{Q_T}(|v|^p+|u|^\kappa)\,\varphi\,dt\,dx+\frac{1}{\lambda_1}\int_{Q_T}\mathcal{J}(\alpha,(g_1(x),g_2(x))_t^C D_T^\alpha\varphi\,dt\,dx\\
&+\frac{1}{\lambda_1}\int_{Q_T}\mathcal{J}(\beta,(h_1(x),h_2(x))_t^C D_T^\beta\varphi\,dt\,dx\\
&\le\frac{\varepsilon}{|\lambda_1|}\int_{Q_T}(|v|^p+|u|^\kappa)\,\varphi\,dt\,dx+I_1+I_2+J_1+J_2,
\end{aligned}
$$

which yields

$$\begin{aligned}&\Big(1-\frac{\varepsilon}{|\lambda_1|}-\frac{\varepsilon|\mu_1|}{|\lambda_1|}\Big)\int_{Q_T}(|v|^p+|u|^\kappa)\,\varphi\,dt\,dx\\&+\frac{1}{\lambda_1}\int_{Q_T}\mathcal{J}(\alpha,(g_1(x),g_2(x))_t^C D_T^\alpha\varphi\,dt\,dx\\&+\frac{1}{\lambda_1}\int_{Q_T}\mathcal{J}(\beta,(h_1(x),h_2(x))_t^C D_T^\beta\varphi\,dt\,dx\\&\le I_1+I_2+J_1+J_2.\end{aligned}\tag{3.35}$$

Following similar arguments as in the proof of of Theorem 3.2, we obtain easily that

$$I_1\le C_1\left(T^{1+\frac{(\alpha+\beta)N}{4}-\frac{\alpha\kappa}{\kappa-1}}+T^{1+\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)\kappa}{2(\kappa-1)}}\right),\tag{3.36}$$

$$I_2\le C_2|\mu_1|\left(T^{\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)p}{4(p-q)}+\frac{\tau p}{p-q}+1}+T^{\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)p}{4(p-q)}+\frac{\nu p}{p-q}+1}\right),\tag{3.37}$$

$$J_1\le C_3\left(T^{1+\frac{(\alpha+\beta)N}{4}-\frac{\beta p}{p-1}}+T^{1+\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)p}{2(p-1)}}\right),\tag{3.38}$$

$$J_2\le C_4|\mu_1|\left(T^{\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)\kappa}{4(\kappa-\sigma)}+\frac{\chi\kappa}{\kappa-\sigma}+1}+T^{\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)\kappa}{4(\kappa-\sigma)}+\frac{\ell\kappa}{\kappa-\sigma}+1}\right),\tag{3.39}$$

where $C_j>0$ are certain constants, $j=1,2,3,4$. On the other hand, we have

$$\begin{aligned}&\frac{1}{\lambda_1}\int_{Q_T}\mathcal{J}(\alpha,(g_1(x),g_2(x))_t^C D_T^\alpha\varphi\,dt\,dx\\&=\frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)}T^{1-\alpha}\int_{\mathbb{R}^N}\mathcal{J}(\alpha,(g_1(x),g_2(x))\Phi\left(T^{\frac{-(\alpha+\beta)}{4}}x\right)^\omega\,dx\end{aligned}\tag{3.40}$$

and

$$\begin{aligned}&\frac{1}{\lambda_1}\int_{Q_T}\mathcal{J}(\beta,(h_1(x),h_2(x))_t^C D_T^\beta\varphi\,dt\,dx\\&=\frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\beta)}T^{1-\beta}\int_{\mathbb{R}^N}\mathcal{J}(\beta,(h_1(x),h_2(x))\Phi\left(T^{\frac{-(\alpha+\beta)}{4}}x\right)^\omega\,dx.\end{aligned}\tag{3.41}$$

Taking

$$\varepsilon=\frac{|\lambda_1|}{2(1+|\mu_1|)}$$

in (3.35), using (3.36), (3.37), (3.38), (3.39), (3.40) and (3.41), we obtain

$$\begin{aligned}&\frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)}\int_{\mathbb{R}^N}\mathcal{J}(\alpha,(g_1(x),g_2(x))\Phi\left(T^{\frac{-(\alpha+\beta)}{4}}x\right)^\omega\,dx\\&+\frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\beta)}T^{\alpha-\beta}\int_{\mathbb{R}^N}\mathcal{J}(\beta,(h_1(x),h_2(x))\Phi\Big(T^{\frac{-(\alpha+\beta)}{4}}x\Big)^\omega\,dx\\&\le L_1\Big(T^{\alpha+\frac{(\alpha+\beta)N}{4}-\frac{\alpha\kappa}{\kappa-1}}+T^{\alpha+\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)\kappa}{2(\kappa-1)}}+T^{\alpha+\frac{(\alpha+\beta)N}{4}-\frac{\beta p}{p-1}}\\&\quad+T^{\alpha+\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)p}{2(p-1)}}\Big)+L_2|\mu_1|\Big(T^{\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)p}{4(p-q)}+\frac{\tau p}{p-q}+\alpha}\\&\quad+T^{\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)p}{4(p-q)}+\frac{\nu p}{p-q}+\alpha}+T^{\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)\kappa}{4(\kappa-\sigma)}+\frac{\chi\kappa}{\kappa-\sigma}+\alpha}\\&\quad+T^{\frac{(\alpha+\beta)N}{4}-\frac{(\alpha+\beta)\kappa}{4(\kappa-\sigma)}+\frac{\ell\kappa}{\kappa-\sigma}+\alpha}\Big),\end{aligned}\tag{3.42}$$

where $L_j>0$ are some constants, $j=1,2$.

Suppose now that

$$\alpha=\beta,\ \lambda_1\int_{\mathbb{R}^N}\Big(\mathcal{J}(\alpha,(g_1(x),g_2(x))+\mathcal{J}(\beta,(h_1(x),h_2(x))\Big)\,dx>0. \tag{3.43}$$

We discuss two cases.

Case 1: $\mu_1=0$ and $N<2\min\{\frac{1}{p-1},\frac{1}{\kappa-1}\}$. In this case, passing to the limit as $T\to+\infty$ in (3.42), we obtain a contradiction with (3.43).

Case 2: $\mu_1\neq 0$ and $N<\min_{j=1,\dots,6}\theta_j(\alpha,\alpha)$. Similarly, passing to the limit as $T\to+\infty$ in (3.42), we obtain a contradiction with (3.43).

Suppose now that

$$\beta<\alpha,\quad \lambda_1\int_{\mathbb{R}^N}\mathcal{J}(\beta,(h_1(x),h_2(x))\,dx>0. \tag{3.44}$$

We discuss the following two cases:

Case 1: $\mu_1=0$ and

$$N<\frac{4}{\alpha+\beta}\min\big\{\frac{\beta p}{p-1}-\alpha,\frac{(\alpha+\beta)\kappa}{2(\kappa-1)}-\alpha\big\}.$$

In this case, passing to the limit as $T\to+\infty$ in (3.42), we obtain a contradiction with (3.44).

Case 2: $\mu_1\neq 0$ and $N<\min_{j=1,\dots,6}\theta_j(\alpha,\beta)$. Similarly, passing to the limit as $T\to+\infty$ in (3.42), we obtain a contradiction with (3.44).

Next, we consider the case $\lambda_2\neq 0$. Observe that

$$(U(t,x),V(t,x))=\Big(\frac{u(t,x)}{i},\frac{v(t,x)}{i}\Big),\quad t\geq 0,\ x\in\mathbb{R}^N$$

is a global weak solution to the system

$$\begin{gathered}
i^{\alpha}\,{}_0^C D_t^{\alpha}U+\Delta U=\lambda'|v|^p+\mu' a(x)\cdot\nabla|V|^q,\quad t>0,\ x\in\mathbb{R}^N,\\
i^{\beta}\,{}_0^C D_t^{\beta}V+\Delta V=\lambda'|U|^\kappa+\mu' b(x)\cdot\nabla|U|^\sigma,\quad t>0,\ x\in\mathbb{R}^N,\\
U(0,x)=\widetilde{g(x)},\quad V(0,x)=\widetilde{h(x)}\quad x\in\mathbb{R}^N,
\end{gathered}$$

where

$$\begin{gathered}
\lambda'=\lambda_2+(-\lambda_1)i:=\lambda_1'+i\lambda_2',\\
\mu'=\mu_2+(-\mu_1)i:=\mu_1'+i\mu_2',\\
\widetilde{g(x)}=g_2(x)+(-g_1(x))i:=\widetilde{g}_1(x)+i\widetilde{g}_2(x),\quad x\in\mathbb{R}^N,\\
\widetilde{h(x)}=h_2(x)+(-h_1(x))i:=\widetilde{h}_1(x)+i\widetilde{h}_2(x),\quad x\in\mathbb{R}^N.
\end{gathered}$$

Therefore, from the previous study, if one of the cases (III) or (IV) holds, we obtain a contradiction. □

Remark 3.6. Taking $u=v$, $\alpha=\beta$, $p=\kappa$, $q=\sigma$, $a=b$, and $g=h$, using Theorem 3.5, we obtain the result given by Theorem 3.2 concerning the single Schrödinger equation (1.1).

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