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BOUNDARY REGULARITY FOR QUASI-LINEAR ELLIPTIC EQUATIONS WITH LOWER ORDER TERM

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ABSTRACT. We give results of summability up to the boundary for the solutions to quasi-linear elliptic equations involving the p-Laplace operator and a term depending on the gradient of the solution.

1. INTRODUCTION

In this article we extend the results in [16] to quasilinear elliptic equations involving lower order terms (see for instance [9, 10, 14] and the references quoted there for remarks on applications of these kind of equations). Given $\Omega \subset \mathbb{R}^n$ bounded and with smooth boundary, for fixed $p > 1$ we consider the equation:

$$\begin{gathered} -\Delta_p u + H(x, \nabla u) = f(x, u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where $f = f(x, s) : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is Lipshitz continuous on each compact subset of $\bar{\Omega} \times \mathbb{R}$ and $H = H(x, \xi) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which is also locally Lipshitz continuous with respect to x and C^1 with respect to ξ and satisfies the following property: For each $M > 0$, there exists $k = k(M) > 0$ such that

$$|H_\xi(x, \xi)| \le k|\xi|^{\frac{p-2}{2}} \quad \forall \xi : |\xi| \le M \quad \text{for a.e. } x \in \Omega. \tag{1.2}$$

Let $u \in W_0^{1,p}(\Omega)$ be a weak solution to (1.1), that means u satisfies:

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle dx + \int_\Omega H(x, \nabla u)\varphi dx = \int_\Omega f(x, u)\varphi dx, \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{1.3}$$

By [4, 8, 20] we know that $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha < 1$. We focus our attention on the summability of the second derivatives of u. If the term $H(x, \nabla u)$ there is not in (1.1), results on the regularity of the second derivatives (in the interior of Ω) can be found in [1, 17, 18]. Moreover, if the Hopf's boundary Lemma can be applied, the results hold up to the boundary (in fact the solutions have no critical points there and therefore the equation is no more degenerate). Therefore we consider cases where Hopf's Lemma cannot be applied. Our main result is the following.

Theorem 1.1. *Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a weak solution to* (1.1). *We have*

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(i) *if* $p \leq 2$, *then* $u \in W^{2,2}(\Omega)$
(ii) *if* $p > 2$, *then* $|\nabla u|^{p-1} \in W^{1,2}(\Omega)$.

The main tool in our proofs is a weighted estimate for the second derivatives of the solution, achieved by means of the linearized operator of a transformed equation (using the techniques developed in [1, 3, 11, 17, 18]).

Regularity results in cases, where Hopf's Lemma does not apply, can be found also in [11], where the summability of the second derivatives is studied for p close to two. An example of application of the above techniques to cases where the Hopf's Lemma fails, can be found also in [9, 14]. Note that the study of the regularity of solutions to p-Laplace equations, which is interesting in itself, is also very much related to the study of the qualitative properties of the solutions (see for instance [1, 2, 5, 6, 13, 19]).

2. Notation and preliminary results

Given a matrix A, A^T is the transposed of A and we set $|A| = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2}$. For $r > 0$ and $\overline{x} \in \mathbb{R}^n$ we set $B_r(\overline{x}) = \{x \in \mathbb{R}^n : |x - \overline{x}| < r\}$. For $\overline{x} \in \partial\Omega$ and $r > 0$ small enough, we set $B^+ := B_r(\overline{x}) \cap \Omega$. We consider a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ (also called *flattening operator*) such that $\Phi^{-1}(B_r(\overline{x}) \cap \partial\Omega) \subset \{y_n = 0\}$. We set: $\mathcal{B}^+ = \Phi^{-1}(B^+)$ and we denote by $\mathcal{B}^-$ the reflection of $\mathcal{B}^+$ with respect to the hyperplane $\{y_n = 0\}$. We set $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^- \cup \Phi^{-1}(\partial\Omega \cap B_r(\overline{x}))$.

We set $\Psi = \Phi^{-1}$ and we use the change of variable: $x = \Phi(y)$, $y = \Psi(x)$, so that we have
$$w(y) = u(\Phi(y)) \quad \text{in} \quad \mathcal{B}^+ .$$
Denoting by $J\Phi(y)$ the Jacobian matrix of Φ, equation (1.3) becomes
$$\begin{aligned}
&\int_{\mathcal{B}^+} |\nabla_x u(\Phi(y))|^{p-2} \langle \nabla_x u(\Phi(y)), \nabla_x \varphi(\Phi(y))\rangle |\det J\Phi(y)| dy \\
&+ \int_{\mathcal{B}^+} H(\Phi(y), \nabla_x u(\Phi(y)))\psi(y)|\det J\Phi(y)| dy \\
&= \int_{\mathcal{B}^+} g(y, w(y))\psi(y)|\det J\Phi(y)| dy
\end{aligned} \tag{2.1}$$
where $\det J\Phi(y)$ denotes the determinant of $J\Phi(y)$ and we have set
$$g(y, w) = f(\Phi(y), w), \quad \psi(y) = \varphi(\Phi(y)) .$$
Hence we have
$$\nabla_y w(y) = J\Phi(y)^T \nabla_x u(\Phi(y)) , \quad \nabla_y \psi(y) = J\Phi(y)^T \nabla_x \varphi(\Phi(y))$$
and (2.1) gives
$$\begin{aligned}
&\int_{\mathcal{B}^+} \left|[J\Phi(y)^T]^{-1} \nabla w(y)\right|^{p-2} \left\langle \left([J\Phi(y)^T]^{-1}\right)^T [J\Phi(y)^T]^{-1} \nabla w(y), \nabla \psi(y)\right\rangle \\
&\times |\det J\Phi(y)| dy + \int_{\mathcal{B}^+} L(y, \nabla w(y))\psi(y)|\det J\Phi(y)| dy \\
&= \int_{\mathcal{B}^+} g(y, w(y))\psi(y)|\det J(\Phi(y))| dy ,
\end{aligned} \tag{2.2}$$
where we have set
$$L(y, \nabla w) = H\left(\Phi(y), [J(\Phi(y))^T]^{-1} \nabla w\right) .$$

Remark 2.1. From the properties of Φ it follows that L enjoys the same regularity properties of H and in particular it satisfies (1.2).

Setting

$$A(y) = [J(\Phi(y))^T]^{-1}, \quad K(y) = A(y)^T A(y), \quad \rho(y) = |\det J(\Phi(y))| \tag{2.3}$$

it follows that $w(y)$ weakly satisfies

$$-\operatorname{div}\big(\rho(y)|A(y)\nabla w(y)|^{p-2}K(y)\nabla w(y)\big) + L(y, \nabla w(y))\rho(y) = g(y, w(y))\rho(y) \tag{2.4}$$

in $\mathcal{B}^+$.

We define the odd extension of $w(y)$ and the even extension of $\rho(y)$ and $A(y)$ (and hence of K) as follows:

$$\bar{w}(y) = \begin{cases} w(y), & \text{if } y_n \geq 0, \\ -w(y_1, \dots, y_{n-1}, -y_n), & \text{if } y_n < 0; \end{cases} \tag{2.5}$$

$$\bar{\rho}(y) = \begin{cases} \rho(y), & \text{if } y_n \geq 0, \\ \rho(y_1, \dots, y_{n-1}, -y_n), & \text{if } y_n < 0; \end{cases} \tag{2.6}$$

$$\bar{A}(y) = \begin{cases} A(y), & \text{if } y_n \geq 0, \\ A(y_1, \dots, y_{n-1}, -y_n), & \text{if } y_n < 0. \end{cases} \tag{2.7}$$

For the function $g(y, t)$ we consider the mixed extension (odd with respect to t and even with respect to y_n):

$$\bar{g}(y, t) = \begin{cases} g(y, t), & \text{if } y_n \geq 0,\ t \geq 0, \\ -g(y, -t), & \text{if } y_n \geq 0,\ t < 0, \\ g(y_1, \dots, y_{n-1}, -y_n, t), & \text{if } y_n < 0,\ t \geq 0, \\ -g(y_1, \dots, y_{n-1}, -y_n, -t), & \text{if } y_n < 0,\ t < 0\,. \end{cases} \tag{2.8}$$

For $L(y, \xi)$ we consider the extension (even with respect to y_n):

$$\bar{L}(y) = \begin{cases} L(y, \xi), & \text{if } y_n \geq 0, \\ L(y_1, \dots, y_{n-1}, -y_n, \xi), & \text{if } y_n < 0. \end{cases} \tag{2.9}$$

In this way, $\bar{w}(y)$ satisfies the equation

$$-\operatorname{div}(\bar{\rho}(y)|\bar{A}(y)\nabla \bar{w}(y)|^{p-2}\bar{K}(y)\nabla \bar{w}(y)) + L(y, \nabla \bar{w}(y))\bar{\rho}(y) = \bar{g}(y, \bar{w})\bar{\rho}(y) \tag{2.10}$$

in $\mathcal{B}$.

We remark that Standard $C^{1,\alpha}$ regularity results (in $\mathcal{B}$) (see for instance [4, 8, 20]) can be applied to (2.10).

We assume that $\partial\Omega$ is smooth enough and, without loss of generality, we can assume that $\Psi(\bar{x}) = 0$. Therefore we can construct Φ in such a way that:

$$\Phi(y) = y + F(y) \tag{2.11}$$

with F such that $F(0) = \bar{x}$ and $\sup_{|y|<\tau_1} |JF(y)| < \tau_2$ for suitable τ_1 and τ_2 small enough. An explicit representation of Φ can be found for instance in [7]. By (2.11) we have:

$$J\Phi(y) = I + JF(y), \tag{2.12}$$

where I is the identity matrix and JF is the Jacobian matrix of F.

By classical results of linear algebra and the regularity properties of Φ it follows that, there exist $\delta = \delta(\tau_1)$ and $c_1 > 0, c_2 > 0, c_3 > 0, c_4 > 0$ such that:

$$c_1|v| \le |A(y)v| \le c_2|v| \quad \forall v \in \mathbb{R}^n\,, \forall y \in \mathcal{B}_\delta(0), \tag{2.13}$$

$$c_3|v| \le |K(y)v| \le c_4|v| \quad \forall v \in \mathbb{R}^n\,, \forall y \in \mathcal{B}_\delta(0)\,. \tag{2.14}$$

3. Main results

We first compute a linearized version of (2.10). To simplify notation, we will omit the bar over the functions defined in (2.5), (2.6),(2.7), (2.8), (2.9). For a function $v = v(y)$ in the sequel v_j is the derivative with respect to y_j and for the functions $g = g(y,t)$ and $L(y,\xi)$ we denote by g_j and L_j the derivative with respect to y_j and by $g'(y,t)$ and L_ξ the derivative of g with respect to t and the gradient of L with respect to ξ.

We briefly recall definition and basic properties about some weighted Sobolev spaces useful to write a linearized equation associated to equation (2.10) (for more details see for instance [12, 15, 21]).

Let $U \subset \mathbb{R}^n$ be a bounded smooth domain. For $\mu \in L^1(U)$ the weighted Sobolev space $H^1_\mu(U)$ (with respect to the weight μ) is defined as the completion of $C^\infty(\overline{U})$ with respect to the norm

$$\|v\| = \Big(\int_U |v|^2\Big)^{1/2} + \Big(\int_U |\nabla v|^2\mu\Big)^{1/2}, \tag{3.1}$$

where ∇v is the distributional derivative. The space $H^1_{0,\mu}(U)$ is defined as the closure of $C^\infty_c(U)$ in $H^1_\mu(U)$.

We set $Z = \{x \in \mathcal{B} : \nabla w(x) = 0\}$ and we consider $\psi \in C^\infty_c(\mathcal{B} \setminus Z)$. For any $j = 1,\dots,n$, we use ψ_j as test function in the weak formulation of (2.10) and, since $w \in C^2(\mathcal{B}\setminus Z)$, we can integrate by parts obtaining:

$$\begin{aligned}
&\int_{\mathcal{B}} \rho_j(y)|A(y)\nabla w(y)|^{p-2}\langle K(y)\nabla w(y), \nabla\psi(y)\rangle dy\\
&+ (p-2)\int_{\mathcal{B}} \rho(y)|A(y)\nabla w(y)|^{p-4}\langle A_j(y)^T A(y)\nabla w(y), \nabla w(y)\rangle\\
&\times \langle K(y)\nabla w(y), \nabla\psi(y)\rangle dy\\
&+ (p-2)\int_{\mathcal{B}} \rho(y)|A(y)\nabla w(y)|^{p-4}\langle K(y)\nabla w(y), \nabla w_j(y)\rangle\\
&\times \langle K(y)\nabla w(y), \nabla\psi(y)\rangle dy\\
&+ \int_{\mathcal{B}} \rho(y)|A(y)\nabla w(y)|^{p-2}\langle K_j(y)\nabla w(y), \nabla\psi(y)\rangle dy\\
&+ \int_{\mathcal{B}} \rho(y)|A(y)\nabla w(y)|^{p-2}\langle K(y)\nabla w_j(y), \nabla\psi(y)\rangle dy\\
&+ \int_{\mathcal{B}} \rho(y)L_j(y,\nabla w(y))\psi(y)dy + \int_{\mathcal{B}} \rho(y)\langle L_\xi(y,\nabla w(y)), \nabla w_j(y)\rangle\psi(y)dy\\
&= \int_{\mathcal{B}} [g_j(y,w(y))\rho(y) + g'(y,w(y))w_j(y)\rho(y) + g(y,w(y))\rho_j(y)]\,\psi dy\,.
\end{aligned} \tag{3.2}$$

By a density argument (3.2) holds for any $\psi \in H^1_\mu(\mathcal{B}) \cap L^\infty(\mathcal{B})$ with compact support in $\mathcal{B} \setminus Z$. The main tool to achieve our estimates is the following result.

Proposition 3.1 (Hessian estimate). *For $p \in (1,\infty)$ fixed we consider a weak solution $w \in W^{1,\infty}_{\rm loc}(\mathcal{B})$ of (2.10). For $y_0 \in \mathcal{B}$, let $r > 0$ be such that $B_{2r}(y_0) \subset \mathcal{B}$. It holds*

$$\int_{B_r(y_0)} |\nabla w|^{p-2}|D^2 w|^2 dy \le C\,, \tag{3.3}$$

where $C = C(y_0, r, p, n, \|w\|_{W^{1,\infty}}, g, L)$.

Proof. Let $G_\alpha : \mathbb{R} \to \mathbb{R}$ be defined as

$$G_\alpha(s) = \begin{cases} s & \text{if } |s| \ge 2\alpha, \\ 2[s - \alpha\frac{s}{|s|}] & \text{if } \alpha < |s| < 2\alpha, \\ 0 & \text{if } |s| \le \alpha, \end{cases}$$

and let φ be a cut-off function such that

$$\varphi \in C^\infty_c(B_{2r}(y_0)),\ \varphi \equiv 1 \text{ in } B_r(y_0) \quad \text{and} \quad |D\varphi| \le \frac{2}{r}, \tag{3.4}$$

with $2r < \operatorname{dist}(y_0, \partial\Omega)$. We set

$$\psi(y) = G_\varepsilon(w_j(y))\varphi^2(y). \tag{3.5}$$

In the sequel we omit the dependence on y. We put ψ as test function in (3.2), and we obtain

$$\begin{aligned}
&\int_{\mathcal{B}} \rho_j |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j)\langle K\nabla w, \nabla\varphi\rangle\, dy \\
&+ \int_{\mathcal{B}} \rho_j |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j)\langle K\nabla w, \nabla w_j\rangle\, dy \\
&+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} 2\varphi G_\varepsilon(w_j)\langle A_j^T A\nabla w, \nabla w\rangle \cdot \langle K\nabla w, \nabla\varphi\rangle\, dy \\
&+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G'_\varepsilon(w_j)\langle A_j^T A\nabla w, \nabla w\rangle \cdot \langle K\nabla w, \nabla w_j\rangle\, dy \\
&+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} 2\varphi G_\varepsilon(w_j)\langle K\nabla w, \nabla w_j\rangle \cdot \langle K\nabla w, \nabla\varphi\rangle\, dy \\
&+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G'_\varepsilon(w_j)\langle K\nabla w, \nabla w_j\rangle^2\, dy \\
&+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j)\langle K_j\nabla w, \nabla\varphi\rangle\, dy \\
&+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j)\langle K_j\nabla w, \nabla w_j\rangle\, dy \\
&+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j)\langle K\nabla w_j, \nabla\varphi\rangle\, dy \\
&+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j)\langle K\nabla w_j, \nabla w_j\rangle\, dy \\
&+ \int_{\mathcal{B}} \rho\,[L_j(y, D\nabla w) + \langle L_\xi(y,\nabla w), \nabla w_j\rangle]\ G_\varepsilon(w_j)\varphi^2 dy \\
&= \int_{\mathcal{B}} [g_j\rho + g' w_j \rho + g\rho_j]\,\varphi^2 G_\varepsilon(w_j)\, dy\,.
\end{aligned} \tag{3.6}$$

In the sequel c and C will be positive constants (possibly depending on r, y_0, $\|w\|_{W^{1,\infty}(B_{2r}(\bar{x}))}$) whose value can varies from line to line.

We set:

$$\begin{aligned} I_1 = &\int_{\mathcal{B}} \rho_j |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j)\langle K\nabla w, \nabla\varphi\rangle\, dy \\ &+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} 2\varphi G_\varepsilon(w_j)\langle A_j^T A\nabla w, \nabla w\rangle \cdot \langle K\nabla w, \nabla\varphi\rangle\, dy \\ &+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j)\langle K_j\nabla w, \nabla\varphi\rangle\, dy\,; \end{aligned} \tag{3.7}$$

$$\begin{aligned} I_2 = (p-2)&\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} 2\varphi G_\varepsilon(w_j)\langle K\nabla w, \nabla w_j\rangle\langle K\nabla w, \nabla\varphi\rangle\, dy \\ &+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} 2\varphi G_\varepsilon(w_j)\langle K\nabla w_j, \nabla\varphi\rangle\, dy\,; \end{aligned} \tag{3.8}$$

$$\begin{aligned} I_3 = &\int_{\mathcal{B}} \rho_j |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j)\langle K\nabla w, \nabla w_j\rangle\, dy \\ &+ (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G'_\varepsilon(w_j)\langle A_j^T A\nabla w, \nabla w\rangle\langle K\nabla w, \nabla w_j\rangle\, dy \\ &+ \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j)\langle K_j\nabla w, \nabla w_j\rangle\, dy\,; \end{aligned} \tag{3.9}$$

$$I_4 = (p-2)\int_{\mathcal{B}} \rho |A\nabla w|^{p-4} \varphi^2 G'_\varepsilon(w_j)\langle K\nabla w, \nabla w_j\rangle^2\, dy\,; \tag{3.10}$$

$$I_5 = \int_{\mathcal{B}} \rho |A\nabla w|^{p-2} \varphi^2 G'_\varepsilon(w_j)\langle K\nabla w_j, \nabla w_j\rangle\, dy\,; \tag{3.11}$$

$$I_6 = \int_{\mathcal{B}} \rho L_j(y, D\nabla w) G_\varepsilon(w_j)\varphi^2 dy\,; \tag{3.12}$$

$$I_7 = \int_{\mathcal{B}} \rho \langle L_\xi(y,\nabla w), \nabla w_j\rangle G_\varepsilon(w_j)\varphi^2 dy\,; \tag{3.13}$$

$$I_8 = \int_{\mathcal{B}} [g_j\rho + g' w_j \rho + g\rho_j]\, G_\varepsilon(w_j)\varphi^2\, dy\,. \tag{3.14}$$

If $p \geq 2$, then I_4 is positive and hence

$$I_4 + I_5 \geq I_5\,. \tag{3.15}$$

If $p < 2$, then by the definition of K we have

$$|A\nabla w|^{p-4}\langle K\nabla w, \nabla w_j\rangle^2 = |A\nabla w|^{p-4}\langle A\nabla w, A\nabla w_j\rangle^2 \leq |A\nabla w|^{p-2}|A\nabla w_j|^2\,, \tag{3.16}$$

which implies

$$(p-2)|A\nabla w|^{p-4}\langle K\nabla w, \nabla w_j\rangle^2 \geq (p-2)|A\nabla w|^{p-2}|A\nabla w_j|^2 \tag{3.17}$$

and hence

$$I_4 + I_5 \geq (p-1) I_5\,. \tag{3.18}$$

By (3.15) and (3.18) we have that, for every $p > 1$, it holds

$$I_4 + I_5 \geq \min\{1, p-1\} I_5\,. \tag{3.19}$$

By (3.6) and (3.19) we infer

$$\min\{1, p-1\} I_5 \leq I_4 + I_5 \leq \sum_{i=1}^{7} |I_i| + I_8\,. \tag{3.20}$$

By the properties of Φ there exist $c, M > 0$ such that

$$c \le \rho(y) \le M, \quad |\rho_j(y)| \le M \quad \forall y \in \mathcal{B},\ \forall j = 1, \dots, n. \tag{3.21}$$

By (2.13), (2.14), (3.20) and (3.21), estimating the terms in the righthand side of (3.20), we obtain

$$\begin{aligned}
&\int_{\mathcal{B}} |\nabla w|^{p-2} |\nabla w_j|^2 G'_\varepsilon(w_j) H_{\delta,z} \varphi^2 dy \\
&\le c \int_{\mathcal{B}} |G_\varepsilon(w_j)| |\nabla w|^{p-1} |\nabla \varphi| \varphi dy \\
&\quad + c \int_{\mathcal{B}} |G_\varepsilon(w_j)| |\nabla w|^{p-2} |\nabla w_j| |\nabla \varphi| \varphi dy \\
&\quad + c \int_{\mathcal{B}} |G'_\varepsilon(w_j)| |\nabla w|^{p-1} |\nabla w_j| \varphi^2 dy \\
&\quad + c \int_{\mathcal{B}} |L_\xi(y, \nabla w)| |\nabla w_j| |G_\varepsilon(w_j)| \varphi^2 dy \\
&\quad + c \int_{\mathcal{B}} [(|L_j| + |g_j| + |g'||w_j|)\rho + |g\rho_j|] |G_\varepsilon(w_j)| \varphi^2 dy\,.
\end{aligned} \tag{3.22}$$

We set

$$J_1 = \int_{\mathcal{B}} |G_\varepsilon(w_j)| |\nabla w|^{p-1} |\nabla \varphi| \varphi dy$$

$$J_2 = \int_{\mathcal{B}} |G_\varepsilon(w_j)| |\nabla w|^{p-2} |\nabla w_j| |\nabla \varphi| \varphi dy$$

$$J_3 = \int_{\mathcal{B}} |G'_\varepsilon(w_j)| |\nabla w|^{p-1} |\nabla w_j| \varphi^2 dy = \int_{\mathcal{B} \cap \{w_j > \varepsilon\}} |G'_\varepsilon(w_j)| |\nabla w|^{p-1} |\nabla w_j| \varphi^2 dy$$

$$J_4 = \int_{\mathcal{B}} |L_\xi(y, \nabla w)| |\nabla w_j| |G_\varepsilon(w_j)| \varphi^2 dy$$

$$J_5 = \int_{\mathcal{B}} [(|L_j| + |g_j| + |g'||w_j|)\rho + |g\rho_j|] |G_\varepsilon(w_j)| \varphi^2 dy\,.$$

From definition of G_ε it follows that

$$|G_\varepsilon(w_j)| \le 2|w_j|, \tag{3.23}$$

$$|G'_\varepsilon(w_j)| \le C. \tag{3.24}$$

Recalling that g and L are locally Lipshitz continuous, by properties of ρ and the regularity of w we have:

$$J_5 \le C\,. \tag{3.25}$$

We recall that for $a, b \in \mathbb{R}$ and $\theta > 0$ there holds the Young inequality

$$ab \le \theta a^2 + \frac{1}{4\theta} b^2\,. \tag{3.26}$$

Using (3.26), we obtain

$$J_3 \le c \int_{\mathcal{B}} |\nabla w|^{p-2} |\nabla w_j| \varphi^2 \chi_{\{w_j > \varepsilon\}} \le \theta \int_{\mathcal{B}} |\nabla w|^{p-2} |\nabla w_j|^2 \varphi^2 \chi_{\{w_j > \varepsilon\}} dy + C\,. \tag{3.27}$$

Recalling that $|\nabla \varphi| \le \frac{2}{\rho}$, we also have

$$J_1 \le C, \tag{3.28}$$

$$J_2 \le \theta \int_{\mathcal{B}} |\nabla w|^{p-2} |\nabla w_j|^2 \varphi^2 dy + C \,. \tag{3.29}$$

Recalling Remark 2.1 and using (3.26), we obtain

$$J_4 \le c \int_{\mathcal{B}} |\nabla w|^{\frac{p-2}{2}} |\nabla w_j| |G_\varepsilon(w_j)| \varphi^2 dy \le \theta \int_{\mathcal{B}} |\nabla w|^{p-2} |\nabla w_j|^2 G_\varepsilon(w_j) \varphi^2 dy + C \,. \tag{3.30}$$

After setting $\vartheta = c\theta$, we choose θ such that $\vartheta < 1$. By the above estimates we obtain

$$\int_{\mathcal{B}\cap\{w_j>\varepsilon\}} |\nabla w|^{p-2} |\nabla w_j|^2 \left(G'_\varepsilon(w_j) - \vartheta\right) \varphi^2 dy \le c \,. \tag{3.31}$$

From the definition of G_ε it follows that for all $s > 0$ $G'_\varepsilon(s)$ converges to 1 as ε goes to 0. Therefore by Fatou's Lemma we have

$$\int_{\mathcal{B}\setminus\{w_j=0\}} |\nabla w|^{p-2} |\nabla w_j|^2 \varphi^2 dy \le c \tag{3.32}$$

and hence

$$\int_{\mathcal{B}\setminus Z} |\nabla w|^{p-2} |\nabla w_j|^2 \varphi^2 dy \le c \,, \tag{3.33}$$

where c depends on $y_0, r, n, p, g, \Phi, \|w\|_{W^{1,\infty}(B_{2r}(\bar{x}))}$. From the properties of φ we infer

$$\int_{B_r(y_0)\setminus Z} |\nabla w|^{p-2} |D^2 w|^2 dy \le c \,.$$

□

We now prove Theorem 1.1. Since

$$\int_{B_r(y_0)} |\nabla w|^{p-2} |D^2 w|^2 dy \le C,$$

by the properties of Φ the same estimate holds for u in $\Phi(\mathcal{B})$. Hence, recalling that $\overline{\Omega}$ is compact, we get that there exists $C > 0$ such that

$$\int_{\Omega} |\nabla u|^{p-2} |D^2 u|^2 dx \le C \,. \tag{3.34}$$

If $p \le 2$, it immediately follows that

$$\int_{\Omega} |D^2 u|^2 dx \le \int_{\Omega} |\nabla u|^{p-2} |D^2 u|^2 dx$$

and hence by (3.34) the thesis (i) of Theorem 1.1 follows.

If $p > 2$, since for suitable $c > 0$ it holds

$$|\nabla \left(|\nabla u|^{p-1}\right)| \le c |\nabla u|^{p-2} |D^2 u|,$$

recalling that ∇u is bounded, there exists $C > 0$:

$$|\nabla \left(|\nabla u|^{p-1}\right)|^2 \le c^2 |\nabla u|^{2(p-2)} |D^2 u|^2 \le C |\nabla u|^{p-2} |D^2 u|^2$$

and hence statment (ii) of Theorem 1.1 follows by (3.34), provided that, arguing as in [16], it can be shown that, for every $i \in \{1, \dots, n\}$, the i-th generalized derivative of $|\nabla u|^{p-1}$ coincides with the classical one.

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