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IMPULSIVE FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH A WEAKLY CONTINUOUS NONLINEARITY

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Abstract. A general theorem on the local and global existence of solutions is established for an impulsive fractional delay differential equation with Caputo fractional substantial derivative in a separable Hilbert space under the assumption that the nonlinear term is weakly continuous. The uniqueness of solutions is also considered under an additional Lipschitz assumption.

1. Introduction

Fractional differential equations have been used to establish a more accurate model in diverse fields such as engineering, physics, chemistry, signal analysis and economics. It is applied widely in nonlinear oscillations of earthquakes, physical phenomena like seepage flow in porous media and in fluid dynamic traffic models. We refer the reader to [13, 15, 16, 17] for more details on fractional calculus. In 2006, the concept of fractional substantial derivative was presented by Friedrich et al. in [10] when they considered retardation effects in Kramers-Fokker-Planck type equations. Carmi et al. [5] used them to study the distribution of functionals of anomalous diffusion trajectories. The fractional substantial integral is defined by [8, 9]

$$I_s^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-\tau)^{\nu-1} e^{-\beta(x-\tau)} f(\tau) d\tau, \quad \nu > 0,$$

and in the similar way [6], the Caputo fractional substantial derivative is defined as

$$D_s^{\mu} f(x) = I_s^{\nu}[D_s^m f(x)], \quad \nu = m-\mu,$$

where β is a constant or a function independent of x, say $\beta(y)$, m is the smallest integer that exceeds μ, and

$$D_s^m = \Big(\frac{\partial}{\partial x} + \beta\Big)^m = (D+\beta)^m.$$

In the previous decades, the theory of impulsive differential equations has been studied with great interests mainly due to the important role such equations play

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in studying evolution processes that are subject to abrupt changes in their states, such as changes of populations, transmission of diseases, and so on. The reader is referred to [1, 2, 14] for the basic theory of impulsive differential equations.

In this paper we establish some global existence theorems for impulsive fractional delay differential equations on Hilbert spaces. These results will be used by us in [18] to investigate the asymptotic behavior of lattice models involving such equations. The global existence of mild solutions to impulsive fractional functional differential equations was discussed in [6, 11], while in [12] the existence and uniqueness of solutions for impulsive fractional functional differential equations were considered. In addition, the Cauchy problem for fractional impulsive differential equations with delay was addressed in [19]. Moreover, Benchohra and Berhoun [3] investigated the existence of solutions for impulsive fractional differential equations with state-dependent delay.

We consider the global existence of solutions of impulsive functional differential equations with Caputo fractional substantial time derivative

$$\begin{gathered} D_s^\alpha u(t) = f(t, u_t), \quad t \geq 0,\ t \neq t_k, \\ u(s) = \phi(s), \quad \forall s \in [-h, 0], \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), \quad k = 1, 2, \dots \end{gathered} \tag{1.1}$$

in the separable Hilbert space X, where D_s^α is the Caputo fractional substantial derivative with $0 < \alpha < 1$ and $\beta > 0$. We assume that the nonlinear term f is weakly continuous in bounded sets. This concept was given in [4] and delay differential equations in Banach spaces with a classical derivative were treated. In addition, we prove the uniqueness of solutions of (1.1) under Lipschitz conditions.

This article is structured as follows. Notation, some basic definitions and preliminary results are given in the next section, and then, in Section 3 we present theorems of the local and global existence and also uniqueness of solutions for (1.1) in a separable Hilbert space. Proofs of these theorems are then given in Sections 4, 5 and 6.

2. Preliminaries

Let X be a separable Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. Let $PC_t := PC([-h,t];X)$, $h > 0$, $t \geq 0$, be a Banach space of all such functions $u : [-h,t] \to X$, which are continuous everywhere except for a finite number of points t_k, $k = 1, 2, \dots, m$, at which $u(t_k^+)$ and $u(t_k^-)$ exist and $u(t_k) = u(t_k^-)$, endowed with the norm

$$\|u\|_{PC_t} = \sup_{-h \leq s \leq t} \|u(s)\|.$$

For any $u \in PC_T = PC([-h,T];X)$, we denote by u_t the element of $PC_0 = PC([-h,0];X)$ defined by $u_t(\theta) = u(t+\theta)$, $\theta \in [-h,0]$. Here $I_k \in C(X,X)$ for each k, $u(t_k^+) = \lim_{h\to 0} u(t_k + h)$ and $u(t_k^-) = \lim_{h \to 0} u(t_k - h)$ represent the right and left-hand limits of $u(t)$ at $t = t_k$, respectively.

Let X^* be the dual space of X with the pairing between X and X^* denoted by $\langle \cdot,\cdot \rangle$, and let X_w be the space X endowed with the weak topology. We consider the space $PC_{0,w} = PC([-h,0];X_w)$. Let $t \geq 0$ and $\{u_t^n\}_{n=1}^\infty$ be a given sequence. We say that $u_t^n \to u_t \in PC_{0,w}$ in $PC_{0,w}$ if it satisfies

(1) for any $s \in [-h,0]$ with $t + s \neq t_k$, for $k = 1, 2, \dots$,

$$u^n(t + s_n) \to u(t+s) \quad \text{in } X_w \text{ as } n \to \infty$$

for any sequence $\{s_n\}_{n=1}^{\infty}$ with $s_n \to s$;

(2) for any $s \in [-h, 0]$ with $t + s = t_k$ for some $k = 1, 2, \dots$,
$$u^n(t + s_n) \to u(t + s) \quad \text{in } X_w \text{ as } n \to \infty$$
for any sequence $\{s_n\}_{n=1}^{\infty}$ with $s_n \le s$ and $s_n \to s$. We say that the function $f : [0, \infty) \times PC_0 \to X$ is weakly continuous in bounded sets for each $t \in [0, \infty)$ if $u^n \to u$ in $PC_{0,w}$, and $\|u^n\|_{PC_0} \le M$ for all $n \in \mathbb{N}$, imply that $f(t, u^n) \to f(t, u)$ in X_w, and we say that the function $g : X \to X$ is weakly continuous in bounded sets if $v^n \to v$ in X_w and $\|v^n\| \le M$ for all $n \in \mathbb{N}$, imply that $g(v^n) \to g(v)$ in X_w.

Definition 2.1. A function $u \in PC_T$ is called a solution of initial value problem (1.1) if $u(t) = \phi(t)$ for $t \in [-h, 0]$ with $\phi \in PC_0$, and, for $t \in [0, T]$, $u(t)$ satisfies the integral equation
$$u(t) = \begin{cases} \phi(0)e^{-\beta t} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau, & t \in [0, t_1], \\ \big(u(t_1^-) + I_1(u(t_1^-))\big) e^{-\beta(t-t_1)} \\ + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau, & t \in (t_1, t_2], \\ \dots \\ \big(u(t_m^-) + I_m(u(t_m^-))\big) e^{-\beta(t-t_m)} \\ + \frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau, & t \in (t_m, T], \end{cases}$$
where $t_m = \max\{t_k : t_k < T, k = 0, 1, 2, \dots\}$ and $t_0 = 0$. Here and elsewhere Γ denotes the Gamma function.

Lemma 2.2. *A function $u \in PC_T$ is a solution of initial value problem* (1.1) *if and only if*
$$u(t) = \begin{cases} \phi(t), & t \in [-h, 0], \\ \phi(0)e^{-\beta t} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau, & t \in [0, t_1], \\ \phi(0)e^{-\beta t} + I_1(u(t_1^-)) e^{-\beta(t-t_1)} \\ + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau \\ + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau, & t \in (t_1, t_2], \\ \dots \\ \phi(0)e^{-\beta t} + \sum_{k=1}^m I_k(u(t_k^-)) e^{-\beta(t-t_k)} \\ + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^m \int_{t_{k-1}}^{t_k} (t_k-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau \\ + \frac{1}{\Gamma(\alpha)} \int_{t_m}^t (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau, & t \in (t_m, T]. \end{cases} \tag{2.1}$$

Proof. Assume that u is a solution of the initial value problem (1.1). Then by Definition 2.1, we obtain
$$u(t) = \phi(0)e^{-\beta t} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau, \quad \text{if } t \in [0, t_1],$$
$$\begin{aligned} u(t) =& \Big(\phi(0)e^{-\beta t_1} + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-\tau)^{\alpha-1} e^{-\beta(t_1-\tau)} f(\tau, u_\tau) d\tau\Big) e^{-\beta(t-t_1)} \\ &+ I_1(u(t_1^-)) e^{-\beta(t-t_1)} + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau \end{aligned}$$

$$
\begin{aligned}
&=\phi(0)e^{-\beta t}+I_1(u(t_1^-))e^{-\beta(t-t_1)}+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-\tau)^{\alpha-1}e^{-\beta(t-\tau)}f(\tau,u_\tau)d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-\tau)^{\alpha-1}e^{-\beta(t-\tau)}f(\tau,u_\tau)d\tau,\quad \text{if } t\in(t_1,t_2],\\
&\dots\\
u(t)&=\phi(0)e^{-\beta t}+\sum_{k=1}^{m}I_k(u(t_k^-))e^{-\beta(t-t_k)}\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\alpha-1}e^{-\beta(t-\tau)}f(\tau,u_\tau)d\tau \qquad (2.2)\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{t}(t-\tau)^{\alpha-1}e^{-\beta(t-\tau)}f(\tau,u_\tau)d\tau,\quad \text{if } t\in(t_m,T].
\end{aligned}
$$

In a similar way, if (2.1) holds, then we can prove that u is the solution of (1.1), and thus the proof of this lemma is complete. □

Definition 2.3. A set Λ is said to be quasi-equicontinuous in $[0,T]$ if for any $\varepsilon>0$, there exists $\delta'>0$ such that if $u\in\Lambda$, $k\in\mathbb{N}$, $s_1,s_2\in(t_{k-1},t_k]\cap[0,T]$ and $|s_1-s_2|<\delta'$, then $\|u(s_1)-u(s_2)\|<\varepsilon$.

Theorem 2.4 (Leray-Schauder fixed point theorem)**.** *Let F be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$\{x\in X : x=\lambda Fx \text{ for some } 0\le\lambda\le1\}$$

is bounded. Then F has a fixed point.

3. Existence theorems

In this section, we consider the existence and uniqueness of global solutions of the initial value problem (1.1). First we state some assumptions for the functions f and I_k in (1.1).

(H1) The function $f:[0,\infty)\times PC_0\to X$ is weakly continuous in bounded sets for each $t\in[0,\infty)$, and there exist $K_2>0$ and a function $K_1\in L^{1/\gamma}([0,\infty),\mathbb{R}_+)$ with $\gamma<\alpha$ such that
$$\|f(t,\psi)\|\le K_1(t)+K_2\|\psi\|_{PC_0}\quad\text{for all } \psi\in PC_0 \text{ and } t\in[0,\infty).$$

(H2) The functions $I_k:X\to X$ are weakly continuous in bounded sets and there exist $J_1,\ J_2>0$ such that
$$\|I_k(x)\|\le J_1\|x\|+J_2\quad\text{for all } x\in X \text{ and } k\in\mathbb{N}.$$

(H3) $\delta=\sup_{k\in\mathbb{N}}\{t_k-t_{k-1}\}<\infty$, $\eta=\inf_{k\in\mathbb{N}}\{t_k-t_{k-1}\}>0$.

(H4) There exists $M_1>0$ such that
$$\|f(t,\varphi)-f(t,\psi)\|\le M_1\|\varphi-\psi\|_{PC_0}\quad\text{for all } \varphi,\psi\in PC_0 \text{ and } t\in[0,\infty).$$

(H5) There exists $N>0$ such that
$$\|I_k(x)-I_k(y)\|\le N\|x-y\|\quad\text{for all } x,y\in X \text{ and } k\in\mathbb{N}.$$

In the sequel C denotes an arbitrary positive constant, which may be different from line to line and even in the same line. We now state a theorem regarding the local existence of solutions for problem (1.1).

Theorem 3.1. *Assume that X is a separable Hilbert space, and conditions* (H1)–(H3) *are satisfied. Then for every $\phi \in PC_0$, initial value problem* (1.1) *has at least one solution defined on $[0, b]$ with $b > t_1$, where t_1 is given by* (1.1).

Theorem 3.2. *Assume the conditions of Theorem 3.1. Then for every $\phi \in PC_0$, initial value problem* (1.1) *has at least one solution defined on $[0, \infty)$ in the sense of Definition 2.1.*

We will also prove the uniqueness of solutions.

Theorem 3.3. *Assume the hypotheses of Theorem 3.1. Also, suppose that the conditions* (H4), (H5) *are satisfied. Then for every $\phi \in PC_0$, problem* (1.1) *possesses a unique solution $u(\cdot)$ defined on $[0, \infty)$ in the sense of Definition 2.1.*

4. Proof of Theorem 3.1

Since X is separable, there exists a family of elements $\{e_j\}_{j=1}^{\infty}$ of X which are orthonormal in X. Let $X_{(n)} = \operatorname{span}\{e_1, \dots, e_n\}$ in X and $P_n : X \to X_{(n)}$ is an orthonormal projector. Fix some $\phi \in PC_0$, and let $u_n = P_n u$, $\phi_n = P_n \phi$. By Lemma 2.2, for every n we introduce the mapping $T_n : PC_b \to PC_b$ defined by

$$
(T_n u_n)(t) = \begin{cases} \phi_n(t), & t \in [-h, 0], \\ \phi_n(0)e^{-\beta t} + \sum_{0<t_k<t} P_n I_k(u_n(t_k^-))e^{-\beta(t-t_k)} \\ +\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \tau)^{\alpha-1} e^{-\beta(t-\tau)} P_n f(\tau, u_{n\tau}) d\tau \\ +\frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} P_n f(\tau, u_{n\tau}) d\tau, & t \in [0, b], \end{cases} \tag{4.1}
$$

where $t_m = \max\{t_k : t_k < t, k = 0, 1, 2, \dots\}$ and $t_0 = 0$. Now we show that the operator T_n is continuous and completely continuous. Since the proof of the case $m = 0$ is similar, we assume $m \geq 1$.

Step 1: T_n maps bounded sets into bounded sets in PC_b. Indeed, it is enough to show that for any $\rho > 0$, there exists a positive constant ρ' such that for each $u_n \in B(\rho)$ one has $\sup_{t\in[0,b]} \|T_n u_n(t)\| \leq \rho'$, where

$$
B(\rho) = \{u_n \in PC_b \ : \ u_n(t) = \phi_n(t) \text{ on } [-h, 0] \text{ and } \sup_{t\in[0,b]} \|u_n(t)\| \leq \rho\}.
$$

Let $u_n \in B(\rho)$, by (H1) and (H2) we obtain for each $t \in [0, b]$,

$$
\begin{aligned}
&\|(T_n u_n)(t)\| \\
&\leq \|\phi_n(0)\|e^{-\beta t} + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \tau)^{\alpha-1} e^{-\beta(t-\tau)} \|P_n f(\tau, u_{n\tau})\| d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} \|P_n f(\tau, u_{n\tau})\| d\tau \\
&\quad + \sum_{0<t_k<t} \|P_n I_k(u_n(t_k^-))\| e^{-\beta(t-t_k)} \\
&\leq \|\phi(0)\| + \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - \tau)^{\alpha-1} e^{-\beta(t-\tau)} \Big(K_1(\tau) + K_2\rho\Big) d\tau \\
&\quad + \sum_{k=1}^{m} \Big(J_1\rho + J_2\Big) e^{-\beta(t-t_k)}
\end{aligned}
$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{t}(t-\tau)^{\alpha-1}e^{-\beta(t-\tau)}\Big(K_1(\tau)+K_2\rho\Big)d\tau. \tag{4.2}$$

We proceed to estimate the three last terms in (4.2). First, by (H3) we have

$$\begin{aligned}
&\frac{K_2\rho}{\Gamma(\alpha)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\alpha-1}e^{-\beta(t-\tau)}d\tau+\int_{t_m}^{t}(t-\tau)^{\alpha-1}e^{-\beta(t-\tau)}d\tau\Big)\\
&=-\frac{K_2\rho}{\Gamma(\alpha+1)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}e^{-\beta(t-\tau)}d(t_k-\tau)^{\alpha}+\int_{t_m}^{t}e^{-\beta(t-\tau)}d(t-\tau)^{\alpha}\Big)\\
&\le\sum_{k=1}^{m}\frac{K_2\rho}{\Gamma(\alpha+1)}\Big((t_k-t_{k-1})^{\alpha}e^{-\beta(t-t_{k-1})}+\beta\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\alpha}e^{-\beta(t-\tau)}d\tau\Big)\\
&\quad+\frac{K_2\rho}{\Gamma(\alpha+1)}\Big((t-t_m)^{\alpha}e^{-\beta(t-t_m)}+\beta\int_{t_m}^{t}(t-\tau)^{\alpha}e^{-\beta(t-\tau)}d\tau\Big) \qquad (4.3)\\
&\le\frac{K_2\rho}{\Gamma(\alpha+1)}\Big(\sum_{k=1}^{m}(t_k-t_{k-1})^{\alpha}e^{-\beta(m-k+1)\eta}+(t-t_m)^{\alpha}\\
&\quad+\beta\sum_{k=1}^{m}\int_{t-t_k}^{t-t_{k-1}}z^{\alpha}e^{-\beta z}dz+\beta\int_{0}^{t-t_m}z^{\alpha}e^{-\beta z}dz\Big)\\
&\le\frac{K_2\rho\delta^{\alpha}e^{\beta\eta}}{\Gamma(\alpha+1)(e^{\beta\eta}-1)}+\frac{K_2\rho}{\beta^{\alpha}},
\end{aligned}$$

and in the similar way,

$$\sum_{k=1}^{m}(J_1\rho+J_2)e^{-\beta(t-t_k)}\le(J_1\rho+J_2)\sum_{k=1}^{m}e^{-\beta(m-k)\eta}\le(J_1\rho+J_2)\frac{e^{\beta\eta}}{e^{\beta\eta}-1}. \tag{4.4}$$

Define $K_1^*=\Big(\int_0^{\infty}(K_1(t))^{1/\gamma}dt\Big)^{\gamma}$. Then, using Hölder's inequality, (H1) and (H3), we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\alpha-1}e^{-\beta(t-\tau)}K_1(\tau)d\tau+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{t}(t-\tau)^{\alpha-1}e^{-\beta(t-\tau)}K_1(\tau)d\tau\\
&\le\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\Big(\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\frac{\alpha-1}{1-\gamma}}e^{\frac{-\beta(t-\tau)}{1-\gamma}}d\tau\Big)^{1-\gamma}\Big(\int_{t_{k-1}}^{t_k}(K_1(\tau))^{1/\gamma}d\tau\Big)^{\gamma}\\
&\quad+\frac{1}{\Gamma(\alpha)}\Big(\int_{t_m}^{t}(t-\tau)^{\frac{\alpha-1}{1-\gamma}}e^{\frac{-\beta(t-\tau)}{1-\gamma}}d\tau\Big)^{1-\gamma}\Big(\int_{t_m}^{t}(K_1(\tau))^{1/\gamma}d\tau\Big)^{\gamma}\\
&\le\frac{1}{\Gamma(\alpha)}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\frac{\alpha-1}{1-\gamma}}e^{\frac{-\beta(t-\tau)}{1-\gamma}}d\tau+\int_{t_m}^{t}(t-\tau)^{\frac{\alpha-1}{1-\gamma}}e^{\frac{-\beta(t-\tau)}{1-\gamma}}d\tau\Big)^{1-\gamma}\\
&\quad\times\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(K_1(\tau))^{1/\gamma}d\tau+\int_{t_m}^{t}(K_1(\tau))^{\frac{1}{\gamma}}d\tau\Big)^{\gamma}\\
&\le\frac{K_1^*}{\Gamma(\alpha)}\Big(-\sum_{k=1}^{m}\frac{1-\gamma}{\alpha-\gamma}\int_{t_{k-1}}^{t_k}e^{\frac{-\beta(t-\tau)}{1-\gamma}}d(t_k-\tau)^{\frac{\alpha-\gamma}{1-\gamma}}\\
&\quad-\frac{1-\gamma}{\alpha-\gamma}\int_{t_m}^{t}e^{\frac{-\beta(t-\tau)}{1-\gamma}}d(t-\tau)^{\frac{\alpha-\gamma}{1-\gamma}}\Big)^{1-\gamma}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{K_1^*(\frac{1-\gamma}{\alpha-\gamma})^{1-\gamma}}{\Gamma(\alpha)}\Big(\sum_{k=1}^{m}(t_k-t_{k-1})^{\frac{\alpha-\gamma}{1-\gamma}}e^{\frac{-\beta(m+1-k)\eta}{1-\gamma}}+(t-t_m)^{\frac{\alpha-\gamma}{1-\gamma}}\\
&\quad+\frac{\beta}{1-\gamma}\Big(\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\frac{\alpha-\gamma}{1-\gamma}}e^{\frac{-\beta(t-\tau)}{1-\gamma}}d\tau+\int_{t_m}^{t}(t-\tau)^{\frac{\alpha-\gamma}{1-\gamma}}e^{\frac{-\beta(t-\tau)}{1-\gamma}}d\tau\Big)\Big)^{1-\gamma}\\
&\leq \frac{K_1^*(\frac{1-\gamma}{\alpha-\gamma})^{1-\gamma}}{\Gamma(\alpha)}\Big(\sum_{k=1}^{m}\delta^{\frac{\alpha-\gamma}{1-\gamma}}e^{\frac{-\beta(m+1-k)\eta}{1-\gamma}}+\delta^{\frac{\alpha-\gamma}{1-\gamma}}\\
&\quad+\frac{\beta}{1-\gamma}\sum_{k=1}^{m}\int_{t-t_k}^{t-t_{k-1}}z^{\frac{\alpha-\gamma}{1-\gamma}}e^{\frac{-\beta z}{1-\gamma}}dz+\frac{\beta}{1-\gamma}\int_0^{t-t_m}z^{\frac{\alpha-\gamma}{1-\gamma}}e^{\frac{-\beta z}{1-\gamma}}dz\Big)^{1-\gamma}\\
&\leq \frac{K_1^*}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}\delta^{\frac{\alpha-\gamma}{1-\gamma}}\frac{e^{\frac{\beta\eta}{1-\gamma}}}{e^{\frac{\beta\eta}{1-\gamma}}-1}+\frac{\beta}{\alpha-\gamma}\frac{\Gamma(\frac{\alpha+1-2\gamma}{1-\gamma})}{(\frac{\beta}{1-\gamma})^{\frac{\alpha+1-2\gamma}{1-\gamma}}}\Big)^{1-\gamma}.
\end{aligned}
\tag{4.5}
$$

It then follows from (4.2)–(4.5) that for any $n\in\mathbb{N}$, $u_n\in B(\rho)$ and $t\in[0,b]$,

$$
\begin{aligned}
\|(T_nu_n)(t)\| \leq& \|\phi(0)\|+\Big(\frac{1-\gamma}{\alpha-\gamma}\delta^{\frac{\alpha-\gamma}{1-\gamma}}\frac{e^{\frac{\beta\eta}{1-\gamma}}}{e^{\frac{\beta\eta}{1-\gamma}}-1}+\frac{\beta}{\alpha-\gamma}\frac{\Gamma(\frac{\alpha+1-2\gamma}{1-\gamma})}{(\frac{\beta}{1-\gamma})^{\frac{\alpha+1-2\gamma}{1-\gamma}}}\Big)^{1-\gamma}\frac{K_1^*}{\Gamma(\alpha)}\\
&+\frac{K_2\rho\delta^{\alpha}e^{\beta\eta}}{\Gamma(\alpha+1)(e^{\beta\eta}-1)}+\frac{K_2\rho}{\beta^{\alpha}}+\frac{J_1\rho e^{\beta\eta}}{e^{\beta\eta}-1}+\frac{J_2e^{\beta\eta}}{e^{\beta\eta}-1}=\rho'.
\end{aligned}
$$

Therefore, $T_nu_n\in B(\rho')$.

Step 2: T_n maps bounded sets into quasi-equicontinuous sets of PC_b. Let $B(\rho)$ be a bounded set of PC_b as in Step 1. We show that $T_n(B(\rho))=\{T_nu_n : u_n\in B(\rho)\}$ is a quasi-equicontinuous family of functions, that is, for any $\varepsilon>0$ there exists $\delta'>0$ such that if $n\in\mathbb{N}$, $s_1,s_2\in(t_{k-1},t_k]\cap[-h,b]$ and $|s_1-s_2|<\delta'$, then $\|(T_nu_n)(s_2)-(T_nu_n)(s_1)\|<\varepsilon$.

Since the proof of the case $m=0$ is similar, we assume that $t_m<s_1<s_2\leq t_{m+1}$ for some $m\in\{1,2,\dots\}$. Then from (H1)-(H3) and (4.1), for all $n\in\mathbb{N}$ and $u_n\in B(\rho)$ we obtain that

$$
\begin{aligned}
&\|(T_nu_n)(s_2)-(T_nu_n)(s_1)\|\\
&\leq \|\phi_n(0)\||e^{-\beta s_2}-e^{-\beta s_1}|+\sum_{k=1}^{m}\|P_nI_k(u_n(t_k^-))\|e^{-\beta(s_2-t_k)}|1-e^{-\beta(s_1-s_2)}|\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\alpha-1}e^{-\beta(s_2-\tau)}|1-e^{-\beta(s_1-s_2)}|\|P_nf(\tau,u_{n\tau})\|d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{s_1}\Big((s_1-\tau)^{\alpha-1}e^{-\beta(s_1-\tau)}-(s_2-\tau)^{\alpha-1}e^{-\beta(s_2-\tau)}\Big)\|P_nf(\tau,u_{n\tau})\|d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_2-\tau)^{\alpha-1}e^{-\beta(s_2-\tau)}\|P_nf(\tau,u_{n\tau})\|d\tau\\
&\leq \rho|e^{-\beta s_2}-e^{-\beta s_1}|+(J_1\rho+J_2)|1-e^{-\beta(s_1-s_2)}|\sum_{k=1}^{m}e^{-\beta(m-k)\eta}\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\alpha-1}e^{-\beta(s_2-\tau)}(e^{-\beta(s_1-s_2)}-1)\Big(K_1(\tau)+K_2\rho\Big)d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{s_1}\Big((s_1-\tau)^{\alpha-1}e^{-\beta(s_1-\tau)}-(s_2-\tau)^{\alpha-1}e^{-\beta(s_2-\tau)}\Big)\Big(K_1(\tau)+K_2\rho\Big)d\tau\\
&+\frac{1}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_2-\tau)^{\alpha-1}e^{-\beta(s_2-\tau)}\Big(K_1(\tau)+K_2\rho\Big)d\tau\\
&\le \rho|e^{-\beta s_2}-e^{-\beta s_1}|+(J_1\rho+J_2)|1-e^{-\beta(s_1-s_2)}|\frac{e^{\beta\eta}}{e^{\beta\eta}-1}+E_1+E_2+E_3. \qquad (4.6)
\end{aligned}
$$

For E_1, by the similar argument as in (4.3) and (4.5), we have

$$
E_1\le C\Big(e^{-\beta(s_1-s_2)}-1\Big)\to 0 \quad \text{as } s_2\to s_1. \tag{4.7}
$$

For E_2, by Hölder's inequality and (H1), we obtain

$$
\begin{aligned}
E_2&\le \frac{1}{\Gamma(\alpha)}\int_{t_m}^{s_1}\Big((s_1-\tau)^{\alpha-1}-(s_2-\tau)^{\alpha-1}\Big)\Big(K_1(\tau)+K_2\rho\Big)d\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{s_1}(s_2-\tau)^{\alpha-1}\Big(e^{-\beta(s_1-\tau)}-e^{-\beta(s_2-\tau)}\Big)\Big(K_1(\tau)+K_2\rho\Big)d\tau\\
&\le \frac{1}{\Gamma(\alpha)}\Big(\int_{t_m}^{s_1}\Big((s_1-\tau)^{\alpha-1}-(s_2-\tau)^{\alpha-1}\Big)^{\frac{1}{1-\gamma}}d\tau\Big)^{1-\gamma}\Big(\int_{t_m}^{s_1}(K_1(\tau))^{1/\gamma}d\tau\Big)^{\gamma}\\
&\quad+\frac{1}{\Gamma(\alpha)}\Big(\int_{t_m}^{s_1}(s_2-\tau)^{\frac{\alpha-1}{1-\gamma}}\Big(e^{-\beta(s_1-\tau)}-e^{-\beta(s_2-\tau)}\Big)^{\frac{1}{1-\gamma}}d\tau\Big)^{1-\gamma}\\
&\quad\times\Big(\int_{t_m}^{s_1}(K_1(\tau))^{1/\gamma}d\tau\Big)^{\gamma}+\frac{K_2\rho}{\Gamma(\alpha)}\int_{t_m}^{s_1}(s_1-\tau)^{\alpha-1}-(s_2-\tau)^{\alpha-1}d\tau\\
&\quad+\frac{K_2\rho}{\Gamma(\alpha)}\int_{t_m}^{s_1}(s_2-\tau)^{\alpha-1}\Big(e^{-\beta(s_1-\tau)}-e^{-\beta(s_2-\tau)}\Big)d\tau\\
&\le \frac{K_1^*}{\Gamma(\alpha)}\Big(\int_{t_m}^{s_1}\Big((s_1-\tau)^{\frac{\alpha-1}{1-\gamma}}-(s_2-\tau)^{\frac{\alpha-1}{1-\gamma}}\Big)d\tau\Big)^{1-\gamma}+\frac{K_2\rho}{\Gamma(\alpha+1)}(s_2-s_1)^{\alpha}\\
&\quad+\frac{K_1^*(s_2-s_1)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_{t_m}^{s_1}\Big(e^{\frac{-\beta(s_1-\tau)}{1-\gamma}}-e^{\frac{-\beta(s_2-\tau)}{1-\gamma}}\Big)d\tau\Big)^{1-\gamma} \qquad (4.8)\\
&\quad+\frac{K_2\rho}{\beta\Gamma(\alpha)}(s_2-s_1)^{\alpha-1}(1-e^{-\beta(s_2-s_1)})\\
&\le \frac{K_1^*}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}(s_1-t_m)^{\frac{\alpha-\gamma}{1-\gamma}}-\frac{1-\gamma}{\alpha-\gamma}(s_2-t_m)^{\frac{\alpha-\gamma}{1-\gamma}}+\frac{1-\gamma}{\alpha-\gamma}(s_2-s_1)^{\frac{\alpha-\gamma}{1-\gamma}}\Big)^{1-\gamma}\\
&\quad+\frac{K_1^*}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\beta}\Big)^{1-\gamma}(s_2-s_1)^{\alpha-1}\Big(1-e^{\frac{-\beta(s_2-s_1)}{1-\gamma}}\Big)^{1-\gamma}+\frac{K_2\rho}{\Gamma(\alpha+1)}(s_2-s_1)^{\alpha}\\
&\quad+\frac{K_2\rho}{\beta\Gamma(\alpha)}(s_2-s_1)^{\alpha-1}(1-e^{-\beta(s_2-s_1)})\\
&\le \frac{K_1^*}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}(s_2-s_1)^{\frac{\alpha-\gamma}{1-\gamma}}\Big)^{1-\gamma}+\frac{K_2\rho}{\Gamma(\alpha)}(s_2-s_1)^{\alpha}\\
&\quad+\frac{K_2\rho}{\beta\Gamma(\alpha)}(s_2-s_1)^{\alpha-1}(\beta(s_2-s_1)+o(s_2-s_1))\\
&\quad+\frac{K_1^*}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\beta}\Big)^{1-\gamma}(s_2-s_1)^{\alpha-1}\Big(\frac{\beta(s_2-s_1)}{1-\gamma}+o(s_2-s_1)\Big)^{1-\gamma}\\
&\to 0 \quad \text{as } s_2\to s_1,
\end{aligned}
$$

where $\lim_{s_2-s_1\to 0}\left(\frac{o(s_2-s_1)}{s_2-s_1}\right)=0$.

For E_3, by Hölder's inequality and (H1), we find that

$$\begin{aligned} E_3 &\le \frac{1}{\Gamma(\alpha)}\Big(\int_{s_1}^{s_2}(s_2-\tau)^{\frac{\alpha-1}{1-\gamma}}e^{\frac{-\beta(s_2-\tau)}{1-\gamma}}d\tau\Big)^{1-\gamma}\Big(\int_{s_1}^{s_2}(K_1(\tau))^{1/\gamma}d\tau\Big)^{\gamma}\\ &\quad+\frac{K_2\rho}{\Gamma(\alpha)}\int_{s_1}^{s_2}(s_2-\tau)^{\alpha-1}e^{-\beta(s_2-\tau)}d\tau\\ &\le \frac{K_1^*}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}\Big)^{1-\gamma}(s_2-s_1)^{\alpha-\gamma}+\frac{K_2\rho}{\Gamma(\alpha+1)}(s_2-s_1)^{\alpha}\\ &\to 0 \quad \text{as } s_2\to s_1. \end{aligned} \tag{4.9}$$

Therefore, (4.6)-(4.9) imply that $T_n(B(\rho))$ is quasi-equicontinuous.

Step 3: T_n is continuous. Let $\{u_n^j\}_{j=1}^{\infty}$ be a sequence such that $u_n^j\to v$ in PC_b as $j\to\infty$. Since for any $\tau\in[0,b]$,

$$\begin{aligned} \|u_{n\tau}^j-v_\tau\|_{PC_0} &= \sup_{\theta\in[-h,0]}\|u_n^j(\tau+\theta)-v(\tau+\theta)\|\\ &\le \sup_{t\in[0,b]}\|u_n^j(t)-v(t)\|\to 0 \quad \text{as } j\to\infty, \end{aligned} \tag{4.10}$$

by the weak continuity of the nonlinear terms f and I_k, we obtain that for any $\tau\in[0,b]$,

$$\lim_{j\to\infty}P_nf(\tau,u_{n\tau}^j)=P_nf(\tau,v_\tau), \tag{4.11}$$

and for each $k\in\mathbb{N}$,

$$\lim_{j\to\infty}P_nI_k(u_n^j(t_k^-))=P_nI_k(v(t_k^-)). \tag{4.12}$$

On the other hand, by (H1) and (4.10) we conclude that for all $\tau\in[0,b]$ and j sufficiently large,

$$\begin{aligned} \|P_nf(\tau,u_{n\tau}^j)-P_nf(\tau,v_\tau)\| &\le 2K_1(\tau)+K_2\|u_{n\tau}^j\|_{PC_0}+K_2\|v_\tau\|_{PC_0}\\ &\le 2K_1(\tau)+C+2K_2\|v\|_{PC_b}. \end{aligned} \tag{4.13}$$

Then, by (4.3)-(4.5), (4.11) and (4.13), we deduce from Lebesgue's theorem that

$$\lim_{j\to\infty}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\alpha-1}e^{-\beta(t-\tau)}\|P_nf(\tau,u_{n\tau}^j)-P_nf(\tau,v_\tau)\|=0, \tag{4.14}$$

and

$$\lim_{j\to\infty}\int_{t_m}^{t}(t-\tau)^{\alpha-1}e^{-\beta(t-\tau)}\|P_nf(\tau,u_{n\tau}^j)-P_nf(\tau,v_\tau)\|=0. \tag{4.15}$$

Then by (4.1), (4.12) and (4.14)-(4.15), we find that for any $t\in[0,b]$,

$$\begin{aligned} &\|(T_nu_n^j)(t)-(T_nv)(t)\|\\ &\le \sum_{0<t_k<t}\|P_nI_k(u_n^j(t_k^-))-P_nI_k(v(t_k^-))\|\\ &\quad+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-\tau)^{\alpha-1}e^{-\beta(t-\tau)}\|P_nf(\tau,u_{n\tau}^j)-P_nf(\tau,v_\tau)\|d\tau\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{t}(t-\tau)^{\alpha-1}e^{-\beta(t-\tau)}\|P_nf(\tau,u_{n\tau}^j)-P_nf(\tau,v_\tau)\|d\tau\to 0 \end{aligned} \tag{4.16}$$

as $j \to \infty$. By the proof of Step 2, we see that $\{T_n u_n^j\}_{j=1}^{\infty}$ is a quasi-equicontinuous family of functions. Hence, the Arzelà-Ascoli theorem yields $T_n u_n^j \to T_n v$ in PC_b.

As a consequence of Steps 1-3, and the Arzelà-Ascoli theorem, we can conclude that $T_n : PC([-h, b]; X_{(n)}) \to PC([-h, b]; X_{(n)})$ is continuous and completely continuous.

Step 4: A priori bounds. We show there exists an open set $U \subseteq PC([-h, b]; X_{(n)})$ with $u_n \neq \lambda T_n u_n$ for $\lambda \in (0, 1)$ and $u_n \in \partial U$. Let $u_n \in PC([-h, b]; X_{(n)})$ and $u_n = \lambda T_n u_n$ for some $0 < \lambda < 1$. Then for each $t \in [0, b]$ we have

$$\begin{aligned} u_n(t) = \lambda\Big\{&\phi_n(0)e^{-\beta t} + \sum_{0<t_k<t} P_n I_k(u_n(t_k^-))e^{-\beta(t-t_k)} \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \tau)^{\alpha-1} e^{-\beta(t-\tau)} P_n f(\tau, u_{n\tau}) d\tau \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t - \tau)^{\alpha-1} e^{-\beta(t-\tau)} P_n f(\tau, u_{n\tau}) d\tau\Big\}, \end{aligned} \tag{4.17}$$

where $t_m = \max\{t_k : t_k < t, k = 0, 1, 2, \dots\}$. For $t \in [0, t_1]$, replacing t by $t + \theta$ (where $\theta \in [-h, 0]$) in (4.17), and arguing as in the proof of (4.5), in view of (H1), (H3) and Hölder's inequality, we obtain

$$\begin{aligned} &\|u_n(t + \theta)\| \\ &\le \|\phi(0)\|e^{-\beta(t+\theta)} + \frac{1}{\Gamma(\alpha)} \int_0^{t+\theta} (t + \theta - \tau)^{\alpha-1} e^{-\beta(t+\theta-\tau)} \Big(K_1(\tau) + K_2\|u_{n\tau}\|_{PC_0}\Big) d\tau \\ &\le \|\phi(0)\|e^{-\beta(t+\theta)} + \frac{1}{\Gamma(\alpha)}\Big(\int_0^{t+\theta} (t + \theta - \tau)^{\frac{\alpha-1}{1-\gamma}} e^{\frac{-\beta(t+\theta-\tau)}{1-\gamma}} d\tau\Big)^{1-\gamma} \\ &\quad \times \Big(\int_0^{t+\theta} (K_1(\tau))^{1/\gamma} d\tau\Big)^{\gamma} \\ &\quad + \frac{K_2}{\Gamma(\alpha)}\Big(\int_0^{t+\theta} (t + \theta - \tau)^{\frac{\alpha-1}{1-\gamma}} d\tau\Big)^{1-\gamma} \Big(\int_0^{t+\theta} e^{\frac{-\beta(t+\theta-\tau)}{\gamma}} \|u_{n\tau}\|_{PC_0}^{1/\gamma} d\tau\Big)^{\gamma} \\ &\le \|\phi(0)\|e^{-\beta(t+\theta)} + C_1^* + C_2^*\Big(\int_0^{t+\theta} e^{\frac{-\beta(t+\theta-\tau)}{\gamma}} \|u_{n\tau}\|_{PC_0}^{1/\gamma} d\tau\Big)^{\gamma}, \end{aligned} \tag{4.18}$$

where we have used the notation

$$C_1^* := \frac{K_1^*}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}\delta^{\frac{\alpha-\gamma}{1-\gamma}} + \frac{\beta}{\alpha-\gamma}\frac{\Gamma(\frac{\alpha+1-2\gamma}{1-\gamma})}{(\frac{\beta}{1-\gamma})^{\frac{\alpha+1-2\gamma}{1-\gamma}}}\Big)^{1-\gamma},$$

$$C_2^* := \frac{K_2}{\Gamma(\alpha)}\Big(\frac{1-\gamma}{\alpha-\gamma}\Big)^{1-\gamma}\delta^{\alpha-\gamma}.$$

Note that if $t + \theta < 0$, then

$$\|u_n(t + \theta)\| = \|\phi_n(t + \theta)\| \le \|\phi\|_{PC_0} \le \|\phi\|_{PC_0} e^{-\beta(t+\theta)}.$$

Therefore,

$$\|u_{nt}\|_{PC_0} \le e^{\beta h}\|\phi\|_{PC_0} e^{-\beta t} + C_1^* + C_2^* e^{\beta h}\Big(\int_0^t e^{\frac{-\beta(t-\tau)}{\gamma}} \|u_{n\tau}\|_{PC_0}^{1/\gamma} d\tau\Big)^{\gamma},$$

and

$$e^{\frac{\beta t}{\gamma}}\|u_{nt}\|_{PC_0}^{1/\gamma} \le 3^{\frac{1-\gamma}{\gamma}} e^{\frac{\beta h}{\gamma}}\|\phi\|_{PC_0}^{1/\gamma} + 3^{\frac{1-\gamma}{\gamma}}(C_1^*)^{1/\gamma}e^{\frac{\beta t}{\gamma}} + C_3^*\int_0^t e^{\frac{\beta\tau}{\gamma}}\|u_{n\tau}\|_{PC_0}^{1/\gamma}d\tau,$$

where we have used the notation

$$C_3^* := 3^{\frac{1-\gamma}{\gamma}}(C_2^*)^{1/\gamma}e^{\frac{\beta h}{\gamma}}.$$

Applying Gronwall's inequality, we have for $t \in [0, t_1]$,

$$\begin{aligned}\|u_{nt}\|_{PC_0}^{1/\gamma} &\le 2\times 3^{\frac{1-\gamma}{\gamma}} e^{\frac{\beta h}{\gamma}}\|\phi\|_{PC_0}^{1/\gamma}e^{(C_3^*-\frac{\beta}{\gamma})t} \\ &\quad + 3^{\frac{1-\gamma}{\gamma}}(C_1^*)^{1/\gamma} + 3^{\frac{1-\gamma}{\gamma}}(C_1^*)^{1/\gamma}C_3^*\frac{1+e^{|C_3^*-\frac{\beta}{\gamma}|t}}{|\frac{\beta}{\gamma}-C_3^*|},\end{aligned} \tag{4.19}$$

and thus

$$\begin{aligned}\|u_n(t_1)\|^{1/\gamma} &\le 2\times 3^{\frac{1-\gamma}{\gamma}} e^{\frac{\beta h}{\gamma}}\|\phi\|_{PC_0}^{1/\gamma}e^{(C_3^*-\frac{\beta}{\gamma})t_1} \\ &\quad + 3^{\frac{1-\gamma}{\gamma}}(C_1^*)^{1/\gamma} + 3^{\frac{1-\gamma}{\gamma}}(C_1^*)^{1/\gamma}C_3^*\frac{1+e^{|C_3^*-\frac{\beta}{\gamma}|t_1}}{|\frac{\beta}{\gamma}-C_3^*|} = D_1^*.\end{aligned} \tag{4.20}$$

For $t \in (t_1, t_2]$, similar to (4.18) and (4.19), in view of (H1)-(H3), we find for $t+\theta > t_1$ (where $\theta \in [-h, 0]$) that

$$\begin{aligned}\|u_n(t+\theta)\| &\le \|u_n(t_1^-) + I_1(u_n(t_1^-))\|e^{-\beta(t+\theta-t_1)} \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t+\theta}(t+\theta-\tau)^{\alpha-1}e^{-\beta(t+\theta-\tau)}\|f(\tau,u_{n\tau})\|d\tau \\ &\le (1+J_1)\|u_n(t_1^-)\|e^{-\beta(t+\theta-t_1)} + J_2e^{-\beta(t+\theta-t_1)} \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t+\theta}(t+\theta-\tau)^{\alpha-1}e^{-\beta(t+\theta-\tau)}\Big(K_1(\tau)+K_2\|u_{n\tau}\|_{PC_0}\Big)d\tau \\ &\le (1+J_1)\|u_n(t_1^-)\|e^{-\beta(t+\theta-t_1)} + J_2e^{-\beta(t+\theta-t_1)} + C_1^* \\ &\quad + C_2^*\Big(\int_{t_1}^{t+\theta} e^{\frac{-\beta(t+\theta-\tau)}{\gamma}}\|u_{n\tau}\|_{PC_0}^{1/\gamma}d\tau\Big)^{\gamma}.\end{aligned} \tag{4.21}$$

Hence,

$$\begin{aligned}\|u_n(t+\theta)\|^{1/\gamma} &\le 3^{\frac{1-\gamma}{\gamma}}(1+J_1)^{1/\gamma}e^{\frac{\beta h}{\gamma}}\|u_n(t_1^-)\|^{1/\gamma}e^{-\frac{\beta(t-t_1)}{\gamma}} \\ &\quad + 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma} + C_3^*\int_{t_1}^t e^{-\frac{\beta(t-\tau)}{\gamma}}\|u_{n\tau}\|_{PC_0}^{1/\gamma}d\tau.\end{aligned} \tag{4.22}$$

It follows from (4.19) and (4.20) that if $t \in (t_1, t_2]$ and $t+\theta \le t_1$, then we have

$$\begin{aligned}\|u_n(t+\theta)\|^{1/\gamma} &\le 2\times 3^{\frac{1-\gamma}{\gamma}} e^{\frac{\beta h}{\gamma}}\|\phi\|_{PC_0}^{1/\gamma}e^{(C_3^*-\frac{\beta}{\gamma})(t+\theta)} + 3^{\frac{1-\gamma}{\gamma}}(C_1^*)^{1/\gamma} \\ &\quad + 3^{\frac{1-\gamma}{\gamma}}(C_1^*)^{1/\gamma}C_3^*\frac{1+e^{|C_3^*-\frac{\beta}{\gamma}|(t+\theta)}}{|\frac{\beta}{\gamma}-C_3^*|} \\ &\le D_1^*e^{-\frac{\beta}{\gamma}(t+\theta-t_1)}.\end{aligned} \tag{4.23}$$

Using $\theta \in [-h, 0]$ we get from (4.22) and (4.23) that

$$\begin{aligned} e^{\frac{\beta}{\gamma}t}\|u_{nt}\|_{PC_0}^{1/\gamma} \le 3^{\frac{1-\gamma}{\gamma}}(1+J_1)^{1/\gamma}D_1^* e^{\frac{\beta h}{\gamma}} e^{\frac{\beta t_1}{\gamma}} + 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma}e^{\frac{\beta}{\gamma}t} \\ + C_3^*\int_{t_1}^t e^{\frac{\beta\tau}{\gamma}}\|u_{n\tau}\|_{PC_0}^{1/\gamma}d\tau. \end{aligned} \tag{4.24}$$

By using Gronwall's inequality, we have that for $t \in (t_1, t_2]$,

$$\begin{aligned} \|u_{nt}\|_{PC_0}^{1/\gamma} \le 2\times 3^{\frac{1-\gamma}{\gamma}}(1+J_1)^{1/\gamma}D_1^* e^{\frac{\beta h}{\gamma}} e^{(C_3^*-\frac{\beta}{\gamma})(t-t_1)} + 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma} \\ + 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma}C_3^*\frac{1+e^{|C_3^*-\frac{\beta}{\gamma}|(t-t_1)}}{|\frac{\beta}{\gamma}-C_3^*|}, \end{aligned} \tag{4.25}$$

and consequently,

$$\begin{aligned} &\|u_n(t_2)\|^{1/\gamma} \\ &\le 2\times 3^{\frac{1-\gamma}{\gamma}}(1+J_1)^{1/\gamma}D_1^* e^{\frac{\beta h}{\gamma}} e^{(C_3^*-\frac{\beta}{\gamma})(t_2-t_1)} \\ &\quad + 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma} + 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma}C_3^*\frac{1+e^{|C_3^*-\frac{\beta}{\gamma}|(t_2-t_1)}}{|\frac{\beta}{\gamma}-C_3^*|} = D_2^*. \end{aligned} \tag{4.26}$$

In a similar way as above, we obtain that for $t \in (t_m, t_{m+1}]$ with $m \ge 2$,

$$\begin{aligned} \|u_{nt}\|_{PC_0}^{1/\gamma} \le 2\times 3^{\frac{1-\gamma}{\gamma}}(1+J_1)^{1/\gamma}D_m^* e^{\frac{\beta h}{\gamma}} e^{(C_3^*-\frac{\beta}{\gamma})(t-t_m)} + 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma} \\ + 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma}C_3^*\frac{1+e^{|C_3^*-\frac{\beta}{\gamma}|(t-t_m)}}{|\frac{\beta}{\gamma}-C_3^*|}, \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} &\|u_n(t_{m+1})\|^{1/\gamma} \\ &\le 2\times 3^{\frac{1-\gamma}{\gamma}}(1+J_1)^{1/\gamma}D_m^* e^{\frac{\beta h}{\gamma}} e^{(C_3^*-\frac{\beta}{\gamma})(t_{m+1}-t_m)} + 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma} \\ &\quad + 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma}C_3^*\frac{1+e^{|C_3^*-\frac{\beta}{\gamma}|(t_{m+1}-t_m)}}{|\frac{\beta}{\gamma}-C_3^*|} = D_{m+1}^*. \end{aligned} \tag{4.28}$$

For convenience, let

$$\begin{gathered} B_1^* = 2\times 3^{\frac{1-\gamma}{\gamma}}(1+J_1)^{1/\gamma}e^{\frac{\beta h}{\gamma}}, \\ B_2^* = 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma} + 3^{\frac{1-\gamma}{\gamma}}(J_2+C_1^*)^{1/\gamma}C_3^*\frac{1+e^{|C_3^*-\frac{\beta}{\gamma}|\delta}}{|\frac{\beta}{\gamma}-C_3^*|}. \end{gathered}$$

Then by using mathematical induction, we find for $m \ge 2$ that

$$\begin{aligned} D_m^* &\le B_1^* D_{m-1}^* e^{(C_3^*-\frac{\beta}{\gamma})(t_m-t_{m-1})} + B_2^* \\ &\le (B_1^*)^{m-1}e^{(C_3^*-\frac{\beta}{\gamma})(t_m-t_1)}D_1^* + B_2^*\sum_{k=2}^m e^{(C_3^*-\frac{\beta}{\gamma})(t_m-t_k)}(B_1^*)^{m-k}. \end{aligned} \tag{4.29}$$

Note that (H3) implies $(m-1)\eta \le t_m - t_1 \le (m-1)\delta$ and $(m-k)\eta \le t_m - t_k \le (m-k)\delta$. It follows from (4.29) that

$$
\begin{aligned}
D_m^* &\le (B_1^*)^{\frac{t_m - t_1}{\eta}} e^{(C_3^* - \frac{\beta}{\gamma})(t_m - t_1)} D_1^* + B_2^* \sum_{k=2}^{m} e^{(C_3^* - \frac{\beta}{\gamma})(t_m - t_k)} (B_1^*)^{\frac{t_m - t_k}{\eta}} \\
&= e^{(C_3^* - \frac{\beta}{\gamma} + \frac{\ln B_1^*}{\eta})(t_m - t_1)} D_1^* + B_2^* \sum_{k=2}^{m} e^{|C_3^* - \frac{\beta}{\gamma} + \frac{\ln B_1^*}{\eta}|(t_m - t_k)} \\
&\le e^{(C_3^* - \frac{\beta}{\gamma} + \frac{\ln B_1^*}{\eta})(t_m - t_1)} D_1^* + B_2^* \sum_{k=2}^{m} e^{|C_3^* - \frac{\beta}{\gamma} + \frac{\ln B_1^*}{\eta}|(m-k)\delta} \\
&\le e^{\left(C_3^* - \frac{\beta}{\gamma} + \frac{\ln B_1^*}{\eta}\right)(t_m - t_1)} D_1^* + B_2^* \frac{e^{|C_3^* - \frac{\beta}{\gamma} + \frac{\ln B_1^*}{\eta}| \frac{\delta}{\eta}(t_m - t_2)}}{1 - e^{-|C_3^* - \frac{\beta}{\gamma} + \frac{\ln B_1^*}{\eta}|\delta}}.
\end{aligned}
\tag{4.30}
$$

Therefore, by (4.27) and (4.30) we deduce that for $t \in (t_m, t_{m+1}]$ with $m \ge 2$,

$$
\begin{aligned}
\|u_{nt}\|_{PC_0}^{1/\gamma} &\le B_1^* D_m^* e^{(C_3^* - \frac{\beta}{\gamma})(t - t_m)} + B_2^* \\
&\le C e^{(C_3^* - \frac{\beta}{\gamma} + \frac{\ln B_1^*}{\eta})(t - t_1)} D_1^* + C e^{|C_3^* - \frac{\beta}{\gamma} + \frac{\ln B_1^*}{\eta}| \frac{\delta}{\eta}(t - t_2)} + C.
\end{aligned}
$$

Combining this with (4.19), (4.20) and (4.25), we obtain that for all $t \ge 0$,

$$
\|u_{nt}\|_{PC_0}^{1/\gamma} \le C \|\phi\|_{PC_0}^{1/\gamma} e^{(C_3^* - \frac{\beta}{\gamma} + \frac{\ln B_1^*}{\eta})t} + C e^{Ct} + C. \tag{4.31}
$$

Hence, we can find a $W^* > 0$ such that for all $t \in [0, b]$,

$$
\|u_{nt}\|_{PC_0} \le W^*.
$$

Set

$$
U = \{u_n \in PC([-h, b]; X_{(n)}) : \|u_n\|_{PC_b} < W^* + 1\}.
$$

Note that $T_n : \overline{U} \to PC([-h, b]; X_{(n)})$ is continuous and completely continuous. From the choice of U, there is no $u_n \in \partial U$ such that $u_n = \lambda T_n(u_n)$ for $\lambda \in (0, 1)$. As a consequence of Theorem 2.4, we deduce that T_n has a fixed point u_n in U, which is a local solution of (1.1) in $X_{(n)}$.

Step 5: Existence of local solutions for (1.1) **in** X**.** We pass now to the case of a general separable Hilbert space X. We form the approximating equations

$$
\begin{gathered}
D_s^\alpha u_n(t) = P_n(f(t, u_{nt})), \quad t \ge 0,\ t \ne t_k, \\
u_n(s) = \phi_n(s), \quad \forall s \in [-h, 0], \\
u_n(t_k^+) - u_n(t_k^-) = P_n(I_k(u_n(t_k^-))), \quad k = 1, 2, \dots.
\end{gathered}
\tag{4.32}
$$

It follows from the preceding discussion that we may find a solution u_n of the approximating equation on $0 \le t \le b$ such that for all $n \in \mathbb{N}$,

$$
\sup_{t \in [0,b]} \|u_n(t)\| \le C, \tag{4.33}
$$

and for $t, s \in (t_m, t_{m+1}]$ for each $m \in \{0, 1, 2, \dots\}$, we have

$$
\|u_n(t) - u_n(s)\| \le C |e^{-\beta t} - e^{-\beta s}| + C |t - s|^{\alpha - \gamma}. \tag{4.34}
$$

Since X is a Hilbert space, from (4.33) we deduce that for any $t \in [0,b]$, $\{u_n(t)\}_{n=1}^{\infty}$ is relatively compact in X_w. Using the diagonal method one can choose a subsequence of $\{u_n(\cdot)\}_{n=1}^{\infty}$ and a function $u : \big(\mathbb{Q} \cup \{t_k\}_{k=1}^{\infty}\big) \cap [0,b] \to X$ such that $u_n(t) \to u(t)$ in X_w for any $t \in \big(\mathbb{Q} \cup \{t_k\}_{k=1}^{\infty}\big) \cap [0,b]$. Since

$$\|u(t) - u(s)\| \le \liminf \|u^n(t) - u^n(s)\| \le C|e^{-\beta t} - e^{-\beta s}| + C|t-s|^{\alpha-\gamma}$$

for all $t, s \in \mathbb{Q} \cap (t_m, t_{m+1}]$ and for each $m \in \{0,1,2,\dots\}$, the function u can be extended to a piecewise continuous function (denote again $u : [0,b] \to X$) such that

$$\|u(t) - u(s)\| \le C|e^{-\beta t} - e^{-\beta s}| + C|t-s|^{\alpha-\gamma} \tag{4.35}$$

for all $t, s \in (t_m, t_{m+1}]$ and for each $m \in \{0,1,2,\dots\}$.

We shall prove that $u_n(s_0) \to u(s_0)$ in X_w. Indeed, for any $s_0 \in \big((t_m, t_{m+1}) \setminus \mathbb{Q}\big) \cap [0,b]$ for some $m \in \{0,1,2,\dots\}$ and $v \in X^*$, we have

$$\langle u_n(s_0) - u(s_0), v\rangle = \langle u_n(s_0) - u_n(s_m), v\rangle + \langle u_n(s_m) - u(s_m), v\rangle + \langle u(s_m) - u(s_0), v\rangle,$$

where $s_m \in \mathbb{Q}$ are such that $s_m \to s_0$. For any $\varepsilon > 0$ there exist $m(\varepsilon)$ and $N(m(\varepsilon), \varepsilon)$ such that for all $n \ge N$,

$$\begin{gathered}
\big|\langle u_n(s_0) - u_n(s_m), v\rangle\big| \le \|u_n(s_0) - u_n(s_m)\|\|v\| < \frac{\varepsilon}{3}, \\
\big|\langle u(s_m) - u(s_0), v\rangle\big| \le \|u(s_m) - u(s_0)\|\|v\| < \frac{\varepsilon}{3}, \\
\big|\langle u_n(s_m) - u(s_m), v\rangle\big| < \frac{\varepsilon}{3}.
\end{gathered}$$

Thus, $|\langle u_n(s_0) - u(s_0), v\rangle| < \varepsilon$, and consequently $u_n(s_0) \to u(s_0)$ in X_w. In fact, we have that for any $s_0 \in [0,b] \setminus \{t_k\}_{k=1}^{\infty}$,

$$u_n(s_n) \to u(s_0) \quad \text{in } X_w \text{ if } s_n \to s_0, \tag{4.36}$$

and for $s_0 = t_k \cap [0,b]$ for some $k = 1,2,\dots$,

$$u_n(s_n) \to u(s_0) \quad \text{in } X_w \text{ if } s_n \le s_0 \text{ and } s_n \to s_0. \tag{4.37}$$

By a similar argument, (4.36) and (4.37) can be obtained from the equality

$$\langle u_n(s_n) - u(s_0), v\rangle = \langle u_n(s_n) - u_n(s_0), v\rangle + \langle u_n(s_0) - u(s_0), v\rangle.$$

Then (4.36) and (4.37) imply that for any $\tau \in [0,b]$ and $s \in [-h,0]$, $u_{n\tau}(s_n) = u_n(\tau + s_n) \to u(\tau + s) = u_\tau(s)$ in X_w for $\tau + s_n \to \tau + s \in [0,b] \setminus \{t_k\}_{k=1}^{\infty}$ or $\tau + s_n \to \tau + s$ with $\tau + s \in \{t_k\}_{k=1}^{\infty}$ and $s_n \le s$, so that $u_{n\tau} \to u_\tau$ in $PC_{0,w}$.

Finally, we show that $u(\cdot)$ is a solution of (1.1). For this aim we will pass to the limit in the integral

$$u_n(t) = \begin{cases} \phi_n(t), & t \in [-h,0], \\ \phi_n(0)e^{-\beta t} + \sum_{0<t_k<t} P_n I_k(u_n(t_k^-))e^{-\beta(t-t_k)} \\ +\frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \tau)^{\alpha-1} e^{-\beta(t-\tau)} P_n f(\tau, u_{n\tau}) d\tau \\ +\frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} P_n f(\tau, u_{n\tau}) d\tau, & t \in [0,b], \end{cases}$$

where $t_m = \max\{t_k : k = 0,1,2,\dots, t_k < t\}$. Since f and I_k are weakly continuous in bounded sets, for any $\tau \in [0,t]$ we have

$$f(\tau, u_{n\tau}) \to f(\tau, u_\tau) \quad \text{in } X_w \text{ as } n \to \infty, \tag{4.38}$$

and for each k,

$$I_k(u_n(t_k^-)) \to I_k(u(t_k^-)) \quad \text{in } X_w \text{ as } n \to \infty. \tag{4.39}$$

Using the Riesz representation theorem, we obtain that for any $v \in X^*$, there exists an element $w \in X$ corresponding to v such that

$$\langle u, v\rangle = (u, w) \quad \text{for all } u \in X,$$

in view of $\|f(\tau, u_{n\tau})\| \le K_1(\tau) + K_2 C$ and $\|I_k(u_n(t_k^-))\| \le J_1 C + J_2$, we have

$$\begin{aligned} &|\langle P_n f(\tau, u_{n\tau}), v\rangle - \langle f(\tau, u_{n\tau}), v\rangle| = |(P_n f(\tau, u_{n\tau}) - f(\tau, u_{n\tau}), w)| \\ &= |(f(\tau, u_{n\tau}), (I - P_n)w)| \le (K_1(\tau) + K_2 C)\|(I - P_n)w\| \to 0 \end{aligned} \tag{4.40}$$

as $n \to \infty$, and

$$|\langle P_n I_k(u_n(t_k^-)), v\rangle - \langle I_k(u_n(t_k^-)), v\rangle| \le (J_1 C + J_2)\|(I - P_n)w\| \to 0 \tag{4.41}$$

as $n \to \infty$. Then by (4.38), (4.40) and Lebesgue's theorem we obtain for any $v \in X^*$ that

$$\begin{aligned} &\Big\langle \int_{t_{k-1}}^{t_k} (t_k - \tau)^{\alpha-1} e^{-\beta(t-\tau)} P_n f(\tau, u_{n\tau}) d\tau, v \Big\rangle \\ &= \int_{t_{k-1}}^{t_k} (t_k - \tau)^{\alpha-1} e^{-\beta(t-\tau)} \langle P_n f(\tau, u_{n\tau}), v\rangle d\tau \\ &\to \int_{t_{k-1}}^{t_k} (t_k - \tau)^{\alpha-1} e^{-\beta(t-\tau)} \langle f(\tau, u_\tau), v\rangle d\tau \\ &= \Big\langle \int_{t_{k-1}}^{t_k} (t_k - \tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau, v \Big\rangle, \end{aligned} \tag{4.42}$$

and in a similar way,

$$\begin{aligned} &\Big\langle \int_{t_m}^{t} (t - \tau)^{\alpha-1} e^{-\beta(t-\tau)} P_n f(\tau, u_{n\tau}) d\tau, v \Big\rangle \\ &\to \Big\langle \int_{t_m}^{t} (t - \tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau, v \Big\rangle. \end{aligned} \tag{4.43}$$

Therefore, (4.39) and (4.41)-(4.43) imply that for any $v \in X^*$,

$$\begin{aligned} \langle u(t), v\rangle &= \langle u(0) e^{-\beta t}, v\rangle + \sum_{0<t_k<t} \langle I_k(u(t_k^-)) e^{-\beta(t-t_k)}, v\rangle \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \Big\langle \int_{t_{k-1}}^{t_k} (t_k - \tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau, v \Big\rangle \\ &\quad + \frac{1}{\Gamma(\alpha)} \Big\langle \int_{t_m}^{t} (t - \tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau, v \Big\rangle. \end{aligned}$$

As $v \in X^*$ is arbitrary, we get the equality

$$\begin{aligned} u(t) &= \phi(0) e^{-\beta t} + \sum_{0<t_k<t} I_k(u(t_k^-)) e^{-\beta(t-t_k)} \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{t} (t - \tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau) d\tau \quad \text{for all } t \in [0, b]. \end{aligned}$$

This completes the proof of Theorem 3.1.

5. Proof of Theorem 3.2

Proof. Thanks to Theorem 3.1, we there exists at least one solution $u^{(1)}(t)$ such that

$$u^{(1)}(t) = \begin{cases} \phi(t), & t \in [-h, 0], \\ \phi(0)e^{-\beta t} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau^{(1)}) d\tau, & t \in [0, t_1]. \end{cases} \tag{5.1}$$

Arguing as in the proof of Theorem 3.1, we obtain the existence of $u^{(2)}(t)$ satisfying

$$u^{(2)}(t) = \begin{cases} u^{(1)}(t), & t \in [t_1 - h, t_1], \\ (u^{(1)}(t_1^-) + I_1(u^{(1)}(t_1^-)))e^{-\beta(t-t_1)} & \\ + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} f(\tau, u_\tau^{(2)}) d\tau, & t \in (t_1, t_2]. \end{cases} \tag{5.2}$$

Continuing in this way, we obtain a global solution of (1.1) in the sense of Definition 2.1. □

6. Proof of Theorem 3.3

By Theorem 3.2 there exists at least one solution defined on $[0, \infty)$. We will show that this solution is unique.

If $u(\cdot)$, $v(\cdot)$ are two solutions of problem (1.1) with the initial value ϕ, then for $t \in [0, t_1]$, by Definition 2.1 and (H4), we get

$$\|u(t) - v(t)\| \leq \frac{M_1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} e^{-\beta(t-\tau)} \|u_\tau - v_\tau\|_{PC_0} d\tau. \tag{6.1}$$

Let $\theta \in [-h, 0]$. Replacing t by $t+\theta$ in (6.1), noticing that $\|u(t+\theta) - v(t+\theta)\| = 0$ if $t + \theta < 0$, and for $t + \theta \geq 0$, using Hölder's inequality we have

$$\begin{aligned} &\|u(t+\theta) - v(t+\theta)\| \\ &\leq \frac{M_1}{\Gamma(\alpha)} \Big(\int_0^{t+\theta} (t+\theta-\tau)^{(\alpha-1)p} d\tau \Big)^{1/p} \Big(\int_0^{t+\theta} e^{-\beta(t+\theta-\tau)q} \|u_\tau - v_\tau\|_{PC_0}^q d\tau \Big)^{1/q}, \end{aligned}$$

where p, $q > 1$, $(\alpha - 1)p > -1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Thus,

$$\begin{aligned} \|u_t - v_t\|_{PC_0}^q &= \sup_{\theta \in [-h,0]} \|u(t+\theta) - v(t+\theta)\|^q \\ &\leq \frac{M_1^q \delta^{\alpha q - 1}}{\Gamma^q(\alpha)(p\alpha - p + 1)^{q/p}} e^{\beta q h} \int_0^t e^{-\beta q(t-\tau)} \|u_\tau - v_\tau\|_{PC_0}^q d\tau. \end{aligned}$$

By Gronwall's inequality we find that

$$\|u_t - v_t\|_{PC_0} = 0 \quad \text{for all } t \in [0, t_1]. \tag{6.2}$$

By using mathematical induction and arguing in a similar way as above, in view of (H5) and (6.2), we obtain that for $t \in (t_m, t_{m+1}]$ with $m \geq 1$,

$$\|u_t - v_t\|_{PC_0}^q \leq \frac{M_1^q \delta^{\alpha q - 1}}{\Gamma^q(\alpha)(p\alpha - p + 1)^{q/p}} e^{\beta q h} \int_{t_m}^t e^{-\beta q(t-\tau)} \|u_\tau - v_\tau\|_{PC_0}^q d\tau.$$

Again by Gronwall's inequality, we have that

$$\|u_t - v_t\|_{PC_0} = 0 \quad \text{for all } t \in (t_m, t_{m+1}]. \tag{6.3}$$

Since m is arbitrary, we deduce that $u \equiv v$.

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