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NOTE ON BOUNDED SOLUTIONS TO NONHOMOGENOUS LINEAR DIFFERENCE EQUATIONS

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Abstract. By using a solvability method along with the contraction mapping principle quite recently has been presented an interesting method for showing the existence of a unique bounded solution to a nonhomogenous linear second-order difference equation on the set of nonnegative integers. It is a natural question if the combination of the method and principle can be applied in showing the existence of bounded solutions to some higher-order generalizations of the equation. Here, among others, we give a positive answer to the question for the case of a nonhomogenous linear difference equation of third order. Moreover, the equation is studied on the whole integer domain $\mathbb{Z}$.

1. Introduction

Let $\mathbb{N}_k$ be the set all nonnegative integers n such that $n \geq k$, where $k \geq 0$, and $\mathbb{Z}$ be the set of all integers. There are many methods for studying difference equations and systems of difference equations (see, for example, [1]-[11], [13]-[32] and the references therein). Studying the solvability of the equations and systems is one of the oldest topics. The aim of the solvability methods is finding closed form formulas for solutions to the equations and systems, which can be afterwards used in studying of the long-term behaviour of their solutions. Recall also that a great majority of the equations and systems are practically not solvable in closed form, so that any new solvable equation or system is of some interest, if nothing because of its solvability. Many classical methods for solving the equations and systems can be found, for example, in books [1, 8, 14, 15, 16, 19, 20]. For a renewed interest in the topic, see, for example, recent papers [7, 21, 25, 26], [29]-[32], as well as the numerous related references therein. For some related results or applications of some solvable equations and systems see also [2, 5, 6, 13, 27, 28].

It is interesting to mention that one of the methods by Stević from 2004 for solving a class of nonlinear difference equations has attracted considerable attention (see, for example, [25, 31], the references therein, including the original sources). Namely, the difference equations investigated therein are some extensions of one of those equations which, by using a suitable change of variables, can be transformed

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to a special case of the linear first-order difference equation, that is, of the following equation

$$x_{n+1} = q_n x_n + f_n, \quad n \in \mathbb{N}_0, \tag{1.1}$$

which is solvable and one of the most important difference equations.

Since that time has been shown that there are many related equations and systems which can be solved by using closely related changes of variables (see, for example, [21, 32] and the references therein). Moreover, there are many related classes of systems of difference equations, which can be solved in closed form in a similar way (see, for example, [7, 30]). However, there are some other equations and systems which cannot be solved by closely related methods, although a detailed analysis can show that they also essentially use the solvability of equation (1.1), which also shows the importance of the equation (see, for example, [26] where an interesting class of difference equations with several parameters was studied, [29] where is investigated a two-dimensional product-type system of difference equations on the complex domain, as well as the references therein).

Since a majority of difference equations and systems is not solvable, some other methods for their study are needed. The methods from fixed point theory are suitable in showing the existence of specific types of solutions (see, for example, [1, 9, 10, 11, 17, 18, 24] and the references therein). One of the basic results in the theory is the *contraction mapping principle*, or the Banach fixed-point theorem [4]. A natural idea is to use the solvability methods in getting some formulas which can serve as natural motivations for introducing some operators on spaces of sequences, by which along with some of the theorems from the fixed point theory can be proved the existence of specific types of solutions to the equations and systems (bounded, positive, periodic etc.).

Recently, in [28], Stević has used a classical solvability method along with the contraction mapping principle, to show in an elegant and unified way the existence of a unique bounded solution to the following linear second-order difference equation

$$x_{n+2} - q_n x_n = f_n, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are bounded sequences, and the sequence $(q_n)_{n\in\mathbb{N}_0}$ satisfies some other conditions. Another, more elementary and solvability oriented, but not unified, method for dealing with the problem was also mentioned there. The successful application of the combination of the solvability method and the contraction principle naturally imposes the following problem.

Problem 1.1. Try to apply a version/modification of the combination of the solvability method and the contraction mapping principle for getting some other results on the existence of bounded solutions to some higher-order generalizations of difference equation (1.2).

This problem is concrete and, of course, one can naturally think of solving a more general problem where the solvability method can be any of such ones, and the contraction mapping principle can be any of the fixed point theorems on Banach spaces of sequences. In fact, this is something which, in this or that way, is used in showing the existence of specific types of solutions to the equations and systems, although frequently it is not explicitly said [1, 9, 10, 11, 17, 18, 24].

Here we tackle the problem in a natural way by considering the following difference equation

$$x_{n+3} - q_n x_n = f_n, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $(q_n)_{n\in\mathbb{N}_0}$ and $(f_n)_{n\in\mathbb{N}_0}$ are sequences of real or complex numbers, which is one of the simplest nonhomogenous linear third-order difference equations related to equations (1.1) and (1.2).

First note that if the sequence q_n is constant, that is,

$$q_n = q \neq 0, \quad n \in \mathbb{N}_0, \tag{1.4}$$

and $f_n = 0$, $n \in \mathbb{N}_0$, then equation (1.3) becomes

$$x_{n+3} - qx_n = 0, \quad n \in \mathbb{N}_0. \tag{1.5}$$

It is well-known that equation (1.5) has the general solution in the form

$$x_n = c_1(\sqrt[3]{q})^n + c_2(\sqrt[3]{q}\varepsilon)^n + c_3(\sqrt[3]{q}\bar{\varepsilon})^n = (\sqrt[3]{q})^n(c_1 + c_2\varepsilon^n + c_3\bar{\varepsilon}^n), \quad n \in \mathbb{N}_0, \tag{1.6}$$

where

$$\varepsilon = \frac{-1 + i\sqrt{3}}{2} = e^{\frac{2\pi i}{3}},$$

and c_j, $j = \overline{1,3}$, are arbitrary constants.

Since $\varepsilon^3 = \bar{\varepsilon}^3 = 1$ and $\varepsilon\bar{\varepsilon} = 1$, note that the following useful equalities hold:

$$\begin{gathered} \varepsilon^2 = \bar{\varepsilon}, \quad \bar{\varepsilon}^2 = \varepsilon, \quad \varepsilon - 1 = \varepsilon(1 - \bar{\varepsilon}), \\ \bar{\varepsilon} - 1 = \bar{\varepsilon}(1 - \varepsilon), \quad \varepsilon - \bar{\varepsilon} = \varepsilon(1 - \varepsilon) = \bar{\varepsilon}(\bar{\varepsilon} - 1). \end{gathered} \tag{1.7}$$

Now note that from (1.6), it follows that

$$|x_n| \leq (\sqrt[3]{|q|})^n(|c_1| + |c_2| + |c_3|),$$

from which we have that for $|q| < 1$ all the solutions to equation (1.5) converge geometrically (exponentially) to zero, while if $|q| = 1$ all the solutions to the equation are bounded. On the other hand, if $|q| > 1$, then all the solutions to the equation are unbounded, except the trivial one, that is, $x_n = 0$, $n \in \mathbb{N}_0$.

Let $S \subset \mathbb{Z}$ be an unbounded set. By $l^\infty(S)$ we denote the Banach space containing all bounded sequences $u = (u_n)_{n\in S}$ with the supremum norm

$$\|u\|_{\infty,S} = \sup_{n\in S} |u_n|.$$

Since the choice of set S will be clear from the context and will not influence on the proofs of our results, we will simply use the notations l^∞ and $\|\cdot\|_\infty$, not emphasizing the set.

We show that the methods in [28] can be applied for the case of equation (1.3), but not only on domain $\mathbb{N}_0$. Namely, motivated by recent paper [27], we will also consider the equation on the set $\mathbb{Z} \setminus \mathbb{N}_3$ and consequently on the whole $\mathbb{Z}$. Since the method and principle are classical we regard that some of the results presented here are essentially folklore, but nevertheless, in several cases we present the proofs of the results for the completeness and benefit of the reader. At the end of the paper we give some suggestions for further investigation in the direction.

Before formulating and proving the main results in the paper, we quote a well-known formula which we use. Let $W_3(a_1, a_2, a_3)$ be the following determinant:

$$W_3(a_1, a_2, a_3) := \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix},$$

where $a_j \in \mathbb{C}$, $j = \overline{1,3}$. Then

$$W_3(a_1, a_2, a_3) = (a_2 - a_1)(a_3 - a_1)(a_3 - a_2). \tag{1.8}$$

2. Main results

The main results in the paper are formulated and proved in this section. We follow the idea in [28] on using a combination of the method of solvability and application of the contraction mapping principle to suitable chosen operators naturally arising during consideration of some nonhomogeneous difference equations with constant coefficients. First, we consider (1.3) on the domain $\mathbb{N}_0$, and after that on domain $\mathbb{Z} \setminus \mathbb{N}_3$.

2.1. **Equation** (1.3) **with nonnegative indices.**

The first result is standard and is proved by a known method. It is devoted to the case when the sequence $(q_n)_{n\in\mathbb{N}_0}$ is constant, and gives a closed form formula for the general solution to equation (1.3) in the case.

Proposition 2.1. *Consider the equation*

$$x_{n+3} - qx_n = f_n, \quad n \in \mathbb{N}_0, \tag{2.1}$$

where $q \in \mathbb{C} \setminus \{0\}$, and $(f_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$. Then the general solution to the equation is

$$\begin{aligned} x_n = (\sqrt[3]{q})^n \Big(a_0 + \sum_{k=0}^{n-1} \frac{f_k}{3(\sqrt[3]{q})^{k+3}}\Big) + (\sqrt[3]{q}\varepsilon)^n \Big(b_0 + \sum_{k=0}^{n-1} \frac{\bar{\varepsilon}^k f_k}{3(\sqrt[3]{q})^{k+3}}\Big) \\ + (\sqrt[3]{q}\bar{\varepsilon})^n \Big(c_0 + \sum_{k=0}^{n-1} \frac{\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}}\Big), \quad n \in \mathbb{N}_0, \end{aligned} \tag{2.2}$$

where a_0, b_0 and c_0 are arbitrary complex numbers, and $\sqrt[3]{q}$ is one of the three possible third roots of q.

Proof. To show formula (2.2) we employ the version of the method of undetermined coefficients for the liner difference equations. Namely, based on (1.6), we assume that the general solution to equation (2.1) has the form

$$x_n = a_n(\sqrt[3]{q})^n + b_n(\sqrt[3]{q}\varepsilon)^n + c_n(\sqrt[3]{q}\bar{\varepsilon})^n, \quad n \in \mathbb{N}_0, \tag{2.3}$$

for some (undetermined) sequences $(a_n)_{n\in\mathbb{N}_0}$, $(b_n)_{n\in\mathbb{N}_0}$ and $(c_n)_{n\in\mathbb{N}_0}$, and pose the following two conditions:

$$\begin{aligned} x_{n+1} &= a_{n+1}(\sqrt[3]{q})^{n+1} + b_{n+1}(\sqrt[3]{q}\varepsilon)^{n+1} + c_{n+1}(\sqrt[3]{q}\bar{\varepsilon})^{n+1} \\ &= a_n(\sqrt[3]{q})^{n+1} + b_n(\sqrt[3]{q}\varepsilon)^{n+1} + c_n(\sqrt[3]{q}\bar{\varepsilon})^{n+1}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} x_{n+2} &= a_{n+1}(\sqrt[3]{q})^{n+2} + b_{n+1}(\sqrt[3]{q}\varepsilon)^{n+2} + c_{n+1}(\sqrt[3]{q}\bar{\varepsilon})^{n+2} \\ &= a_n(\sqrt[3]{q})^{n+2} + b_n(\sqrt[3]{q}\varepsilon)^{n+2} + c_n(\sqrt[3]{q}\bar{\varepsilon})^{n+2}, \end{aligned} \tag{2.5}$$

for $n \in \mathbb{N}_0$.

Conditions (2.4) and (2.5) can be obviously written as

$$(a_{n+1} - a_n)(\sqrt[3]{q})^{n+1} + (b_{n+1} - b_n)(\sqrt[3]{q}\varepsilon)^{n+1} + (c_{n+1} - c_n)(\sqrt[3]{q}\bar{\varepsilon})^{n+1} = 0, \tag{2.6}$$

$$(a_{n+1} - a_n)(\sqrt[3]{q})^{n+2} + (b_{n+1} - b_n)(\sqrt[3]{q}\varepsilon)^{n+2} + (c_{n+1} - c_n)(\sqrt[3]{q}\bar{\varepsilon})^{n+2} = 0, \tag{2.7}$$

for $n \in \mathbb{N}_0$.

From (2.1), (2.3), and (2.5) where n is replaced by $n + 1$, it follows that

$$(a_{n+1} - a_n)(\sqrt[3]{q})^{n+3} + (b_{n+1} - b_n)(\sqrt[3]{q}\varepsilon)^{n+3} + (c_{n+1} - c_n)(\sqrt[3]{q}\bar{\varepsilon})^{n+3} = f_n, \tag{2.8}$$

for $n \in \mathbb{N}_0$.

Since $q \neq 0$, equations (2.6)-(2.8) together, can be written as the following linear system of equations:

$$\begin{aligned} a_{n+1}-a_n+(b_{n+1}-b_n)\varepsilon^{n+1}+(c_{n+1}-c_n)\bar{\varepsilon}^{n+1}&=0\\ a_{n+1}-a_n+(b_{n+1}-b_n)\varepsilon^{n+2}+(c_{n+1}-c_n)\bar{\varepsilon}^{n+2}&=0\\ a_{n+1}-a_n+(b_{n+1}-b_n)\varepsilon^{n+3}+(c_{n+1}-c_n)\bar{\varepsilon}^{n+3}&=\frac{f_n}{(\sqrt[3]{q})^{n+3}}, \end{aligned} \tag{2.9}$$

for each fixed $n \in \mathbb{N}_0$, that is, as a three-dimensional linear system in variables $a_{n+1}-a_n$, $b_{n+1}-b_n$ and $c_{n+1}-c_n$.

In what follows we will use the following determinant, which appears, among others, in solving the linear system,

$$W_3(1,\varepsilon,\bar{\varepsilon}) := \begin{vmatrix} 1 & 1 & 1\\ 1 & \varepsilon & \bar{\varepsilon}\\ 1 & \varepsilon^2 & \bar{\varepsilon}^2 \end{vmatrix}.$$

From (1.8), we have

$$W_3(1,\varepsilon,\bar{\varepsilon}) = (\varepsilon-1)(\bar{\varepsilon}-1)(\bar{\varepsilon}-\varepsilon) = -3\sqrt{3}i. \tag{2.10}$$

Using (2.10), we see that the determinant of system (2.9) is

$$\Delta_n = \begin{vmatrix} 1 & \varepsilon^{n+1} & \bar{\varepsilon}^{n+1}\\ 1 & \varepsilon^{n+2} & \bar{\varepsilon}^{n+2}\\ 1 & \varepsilon^{n+3} & \bar{\varepsilon}^{n+3} \end{vmatrix} = (\varepsilon\bar{\varepsilon})^{n+1} W_3(1,\varepsilon,\bar{\varepsilon}) = -3\sqrt{3}i, \tag{2.11}$$

for every $n \in \mathbb{N}_0$.

By using Cramer's rule, (1.7), (2.11), and some calculation, we have that the solution to linear system (2.9) is given by

$$\begin{aligned} a_{n+1}-a_n &= -\frac{1}{3\sqrt{3}i}\begin{vmatrix} 0 & \varepsilon^{n+1} & \bar{\varepsilon}^{n+1}\\ 0 & \varepsilon^{n+2} & \bar{\varepsilon}^{n+2}\\ \frac{f_n}{(\sqrt[3]{q})^{n+3}} & \varepsilon^{n+3} & \bar{\varepsilon}^{n+3} \end{vmatrix}\\ &= -\frac{(\varepsilon\bar{\varepsilon})^{n+1}(\bar{\varepsilon}-\varepsilon)f_n}{3\sqrt{3}i(\sqrt[3]{q})^{n+3}} = \frac{f_n}{3(\sqrt[3]{q})^{n+3}}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} b_{n+1}-b_n &= -\frac{1}{3\sqrt{3}i}\begin{vmatrix} 1 & 0 & \bar{\varepsilon}^{n+1}\\ 1 & 0 & \bar{\varepsilon}^{n+2}\\ 1 & \frac{f_n}{(\sqrt[3]{q})^{n+3}} & \bar{\varepsilon}^{n+3} \end{vmatrix}\\ &= \frac{\bar{\varepsilon}^{n+1}(\bar{\varepsilon}-1)f_n}{3\sqrt{3}i(\sqrt[3]{q})^{n+3}} = \frac{\bar{\varepsilon}^{n} f_n}{3(\sqrt[3]{q})^{n+3}}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} c_{n+1}-c_n &= -\frac{1}{3\sqrt{3}i}\begin{vmatrix} 1 & \varepsilon^{n+1} & 0\\ 1 & \varepsilon^{n+2} & 0\\ 1 & \varepsilon^{n+3} & \frac{f_n}{(\sqrt[3]{q})^{n+3}} \end{vmatrix}\\ &= -\frac{\varepsilon^{n+1}(\varepsilon-1)f_n}{3\sqrt{3}i(\sqrt[3]{q})^{n+3}} = \frac{\varepsilon^{n} f_n}{3(\sqrt[3]{q})^{n+3}}, \end{aligned} \tag{2.14}$$

for $n \in \mathbb{N}_0$.

From (2.12)–(2.14) it easily follows that

$$a_n = a_0 + \sum_{k=0}^{n-1} \frac{f_k}{3(\sqrt[3]{q})^{k+3}}, \tag{2.15}$$

$$b_n = b_0 + \sum_{k=0}^{n-1} \frac{\bar{\varepsilon}^k f_k}{3(\sqrt[3]{q})^{k+3}}, \tag{2.16}$$

$$c_n = c_0 + \sum_{k=0}^{n-1} \frac{\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}}, \tag{2.17}$$

for $n \in \mathbb{N}_0$.

Using the equalities (2.15)–(2.17) in (2.3), it is obtained that formula (2.2) holds for every $n \in \mathbb{N}_0$.

Now note that x_n in (2.2) can be written in the form

$$x_n = x_n^h + x_n^p,$$

where

$$x_n^h = a_0(\sqrt[3]{q})^n + b_0(\sqrt[3]{q}\varepsilon)^n + c_0(\sqrt[3]{q}\bar{\varepsilon})^n$$

and

$$\begin{aligned} x_n^p =& (\sqrt[3]{q})^n \sum_{k=0}^{n-1} \frac{f_k}{3(\sqrt[3]{q})^{k+3}} + (\sqrt[3]{q}\varepsilon)^n \sum_{k=0}^{n-1} \frac{\bar{\varepsilon}^k f_k}{3(\sqrt[3]{q})^{k+3}} + (\sqrt[3]{q}\bar{\varepsilon})^n \sum_{k=0}^{n-1} \frac{\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}} \\ =& (\sqrt[3]{q})^n \sum_{k=0}^{n-1} \frac{(1 + \varepsilon^{n-k} + \bar{\varepsilon}^{n-k}) f_k}{3(\sqrt[3]{q})^{k+3}}, \end{aligned}$$

for $n \in \mathbb{N}_0$.

Since $(x_n^h)_{n\in\mathbb{N}_0}$ is obviously the general solution to the corresponding homogeneous difference equation, that is, to equation (1.5), while $(x_n^p)_{n\in\mathbb{N}_0}$ is a particular solution to equation (2.1), which is easily checked by some calculation, by a well-known result [1, 14, 16, 20], it follows that formula (2.2) really presents the general solution to nonhomogeneous difference equation (2.1), completing the proof of the result. □

Remark 2.2. It is known that the method of undetermined coefficients used in the proof of Proposition 2.1 can be applied to any nonhomogeneous linear difference equation with constant coefficients. However, the formula in the general case is of theoretical importance, since by the Abel-Ruffini theorem the polynomials of degree greater than or equal to five need not be solvable by radicals. In the case of equation (2.1) the associated characteristic polynomial to the corresponding homogeneous equation is of the third-order, so solvable by radicals. In fact, the polynomial is a special case of the following one: $P_k(\lambda) = \lambda^k + a$, $k \in \mathbb{N}$, which is solvable by radicals. Besides, the formulas in the general case seem complicated for calculating in some reasonably simple closed forms.

Corollary 2.3. *The solution to equation* (2.1) *with the initial values* $x_j \in \mathbb{C}$, $j = \overline{0,2}$, *where* $q \in \mathbb{C} \setminus \{0\}$ *and* $(f_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$ *is given by*

$$\begin{aligned} x_n =&(\sqrt[3]{q})^n\Big(\frac{qx_0 + (\sqrt[3]{q})^2x_1 + \sqrt[3]{q}x_2}{3q} + \sum_{k=0}^{n-1}\frac{f_k}{3(\sqrt[3]{q})^{k+3}}\Big)\\ &+ (\sqrt[3]{q}\varepsilon)^n\Big(\frac{qx_0 + \bar\varepsilon(\sqrt[3]{q})^2x_1 + \varepsilon\sqrt[3]{q}x_2}{3q} + \sum_{k=0}^{n-1}\frac{\bar\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}}\Big)\\ &+ (\sqrt[3]{q}\bar\varepsilon)^n\Big(\frac{qx_0 + \varepsilon(\sqrt[3]{q})^2x_1 + \bar\varepsilon\sqrt[3]{q}x_2}{3q} + \sum_{k=0}^{n-1}\frac{\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}}\Big), \quad n \in \mathbb{N}_0. \end{aligned} \tag{2.18}$$

Proof. By using formula (2.2) and some calculation, it follows that for the solution to equation (2.1) with the initial values $x_j \in \mathbb{C}$, $j = \overline{0,2}$, the following equalities must hold:

$$\begin{aligned} x_0 &= a_0 + b_0 + c_0,\\ x_1 &= \sqrt[3]{q}a_0 + \sqrt[3]{q}\varepsilon b_0 + \sqrt[3]{q}\bar\varepsilon c_0,\\ x_2 &= (\sqrt[3]{q})^2a_0 + (\sqrt[3]{q})^2\varepsilon^2 b_0 + (\sqrt[3]{q})^2\bar\varepsilon^2 c_0. \end{aligned} \tag{2.19}$$

Let Δ be the determinant of system (2.19). Then it is easy to see that

$$\Delta = qW_3(1, \varepsilon, \bar\varepsilon).$$

From this, by some calculation and (1.7), we have

$$\begin{aligned} a_0 =&\frac{1}{qW_3(1,\varepsilon,\bar\varepsilon)}\begin{vmatrix} x_0 & 1 & 1\\ x_1 & \sqrt[3]{q}\varepsilon & \sqrt[3]{q}\bar\varepsilon\\ x_2 & (\sqrt[3]{q})^2\varepsilon^2 & (\sqrt[3]{q})^2\bar\varepsilon^2 \end{vmatrix}\\ =&\frac{(\bar\varepsilon-\varepsilon)(qx_0 + (\sqrt[3]{q})^2x_1 + \sqrt[3]{q}x_2)}{q(\varepsilon-1)(\bar\varepsilon-1)(\bar\varepsilon-\varepsilon)}\\ =&\frac{qx_0 + (\sqrt[3]{q})^2x_1 + \sqrt[3]{q}x_2}{3q}, \end{aligned} \tag{2.20}$$

$$\begin{aligned} b_0 =&\frac{1}{qW_3(1,\varepsilon,\bar\varepsilon)}\begin{vmatrix} 1 & x_0 & 1\\ \sqrt[3]{q} & x_1 & \sqrt[3]{q}\bar\varepsilon\\ (\sqrt[3]{q})^2 & x_2 & (\sqrt[3]{q})^2\bar\varepsilon^2 \end{vmatrix}\\ =&\frac{(\bar\varepsilon-\varepsilon)(qx_0 + \bar\varepsilon(\sqrt[3]{q})^2x_1 + \varepsilon\sqrt[3]{q}x_2)}{q(\varepsilon-1)(\bar\varepsilon-1)(\bar\varepsilon-\varepsilon)}\\ =&\frac{qx_0 + \bar\varepsilon(\sqrt[3]{q})^2x_1 + \varepsilon\sqrt[3]{q}x_2}{3q}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} c_0 =&\frac{1}{qW_3(1,\varepsilon,\bar\varepsilon)}\begin{vmatrix} 1 & 1 & x_0\\ \sqrt[3]{q} & \sqrt[3]{q}\varepsilon & x_1\\ (\sqrt[3]{q})^2 & (\sqrt[3]{q})^2\varepsilon^2 & x_2 \end{vmatrix}\\ =&\frac{(\bar\varepsilon-\varepsilon)(qx_0 + \varepsilon(\sqrt[3]{q})^2x_1 + \bar\varepsilon\sqrt[3]{q}x_2)}{q(\varepsilon-1)(\bar\varepsilon-1)(\bar\varepsilon-\varepsilon)}\\ =&\frac{qx_0 + \varepsilon(\sqrt[3]{q})^2x_1 + \bar\varepsilon\sqrt[3]{q}x_2}{3q}. \end{aligned} \tag{2.22}$$

Using (2.20)-(2.22) in (2.2), is obtained formula (2.18). $\square$

The following two theorems show how formula (2.2) can be effectively used in getting some results on the boundedness of solutions to equation (1.3) for the case when the sequence $(q_n)_{n\in\mathbb{N}_0}$ is constant, but such that the modulus of the constant is different from 1, while the sequence $(f_n)_{n\in\mathbb{N}_0}$ is bounded.

The first theorem deals with the case $0<|q|<1$ and is certainly folklore. It can be obtained in a direct, but not unified, way by iterating the following simple consequence of (2.1)

$$|x_{n+3}|\le |q||x_n|+|f_n|,$$

using the conditions posed in the theorem, and some simple inequalities and summation formulas (this is one of the basic estimates concerning linear difference equations, with enormous applications to many classes of difference equations). However, the proof which we present here is straightforward, so, more elegant.

Theorem 2.4. *Consider equation* (2.1)*, where $q\in\mathbb{C}$, $0<|q|<1$ and $(f_n)_{n\in\mathbb{N}_0}\subset\mathbb{C}$ is a bounded sequence. Then every solution to the equation is bounded.*

Proof. By using formula (2.2), the conditions of the theorem and some simple estimates, we have

$$\begin{aligned}
|x_n| \le& |\sqrt[3]{q}|^n\Big|a_0+\sum_{k=0}^{n-1}\frac{f_k}{3(\sqrt[3]{q})^{k+3}}\Big|+|\sqrt[3]{q}\varepsilon|^n\Big|b_0+\sum_{k=0}^{n-1}\frac{\bar\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}}\Big|\\
&+|\sqrt[3]{q}\bar\varepsilon|^n\Big|c_0+\sum_{k=0}^{n-1}\frac{\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}}\Big|\\
\le& |\sqrt[3]{q}|^n\Big(|a_0|+\sum_{k=0}^{n-1}\frac{|f_k|}{3|\sqrt[3]{q}|^{k+3}}\Big)+|\sqrt[3]{q}|^n\Big(|b_0|+\sum_{k=0}^{n-1}\frac{|f_k|}{3|\sqrt[3]{q}|^{k+3}}\Big)\\
&+|\sqrt[3]{q}|^n\Big(|c_0|+\sum_{k=0}^{n-1}\frac{|f_k|}{3|\sqrt[3]{q}|^{k+3}}\Big)\\
\le& |\sqrt[3]{q}|^n\big(|a_0|+|b_0|+|c_0|\big)+\frac{\|f\|_\infty}{|\sqrt[3]{q}|^2}\sum_{k=0}^{n-1}|\sqrt[3]{q}|^{n-k-1}\\
\le& |a_0|+|b_0|+|c_0|+\frac{\|f\|_\infty}{|\sqrt[3]{q}|^2(1-|\sqrt[3]{q}|)},
\end{aligned}$$

for every $n\in\mathbb{N}_0$, from which the theorem follows. □

The following result solves the problem of the unique existence of a bounded solution to equation (1.3) for the case $q_n=q$, $n\in\mathbb{N}_0$, $|q|>1$, in a unified way.

Theorem 2.5. *Consider equation* (2.1)*, where $q\in\mathbb{C}$, $|q|>1$ and $(f_n)_{n\in\mathbb{N}_0}\subset\mathbb{C}$ is a bounded sequence. Then there is a unique bounded solution to the equation.*

Proof. By Proposition 2.1 the general solution to (2.1) is given by (2.2). Hence

$$x_{3n}=q^n\Big(a_0+b_0+c_0+\sum_{k=0}^{3n-1}\frac{(1+\varepsilon^k+\bar\varepsilon^k)f_k}{3(\sqrt[3]{q})^{k+3}}\Big), \tag{2.23}$$

$$x_{3n+1}=(\sqrt[3]{q})^{3n+1}\Big(a_0+b_0\varepsilon+c_0\bar\varepsilon+\sum_{k=0}^{3n}\frac{(1+\varepsilon^{k-1}+\bar\varepsilon^{k-1})f_k}{3(\sqrt[3]{q})^{k+3}}\Big), \tag{2.24}$$

$$x_{3n+2} = (\sqrt[3]{q})^{3n+2}\Big(a_0 + b_0\varepsilon^2 + c_0\bar{\varepsilon}^2 + \sum_{k=0}^{3n+1} \frac{(1+\varepsilon^{k-2}+\bar{\varepsilon}^{k-2})f_k}{3(\sqrt[3]{q})^{k+3}}\Big), \tag{2.25}$$

for every $n \in \mathbb{N}_0$.

Since $|q| > 1$ and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence, by using some elementary inequalities, we have

$$\Big|\sum_{k=0}^{\infty} \frac{(1+\varepsilon^{k-j}+\bar{\varepsilon}^{k-j})f_k}{3(\sqrt[3]{q})^{k+3}}\Big| \le \sum_{k=0}^{\infty} \frac{\|f\|_\infty}{|\sqrt[3]{q}|^{k+3}} = \frac{\|f\|_\infty}{|\sqrt[3]{q}|^2(|\sqrt[3]{q}|-1)} < \infty, \tag{2.26}$$

for $j = \overline{0,2}$, which shows that the sums appearing in equalities (2.23)-(2.25) are absolutely convergent.

Using this fact, along with equalities (2.23)-(2.25) and the assumption $|q| > 1$, we see that if $(x_n)_{n\in\mathbb{N}_0}$ is a bounded solution to equation (2.1), the following equalities must hold:

$$a_0 + b_0 + c_0 = -\sum_{k=0}^{\infty} \frac{(1+\varepsilon^{k}+\bar{\varepsilon}^{k})f_k}{3(\sqrt[3]{q})^{k+3}} =: S_1, \tag{2.27}$$

$$a_0 + b_0\varepsilon + c_0\bar{\varepsilon} = -\sum_{k=0}^{\infty} \frac{(1+\varepsilon^{k-1}+\bar{\varepsilon}^{k-1})f_k}{3(\sqrt[3]{q})^{k+3}} =: S_2, \tag{2.28}$$

$$a_0 + b_0\varepsilon^2 + c_0\bar{\varepsilon}^2 = -\sum_{k=0}^{\infty} \frac{(1+\varepsilon^{k-2}+\bar{\varepsilon}^{k-2})f_k}{3(\sqrt[3]{q})^{k+3}} =: S_3. \tag{2.29}$$

Indeed, since $|q| > 1$, we have $|\sqrt[3]{q}|^{3n+j} \to +\infty$, as $n \to +\infty$, for each $j \in \{0,1,2\}$, from which by letting $n \to +\infty$ in (2.23)-(2.25), we see that a solution $(x_n)_{n\in\mathbb{N}_0}$ to equation (2.1) is bounded only if equalities (2.27)-(2.29) hold.

The equalities can be regarded as a three-dimensional linear system in variables a_0, b_0 and c_0. The system can be solved, so, by using (2.10) and by some calculation, it follows that

$$\begin{aligned} a_0 &= \frac{1}{W_3(1,\varepsilon,\bar{\varepsilon})}\begin{vmatrix} S_1 & 1 & 1 \\ S_2 & \varepsilon & \bar{\varepsilon} \\ S_3 & \varepsilon^2 & \bar{\varepsilon}^2 \end{vmatrix} = \frac{(\bar{\varepsilon}-\varepsilon)(S_1+S_2+S_3)}{W_3(1,\varepsilon,\bar{\varepsilon})} = \frac{S_1+S_2+S_3}{3} \\ &= -\sum_{k=0}^{\infty}\sum_{j=0}^{2} \frac{(1+\varepsilon^{k-j}+\bar{\varepsilon}^{k-j})f_k}{9(\sqrt[3]{q})^{k+3}} = -\sum_{k=0}^{\infty} \frac{f_k}{3(\sqrt[3]{q})^{k+3}}, \end{aligned} \tag{2.30}$$

$$\begin{aligned} b_0 &= \frac{1}{W_3(1,\varepsilon,\bar{\varepsilon})}\begin{vmatrix} 1 & S_1 & 1 \\ 1 & S_2 & \bar{\varepsilon} \\ 1 & S_3 & \bar{\varepsilon}^2 \end{vmatrix} = \frac{(\bar{\varepsilon}-\varepsilon)(S_1+\bar{\varepsilon}S_2+\varepsilon S_3)}{W_3(1,\varepsilon,\bar{\varepsilon})} = \frac{S_1+\bar{\varepsilon}S_2+\varepsilon S_3}{3} \\ &= -\sum_{k=0}^{\infty}\sum_{j=0}^{2} \frac{\bar{\varepsilon}^j(1+\varepsilon^{k-j}+\bar{\varepsilon}^{k-j})f_k}{9(\sqrt[3]{q})^{k+3}} = -\sum_{k=0}^{\infty} \frac{\bar{\varepsilon}^k f_k}{3(\sqrt[3]{q})^{k+3}}, \end{aligned} \tag{2.31}$$

$$\begin{aligned} c_0 &= \frac{1}{W_3(1,\varepsilon,\bar{\varepsilon})}\begin{vmatrix} 1 & 1 & S_1 \\ 1 & \varepsilon & S_2 \\ 1 & \varepsilon^2 & S_3 \end{vmatrix} = \frac{(\bar{\varepsilon}-\varepsilon)(S_1+\varepsilon S_2+\bar{\varepsilon}S_3)}{W_3(1,\varepsilon,\bar{\varepsilon})} = \frac{S_1+\varepsilon S_2+\bar{\varepsilon}S_3}{3} \\ &= -\sum_{k=0}^{\infty}\sum_{j=0}^{2} \frac{\varepsilon^j(1+\varepsilon^{k-j}+\bar{\varepsilon}^{k-j})f_k}{9(\sqrt[3]{q})^{k+3}} = -\sum_{k=0}^{\infty} \frac{\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}}. \end{aligned} \tag{2.32}$$

When we use (2.30)-(2.32) in (2.2), we obtain

$$\begin{aligned} x_n =& (\sqrt[3]{q})^n \Big(\frac{S_1+S_2+S_3}{3} + \sum_{k=0}^{n-1} \frac{f_k}{3(\sqrt[3]{q})^{k+3}} \Big) \\ &+ (\sqrt[3]{q}\varepsilon)^n \Big(\frac{S_1+\bar\varepsilon S_2+\varepsilon S_3}{3} + \sum_{k=0}^{n-1} \frac{\bar\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}} \Big) \\ &+ (\sqrt[3]{q}\bar\varepsilon)^n \Big(\frac{S_1+\varepsilon S_2+\bar\varepsilon S_3}{3} + \sum_{k=0}^{n-1} \frac{\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}} \Big) \\ =& - (\sqrt[3]{q})^n \sum_{k=n}^{\infty} \frac{f_k}{3(\sqrt[3]{q})^{k+3}} - (\sqrt[3]{q}\varepsilon)^n \sum_{k=n}^{\infty} \frac{\bar\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}} \\ &- (\sqrt[3]{q}\bar\varepsilon)^n \sum_{k=n}^{\infty} \frac{\varepsilon^k f_k}{3(\sqrt[3]{q})^{k+3}} \\ =& - (\sqrt[3]{q})^n \sum_{k=n}^{\infty} \frac{(1+\varepsilon^{k-n}+\bar\varepsilon^{k-n}) f_k}{3(\sqrt[3]{q})^{k+3}}, \end{aligned} \tag{2.33}$$

for $n \in \mathbb{N}_0$.

That sequence $(x_n)_{n\in\mathbb{N}_0}$ given by (2.33) is a solution to equation (2.1) is easily checked by some calculation. On the other hand, similar to (2.26), it is proved the following estimate:

$$|x_n| \le \frac{\|f\|_\infty}{|\sqrt[3]{q}|^2(|\sqrt[3]{q}|-1)} < \infty, \quad n \in \mathbb{N}_0,$$

which means that the sequence is bounded. That the sequence presents a unique bounded solution to equation (2.1) follows from the unique choice of constants a_0, b_0 and c_0, in (2.30)–(2.32).

Moreover, since

$$a_0 + b_0 + c_0 = x_0, \tag{2.34}$$

$$a_0 + \varepsilon b_0 + \bar\varepsilon c_0 = \frac{x_1}{\sqrt[3]{q}}, \tag{2.35}$$

$$a_0 + \bar\varepsilon b_0 + \varepsilon c_0 = \frac{x_2}{(\sqrt[3]{q})^2}, \tag{2.36}$$

from (2.27)-(2.29), we obtain that

$$x_0 = S_1, \quad x_1 = \sqrt[3]{q} S_2, \quad x_2 = (\sqrt[3]{q})^2 S_3, \tag{2.37}$$

are the initial values for which is obtained the bonded solution to (2.1) in the case. □

Remark 2.6. The case $|q| = 1$ is a boundary one, and, as usual, in such situations, there is no a unique result regarding the boundedness character of the solutions to equation (2.1) in the case. For example, if $q = 1$, then, depending on the choice of a bounded sequence $(f_n)_{n\in\mathbb{N}_0}$, sequence $(x_n)_{n\in\mathbb{N}_0}$ can converge, or goes to infinity, or its limit set can be the whole closed interval $[\liminf_{n\to\infty} x_n, \limsup_{n\to\infty} x_n]$, or it can be even a more complicated set ([19, 22]). For some related results in metric spaces see [3].

The following result considers the case when $(q_n)_{n\in\mathbb{N}_0}$ is not a constant sequence, but is bounded. It solves Problem 1.1 for the case of equation (1.3) on its original domain $\mathbb{N}_0$.

Theorem 2.7. *Consider equation* (1.3), *where*

$$1 < a \le q_n \le b, \quad n \in \mathbb{N}_0, \tag{2.38}$$

or

$$-b \le q_n \le -a < -1, \quad n \in \mathbb{N}_0, \tag{2.39}$$

for some positive numbers a and b, and $(f_n)_{n\in\mathbb{N}_0}$ is a bounded sequence of complex numbers. Then the equation has a unique bounded solution.

Proof. We prove the theorem in the case when condition (2.38) holds. The case when condition (2.39) holds is dealt with analogously. Hence, the proof in the case is omitted.

Let q be a positive number such that

$$q \in \big(\max\{a, (b+1)/2\}, b). \tag{2.40}$$

Write equation (1.3) in the form

$$x_{n+3} - qx_n = (q_n - q)x_n + f_n, \quad n \in \mathbb{N}_0. \tag{2.41}$$

Let A be an operator defined on l^∞, as follows

$$A(u) = \Big(-(\sqrt[3]{q})^n \frac{1}{3}\sum_{k=n}^{\infty} \frac{((1+\varepsilon^{k-n}+\bar{\varepsilon}^{k-n})((q_k-q)u_k+f_k)}{(\sqrt[3]{q})^{k+3}}\Big)_{n\in\mathbb{N}_0}. \tag{2.42}$$

Assume that $u \in l^\infty$. Then, from (2.42) we have

$$\begin{aligned}
\|A(u)\|_\infty &= \sup_{n\in\mathbb{N}_0}\Big| -(\sqrt[3]{q})^n \frac{1}{3}\sum_{k=n}^{\infty} \frac{(1+\varepsilon^{k-n}+\bar{\varepsilon}^{k-n})((q_k-q)u_k+f_k)}{(\sqrt[3]{q})^{k+3}}\Big| \\
&\le \sup_{n\in\mathbb{N}_0}\sum_{k=n}^{\infty}\frac{(q_k+q)|u_k|+|f_k|}{|\sqrt[3]{q}|^{k+3-n}} \\
&\le \frac{(b+q)\|u\|_\infty + \|f\|_\infty}{|\sqrt[3]{q}|^2(|\sqrt[3]{q}|-1)} < \infty,
\end{aligned} \tag{2.43}$$

which means that $A(u) \in l^\infty$.

If $u, v \in l^\infty$, then

$$\begin{aligned}
\|A(u)-A(v)\|_\infty &= \sup_{n\in\mathbb{N}_0}\Big|(\sqrt[3]{q})^n \frac{1}{3}\sum_{k=n}^{\infty}\frac{(1+\varepsilon^{k-n}+\bar{\varepsilon}^{k-n})(q_k-q)(u_k-v_k)}{(\sqrt[3]{q})^{k+3}}\Big| \\
&= \sup_{n\in\mathbb{N}_0}\Big|(\sqrt[3]{q})^n \sum_{j=0}^{\infty}\frac{(q_{n+3j}-q)(u_{n+3j}-v_{n+3j})}{(\sqrt[3]{q})^{n+3j+3}}\Big| \\
&\le \sup_{n\in\mathbb{N}_0}\sum_{j=0}^{\infty}\frac{|q_{n+3j}-q||u_{n+3j}-v_{n+3j}|}{q^{j+1}} \\
&\le \frac{\max\{q-a, b-q\}}{q-1}\|u-v\|_\infty = q_1\|u-v\|_\infty.
\end{aligned} \tag{2.44}$$

From $a > 1$ and (2.40), it follows that $q_1 \in (0,1)$. Hence, A is a contraction from l^∞ into itself.

By the contraction mapping principle we obtain that A has a unique fixed point, say $x^*=(x_n^*)_{n\in\mathbb{N}_0}\in l^\infty$. Hence

$$x_n^* = -(\sqrt[3]{q})^n\frac{1}{3}\sum_{k=n}^{\infty}\frac{(1+\varepsilon^{k-n}+\bar{\varepsilon}^{k-n})((q_k-q)x_k^*+f_k)}{(\sqrt[3]{q})^{k+3}}, \tag{2.45}$$

for every $n\in\mathbb{N}_0$.

It is easy to verify that (2.45) is a solution to equation (1.3), so that such obtained x^* is its unique (bounded) solution. □

2.2. **Equation** (1.3) **with negative indices.** If $q_n\neq 0$, $n\in\mathbb{N}_0$, then equation (1.3) can be written in the form

$$x_n=\frac{x_{n+3}-f_n}{q_n},\quad n\in\mathbb{N}_0, \tag{2.46}$$

from which it follows that if the sequences q_n and f_n are also defined for $n\le -1$ and $q_n\neq 0$, $n\le -1$, then each solution to equation (1.3) can be also calculated/defined for every $n\le -1$, and consequently on the whole $\mathbb{Z}$.

In this case equation (2.46) can be written as follows

$$x_{-(n+3)}=\frac{1}{q_{-(n+3)}}x_{-n}-\frac{f_{-(n+3)}}{q_{-(n+3)}},\quad n\ge -2, \tag{2.47}$$

which, by using the change of variables

$$y_n=x_{-n}, \tag{2.48}$$

and the notation

$$\hat{q}_n:=\frac{1}{q_{-(n+3)}}\quad\text{and}\quad \hat{f}_n:=-\frac{f_{-(n+3)}}{q_{-(n+3)}}, \tag{2.49}$$

becomes

$$y_{n+3}=\hat{q}_ny_n+\hat{f}_n, \tag{2.50}$$

which is an equation of the form in (1.3), but defined on a slightly bigger domain.

If we consider difference equation (2.50) not on the whole domain $n\ge -2$, but on the restricted one $\mathbb{N}_0$, then by employing Proposition 2.1 to the equation, and using the change of variables and notations in (2.48) and (2.49), the following result is obtained.

Proposition 2.8. *Consider difference equation* (2.47)*. Assume that $q_{-n}=q\in\mathbb{C}\setminus\{0\}$, $n\ge 3$, and $(f_{-n})_{n\ge 3}\subset\mathbb{C}$. Then the general solution to the equation on $\mathbb{Z}\setminus\mathbb{N}$, is given by*

$$\begin{aligned}x_{-n}=&\frac{1}{(\sqrt[3]{q})^n}\Big(\hat{a}_0-\frac{1}{3}\sum_{k=0}^{n-1}(\sqrt[3]{q})^kf_{-(k+3)}\Big)\\ &+\Big(\frac{\varepsilon}{\sqrt[3]{q}}\Big)^n\Big(\hat{b}_0-\frac{1}{3}\sum_{k=0}^{n-1}(\sqrt[3]{q})^k\bar{\varepsilon}^kf_{-(k+3)}\Big)\\ &+\Big(\frac{\bar{\varepsilon}}{\sqrt[3]{q}}\Big)^n\Big(\hat{c}_0-\frac{1}{3}\sum_{k=0}^{n-1}(\sqrt[3]{q})^k\varepsilon^kf_{-(k+3)}\Big),\quad n\in\mathbb{N}_0,\end{aligned} \tag{2.51}$$

where $\hat{a}_0$, $\hat{b}_0$ and $\hat{c}_0$ are arbitrary complex numbers, and $\sqrt[3]{q}$ is one of the three possible third roots of q.

By using formula (2.51), the following result is proved similar to Theorem 2.4. The proof is omitted for the similarity/duality.

Theorem 2.9. *Consider the equation*

$$x_{-(n+3)} = \frac{x_{-n}}{q} - \frac{f_{-(n+3)}}{q}, \quad n \in \mathbb{N}_0, \tag{2.52}$$

where $q \in \mathbb{C}$, $|q| > 1$ *and* $(f_{-n})_{n\geq 3} \subset \mathbb{C}$ *is a bounded sequence. Then every solution to the equation is bounded on* $\mathbb{N}_0$.

The following theorem corresponds to Theorem 2.5 and is essentially its dual statement on domain $\mathbb{Z} \setminus \mathbb{N}$. It can be obtained from Theorem 2.5 by using (2.48) and (2.49), so its detailed proof is omitted.

Theorem 2.10. *Consider equation* (2.52)*, where* $q \in \mathbb{C}$, $0 < |q| < 1$ *and* $(f_{-n})_{n\geq 3} \subset \mathbb{C}$ *is a bounded sequence. Then there is a unique bounded solution to the equation on* $\mathbb{N}_0$.

A problem with formula (2.51) is that it is given on domain $\mathbb{N}_0$. What we need is a formula for solution to equation (2.1) on domain $\mathbb{Z}\setminus\mathbb{N}_3$, which would patch well with formula (2.18), in the sense that the equation on domains $\mathbb{N}_0$ and $\mathbb{Z} \setminus \mathbb{N}_3$ has the same initial/end values. To overcome the problem we will get now the solution to equation (2.52) on domain $n \geq -2$, with initial/end values x_j, $j = \overline{0,2}$, by using the decomposition method. In fact, we will solve a more general linear difference equation of third-order.

The nonhomogeneous linear difference equation of third-order with constant coefficients has the form

$$x_{n+3} + px_{n+2} + qx_{n+1} + rx_n = f_n. \tag{2.53}$$

We will consider the equation on domain $\mathbb{Z} \setminus \mathbb{N}_3$. So, we choose that the initial/end values on the domain are x_0, x_1 and x_2. We assume that $r \neq 0$, since otherwise it becomes an equation of smaller order.

Let λ_j, $j = \overline{1,3}$, be the zeros of the characteristic polynomial

$$P_3(\lambda) = \lambda^3 + p\lambda^2 + q\lambda + r, \tag{2.54}$$

which is associated to the homogeneous equation

$$x_{n+3} + px_{n+2} + qx_{n+1} + rx_n = 0.$$

We additionally assume that the zeros are distinct, that is, $\lambda_i \neq \lambda_j$, $i \neq j$, $i, j \in \{1, 2, 3\}$, which happens if and only if the discriminant of the equation $P_3(\lambda) = 0$ is equal to zero [12].

Hence, if $n \leq -1$, then the equation can be written as follows

$$x_n + \frac{q}{r}x_{n+1} + \frac{p}{r}x_{n+2} + \frac{1}{r}x_{n+3} = \frac{f_n}{r}, \quad n \leq -1,$$

which can be rewritten in the form

$$\begin{aligned} x_{-n} &- \Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\Big)x_{-(n-1)} + \Big(\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_3\lambda_1}\Big)x_{-(n-2)} \\ &- \frac{x_{-(n-3)}}{\lambda_1\lambda_2\lambda_3} \\ &= -\frac{f_{-n}}{\lambda_1\lambda_2\lambda_3}, \quad n \in \mathbb{N}. \end{aligned} \tag{2.55}$$

We will solve equation (2.55) by the method of decomposition (see, for example, [15, 20]). Let

$$y_{-n} = x_{-n} - \frac{1}{\lambda_3} x_{-(n-1)}, \quad n \geq -1. \tag{2.56}$$

Then (2.55) can be written as follows:

$$y_{-n} - \Big(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\Big) y_{-(n-1)} + \frac{y_{-(n-2)}}{\lambda_1\lambda_2} = \frac{\widehat{f}_{-n}}{\lambda_1\lambda_2}, \quad n \in \mathbb{N}, \tag{2.57}$$

where

$$\widehat{f}_{-n} = -\frac{f_{-n}}{\lambda_3}, \quad n \in \mathbb{N}.$$

If we write (2.57) in the form

$$y_{-n} - \frac{y_{-(n-1)}}{\lambda_1} = \frac{1}{\lambda_2}\Big(y_{-(n-1)} - \frac{y_{-(n-2)}}{\lambda_1}\Big) + \frac{\widehat{f}_{-n}}{\lambda_1\lambda_2}, \quad n \in \mathbb{N}, \tag{2.58}$$

and multiply the equation

$$y_{-j} - \frac{y_{-(j-1)}}{\lambda_1} = \frac{1}{\lambda_2}\Big(y_{-(j-1)} - \frac{y_{-(j-2)}}{\lambda_1}\Big) + \frac{\widehat{f}_{-j}}{\lambda_1\lambda_2} \tag{2.59}$$

by $\lambda_2^{-(n-j)}$, $j = \overline{1,n}$, and summing up such obtained equalities, we obtain

$$y_{-n} = \frac{y_{-(n-1)}}{\lambda_1} + \frac{1}{\lambda_2^n}\Big(y_0 - \frac{y_1}{\lambda_1}\Big) + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{n} \frac{\widehat{f}_{-j}}{\lambda_2^{n-j}}, \tag{2.60}$$

for $n \in \mathbb{N}$.

Multiplying the following equality

$$y_{-i} = \frac{y_{-(i-1)}}{\lambda_1} + \frac{1}{\lambda_2^i}\Big(y_0 - \frac{y_1}{\lambda_1}\Big) + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{i} \frac{\widehat{f}_{-j}}{\lambda_2^{i-j}}, \tag{2.61}$$

by $\lambda_1^{-(n-i)}$, $i = \overline{1,n}$, and summing up such obtained equalities, we obtain

$$\begin{aligned}
y_{-n} =& \frac{y_0}{\lambda_1^n} + \frac{1}{\lambda_2}\Big(y_0 - \frac{y_1}{\lambda_1}\Big)\sum_{j=0}^{n-1} \frac{1}{\lambda_1^j \lambda_2^{n-1-j}} + \frac{1}{\lambda_1\lambda_2}\sum_{i=1}^{n} \frac{1}{\lambda_1^{n-i}} \sum_{j=1}^{i} \frac{\widehat{f}_{-j}}{\lambda_2^{i-j}} \qquad (2.62)\\
=& y_0 \frac{\lambda_1^{-(n+1)} - \lambda_2^{-(n+1)}}{\lambda_1^{-1} - \lambda_2^{-1}} - \frac{y_1}{\lambda_1\lambda_2}\frac{\lambda_1^{-n} - \lambda_2^{-n}}{\lambda_1^{-1} - \lambda_2^{-1}} + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{n} \frac{\widehat{f}_{-j}}{\lambda_1^n}\lambda_2^j \sum_{i=j}^{n}\Big(\frac{\lambda_1}{\lambda_2}\Big)^i\\
=& y_0 \frac{\lambda_1^{-(n+1)} - \lambda_2^{-(n+1)}}{\lambda_1^{-1} - \lambda_2^{-1}} - \frac{y_1}{\lambda_1\lambda_2}\frac{\lambda_1^{-n} - \lambda_2^{-n}}{\lambda_1^{-1} - \lambda_2^{-1}} + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{n} \widehat{f}_{-j}\frac{\lambda_1^{j-n-1} - \lambda_2^{j-n-1}}{\lambda_1^{-1} - \lambda_2^{-1}}\\
=& \frac{\lambda_1^{-(n+1)}(y_0 - y_1\lambda_2^{-1} + (\lambda_1\lambda_2)^{-1}\sum_{j=1}^{n} \widehat{f}_{-j}\lambda_1^j)}{\lambda_1^{-1} - \lambda_2^{-1}}\\
&- \frac{\lambda_2^{-(n+1)}(y_0 - y_1\lambda_1^{-1} + (\lambda_1\lambda_2)^{-1}\sum_{j=1}^{n} \widehat{f}_{-j}\lambda_2^j)}{\lambda_1^{-1} - \lambda_2^{-1}}\\
=& \frac{\lambda_1^{-n}(\lambda_2 y_0 - y_1 + \sum_{j=1}^{n} \widehat{f}_{-j}\lambda_1^{j-1}) - \lambda_2^{-n}(\lambda_1 y_0 - y_1 + \sum_{j=1}^{n} \widehat{f}_{-j}\lambda_2^{j-1})}{\lambda_2 - \lambda_1},
\end{aligned} \tag{2.63}$$

for $n \in \mathbb{N}_0$.

Combining (2.56) and (2.63), we obtain

$$\begin{aligned} x_{-n} =& \frac{x_{-(n-1)}}{\lambda_3} + \frac{\lambda_1^{-n}(\lambda_3^{-1}(x_2-(\lambda_2+\lambda_3)x_1+\lambda_2\lambda_3x_0)+\sum_{j=1}^n \widehat{f}_{-j}\lambda_1^{j-1})}{\lambda_2-\lambda_1} \\ &- \frac{\lambda_2^{-n}(\lambda_3^{-1}(x_2-(\lambda_1+\lambda_3)x_1+\lambda_1\lambda_3x_0)+\sum_{j=1}^n \widehat{f}_{-j}\lambda_2^{j-1})}{\lambda_2-\lambda_1}, \end{aligned} \tag{2.64}$$

for $n\in\mathbb{N}$.

Multiplying the following equality

$$\begin{aligned} x_{-i} =& \frac{x_{-(i-1)}}{\lambda_3} + \frac{\lambda_1^{-i}(\lambda_3^{-1}(x_2-(\lambda_2+\lambda_3)x_1+\lambda_2\lambda_3x_0)+\sum_{j=1}^i \widehat{f}_{-j}\lambda_1^{j-1})}{\lambda_2-\lambda_1} \\ &- \frac{\lambda_2^{-i}(\lambda_3^{-1}(x_2-(\lambda_1+\lambda_3)x_1+\lambda_1\lambda_3x_0)+\sum_{j=1}^i \widehat{f}_{-j}\lambda_2^{j-1})}{\lambda_2-\lambda_1}, \end{aligned} \tag{2.65}$$

by $\lambda_3^{-(n-i)}$, $i=\overline{1,n}$, and summing up such obtained equalities, we obtain

$$\begin{aligned} x_{-n} =& \frac{x_0}{\lambda_3^n} + \frac{\frac{x_2-(\lambda_2+\lambda_3)x_1+\lambda_2\lambda_3x_0}{\lambda_3}\sum_{i=1}^n \frac{1}{\lambda_3^{n-i}\lambda_1^i} + \sum_{i=1}^n \lambda_3^{i-n}\lambda_1^{-i}\sum_{j=1}^i \widehat{f}_{-j}\lambda_1^{j-1}}{\lambda_2-\lambda_1} \\ &- \frac{\frac{x_2-(\lambda_1+\lambda_3)x_1+\lambda_1\lambda_3x_0}{\lambda_3}\sum_{i=1}^n \frac{1}{\lambda_3^{n-i}\lambda_2^i} + \sum_{i=1}^n \lambda_3^{i-n}\lambda_2^{-i}\sum_{j=1}^i \widehat{f}_{-j}\lambda_2^{j-1}}{\lambda_2-\lambda_1} \\ =& \frac{x_0}{\lambda_3^n} + \frac{\frac{x_2-(\lambda_2+\lambda_3)x_1+\lambda_2\lambda_3x_0}{\lambda_3-\lambda_1}\left(\frac{1}{\lambda_1^n}-\frac{1}{\lambda_3^n}\right) + \sum_{j=1}^n f_{-j}\frac{\lambda_3^{j-n-1}-\lambda_1^{j-n-1}}{\lambda_3-\lambda_1}}{\lambda_2-\lambda_1} \\ &- \frac{\frac{x_2-(\lambda_1+\lambda_3)x_1+\lambda_1\lambda_3x_0}{\lambda_3-\lambda_2}\left(\frac{1}{\lambda_2^n}-\frac{1}{\lambda_3^n}\right) + \sum_{i=j}^n f_{-j}\frac{\lambda_3^{j-n-1}-\lambda_2^{j-n-1}}{\lambda_3-\lambda_2}}{\lambda_2-\lambda_1} \\ =& \frac{1}{\lambda_1^n}\frac{x_2-(\lambda_2+\lambda_3)x_1+\lambda_2\lambda_3x_0-\sum_{j=1}^n f_{-j}\lambda_1^{j-1}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} \\ &+ \frac{1}{\lambda_2^n}\frac{x_2-(\lambda_1+\lambda_3)x_1+\lambda_1\lambda_3x_0-\sum_{j=1}^n f_{-j}\lambda_2^{j-1}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} \\ &+ \frac{1}{\lambda_3^n}\frac{x_2-(\lambda_2+\lambda_1)x_1+\lambda_2\lambda_1x_0-\sum_{j=1}^n f_{-j}\lambda_3^{j-1}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}, \end{aligned} \tag{2.66}$$

for $n\ge -2$. From the above analysis which leads to formula (2.66) we see that the following result holds.

Proposition 2.11. *Consider equation* (2.53)*, where $p,q\in\mathbb{C}$, $r\in\mathbb{C}\setminus\{0\}$ and $(f_n)_{n\le -1}\subset\mathbb{C}$. Assume that the zeros λ_j, $j=\overline{1,3}$, of the characteristic polynomial* (2.54) *are distinct. Then the solution to the equation with the initial/end values $x_j\in\mathbb{C}$, $j=\overline{0,2}$, on domain $\mathbb{Z}\setminus\mathbb{N}_3$, is given by formula* (2.66)*.*

By using formula (2.66) the following theorem is easily proved similar to Theorem 2.4. Hence, we also omit the proof.

Theorem 2.12. *Consider equation* (2.53)*, where $p,q\in\mathbb{C}$, $r\in\mathbb{C}\setminus\{0\}$ and the sequence $(f_n)_{n\le -1}\subset\mathbb{C}$ is bounded. If the roots of the polynomial* (2.54) *are distinct and satisfy the condition $\min_{j=\overline{1,3}}|\lambda_j|>1$, then every solution to the equation is bounded on domain $\mathbb{Z}\setminus\mathbb{N}_3$.*

Corollary 2.13. *Consider equation (2.1). If $|q| > 1$ and $(f_n)_{n\le -1} \subset \mathbb{C}$ is a bounded sequence. Then every solution to the equation is bounded on domain $\mathbb{Z}\setminus\mathbb{N}_3$.*

Theorem 2.14. *Consider equation (2.53), where $p, q \in \mathbb{C}$, $r \in \mathbb{C}\setminus\{0\}$ and the sequence $(f_n)_{n\le -1} \subset \mathbb{C}$ is a bounded. If the roots of polynomial (2.54) are distinct and satisfy the condition*

$$\max_{j=\overline{1,3}} |\lambda_j| < 1, \tag{2.67}$$

then there is a unique bounded solution to the equation on domain $\mathbb{Z}\setminus\mathbb{N}_3$.

Proof. If $(x_n)_{n\le 2}$ is a bounded solution to equation (2.53), then from (2.66) we have that it must be

$$x_2 - (\lambda_2+\lambda_3)x_1 + \lambda_2\lambda_3 x_0 = \sum_{j=1}^{\infty} f_{-j}\lambda_1^{j-1} =: \widetilde{S}_1, \tag{2.68}$$

$$x_2 - (\lambda_1+\lambda_3)x_1 + \lambda_1\lambda_3 x_0 = \sum_{j=1}^{\infty} f_{-j}\lambda_2^{j-1} =: \widetilde{S}_2, \tag{2.69}$$

$$x_2 - (\lambda_1+\lambda_2)x_1 + \lambda_1\lambda_2 x_0 = \sum_{j=1}^{\infty} f_{-j}\lambda_3^{j-1} =: \widetilde{S}_3. \tag{2.70}$$

Indeed, assume without loss of generality that $|\lambda_1| < |\lambda_2| < |\lambda_3|$ (the other five cases are obtained from this one by permutations of some letters), and note that

$$\Big|\sum_{j=1}^{\infty} f_{-j}\lambda_i^{j-1}\Big| \le \frac{\|f\|_\infty}{1-|\lambda_i|}, \quad i=\overline{1,3}, \tag{2.71}$$

which implies that the quantities $\widetilde{S}_j$, $j=\overline{1,3}$, are finite.

Then, if (2.70) were not hold from (2.66), (2.67) and (2.71) we would have

$$|x_{-n}| \succeq O\Big(\frac{1}{|\lambda_3|^n}\Big),$$

which would contradict the boundedness of $(x_n)_{n\le 2}$ (note that condition (2.67) implies the unboundedness of the right-hand side of the last relation).

Hence, equality (2.70) holds, so by using the equality in the last addend in (2.66), it follows that

$$\begin{aligned}
&\Big|\frac{x_2-(\lambda_2+\lambda_1)x_1+\lambda_2\lambda_1x_0-\sum_{j=1}^n f_{-j}\lambda_3^{j-1}}{\lambda_3^n(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\Big|\\
&= \Big|\frac{\sum_{j=n+1}^{\infty} f_{-j}\lambda_3^{j-1-n}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\Big| \le \frac{\|f\|_\infty}{|(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)|(1-|\lambda_3|)} < \infty.
\end{aligned} \tag{2.72}$$

If (2.69) were not hold, then from (2.66), (2.67), (2.71) and (2.72) we would have

$$|x_{-n}| \succeq O\Big(\frac{1}{|\lambda_2|^n}\Big),$$

which would contradict the boundedness of $(x_n)_{n\le 2}$ (as above condition (2.67) implies the unboundedness of the right-hand side of the last relation).

Hence, (2.69) holds, so by using it in the last but one addend in (2.66) we obtain

$$\begin{aligned} &\Big|\frac{x_2-(\lambda_1+\lambda_3)x_1+\lambda_1\lambda_3x_0-\sum_{j=1}^{n}f_{-j}\lambda_2^{j-1}}{\lambda_2^n(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\Big| \\ &=\Big|\frac{\sum_{j=n+1}^{\infty}f_{-j}\lambda_2^{j-1-n}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}\Big| \le \frac{\|f\|_\infty}{|(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)|(1-|\lambda_2|)}<\infty. \end{aligned} \tag{2.73}$$

If (2.68) were not hold, then from (2.66), (2.67), (2.71), (2.72) and (2.73) we would have

$$|x_{-n}| \asymp O\Big(\frac{1}{|\lambda_1|^n}\Big),$$

which would contradict the boundedness of $(x_n)_{n\le 2}$.

Hence, (2.68) holds, so by using it in the first addend from the right-hand side of the equality in (2.66) we obtain

$$\begin{aligned} &\Big|\frac{x_2-(\lambda_2+\lambda_3)x_1+\lambda_2\lambda_3x_0-\sum_{j=1}^{n}f_{-j}\lambda_1^{j-1}}{\lambda_1^n(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\Big| \\ &=\Big|\frac{\sum_{j=n+1}^{\infty}f_{-j}\lambda_1^{j-1-n}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\Big| \\ &\le \frac{\|f\|_\infty}{|(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)|(1-|\lambda_1|)}<\infty. \end{aligned} \tag{2.74}$$

Hence, if there is a bounded solution to (2.53) in this case, it is given by

$$x_{-n}=\frac{\sum_{j=n+1}^{\infty}f_{-j}\lambda_1^{j-1-n}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{\sum_{j=n+1}^{\infty}f_{-j}\lambda_2^{j-1-n}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{\sum_{j=n+1}^{\infty}f_{-j}\lambda_3^{j-1-n}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}. \tag{2.75}$$

A direct calculation shows that (2.75) is really a solution to (2.53). Moreover, since

$$\Delta_1=\begin{vmatrix}\lambda_2\lambda_3 & -(\lambda_2+\lambda_3) & 1\\ \lambda_3\lambda_1 & -(\lambda_3+\lambda_1) & 1\\ \lambda_1\lambda_2 & -(\lambda_1+\lambda_2) & 1\end{vmatrix}=(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)\ne 0,$$

we also see that three-dimensional system (2.68)-(2.70) has a unique solution in variables x_0, x_1 and x_2.

Hence, the bounded solution to equation (2.53) on $\mathbb{Z}\setminus\mathbb{N}_3$, is obtained for

$$x_0=\frac{1}{\Delta_1}\begin{vmatrix}\widetilde{S}_1 & -(\lambda_2+\lambda_3) & 1\\ \widetilde{S}_2 & -(\lambda_3+\lambda_1) & 1\\ \widetilde{S}_3 & -(\lambda_1+\lambda_2) & 1\end{vmatrix}, \tag{2.76}$$

$$x_1=\frac{1}{\Delta_1}\begin{vmatrix}\lambda_2\lambda_3 & \widetilde{S}_1 & 1\\ \lambda_3\lambda_1 & \widetilde{S}_2 & 1\\ \lambda_1\lambda_2 & \widetilde{S}_3 & 1\end{vmatrix}, \tag{2.77}$$

$$x_2=\frac{1}{\Delta_1}\begin{vmatrix}\lambda_2\lambda_3 & -(\lambda_2+\lambda_3) & \widetilde{S}_1\\ \lambda_3\lambda_1 & -(\lambda_3+\lambda_1) & \widetilde{S}_2\\ \lambda_1\lambda_2 & -(\lambda_1+\lambda_2) & \widetilde{S}_3\end{vmatrix}, \tag{2.78}$$

which completes the proof. □

Since the characteristic polynomial associated with (1.5) has three different zeros, from Theorem 2.14 we obtain the following corollary.

Corollary 2.15. *Consider equation* (2.1). *If* $|q| < 1$ *and* $(f_n)_{n\le -1} \subset \mathbb{C}$ *is a bounded sequence, then there is a unique bounded solution to the equation on domain* $\mathbb{Z}\setminus\mathbb{N}_3$*, which is given by*

$$x_{-n} = \frac{1}{3}\sum_{j=n}^{\infty}(1+\varepsilon^{j-n}+\bar{\varepsilon}^{j-n})(\sqrt[3]{q})^{j-n} f_{-(j+3)}, \quad n \ge -2. \tag{2.79}$$

Proof. From the proof of Theorem 2.14 we see that the bounded solution to (2.1) on the domain $\mathbb{Z}\setminus\mathbb{N}_3$ is given by (2.75) with $\lambda_j = \varepsilon^{j-1}\sqrt[3]{q}$, $j=\overline{1,3}$. From this, by using (1.7), and by some calculation formula (2.79) is obtained. $\square$

From Theorems 2.4 and 2.5, and Corollaries 2.13 and 2.15, we obtain the following result.

Corollary 2.16. *Consider equation* (2.1). *If* $|q|\ne 1$ *and* $(f_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ *is a bounded sequence. Then, the following statements are true.*

(a) *There is a unique bounded solution to the equation on domain* $\mathbb{Z}$.
(b) *If* $|q|>1$*, then the bounded solution is obtained for the initial values given in* (2.37).
(c) *If* $|q|<1$*, then the bounded solution is obtained for the initial values given in* (2.76)-(2.78)*, with* $\lambda_j = \sqrt[3]{q}\varepsilon^{j-1}$, $j=\overline{1,3}$.

Now we consider equation (1.3) on domain $\mathbb{Z}\setminus\mathbb{N}_3$ when the sequence q_n is not constant.

Theorem 2.17. *Consider equation* (1.3) *on domain* $\mathbb{Z}\setminus\mathbb{N}_3$. *Assume that the sequence* $(q_n)_{n\le -1}$ *satisfies*

$$1 < \hat{a} \le 1/q_{-n} \le \hat{b}, \quad n\in\mathbb{N}, \tag{2.80}$$

or

$$-\hat{b} \le 1/q_{-n} \le -\hat{a} < -1, \quad n\in\mathbb{N}, \tag{2.81}$$

for some positive numbers $\hat{a}$ *and* $\hat{b}$*, and that* $(f_n)_{n\le -1}$ *is a bounded sequence of complex numbers. Then the difference equation has a unique bounded solution on the domain.*

Proof. We prove the theorem when (2.80) holds. The case (2.81) is treated similarly, so its proof is omitted.

Let q be a positive number such that

$$1/q \in \big(\max\{\hat{a}, (\hat{b}+1)/2\}, \hat{b}\big). \tag{2.82}$$

Write equation (1.3) in the form

$$x_{-(n+3)} - \frac{x_{-n}}{q} = \Big(\frac{1}{q_{-(n+3)}} - \frac{1}{q}\Big)x_{-n} - \frac{f_{-(n+3)}}{q_{-(n+3)}}, \quad n\ge -2.$$

Let A be an operator defined on l^∞, as follows

$$A(u) = \Big(\frac{1}{3}\sum_{k=n}^{\infty}(1+\varepsilon^{k-n}+\bar{\varepsilon}^{k-n})(\sqrt[3]{q})^{k-n+3}\Big(\frac{f_{-(k+3)}}{q_{-(k+3)}} - \Big(\frac{1}{q_{-(k+3)}} - \frac{1}{q}\Big)u_{-k}\Big)\Big)_{n\in\mathbb{N}_0}.$$

Assume that $u\in l^\infty$.

Then, we have

$$
\begin{aligned}
&\|A(u)\|_\infty \\
&= \sup_{n\in\mathbb{N}_0}\Big|\sum_{k=n}^{\infty}\frac{(1+\varepsilon^{k-n}+\bar{\varepsilon}^{k-n})}{3}(\sqrt[3]{q})^{k-n+3}\Big(\frac{f_{-(k+3)}}{q_{-(k+3)}}-\Big(\frac{1}{q_{-(k+3)}}-\frac{1}{q}\Big)u_{-k}\Big)\Big| \\
&\le \sup_{n\in\mathbb{N}_0}\sum_{k=n}^{\infty}\Big(\Big(\frac{1}{q_{-(k+3)}}+\frac{1}{q}\Big)|u_{-k}|+\Big|\frac{f_{-(k+3)}}{q_{-(k+3)}}\Big|\Big)|\sqrt[3]{q}|^{k-n+3} \\
&\le \frac{(q\hat{b}+1)\|u\|_\infty+q\hat{b}\|f\|_\infty}{1-|\sqrt[3]{q}|}<\infty,
\end{aligned}
$$

which means that $A(u)\in l^\infty$.

If $u,v\in l^\infty$, then

$$
\begin{aligned}
&\|A(u)-A(v)\|_\infty \\
&= \sup_{n\in\mathbb{N}_0}\Big|\sum_{k=n}^{\infty}\frac{(1+\varepsilon^{k-n}+\bar{\varepsilon}^{k-n})}{3}(\sqrt[3]{q})^{k-n+3}\Big(\frac{1}{q_{-(k+3)}}-\frac{1}{q}\Big)(v_{-k}-u_{-k})\Big| \\
&= \sup_{n\in\mathbb{N}_0}\Big|\sum_{j=0}^{\infty}(\sqrt[3]{q})^{3j+3}\Big(\frac{1}{q_{-(n+3j)}}-\frac{1}{q}\Big)(u_{-(n+3j)}-v_{-(n+3j)})\Big| \\
&\le \frac{\max\{q^{-1}-\hat{a},\hat{b}-q^{-1}\}}{q^{-1}-1}\|u-v\|_\infty=\hat{q}_1\|u-v\|_\infty.
\end{aligned}
\tag{2.83}
$$

From (2.82) and since $\hat{a}>1$, we have $\hat{q}_1\in(0,1)$. Hence, operator A is a contraction on l^∞.

By the contraction mapping principle it follows that A has a unique fixed point, say $\hat{x}^*=(\hat{x}^*_{-n})_{n\ge -2}\in l^\infty$. Hence, it must be

$$
\hat{x}^*_{-n}=\frac{1}{3}\sum_{k=n}^{\infty}(1+\varepsilon^{k-n}+\bar{\varepsilon}^{k-n})(\sqrt[3]{q})^{k-n+3}\Big(\frac{f_{-(k+3)}}{q_{-(k+3)}}-\Big(\frac{1}{q_{-(k+3)}}-\frac{1}{q}\Big)\hat{x}^*_{-k}\Big), \tag{2.84}
$$

for every $n\ge -2$.

It is easy to verify that (2.84) is a solution to (1.3), so that such obtained $\hat{x}^*$ is its unique (bounded) solution. $\square$

From Theorems 2.7 and 2.17, we obtain the following result.

Corollary 2.18. *Consider equation* (1.3) *on* $\mathbb{Z}$*, where one of conditions* (2.38) *and* (2.39)*, and one of conditions* (2.80) *and* (2.81) *hold, and where* $(f_n)_{n\in\mathbb{Z}}$ *is a bounded sequence of complex numbers. Then the equation has a unique bounded solution on the domain.*

2.3. **A natural extension to equation** (1.3)**.** The linear difference equation

$$
x_{n+4}-q_nx_n=f_n,\quad n\in\mathbb{N}_0, \tag{2.85}
$$

is the fourth-order cousin of the equations in (1.2) and (1.3). The results regarding equation (2.85) which correspond to above mentioned ones can be formulated and proved similarly. Here we want to mention only one of them which corresponds to Proposition 2.1. The result can be proved by using the same method but is technically more complicated than the one of Proposition 2.1.

Proposition 2.19. *Consider the equation*

$$x_{n+4} - qx_n = f_n, \quad n \in \mathbb{N}_0, \tag{2.86}$$

where $q \in \mathbb{C} \setminus \{0\}$, and $(f_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$. Then the general solution to the equation is given by

$$\begin{aligned} x_n =& (\sqrt[4]{q})^n \Big(a_0 + \sum_{k=0}^{n-1} \frac{f_k}{4(\sqrt[4]{q})^{k+4}} \Big) + (-\sqrt[4]{q})^n \Big(b_0 + \sum_{k=0}^{n-1} \frac{(-1)^k f_k}{4(\sqrt[4]{q})^{k+4}} \Big) \\ &+ (\sqrt[4]{q} i)^n \Big(c_0 + \sum_{k=0}^{n-1} \frac{(-i)^k f_k}{4(\sqrt[4]{q})^{k+4}} \Big) + (-\sqrt[4]{q} i)^n \Big(d_0 + \sum_{k=0}^{n-1} \frac{i^k f_k}{4(\sqrt[4]{q})^{k+4}} \Big), \end{aligned} \tag{2.87}$$

for $n \in \mathbb{N}_0$, where a_0, b_0, c_0 and d_0 are arbitrary complex numbers, and $\sqrt[4]{q}$ is one of the four possible fourth roots of q.

Because of the mentioned similarity, the formulations and proofs of the other corresponding results concerning difference equations (2.85) and (2.86) are left to the interested reader as some exercises.

Based on above presented results and corollaries, given remarks and conducted detailed analyses and calculations, we strongly believe that the methods and ideas in this paper can be applied for proving the corresponding results for the general difference equation

$$x_{n+k} - q_n x_n = f_n, \tag{2.88}$$

where $k \in \mathbb{N}$, and q_n and f_n are given sequences satisfying the corresponding conditions. However, there are several technical difficulties which request some much more involved calculations and more complex formulas, which, at the moment, prevent us to give complete proofs of the corresponding results for the case of general difference equation in (2.88). Hence, we leave the problem for a further investigation.

It is also highly expected that some other difference equations and systems of difference equations can be studied by using some modifications of the above combination of solvability methods and the contraction mapping principle, which should be a general problem of some interest.

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