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EXISTENCE OF PERIODIC SOLUTIONS FOR SUBQUADRATIC DISCRETE SYSTEM INVOLVING THE P-LAPLACIAN

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Abstract. An existence theorem on periodic solution is established for a class of nonautonomous discrete system involving the p-Laplacian under a subquadratic growth condition. The conclusion is based on saddle point theorem and variational methods.

1. Introduction and statement of main results

Let $\mathbb{Z}$ be the set of integers. Given $a < b$ in $\mathbb{Z}$, let $\mathbb{Z}[a, b] = \{a, a+1, \dots, b\}$ and $T > 1$ be a positive integer. In this article, we aim at the existence of periodic solution for the nonlinear discrete system involving the p-Laplacian

$$\Delta_p u(t-1) + \nabla F(t, u(t)) = 0, \quad \forall t \in \mathbb{Z} \tag{1.1}$$

where Δ_p is the discrete p-Laplacian operator, i.e.,

$$\Delta_p u(t-1) := \Delta\phi_p(\Delta u(t-1)) = \phi_p(\Delta u(t)) - \phi_p(\Delta u(t-1)),$$

$\phi_p(s) = |s|^{p-2}s (p > 1)$, Δ is the forward difference operator and the function $F : \mathbb{Z} \times \mathbb{R}^N \to R$ is continuously differentiable in x for every $t \in \mathbb{Z}$, $\nabla F(t, x) = \frac{\partial F(t,x)}{\partial x}$.

In recent years, many authors were interested in difference equations involving the discrete p-Laplacian operator and have obtained many significant conclusions, see, for instance, the papers [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 20, 21]. Various methods have been used to deal with the existence of solutions to the discrete boundary value problems, we refer to the fixed point theorems in cones in [14], the lower and upper solution method in [4], the variational method in [2, 3, 5, 7, 8, 9, 10, 11, 12, 14, 15, 18, 20, 21].

The variational approach represents an important advance as it allows to prove multiplicity results as well. When $p > 1$, via dual least principle, system (1.1) under convex condition was investigated in [13]. Recently, some further improved results have been made in [22]. Via Linking theorem, the existence of one nonconstant solutions was established for system (1.1) under superquadratic condition in [16]. In 2007, in [21] the authors constructed a variational setting unlike the one in [11] to study the discrete system (1.1) with $p = 2$ under subquadratic condition via saddle point theorem. The result was obtained under the following assumptions:

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(A1) For a given integer $T > 0$, $F(t+T, x) = F(t, x)$ for all $(t, x) \in \mathbb{Z} \times \mathbb{R}^N$;
(A2) There are constants $G_1 > 0$, $0 < \beta < 2$ such that

$$(x, \nabla F(t, x)) \le \beta F(t, x)$$

for all $(t, x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N$ and $|x| \ge G_1$;
(A3) $F(t, x) \to +\infty$ as $|x| \to \infty$ for $t \in \mathbb{Z}[1, T]$.

Theorem 1.1 ([21])**.** *Suppose that* (A1)–(A3) *are satisfied. Then problem* (1.1) *possesses at least one periodic solution with period T.*

Inspired by [16, 20, 21], in the article, we further investigate periodic solutions for system (1.1) under a new subquadratic condition which is more general than (A2). Here $\mathcal{H}$ denotes the space of continuous function space such that for any $\theta \in \mathcal{H}$ there exists constant $M_1 > 0$ for which

(i) $\theta(t) > 0$ for all $t \in R^+$,
(ii) $\int_{M_1}^t \frac{1}{s\theta(s)} ds \to +\infty$ as $t \to +\infty$.

Our main result is stated using the following assumptions:

(A4) There exist a constant $M_1 > 0$ and a continuous function $\theta(|x|) \in \mathcal{H}$ with $0 < \frac{1}{\theta(|x|)} < p$ such that for all $(t, x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N$ and $|x| \ge M_1$,

$$(x, \nabla F(t, x)) \le \Big(p - \frac{1}{\theta(|x|)}\Big) F(t, x);$$

(A5) $F(t, x) \ge 0$ as $|x| \to +\infty$ for $t \in \mathbb{Z}[1, T]$;
(A6) $\sum_{t=1}^T \frac{F(t,x)}{\theta(|x|)} \to +\infty$ as $|x| \to +\infty$ for $t \in \mathbb{Z}[1, T]$;

Theorem 1.2. *Assume that* (A1), (A4)–(A6) *are satisfed. Then problem* (1.1) *has at least one periodic solution with period T which is a positive integer.*

Remark 1.3. Set $\inf_{|x| \ge M_1} \frac{1}{\theta(|x|)} = l$. Here l is a constant. One points out that

(1) Theorem 1.2 extends Theorem 1.1 completely since (A4) is weaker than (A2) when $l = 0$ even if $p = 2$.

(2) Theorem 1.2 generalizes Theorem 1.1 even if $l > 0$. Indeed, via (A5), when $l > 0$, (A6) implies

(A6') $\sum_{t=1}^T F(t, x) \to +\infty$ as $|x| \to +\infty$.

However, (A5) and (A6') are weaker than (A3).

(3) There are functions F fulfilling the conditions of Theorem 1.2 but not the assumptions in [11, 12, 13, 15, 21, 22]. For example,

$$F(t, x) = g(t) \frac{2 + |x|^p}{\ln(2 + |x|^2)}, \quad \forall (t, x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N.$$

Here

$$g(t) = \begin{cases} \sin(2\pi t/T), & t \in [0, T/2], \\ 0, & t \in [T/2, T]. \end{cases}$$

Put $\theta(|x|) = \ln(2 + |x|^2)$. A simple computation shows that F satisfies (A1) and (A4)–(A6) in Theorem 1.2, but it does not meet the corresponding conditions of Theorem 1.1.

2. Proof of Theorem 1.2

For a given positive integer T, we define

$$H_T = \{u : Z \to \mathbb{R}^N : u(t+T) = u(t),\ t \in Z\}.$$

H_T is equipped with the inner product

$$\langle u, v\rangle = \sum_{t=1}^{T} (u(t), v(t)), \quad \forall u, v \in H_T$$

and the norm

$$\|u\| = \Big(\sum_{t=1}^{T} |u(t)|^p\Big)^{1/p}, \quad \forall u \in H_T.$$

One can easily see that $(H_T, \langle\cdot,\cdot\rangle)$ is a finite dimensional Hilbert space and linear homeomorphic to R^{NT}. Define

$$\|u\|_\infty = \max_{t\in\mathbb{Z}[1,T]} |u(t)|.$$

Then there exists a constant $c > 0$ such that

$$\|u\|_\infty \le c\|u\|. \tag{2.1}$$

For $u \in H_T$, set

$$\tilde{u} = u - \bar{u} \quad \text{and} \quad \tilde{H}_T = \{u \in H_T : \bar{u} = 0\}.$$

Here $\bar{u} = \sum_{t=1}^{T} u(t)$. Then one knows

$$H_T = \tilde{H}_T \oplus \mathbb{R}^N.$$

Furthermore, via [16], one gets

$$\sum_{t=1}^{T} |u(t)|^p \le \frac{(T-1)^{2p-1}}{T^{p-1}} \sum_{t=1}^{T} |\Delta u|^p, \quad \forall u \in \tilde{H}_T. \tag{2.2}$$

From reference [16], it is known that finding T-periodic solution of problem (1.1) is equivalent to seeking the critical point of the following functional φ defined on H_T,

$$\varphi(u) = \frac{1}{p}\sum_{t=1}^{T} |\Delta u(t)|^p - \sum_{t=1}^{T} F(t, u(t)).$$

Subsequently, two important lemmas are stated for the readers convenience.

Lemma 2.1 (saddle point Theorem [18])**.** *Let X be a Banach space with a direct sum decomposition $X = X_1 \oplus X_2$ with $dim X_2 < \infty$ and let φ be a C^1 function on X satisfying the (PS) condition and*

1. *there exist a constant r and a bounded neighborhood U of 0 in X_2, such that $\varphi(u) \le r$ for $u \in U \subset X_2$,*
2. *there exists a constant $\alpha > r$, such that $\varphi(u) \ge \alpha$ for all $u \in X_1$.*

Then φ has at least one critical point.

As we know, a deformation lemma can be proved with Cerami's condition (C) in [6] by replacing the usual (PS) condition. Then the saddle point theorem is tenable under condition (C).

Lemma 2.2. *Under the conditions of Theorem 1.2, we have*

$$F(t,x) \le \frac{M_2}{M_1^p}|x|^p G(|x|) + M_2 \tag{2.3}$$

for all $x \in \mathbb{R}^N$ *and* $t \in \mathbb{Z}[1,T]$*, where*

$$M_2 = \max\{F(t,x) : |x| \le M_1,\ t \in \mathbb{Z}[1,T]\}, \quad G(|x|) = \exp\Big(-\int_{M_1}^{|x|} \frac{1}{s\theta(s)} ds\Big).$$

Proof. Put

$$y(s) = F(t,sx), \quad s \ge \frac{M_1}{|x|}.$$

Via (A4), a simple computation yields

$$\begin{aligned} y'(s) &= \frac{1}{s}(\nabla F(t,sx), sx) \\ &\le \frac{1}{s}(p - \frac{1}{\theta(s|x|)})F(t,sx) \\ &= \frac{1}{s}(p - \frac{1}{\theta(s|x|)})y(s) \end{aligned} \tag{2.4}$$

for all $s \ge M_1/|x|$. Set

$$h(s) := y'(s) - \frac{1}{s}(p - \frac{1}{\theta(s|x|)})y(s). \tag{2.5}$$

Obviously, $h(s) \le 0$ for all $s \ge \frac{M_1}{|x|}$. Solving the order linear ordinary differential equation (2.5), together with the fact $h(s) \le 0$, one derives

$$y(s) \le \frac{y(\frac{M_1}{|x|})}{M_1^p}|x|^p s^p G(s|x|), \quad \forall s \ge \frac{M_1}{|x|}.$$

Then, one has

$$F(t,x) = y(1) \le \frac{F(t, \frac{M_1 x}{|x|})}{M_1^p}|x|^p G(|x|), \quad \forall |x| \ge M_1. \tag{2.6}$$

Furthermore, one can deduce

$$F(t, \frac{M_1 x}{|x|}) \le M_2 \tag{2.7}$$

for all $x \in \mathbb{R}^N$ and $t \in \mathbb{Z}[1,T]$. Then via (2.6) and (2.7), one obtains

$$F(t,x) \le \frac{M_2}{M_1^p}|x|^p G(|x|) + M_2$$

for all $x \in \mathbb{R}^N$ and $t \in \mathbb{Z}[1,T]$. □

Remark 2.3. (1) Employing property (ii) of θ, one knows that $G(|x|) \to 0$ as $|x| \to +\infty$.

(2) The function $t^p G(t)$ is increasing in t since the range of $\frac{1}{\theta}$ and $(t^p G(t))' = t^{p-1}G(t)(p - \frac{1}{\theta(t)}) > 0$.

Proof of Theorem 1.2. The proof relies on Lemma 2.1 with $X = H_T$, $X_1 = \tilde{H}_T$, and $X_2 = \mathbb{R}^N$. Firstly, one proves that φ satisfies condition (C). Indeed, let $\{u_k\} \subset H_T$ be a sequence such that $\{\varphi(u_k)\}$ is bounded and

$$\|\varphi'(u_k)\|(1 + \|u_k\|) \to 0 \quad \text{as } k \to \infty.$$

Then there exists a constant $M_3 > 0$ for which

$$|\varphi(u_k)| \le M_3, \quad \|\varphi'(u_k)\|(1 + \|u_k\|) \le M_3.$$

Via (A4), a straightforward computation yields

$$-M_4 + (x, \nabla F(t,x)) \le (p - \frac{1}{\theta(|x|)})F(t,x)$$

for all $x \in \mathbb{R}^N$ and $t \in \mathbb{Z}[1,T]$. Here $M_4 > 0$. Thus, one has

$$\begin{aligned}
(p+1)M_3 &\ge \|\varphi'(u_k)\|(1 + \|u_k\|) - p\varphi(u_k) \\
&\ge \langle \varphi'(u_k), u_k \rangle - p\varphi(u_k) \\
&= \sum_{t=1}^{T} (pF(t, u_k(t)) - (\nabla F(t, u_k(t)), u_k(t))) \\
&\ge \sum_{t=1}^{T} \frac{F(t, u_k(t))}{\theta(|u_k|)} - M_4 T
\end{aligned}$$

for all $k \in \mathbb{N}$. Then it holds

$$\sum_{t=1}^{T} \frac{F(t, u_k(t))}{\theta(|u_k|)} \le M_5 \tag{2.8}$$

for all $k \in \mathbb{N}$. Here $M_5 = M_4 T + (p+1)M_3$. In addition, employing (2.3), (2.1) and (2) in Remark 2.3, one has

$$\begin{aligned}
M_3 \ge \varphi(u_k) &= \frac{1}{p}\sum_{t=1}^{T} |\Delta u_k(t)|^p - \sum_{t=1}^{T} F(t, u_k(t)) \\
&\ge \frac{1}{p}\sum_{t=1}^{T} |\Delta u_k(t)|^p - \sum_{t=1}^{T} \Big(\frac{M_2}{M_1^p}|u_k(t)|^p G(|u_k(t)|) + M_2\Big) \\
&\ge \frac{1}{p}\sum_{t=1}^{T} |\Delta u_k(t)|^p - \frac{M_2}{M_1^p}\sum_{t=1}^{T} \|u_k\|_\infty^p G(\|u_k\|_\infty) - M_2 T \\
&\ge \frac{1}{p}\sum_{t=1}^{T} |\Delta u_k(t)|^p - M_6\|u_k\|^p G(\|u_k\|) - M_2 T
\end{aligned} \tag{2.9}$$

for all $k \in \mathbb{N}$ and some $M_6 > 0$. Thus by (2.9), for all $k \in \mathbb{N}$, it holds:

$$\frac{M_3}{\|u_k\|^p} \ge \frac{\varphi(u_k)}{\|u_k\|^p} \ge \frac{1}{p}\sum_{t=1}^{T} \frac{|\Delta u_k(t)|^p}{\|u_k\|^p} - M_6 G(\|u_k\|) - \frac{M_2 T}{\|u_k\|^p}. \tag{2.10}$$

Then one claims that $\{u_k\}$ is bounded. Otherwise, there exists a subsequence of $\{u_k\}$, also denoted by $\{u_k\}$, such that

$$\|u_k\| \to \infty \quad \text{as } k \to +\infty. \tag{2.11}$$

Put $v_k = u_k/\|u_k\|$. Obviously, $\|v_k\| = 1$ and $\{v_k\}$ is bounded in the finite dimensional space H_T. Thus there exist a point $v \in H_T$ and a subsequence of $\{v_k\}$, say $\{v_k\}$, such that

$$v_k \to v \quad \text{in } H_T.$$

Then in light of (2.10), (2.11) and (2) of Remark 2.3, one deduces that

$$\sum_{t=1}^{T} |\Delta v_k|^p \to \sum_{t=1}^{T} |\Delta v|^p = 0 \quad \text{as } k \to +\infty. \tag{2.12}$$

This means $|\Delta v(t)| = 0$. Consequently, one has $|v(t)|$ is a constant for all $t \in \mathbb{Z}[1,T]$. Then one easily gets

$$T|v|^p = \sum_{t=1}^{T} |v|^p = \|v\|^p = 1.$$

Thus, it holds $|u_k(t)| \to +\infty$ as $k \to +\infty$ for $t \in \mathbb{Z}[1,T]$. Then via (A6), one deduces

$$\sum_{t=1}^{T} \frac{F(t,u_k(t))}{\theta(|u_k(t)|)} \to +\infty \quad \text{as } |u_k(t)| \to +\infty.$$

This is a contradiction to (2.8). Thus $\{u_k\}$ is bounded. In finite dimensional space H_T, $\{u_k\}$ has a convergent subsequence. Thus φ satisfies condition (C).

Secondly, one proves that φ satisfies (1) and (2) in Lemma 2.1. For $u \in \mathbb{R}^N$, since $0 < \frac{1}{\theta(t)} < p$, one obtains

$$\varphi(u) = -\sum_{t=1}^{T} F(t,u(t)) \le -\frac{1}{p}\sum_{t=1}^{T} \frac{F(t,u(t))}{\theta(|u(t)|)} \to -\infty \quad \text{as } |u| \to \infty.$$

Thus one concludes that $\varphi(u) \to -\infty$ as $\|u\| \to \infty$ in $\mathbb{R}^N$. Thus (1) in Lemma 2.1 is satisfied.

Then, in a similar way to (2.9), from (2.1), (2.2) and (2.3), for any $u \in \tilde{H}_T$, one gets

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\sum_{t=1}^{T} |\Delta u(t)|^p - \sum_{t=1}^{T} F(t,u(t)) \\
&\ge \frac{1}{p}\sum_{t=1}^{T} |\Delta u(t)|^p - \sum_{t=1}^{T} \Big(\frac{M_2}{M_1^p}|u(t)|^p G(|u(t)|) + M_2\Big) \\
&\ge \frac{1}{p}\frac{T^{p-1}}{(T-1)^{2p-1}} \sum_{t=1}^{T} |u(t)|^p - \frac{M_2}{M_1^p}\sum_{t=1}^{T} \|u\|_\infty^p G(\|u\|_\infty) - M_2 T \\
&\ge \frac{T^{p-1}}{p(T-1)^{2p-1}}\|u\|^p - M_6\|u\|^p G\left(\|u\|\right) - M_2 T \\
&= \Big\{\frac{T^{p-1}}{p(T-1)^{2p-1}} - M_6 G(\|u\|)\Big\}\|u\|^p - M_2 T.
\end{aligned} \tag{2.13}$$

By (2) in Remark 2.3, one obtains

$$G(\|u\|) \to 0 \quad \text{as } \|u\| \to +\infty.$$

Then it is easy to get

$$\frac{T^{p-1}}{p(T-1)^{2p-1}} - M_6 G(\|u\|) > 0 \quad \text{as } \|u\| \to +\infty.$$

Hence by (2.13), we get $\varphi(u) \to +\infty$ as $\|u\| \to +\infty$. Thus (2) in Lemma 2.1 holds. In light of Lemma 2.1, Theorem 1.2 is proved. □

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