Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 298, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

DYNAMICAL ANALYSIS OF ONE MACHINE TO INFINITE BUS POWER SYSTEMS UNDER GAUSS TYPE RANDOM EXCITATION

LEI LIU, PING JU, FENG WU

Communicated by Zhaosheng Feng

Abstract. We discuss the asymptotic behavior of the stochastic one machine to infinite bus power systems. Using the exponential martingale inequality and the Borel-Cantelli Lemma, we obtain asymptotic moment estimation and asymptotic pathwise estimation of the stochastic one machine to infinite bus systems. Using the ergodic properties, we give a good explanation of the fluctuation phenomena. By means of the property of periodicity, Hormander's theorem and a detailed balance method, the existence and probability density function of the stationary distribution on the cylindrical are illustrated.

1. Introduction

In the previous decades, dynamics of deterministic power systems has received a lot of attention, see, e.g. [5, 6, 17, 26, 28, 29], and references therein. By nature, a power system is continually experiencing stochastic disturbances, such as, switching events, load level fluctuations, which may have a significant effect on the operation of power systems and quality of electricity. In recent years, many researchers have been carried out to study the dynamics of power systems under random excitations, see, e.g., [8, 9, 10, 13, 21, 24, 27]. The random factors in power systems were classified into three categories in [30], namely, random initial values, random parameters, and random excitation. With the increase integration of the renewable energy generation system and electric vehicles, and the features of randomness into the power system, the dynamics of power system under random excitations has received more and more attentions (see [14, 20, 23]).

Since the Gauss white noise is a well-known mathematical interpretation for the random excitations, the application of Itô stochastic differential equations (SDEs) has been taken into account in the research on power systems. The detailed understanding of SDEs can be founded in [7, 15]. In this paper, we start our analysis by considering the stochastic one machine to infinite bus (OMIB) power system (see Figure 1(top)) perturbed with random excitations. Then the stochastic OMIB

2010 *Mathematics Subject Classification.* 34F05, 60H10, 93E03.
Key words and phrases. Power systems; Random excitation; stochastic fluctuation; stationary distribution; asymptotic pathwise estimation; asymptotic moment estimation.
©2017 Texas State University.
Submitted October 8, 2016. Published November 30, 2017.

system can be described by the Itô equation as follows:

$$\begin{aligned} \mathrm{d}\delta &= (\omega - 1)\mathrm{d}t \\ \mathrm{d}\omega &= \big(-\frac{D}{M}(\omega - 1) + \frac{1}{M}(P_M - P_e)\big)\mathrm{d}t + \sigma \mathrm{d}B(t), \end{aligned} \tag{1.1}$$

where δ is the rotor angle; ω is the rotor rotating speed; $P_e = P_{\max} \sin\delta$ is the electrical power; P_M is the mechanical power and is assumed to be constant, i.e. $P_M = P_{\max} \sin \delta_s$; M is the inertia constant in per unit and taken $M = 2569.8288$.

It is necessary to reveal how the noise affects the stochastic OMIB systems (1.1). The mean stability and mean square stability of the corresponding linear systems of (1.1) were analyzed theoretically by Zhang et al [30]. Using the stochastic averaging methods, [3] has studied the first-passage problem of dynamical power system of (1.1). Recently, Keyou Wang [25] has applied the Fokker-Planck Equation to study the evolution of the probability density function of the system (1.1).

However, the asymptotic properties of stochastic OMIB systems (1.1) has not been fully investigated. In addition, if we make a great number of records of ω to investigate the dynamic behavior of a stochastic OMIB power system (1.1), it can be found that a single record may fluctuate even if the number of records is large. Then two questions arise naturally: (1) Can the fluctuation phenomena be given a explanation? (2)Is there a stationary distribution to the system? As a result, to solve these two problems is the main motivation of this paper. The primary contributions of this paper are as follows:

- The p-th moment estimation and asymptotic pathwise of ω were obtained using the exponential martingale inequality and Gronwall's inequality;
- Utilizing the ergodic property and the comparative method, a good explanation of the fluctuation phenomena is given;
- With the help of the Hormanders theorem and detailed balance method, the existence of the stationary distribution on cylindrical is proved.

2. Notation and problem statement

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t)$ be an one-dimensional Brownian motion defined on the probability space.

In the same way as Mao et al. [15] did, we can show the following result on the existence of a global solution.

Lemma 2.1. *For any given initial data $x_0 \in R^2$, there is a unique global solution $\big(\delta(t, x_0), \omega(t, x_0)\big)$ of system (1.1) on $[0, +\infty)$.*

The dynamics of the deterministic classical OMIB system has been extensively studied; see [2, 12]. These existing literature show clearly that the point $(\delta_s, 1)$ is a stable equilibrium point (SEP). The trajectory will converge to the SEP if the initial conditions are selected to lie in the attraction region of the SEP. Applying a random excitation, the dynamic behavior becomes more complicated than the deterministic classical OMIB system. The simulations show that a single record of $\omega(t)$ may fluctuate even if the number of records is large(see Figure 1(bottom)).

By means of the stochastic analysis techniques and the ergodic properties, some estimation and a good explanation of the fluctuation phenomena will be provided in Section III and Section IV, respectively.

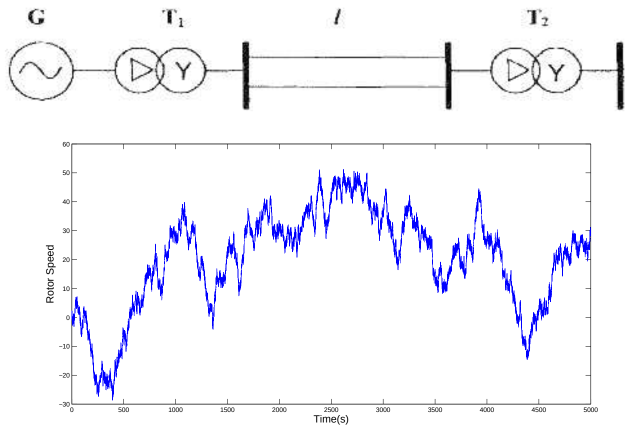

FIGURE 1. (top) One machine infinite bus system. (bottom) Stochastic trajectory of $\omega(t)$ generated by the Heun scheme for time step $h = 2^{-8}$ for (1.1) with $\sigma = 0.6$ on $[0, 5000]$.

3. Estimation on the stochastic fluctuation

The aim of this section is to estimate the stochastic fluctuation for the rotor speed ω.

3.1. **Moment estimation.**

Theorem 3.1. *For any $p > 0$ and initial data $x_0 = (\delta_s, 1)$, there exists a constant $M_p > 0$ such that the global solution $\omega(t)$ of system (1.1) has the property*

$$\sup_{0\le t<\infty} E|\omega(t, x_0) - 1|^p \le M_p.$$

Proof. For any $p > 0$, let $V(\omega, t) = e^{\varepsilon t}|\omega - 1|^p = e^{\varepsilon t}\big((\omega - 1)^2\big)^{\frac{p}{2}}$, where ε is a sufficient small positive number. Let $\omega(t) = \omega(t; x_0)$ for simplicity. Applying Itô formula and Young inequality (see Mao [15]) to $V(\omega, t)$ yields

$$\begin{aligned}
&\mathscr{L}|\omega - 1|^p \\
&= e^{\varepsilon t}\frac{p}{2}|\omega - 1|^{p-2}\big[\big(-\frac{2D}{M} + \frac{2}{p}\varepsilon\big)(\omega - 1)^2 \\
&\quad + \big(\frac{2P_{\max}}{M}(\sin\delta_s - \sin\delta) + 2\big)(\omega - 1) + (p-1)\sigma^2\big] \\
&\le e^{\varepsilon t}\frac{p}{2}|\omega - 1|^{p-2}\big[\big(-\frac{2D}{M} + \frac{2}{p}\varepsilon\big)(\omega - 1)^2 + \big(\frac{4P_{\max}}{M} + 2\big)|\omega - 1| + (p-1)\sigma^2\big] \\
&= e^{\varepsilon t}\frac{p}{2}\big[\big(-\frac{2D}{M} + \frac{2}{p}\varepsilon\big)|\omega - 1|^p + \big(\frac{4P_{\max}}{M} + 2\big)|\omega - 1|^{p-1} + (p-1)\sigma^2|\omega - 1|^{p-2}\big].
\end{aligned}$$

Note that $(-\frac{2D}{M} + \frac{2}{p}\varepsilon) < 0$. Then there exists a constant K_P such that

$$\sup_{0\le u\le\infty}\frac{p}{2}[(-\frac{2D}{M} + \frac{2}{p}\varepsilon)u^p + \frac{4P_{\max}}{M}u^{p-1} + (p-1)\sigma^2 u^{p-2}] \le K_p.$$

This implies

$$Ee^{\varepsilon t}|\omega(t)-1|^p \le E\big((\omega-1)^2(0)\big)^{\frac{p}{2}} + \int_0^t Ke^{\varepsilon s}ds \le E|\omega(0)-1|^p + \frac{K_p}{\varepsilon}(e^{\varepsilon t}-1),$$

which means

$$E|\omega(t)-1|^p \le e^{-\varepsilon t}E|\omega(0)-1|^p + \frac{K_p}{\varepsilon}(1-e^{\varepsilon - t}) \le E|\omega(0)-1|^p + \frac{K_p}{\varepsilon} := M_p$$

We can claim that for any $p>0$ there exists a $M_p>0$ such that

$$\sup_{0\le t<\infty} E|\omega(t)-1|^p \le M_p.$$

The proof is complete. □

3.2. **Asymptotic pathwise estimation.** Theorem 3.1 shows that the p-th moment of $|\omega(t)-1|$ is boundedness. Now we process to derive some asymptotic pathwise estimation of $|\omega(t)-1|$ by applying the exponential martingale inequality and Borel-Cantelli Lemma (see [15]), which shows the solution of (1.1) how to vary in R^2 pathwisely.

Theorem 3.2. *For any $p>0$ and initial data $x_0=(\delta_s,1)$, the global solution $\omega(t,x_0)$ of system* (1.1) *has the property*

$$\lim_{t\to\infty}\sup \frac{1}{t}\int_0^t \big(\omega(s,X_0)-1\big)^2 ds < \Big(\sqrt{\frac{\sigma^2}{D^2}+\frac{2P_{\max}}{D}}+\frac{2P_{\max}}{D}\Big)^2. \tag{3.1}$$

Proof. Let $\omega(t)=\omega(t;x_0)$ for simplicity. Applying Itô's formula to ω^2 yields

$$\begin{aligned}
&\big(\omega(t)-1\big)^2\\
&= \big(\omega(0)-1\big)^2 + \int_0^t \Big(-\frac{2D}{M}\big(\omega(s)-1\big)^2\\
&\quad + \big(\frac{2P_{\max}}{M}(\sin\delta_s - \sin\delta(s)) + \frac{2D}{M}\big)(\omega(s)-1)+\sigma^2\Big)ds + M(t).
\end{aligned} \tag{3.2}$$

where $M(t)=\int_0^t 2(\omega(s))\sigma dB(s)$ is a continuous local martingale. The quadratic variation of $M(t)$ is $\langle M(t)\rangle = \int_0^t 4\sigma^2\big(\omega(t)-1\big)^2 d(s)$. For any $\alpha\in(0,\frac{D}{M\sigma^2})$ and every integer $k\ge 1$, using the exponential martingale inequality (cf. Mao [15, Theorem 1.7.4]) we have

$$\mathbb{P}\Big(\sup_{0\le t\le T}(M(t)-\frac{\alpha}{2}\langle M(t)\rangle) > \frac{1}{\alpha}\log k\Big) \le \frac{1}{k^2}, \quad k=1,2,\dots,$$

An application of the well-known Borel-Cantelli lemma then yields that for almost all $\varpi\in\Omega$ there is a random integer $k_0=k_0(\varpi)\ge 1$ such that

$$M(t) \le \frac{2}{\alpha}\log k + 0.5\alpha\langle M(t)\rangle, \tag{3.3}$$

for $t\in[0,k], k\ge k_0$, almost surely. Substituting (3.3) into (3.2), we obtain that

$$\begin{aligned}
&\big(\omega(t)-1\big)^2\\
&= \big(\omega(0)-1\big)^2 + \int_0^t \Big(-\frac{2D}{M}\big(\omega(s)-1\big)^2 + \big(\frac{2P_{\max}}{M}(\sin\delta_s - \sin\delta(s))\\
&\quad + \frac{2D}{M}\big)(\omega(s)-1)+\sigma^2\Big)ds + \frac{2}{\alpha}\log k + 0.5\alpha\int_0^t 4\sigma^2\big(\omega(s)-1\big)^2 ds
\end{aligned}$$

$$\begin{aligned}&\le \omega(0) + \int_0^t \Big((-\frac{2D}{M} + 2\alpha\sigma^2)\big(\omega(s)-1\big)^2 + \big(\frac{4P_{\max}}{M}\sin\delta_s + \frac{2D}{M}\big)|\omega(s)-1| \\ &\quad + \sigma^2\Big) ds + \frac{2}{\alpha}\log k,\end{aligned}$$

for $t \in [0,k], k \ge k_0$, almost surely. Setting $h(\alpha) = \frac{2D}{M} - 2\alpha\sigma^2$. For any $\eta \in (0,1)$, we have

$$\begin{aligned}\eta h(\alpha)\int_0^t \omega^2(s)ds &\le \omega^2(0) + \int_0^t \Big((1-\eta)h(\alpha)\omega^2(s) \\ &\quad + \big(\frac{4P_{\max}}{M}\sin\delta_0 + \frac{2D}{M}\big)|\omega(s)| + \sigma^2\Big) ds + \frac{2}{\alpha}\log k.\end{aligned}$$

For almost all $\varpi \in \Omega$, if $k \ge k_0$ and $k-1 \le t \le k$, simple computations show that

$$\frac{1}{t}\int_0^t \omega^2(s)ds \le \frac{1}{\eta h(\alpha)(k-1)}\big(\omega^2(0) + (4\frac{P_m^2}{(1-\eta)h(\alpha)M^2} + \sigma^2)k + \frac{2}{\alpha}\log k\big).$$

Letting $t \to \infty$ ($k \to \infty$) and $\alpha \to 0$ yields

$$\frac{1}{t}\int_0^t \omega^2(s)ds \le \frac{P_{\max}^2}{\eta(1-\eta)D^2} + \frac{M\sigma^2}{2D\eta}.$$

Setting $\eta = 1/(1 + P_{\max}/\sqrt{2P_m^2 + \sigma^2 MD})$ yields

$$\lim_{t\to\infty}\sup \frac{1}{t}\int_0^t \big(\omega(s)-1\big)^2 ds < \Big(\sqrt{\frac{P_{\max}^2}{D^2} + \frac{\sigma^2 M}{2D}} + \frac{P_{\max}M}{D}\Big)^2.$$

The proof is complete. □

Theorem 3.3. *For initial data $x_0 = (\delta_s, 1)$, the global solution $\omega(t;x_0)$ of system* (1.1) *has the property*

$$\limsup_{t\to\infty} \frac{|\omega(t,x_0)-1|}{\sqrt{\log t}} \le \sqrt{\frac{Me}{D}}\sigma.$$

Proof. Let $\omega(t) = \omega(t;x_0)$ for simplicity. Simple computation shows that

$$\begin{aligned}(\omega-1)\big(\big(-\frac{D}{M}(\omega-1) + \frac{1}{M}(P_M - P_e)\big)\big) &\le \frac{2P_{\max}}{M}|(\omega-1)| - \frac{D}{M}(\omega-1)^2 \\ &\le -(\frac{D}{M} - \frac{P_{\max}}{M}\epsilon)(\omega-1)^2 + \frac{P_{\max}}{M\epsilon}.\end{aligned}$$

Set

$$\lambda(\epsilon) = \frac{D}{M} - \frac{P_{\max}}{M}\epsilon, \quad b = \frac{P_{\max}}{M\epsilon}.$$

Applying Itô formula and Young inequality (see Mao [15]) to $e^{-2\lambda(\epsilon)t}(\omega-1)^2$ yields

$$\begin{aligned}e^{-2\lambda(\epsilon)t}(\omega(t)-1)^2 &= (\omega-1)^2(0) + \int_0^t e^{-2\lambda(\epsilon)s}(2b+\sigma^2)ds \\ &\quad + 2\sigma\int_0^t e^{-2\lambda(\epsilon)s}(\omega-1)(s)dB(s),\end{aligned} \tag{3.4}$$

where $M(t) = 2\sigma\int_0^t e^{-2\lambda(\epsilon)s}(\omega-1)(s)dB(s)$ is a continuous local martingale with the quadratic variation

$$\langle M(t)\rangle = 4\sigma^2\int_0^t e^{-4\lambda(\epsilon)s}|(\omega-1)(s)|^2 \mathrm{d}s.$$

For each integer $n > 0$, $\varepsilon > 0$, $\gamma > 0$, $\theta > 1$, the exponential martingale inequality (see [23, Theorem 7.4 on page44]) yields

$$\mathbb{P}\big(\sup_{0\le t\le t_{n+1}}(M(t)-\frac{\gamma_n}{2}\langle M(t)\rangle) > \frac{1}{\gamma_n}\log n^\theta\big) \le \frac{\log n^\theta}{\gamma_n}, \quad n=1,2,\dots,$$

where $t_n = n\varepsilon, \gamma_n = \gamma e^{2n\lambda(\epsilon)\varepsilon}$. Noting that $\sum_{n=1}^\infty \frac{1}{n^\theta} < \infty$, by the well-known Borel-Cantelli lemma, there exists $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$, such that for any $\omega \in \Omega_0$, there exists an integer $N(\omega)$ such that for all $n \ge N(\omega)$ and $0 \le t \le t_{n+1}$

$$M(t) \le \frac{\gamma_n}{2}\langle M(t)\rangle + \frac{1}{\gamma_n}\log n^\theta. \tag{3.5}$$

Setting $h(t) = e^{-2\lambda(\epsilon)t}(\omega(t)-1)^2$, Substituting (3.5) into (3.4), with probability one, gives,

$$\begin{aligned}
&h(t)-h(0)\\
&\le \frac{2b+\sigma^2}{2|\lambda(\epsilon)|}e^{-2\lambda(\epsilon)t} + 2\sigma^2\gamma_n\int_0^t e^{-2\lambda(\epsilon)s}h(s)ds + \frac{\log n^\theta}{\gamma_n}\\
&\le \frac{2b+\sigma^2}{2|\lambda(\epsilon)|}e^{-2\lambda(\epsilon)(n+1)\varepsilon} + \frac{\log n^\theta}{\gamma}e^{-2\lambda(\epsilon)n\varepsilon} + 2\sigma^2\gamma e^{2n\lambda(\epsilon)\varepsilon}\int_0^t e^{-2\lambda(\epsilon)s}h(s)ds.
\end{aligned}$$

Then Gronwall's inequality implies

$$h(t) \le (h(0) + \frac{2b+\sigma^2}{2|\lambda(\epsilon)|}e^{-2\lambda(\epsilon)(n+1)\varepsilon}\frac{\log n^\theta}{\gamma}e^{-2\lambda(\epsilon)n\varepsilon})\exp(\frac{\sigma^2}{|\lambda(\epsilon)|}\gamma e^{2n\lambda(\epsilon)\varepsilon}e^{-2\lambda(\epsilon)t}).$$

Consequently, for almost all $\omega \in \Omega_0$, if $k \ge N$ and $n\varepsilon \le t \le (n+1)\varepsilon$,

$$\begin{aligned}
&(\omega(t)-1)^2\\
&\le \Big((\omega(0)-1)^2e^{-2\lambda(\epsilon)t} + \frac{\log n^\theta}{\gamma}e^{-2\lambda(\epsilon)(n\varepsilon-t)} + \frac{2b+\sigma^2}{2|\lambda(\epsilon)|}e^{-2\lambda(\epsilon)((n+1)\varepsilon-t)}\Big)\\
&\quad\times\exp(\frac{\sigma^2}{|\lambda(\epsilon)|}\gamma e^{2n\lambda(\epsilon)\varepsilon}e^{-2\lambda(\epsilon)t})\\
&\le \Big((\omega(0)-1)^2e^{-2\lambda(\epsilon)t} + \frac{2b+\sigma^2}{2|\lambda(\epsilon)|}e^{-2\lambda(\epsilon)((n+1)\varepsilon-t)} + \frac{\theta(\log t-\log\varepsilon)}{\gamma}e^{-2\lambda(\epsilon)(n\varepsilon-t)}\Big)\\
&\quad\times\exp(\frac{\sigma^2}{|\lambda(\epsilon)|}\gamma e^{2n\lambda(\epsilon)\varepsilon}e^{-2\lambda(\epsilon)t}).
\end{aligned}$$

We therefore have

$$\begin{aligned}
\frac{(\omega(t)-1)^2}{\log t} &\le ((\omega(0)-1)^2e^{-2\lambda(\epsilon)t} + \frac{2b+\sigma^2}{2|\lambda(\epsilon)|}e^{-2\lambda(\epsilon)((n+1)\varepsilon-t)}\\
&\quad + \frac{\theta(\log t-\log\varepsilon)}{\log t\gamma}e^{-2\lambda(\epsilon)(n\varepsilon-t)})\exp(\frac{\sigma^2}{|\lambda(\epsilon)|}\gamma e^{-2\lambda(\epsilon)(t-n\varepsilon)}).
\end{aligned}$$

Letting $t\to\infty$ (forcing $k\to\infty$) and $\epsilon\to 0$ yields

$$\limsup_{t\to\infty}\frac{(\omega(t)-1)^2}{\log t} \le \frac{\theta}{\gamma}\exp(\frac{\sigma^2\gamma M}{D}e^{-2\frac{D}{M}\varepsilon}).$$

Letting $\theta\to 1$, $\varepsilon\to 0$, and then setting $\gamma = \frac{D}{M\sigma^2}$ yields

$$\limsup_{t\to\infty}\frac{|\omega(t)|}{\sqrt{\log t}} \le \sqrt{\frac{Me}{D}}\sigma.$$

The proof is complete. $\square$

3.3. **Numerical examples.** An stochastic OMIB power system is used in this paper to study power system dynamics under random excitation. In the OMIB, the transformer reactance $x_{T_1} = 0.138$ pu and $x_{T_2} = 0.122$ pu; the line reactance $x_1 = 0.234$ pu; the generator transient reactance $x'_d = 0.295$ pu; the inertia constant $M = 2569.8288$ pu; and the damping coefficient $D = 2.0$ pu. The initial system power $P_0 = 1.0$ pu, $Q_0 = 0.2$ pu, voltages behind the reactance $E' = 1.41$ pu and rotor angle $\delta_s = 34.46$. Per unit system: $S_B = 220$ MVA, $U_{B(220)} = 209$ kV.

To conform the analytical results above, we use Heun method (see [11]) to simulate the solutions of system (1.1) with given initial value and the parameters. For a certain positive integer K, we have $h = T/K$, $\delta_0 = \delta(0)$, $\omega_0 = \omega(0)$, $\Delta B_i = B((i+1)h) - B(ih) \sim \sqrt{h}N(0,1)$. The corresponding discretization equations are

$$
\begin{gathered}
\hat{\delta} = \delta_i + h(\omega_i - 1), \\
\hat{\omega} = \omega_i + h\big(-\frac{D}{M}(\omega_i - 1) + \frac{P_{\max}}{M}(\sin(\delta_s) - \sin(\delta_i))\big) + \sigma\Delta B_i, \\
\delta_{i+1} = \delta_i + 0.5h(\omega_i + \hat{\omega}), \\
\begin{aligned}
\omega_{i+1} = \omega_i + 0.5h\Big(\big(-\frac{D}{M}(\omega_i - 1) + \frac{P_{\max}}{M}(\sin(\delta_s) - \sin(\delta_i))\big) \\
+ (-\frac{D}{M}(\hat{\omega} - 1) + \frac{P_{\max}}{M}\big(\sin(\delta_s) - \sin(\hat{\delta}))\big)\Big) + \sigma\Delta B_i, \quad i = 0, \cdots, K.
\end{aligned}
\end{gathered}
$$

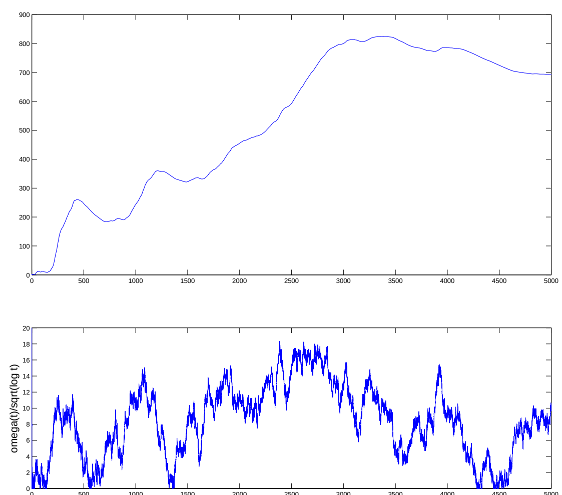

FIGURE 2. (top) Stochastic trajectory of $\frac{1}{t}\int_0^t \omega^2(s)ds$ generated by the Heun scheme for time step $h = 2^{-8}$ for (1.1) with $\sigma = 0.6$ on $[0, 5000]$. (bottom) Stochastic trajectory of $\frac{|\omega(t)|}{\sqrt{\log t}}$ generated by the Heun scheme for time step $h = 2^{-8}$ for (1.1) with $\sigma = 0.6$ on $[0, 5000]$.

4. Stochastic fluctuation

The main aim of this section is to illustrate the fluctuation phenomena. At this end, we consider two auxiliary stochastic differential equations

$$\begin{gathered} \mathrm{d}y_1 = (-\frac{D}{M}(y_1 - 1) + \frac{2P_{\max}}{M}\sin\delta_s \mathrm{d}t + \sigma \mathrm{d}B(t), \\ y_1(0) = \omega(0). \end{gathered} \tag{4.1}$$

and

$$\begin{gathered} \mathrm{d}y_2 = (-\frac{D}{M}(y_2 - 1) - \frac{2P_{\max}}{M}\sin\delta_s \mathrm{d}t + \sigma \mathrm{d}B(t), \\ y_1(0) = \omega(0). \end{gathered} \tag{4.2}$$

Itô's formula implies

$$\begin{gathered} \omega(t) = e^{-\frac{D}{M}t}\big\{\omega(0) + \int_0^t \big[e^{\frac{D}{M}s}\big(\frac{P_{\max}}{M}(\sin\delta_s - \sin\delta)\big) + \frac{D}{M}\big]ds + \int_0^t e^{\frac{D}{M}s}\sigma dB(s)\big\}, \\ y_1(t) = e^{-\frac{D}{M}t}\big\{\omega(0) + \int_0^t \big[e^{\frac{D}{M}s}\big(\frac{2P_{\max}}{M}\sin\delta_s + \frac{D}{M}\big]ds + \int_0^t e^{\frac{D}{M}s}\sigma dB(s)\big\}, \\ y_2(t) = e^{-\frac{D}{M}t}\big\{\omega(0) + \int_0^t e^{\frac{D}{M}s}\big(\frac{D}{M} - \frac{2P_{\max}}{M}\sin\delta_s\big)ds + \int_0^t e^{\frac{D}{M}s}\sigma dB(s)\big\}. \end{gathered}$$

We therefore have that

$$y_2(t) \le \omega(t) \le y_1(t). \tag{4.3}$$

Next we have a well known lemma (see Hasminskii [7, pp. 106-125]). Let $x(t)$ be a homogeneous Markov process in R^n described by the following stochastic differential equation:

$$dx(t) = f(x)dt + g(x)dB(t). \tag{4.4}$$

the drift coefficients and diffusion coefficients of system (4.4) are $a(x) = f(x), \sigma(x) = g(x)g^T(x)$ respectively.

Lemma 4.1 ([7])**.** *We assume that there is a bounded open subset $G \subset R$:*

(i) in the domain G and some neighborhood therefore, the smallest diffusion matrix $g^2(x)$ is bounded away from zero.

(ii) If $x \in R\backslash G$, the mean time τ at which a path issuing from x reaches the set G is finite, and $\sup_{x\in K\backslash G} E_x\tau < +\infty$ for every compact subset $K \in R$.and throughout this paper we set $\inf\emptyset = \infty$. Then the Markov process $x(t)$ of system(4.4) is ergodic and positive recurrent.

Lemma 4.2. *The solution process $(y_1(t))$ of system (4.1) and $(y_2(t))$ of system (4.2) are ergodic and positive recurrent.*

Proof. Without loss of generality, we only provide the proof for system (4.1), the proof for system (4.2) is similar. To validate condition (i) and (ii), it suffices to prove that there exists some neighborhood U and a nonnegative C^2-function V such that σ^2 is uniformly elliptical in U and $\mathscr{L}V \le -1$ for any $x \in R \setminus U$, (details refer to [16] p. 400). Applying Itô's formula to $V(y_1) = y_1^2$ we have

$$\mathscr{L}V(y_1) = -\frac{2D}{M}y_1^2 + (\frac{4P_{\max}}{M}\sin\delta_s + 2\frac{D}{M})y_1 + \sigma^2.$$

Note from $\frac{2D}{M} > 0$, that there is a sufficiently large N, such that

$$\mathscr{L}V(y_1) \le -1, \quad \forall |y_1| > N; \quad \inf_{|y_1|>N} \lambda_{min}(\sigma^2) = \sigma^2 > 0.$$

This immediately implies condition (i) and (ii) in Lemma 4.1. □

Now, we use the recurrence of $(y_1(t))$ and $(y_2(t))$ to explain such fluctuation phenomena. For fixed α_1 and α_2 such that $\alpha_1 > \alpha_2$, Let α_1, α_2 denote the higher and lower levels respectively. Set $\tau_0^i = \inf\{t \geq 0 : y_i(t) \geq \alpha_1\}, \tau_1^i = \inf\{t \geq \tau_0^i : y_i(t) \leq \alpha_2\}, i = 1, 2.$ and $\tau_0 = \inf\{t \geq 0 : \omega(t) \geq \alpha_1\}, \tau_1 = \inf\{t \geq \tau_0 : \omega(t) \leq \alpha_2\}$. For $k \geq 1, i = 1, 2, \dots,$ we define the following sequence of stopping times recursively

$$\begin{aligned} \tau_{2k}^i &= \inf\{t \geq \tau_{2k-1}^i : y_i(t) \leq \alpha_2\}, \quad \tau_{2k+1}^i = \inf\{t \geq \tau_{2k}^i : y_i(t) \geq \alpha_1\}, \\ \tau_{2k} &= \inf\{t \geq \sigma_{2k-1}^i : \omega(t) \leq \alpha_2\}, \quad \tau_{2k+1}^i = \inf\{t \geq \sigma_{2k}^i : \omega(t) \geq \alpha_1\}. \end{aligned}$$

For $i = 1, 2$, the strong Markov property and Lemma 4.2 imply

$$\tau_k^i < \infty, \quad k \geq 0 \quad a.s.$$

For each $k > 0$, it therefore follows from (4.3) that

$$\begin{aligned} &\omega(\tau_{2k}^1) \leq y_1(\tau_{2k}^1) = \alpha_1, \omega(\tau_{2k+1}^1) \leq y_1(\tau_{2k+1}^1) = \alpha_1, \\ &\omega(\tau_{2k}^2) \geq y_2(\tau_{2k}^2) = \alpha_2, \omega(\tau_{2k+1}^2) \geq y_2(\tau_{2k+1}^2) = \alpha_2. \end{aligned}$$

This implies

$$\tau_{2k}^1 \leq \tau_{2k} \leq \tau_{2k}^2 < \infty, \tau_{2k+1}^2 \leq \tau_{2k+1} \leq \tau_{2k+1}^1 < \infty, k \geq 0.$$

This means that the higher and lower levels of $\omega(t)$ occurs, which coincides with the recurrent phenomena in practice.

5. Discussion on stationary distribution

The main aim of this section is to discuss the existence of a stationary distribution of the system (1.1). Letting $x = [x_1, x_2] = [\delta, \omega - 1]$, the stochastic OMIB system (1.1) can be rewritten as the following vector form

$$dx = f(x)dt + g(x)dB(t), \tag{5.1}$$

where

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-D}{M}(x_2) + \frac{P_{\max}}{M}(\sin x_1 - \sin \delta_s) \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ \sigma \end{bmatrix}, \tag{5.2}$$

and the drift coefficients and diffusion coefficients of system (5.1) are

$$a(x) = \begin{bmatrix} x_2 \\ \frac{-D}{M}(x_2) + \frac{P_{\max}}{M}(\sin x_1 - \sin \delta_s) \end{bmatrix}, \quad \sigma(x) = g(x)g^T(x) = \begin{bmatrix} 0 & 0 \\ 0 & \sigma^2 \end{bmatrix}. \tag{5.3}$$

Let $P(t, x_0, A)$ denote the probability measure induced by $x(t, x_0)$; that is,

$$P(t, x_0, A) = \mathbb{P}(x(t, x_0) \in A), \quad A \in \mathscr{B}(E^n),$$

where $\mathscr{B}(E^n)$ is the σ-algebra of all the Borel sets $A \subset E^n$. Now we will show that the solution process $x(t)$ has a transition density function $p(t, x_0, y)$.

Lemma 5.1. *The transition probability measure $P(t, x_0, A)$ of system* (5.1) *has a density $p(t, x_0, y) \in C^\infty((0, \infty) \times R^2 \times R^2)$.*

Proof. Let us now introduce the notation of Lie bracket. If $a(x)$ and $b(x)$ are vector fields on R^d, then the Lie bracket $[a,b]$ is a vector field given by

$$[a,b]_j(x)\sum_{k=1}^{d}(a_k\frac{\partial b_j}{\partial x_k}(x)-b_k\frac{\partial a_j}{\partial x_k}(x)), j=1,2,\cdots,d.$$

For SMIB system, simple computation shows that $[f,g](x)=[-\sigma,\frac{\sigma D}{M}]^T$. Consequently

$$|[f,g](x),g(x)|=\begin{vmatrix}-\sigma & 0\\ \frac{\sigma D}{M} & \sigma\end{vmatrix}=-\sigma^2\neq 0, \tag{5.4}$$

which means that $[f,g],g$ are linearly independent on R^2. Thus for every $(x_1,x_2)\in R^2$, vector $[f,g],g$ span the space R^2. In view of Hormander's Theorem in [1], the transition probability measure $P(t,x_0,A)$ has a density $p(t,x_0,y)\in C^\infty((0,\infty)\times R^2\times R^2)$. The proof is complete. □

For the stochastic OMIB system (5.1), the density function $p(t,x_0,y)$ satisfies the following Fokker-Planck equation (FPE):

$$\frac{\partial p(t,x_0,y)}{\partial t}=-\frac{\partial f_1(y)p(t,x_0,y)}{\partial y_1}-\frac{\partial f_2(y)p(t,x_0,y)}{\partial y_2}+\frac{\sigma^2}{2}\frac{\partial^2 p(t,x_0,y)}{\partial y_2^2} \tag{5.5}$$

The FPE (5.5) is a two-dimensional parabolic partial differential equation, some specified initial conditions and boundary conditions are provided by Wang in [25]. If the process $x(t,x_0)$ of (5.1) has a stationary distribution, then its probability density function $p_s(y)$ satisfies the following stationary Fokker-Planck equation

$$\frac{\partial f_1(y)p_s(y)}{\partial y_1}+\frac{\partial f_2(y)p_s(y)}{\partial y_2}=\frac{\sigma^2}{2}\frac{\partial^2 p_s(y)}{\partial y_2^2}. \tag{5.6}$$

Now we try to solve the solve the stationary FPE by the detailed balance method, which was first applied by van Kampen in [22]. This method classify the components of the response vector as either odd or even variables, according to the transformation from x_j to $\bar{x}_j$ upon the time reversal $t\to -t$. The even variables do not change their signs when time is reversed, whereas the odd variables do. More detailed information, we refer the reader to [22] and [4].

Lemma 5.2. *The stationary Fokker-Planck equation* (4.1) *has a solution as following.*

$$p_s(y_1,y_2)=C\exp\Big(-\frac{2D}{M\sigma^2}(\frac{y_2^2}{2}-\frac{P_{\max}}{M}\sin\delta y_1-\frac{P_{\max}}{M}\cos y_1)\Big) \tag{5.7}$$

Proof. For the second-order differential equations, Risken[18] showed that the first variable is always even and the second variable is odd. To obtain conditions of detailed balance we separate each drift coefficient $a_j(y)$ into reversible and irreversible parts. $a_j(y)=a_j^I(y)+a_j^R(y)$, where

$$a_j^I(y)=\frac{1}{2}(a_j(y)+\Delta_j a_j(\bar{y})), a_j^R(y)=\frac{1}{2}(a_j(y)-\Delta_j a_j(\bar{y})),\quad j=1,2.$$

where $\bar{y}_j=\Delta_j y_j$, $\Delta_j=\pm 1$ corresponds to even $(+)$ or odd $(-)$ variables. Since $p_s(y_1,y_2)$ is nonnegative, it can be expressed in the form

$$p_s(y_1,y_2)=C\exp(-\varphi(y_1,y_2)).$$

Then according to [4] equations to determine φ are

$$\begin{gathered}\sum_j 2a_j^I(y) - \sum_{j,k}\frac{\partial\sigma_{j,k}}{\partial y_k} + \sum_{j,k}\sigma_{j,k}\frac{\partial\varphi}{\partial y_k} = 0,\\ \sum_j \frac{\partial a_j^R(y)}{\partial y_j} + \sum_j a_j^R(y)\frac{\partial\varphi}{\partial y_k} = 0,\\ \sum_{j,k}[\sigma_{jk}(y) - \Delta_j\Delta_k\sigma_{jk}(\bar{y})] = 0.\end{gathered}\tag{5.8}$$

It is easy to show that the reversible and irreversible parts of each drift coefficient of system (5.1) are

$$a_1^I(y) = 0, a_1^R(y) = y_2, \quad a_2^I(y) = -\frac{D}{M}y_2, \quad a_2^R(y) = \frac{P_{\max}}{M}(\sin\delta - \sin y_1)$$

And the corresponding equation (5.8) of stochastic OMIB (5.1) is given by

$$\frac{\partial\varphi}{\partial y_2} = \frac{2D}{M\sigma^2}y_2; \frac{\partial\varphi}{\partial y_1} = \frac{2DP_{\max}}{M^2\sigma^2}(\sin y_1 - \sin\delta).\tag{5.9}$$

Simple computations show that

$$p_s(y_1, y_2) = C\exp\big(-\frac{2D}{M\sigma^2}(\frac{y_2^2}{2} - \frac{P_{\max}}{M}\sin\delta y_1 - \frac{P_{\max}}{M}\cos y_1)\big).\tag{5.10}$$

where C is a constant. The proof is complete. $\square$

Obviously, the solution $p_s(y_1, y_2)$ is periodic in y_1, which implies that the integral $\int_{R^2} p_s(y_1, y_2)dy_1dy_2$ can not be convergent. As a result, the solution can not be viewed as a density function on R^2. On the other hand, the periodic might provide another perspective on this topic. Now we state a lemma to indicate the periodic of the solution process.

Lemma 5.3. *Let $x(t, x_0) = (x_1(t, x_0), x_2(t, x_0))$ be the global solution of system (5.1) with initial data $x_0 = (x_{01}, x_{02})$. Setting $x_0' = (x_{01} + 2\pi, x_{02})$, for any $t > 0$, we can claim that*

$$x_1(t, x_0') - x_1(t, x_0) = 2\pi, \quad x_2(t, x_0') - x_2(t, x_0) = 0.$$

Proof. It follows from the periodicity of $f(x)$ with x_1 that the process $(x_1(t, x_0) + 2\pi, x_2(t, x_0))$ is still the solution of system (5.1).

$$\begin{aligned} x_1(t, x_0') &= x_{01}' + \int_0^t x_2(s, x_0')ds,\\ x_2(t, x_0') &= x_{02}' + \int_0^t \Big(\frac{-D}{M}x_2(s, x_0') + \frac{P_{\max}}{M}(\sin x_1(s, x_0') - \sin\delta)\Big)ds \\ &\quad + \int_0^t \sigma dB(s),\end{aligned}\tag{5.11}$$

$$\begin{aligned} x_1(t, x_0) + 2\pi &= x_{01} + 2\pi + \int_0^t x_2(s, x_0)ds,\\ x_2(t, x_0) &= x_{02} + \int_0^t \Big(\frac{-D}{M}x_2(s, x_0) + \frac{P_{\max}}{M}(\sin(x_1(s, x_0) + 2\pi) \\ &\quad - \sin\delta)\Big)ds + \int_0^t \sigma dB(s).\end{aligned}\tag{5.12}$$

Subtracting (5.11) from (5.12) yields

$$x_1(t,x_0')-x_1(t,x_0)-2\pi=\int_0^t(x_2(s,x_0')-x_2(s,x_0))ds,$$
$$x_2(t,x_0')-x_2(t,x_0)=\int_0^t\Big(\frac{-D}{M}(x_2(s,x_0')-x_2(s,x_0))+\frac{P_{\max}}{M}(\sin(x_1(s,x_0'))-\sin\big(x_1(s,x_0)+2\pi)\big)\Big)ds.$$

Simple computations show that

$$\begin{aligned}&(x_1(t,x_0')-x_1(t,x_0)-2\pi)^2+(x_2(t,x_0')-x_2(t,x_0))^2\\&\le\int_0^t(\frac{P_{\max}^2}{M^2}+\frac{D}{M}+1)\Big((x_1(s,x_0')-x_1(s,x_0)-2\pi)^2+(x_2(s,x_0')\\&\quad-x_2(s,x_0))^2\Big)ds.\end{aligned}$$

The well-known Gronwall inequality yields

$$x_1(t,x_0')\equiv x_1(t,x_0)+2\pi,\quad x_2(t,x_0')\equiv x_2(t,x_0).$$

Therefore we get the desired assertion.

By Lemma 5.3, the state space can be viewed either in planar R^2 or cylindrical $S^1\times R$. The cylindrical space is a more natural space from the physical perspective. And the process $x(t,x_0)=(x_1(t,x_0),x_2(t,x_0))$ on planar R^2 can be mapped as a process $\tilde{x}(t,x_0)=(\tilde{x}_1(t,x_0),\tilde{x}_2(t,x_0))$ on cylindrical space $S^1\times R$, where

$$\tilde{x}_1(t,x_0)=x_1(t,x_0)\pmod{2\pi},\quad \tilde{x}_2(t,x_0)=x_2(t,x_0). \tag{5.13}$$

The corresponding stationary Fokker-Planck equation has the form

$$\frac{\partial f_1(y)\tilde{p}_s(\tilde{y})}{\partial\tilde{y}_1}+\frac{\partial f_2(\tilde{y})\tilde{p}_s(\tilde{y})}{\partial\tilde{y}_2}=\frac{\sigma^2}{2}\frac{\partial^2\tilde{p}_s(\tilde{y})}{\partial\tilde{y}_2^2}. \tag{5.14}$$

In this case, we can choose a constant C_1 such that

$$C_1\int_{S^1\times R}\exp\Big(-\frac{2D}{M\sigma^2}(\frac{\tilde{y}_2^2}{2}-\frac{P_{\max}}{M}\sin\delta_s\tilde{y}_1-\frac{P_{\max}}{M}\cos\tilde{y}_1)\Big)d\tilde{y}_1d\tilde{y}_2=1. \tag{5.15}$$

That is to say, $\tilde{p}_s(\tilde{y})$ can be seen as a density function on $S^1\times R$. Then a interesting question arise naturally: Is there a stationary distribution to system (5.1) on the cylinder? To illustrate this topic, let us present a lemma which is essential to the proof. □

Lemma 5.4 ([19]). *We assume that drift vector $a(x)\in R^n$ and diffusion matrix $\sigma(x)$ of the diffusion process X are continuous on E^n and independent of time t, and*

(i) *The diffusion process X has a transition density function $p(t,x,y)$.*
(ii) *For all $x\in E^n$, $j,k\in 1,2,\cdots,n$, the first order partial derivatives of $p(t,x,y)$ with respect to t; the first order partial derivatives of $b_j(y)p(t,x,y)$ with respect to y_j, the second order partial derivatives of $\sigma_{jk}(y(y)p(t,x,y)$ with respect to y_j and y_k, exist and are all continuously differentiable.*

(iii) *There exists a function $p_s(y)$ from R^n into R satisfying the steady state Fokker-Planck equation on R^n which can be written for all x in E^n as*

$$-\sum_{i=1}^{n}\frac{\partial a_i(y)p_s(y)}{\partial y_i}+\sum_{i,j}^{n}\frac{1}{2}\frac{\partial^2 b_{ij}(y)p_s(y)}{\partial y_i\partial y_j}=0. \tag{5.16}$$

with the positivity and normalization condition $\int_{E^n} p_s(y)dy=1$.

Then the stochastic process $X(t)$ has an invariant measure with $p_s(y)$ as its probability density function.

Obviously, the smoothness of $p(t,x_0,y)$ can be guaranteed by Lemma 5.1, and probability density function can be obtained by detailed balance technique in Lemma 5.2. Making use of the property of periodicity, we can claim that the corresponding probability density function $\tilde{p}(t,x_0,\tilde{y})$ of process $\tilde{x}(t,x_0)$ is also sufficiently smooth on cylindrical space. and the stationary Fokker-Planck equation of process $\tilde{x}(t,x_0)$ has the following solution on cylindrical space

$$\tilde{p}_s(\tilde{y}_1,\tilde{y}_2)=C_1\exp\Big(-\frac{2D}{M\sigma^2}(\frac{\tilde{y}_2^2}{2}-\frac{P_{\max}}{M}\sin\delta_s\tilde{y}_1-\frac{P_{\max}}{M}\cos\tilde{y}_1)\Big). \tag{5.17}$$

where C_1 is the normalizing constant. Hence, a simple application of the Lemma 5.4 implies the main result of this subsection.

Theorem 5.5. *For $x_0=(\delta_s,1)\in S^1\times R$, the process $\tilde{x}(t,x_0)$ on $S^1\times R$ has a stationary distribution.*

Remark 5.6. Denote by $\mu(\cdot)$ the stationary distribution of the process $\tilde{x}(t,x_0)$ and $(\tilde{y}_1,\tilde{y}_2)$ be the random variable to which $\tilde{x}(t,x_0)$ converges in distribution. The proof of Theorem 3.1 implies that the probability density function of $\tilde{y}_1$ is

$$p_1(\tilde{y}_1)=C_2\exp\Big(\frac{2DP_{\max}}{M^2\sigma^2}(\sin\delta\tilde{y}_1+\cos\tilde{y}_1)\Big),\quad \tilde{y}_1\in[-\pi,\pi), \tag{5.18}$$

where C_2 is the normalizing constant. The distribution of $\tilde{y}_2$ is just the normal distribution with its probability density function

$$p_2(\tilde{y}_2)=\frac{1}{\sqrt{\frac{M\Pi}{D}\sigma^2}}\exp\Big(-\frac{2D}{M\sigma^2}(\frac{\tilde{y}_2^2}{2})\Big),\quad -\infty<\tilde{y}_2<\infty.$$

Conclusion. In this paper, we have investigated the asymptotic behavior of the stochastic OMIB systems. Firstly, utilizing stochastic analysis techniques, the asymptotic bound properties of pth moment and asymptotic pathwise estimation of the stochastic OMIB systems have been researched. Secondly, by ergodic property and the strong Markov property, higher and lower rotor speed levels appear infinitely, which may give a good explanation of the fluctuation phenomena. Finally, the existence of stationary distribution on cylinder has been derived by periodicity and some analysis analysis techniques.

Acknowledgements. This research was supported by the National Science Foundation of China (Grant Nos. 61304070, 61773152, 51137002, 51190102), by the National Key Basic Research Program of China (973 Program) (2013CB228204), and by the Fundamental Research Funds for the Central Universities of China. (Grant No. 2015B19814).

References

[1] D. R. Bell; *The Malliavin Calculus*, Dover Publications, New York, 2006.

[2] X. Chen, W. Zhang, W. Zhang; Chaotic and subharmonic oscillations of a nonlinear power system, *IEEE Trans. Circuits Syst. II*, 52(12) (2005), 811-815.

[3] L. Chen, W. Zhu; First passage failure of dynamical power systems under random perturbations, *Science China Technological Sciences,* 53 (9)(2000), 2495-2500.

[4] R. Graham, H. Hakcn; Generalized thermodynamic potential for Markoff systems in detailed balance and far from thermal equilibrium, *Zeitschrift fr Physik A Hadrons and nuclei*, 243(3) (1971), 289-302.

[5] P. Ju, E. Handschin, D. Karlsson; Nonlinear dynamic load modelling: model and parameter estimation, *IEEE Trans. on Power Systems*, 11(4) (1996), 1689-1697.

[6] P. Ju, E. Handschin, F. Reyer; Genetic algorithm aided controller design with application to SVC, *IEE Proc.-Gen. Transm. Distr.* 143(3) (1996), 258-262.

[7] R. Hasminskii; *Stochastic Stability of Differential Equations*, Springer-Verlag, Berlin, Heidelberg, 2011.

[8] I. Hiskens, J. Alseddiqui; Sensitivity, approximation, and uncertainty in power system dynamic simulation, *IEEE Trans. Power Syst.*, 21(4) (2006), 1808-1820.

[9] J. Hockenberr, B. Lesieutre; Evaluation of uncertainty in dynamic simulations of power system models: The probabilistic collocation method, *IEEE Trans. Power Syst.*, 19 (3) (2004), 1483-1491.

[10] P. Ju, G. Wu, Y. Li; Fundamental theorems on probabilistic stability of power system (in Chinese), *Proc CSEE*, 11(6)(1991), 17-25.

[11] P. Kloeden, C. Platen; *Numerical Solution of Stochastic Differential Equations.* Berlin etc., Springer-Verlag, 1992.

[12] P. Kundur; *Power system and control*, New York, McGraw-Hill, 1994.

[13] F. Liu, W. Wei, S. Mei; On expansion of estimated stability region: Theory, methodology, and application to power systems, *Sci China Tech Sci*, 54 (6) (2011), 1394-1406.

[14] Z. Lu, W. Zhao, D. Xie, G. Li; P-moment stability of power system under small Gauss type random excitation, *Chaos, Solitons & Fractals*, 81(2005), 30-37.

[15] X. Mao; *Stochastic Differential Equations and Applications*, Horwood, Chichester, 1997.

[16] X. Mao; Stationary distribution of stochastic population systems, *Syst. & Contr. Lett.*, 60 (2011), 398-405.

[17] X. Pan, P. Ju, F. Wu, Y. Jin; Parameter estimation of drive system in a fixed-speed wind turbine by utilizing turbulence excitations, *IET Generation, Transmission & Distribution,* 7 (7) (2013), 665-673.

[18] H. Risken; *The Fokkcr-Planck Equation. Methods of Solution and Applications*, Springer, Berlin, 1984.

[19] C. Soize; *The Fokker-Planck equation for stochastic dynamical systems and Its explicit Steady Solutions*, World Scientific, London, 1994.

[20] A. Stankovic, B. Lesieutre; A probabilistic approach to aggregate induction machine modeling, *IEEE Trans. Power Syst.,* 11(4)(1996) 1983-1989.

[21] K. Timko, A. Bose, P. M. Anderson; Monte Carlo simulation of power system stability, *IEEE Trans. Power App. Syst.*, PAS-102 (10) (1983) 3453-3459.

[22] N. G. Van Kampen; Derivation of the phenomenological equations from the master equation: II. Even and odd variables, *Physical.* 23(6C10) (1957), 816-824.

[23] J. Vlachogiannis; Probabilistic Constrained Load Flow Considering Integration of Wind Power Generation and Electric Vehicles, *IEEE Trans. Power Syst.,* 24(4) (2009), 1808-1817.

[24] K. Wang, C. Chung, C. Tse, et al.; Improved probabilistic method for power system dynamic stability studies, *IEE P-Gener Transm D*, 147 (1) (2000), 37-43.

[25] K. Wang, L. Mariesa; The Fokker-Planck equation for power system stability probability density function evolution, *IEEE Trans. Power Syst.*, 28(3)(2013), 2994-3001.

[26] F. Wu, P. Ju, X.P. Zhang, H. Huang, C. Qin, J. Fang; Modeling, control strategy, and power conditioning for direct-drive wave energy conversion to operate with power grid, *Proceedings of the IEEE*, 101 (4) (2013), 925-941.

[27] F. Wu, Y. Tsai; Probabilistic dynamic security assessment of power systems: Part I, Basic model; *IEEE Trans. Circuits Syst.*, 30 (3) (1983), 148-159.

[28] F. Wu, X. P. Zhang, K. Godfrey, P. Ju; Small signal stability analysis and optimal control of a wind turbine with doubly fed induction generator, *IET Generation, Transmission& Distribution*, 1(5) (2007), 751-760.
[29] F. Wu, X. P. Zhang, P. Ju, M. J. H. Sterling; Decentralized nonlinear control of wind turbine with doubly fed induction generator, *IEEE Trans. on Power Systems* , 23 (2) (2008), 613-621.
[30] J. Zhang, P. Ju, Y. Yu, F. Wu; Responses and stability of power system under small Gauss type random excitation, *Science China Technological Sciences*, 55 (7) (2012), 1873-1880.

Lei Liu
College of Science, Hohai University, Nanjing, 210098, China
E-mail address: liulei_hust@163.com

Ping Ju (corresponding author)
College of Energy and Electrical Engineering, Hohai University, Nanjing, 210098, China
E-mail address: pju@hhu.edu.cn

Feng Wu
College of Energy and Electrical Engineering, Hohai University, Nanjing, 210098, China
E-mail address: wufeng@hhu.edu.cn