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ANALYSIS OF A RATE-AND-STATE FRICTION PROBLEM WITH VISCOELASTIC MATERIALS

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Abstract. We consider a mathematical model which describes the frictional contact between a viscoelastic body and a foundation. The contact is modelled with normal compliance associated to a rate-and-state version of Coulomb's law of dry friction. We start by presenting a description of the friction law, together with some examples used in geophysics. Then we state the classical formulation of the problem, list the assumptions on the data and derive a variational formulation of the model. It is in a form of a differential variational inequality in which the unknowns are the displacement field and the surface state variable. Next, we prove the unique weak solvability of the problem. The proof is based on arguments of history-dependent variational inequalities and nonlinear implicit integral equations in Banach spaces.

1. Introduction

Phenomena of contact between deformable bodies abound in industry and everyday life. Usually, they give rise to additional phenomena like friction, wear, adhesion, damage and heat generation. Among these additional effects, friction represents the main ingredient on most of the contact problems. Due to their inherent complexity, contact phenomena lead to strongly nonlinear boundary value problems and their mathematical analysis requires tools of nonsmooth functional analysis, including results on variational inequalities and nonlinear differential equations.

Frictional contact is usually modelled with the Coulomb law of dry friction or a version thereof. According to this law, the tangential traction $\boldsymbol{\sigma}_\tau$ can reach a bound H, the so-called friction bound, which is the maximal frictional resistance that the surfaces can generate, and once it has been reached, a relative slip motion commences. Thus,

$$\|\boldsymbol{\sigma}_\tau\| \le H, \quad -\boldsymbol{\sigma}_\tau = H\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|} \quad \text{if} \quad \dot{\mathbf{u}}_\tau \neq \mathbf{0}. \tag{1.1}$$

Here, $\dot{\mathbf{u}}_\tau$ is the relative tangential velocity or slip rate, and once slip starts, the frictional resistance has magnitude H and is opposing the motion. The bound H

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depends on the process variables and, often, especially in engineering publications, is chosen as

$$H = \mu|\sigma_\nu|, \tag{1.2}$$

where μ is the friction coefficient and σ_ν denotes the normal stress on the contact surface.

We observe that the friction coefficient μ is not an intrinsic thermodynamic property of a material, a body or its surface, since it depends on the contact process and the operating conditions. It is defined as the ratio between the normal stress and the modulus of the tangential stress on the contact surface when sliding commences, and there is no theoretical reason for this ratio to be a well defined function. This may explain the difficulties in the experimental measurements of the friction coefficient. The issue is considerably complicated by the following facts. Engineering surfaces are not mathematically smooth surfaces, but contain asperities and various irregularities. Moreover, very often they contain some or all of the following: moisture, lubrication oils, various debris, wear particles, oxide layers, and chemicals and materials that are different from those of the parent body. Therefore, it is not surprising that the friction coefficient is found to depend on the surface characteristics, on the surface geometry and structure, on the relative velocity between the contacting surfaces, on the surface temperature, on the wear or rearrangement of the surface and, therefore, on its history, and other factors which we skip here. A very thorough description of these issues can be found in [18] (see also the survey [26]). However, and it is somewhat surprising, the concept of a friction coefficient is found to be sufficiently useful to be employed almost universally in frictional contact problems. Indeed, there seems to be no generally accepted current alternative to it.

Until recently, mathematical models for frictional contact used a constant friction coefficient, mainly for mathematical reasons. This is rapidly changing, and the dependence of μ on the process parameters has been incorporated into the models in recent publications. The dependence of the friction coefficient μ on the location $\mathbf{x}$ on the contacting surface, when the surface is not homogeneous, is easy to incorporate into the mathematical models, but is rarely made explicit, except for possibly mentioning it in passing. On the other hand, it is well documented that such dependence may be very pronounced. Indeed, in experiments on axisymmetric stretch forming in [27, 28] the friction coefficient was found to vary steeply from a value close to zero at the center to about 0.3 at the edge, with a very sharp transition region in between which was found to depend on the forming speed.

General models which take into consideration the dependence of the coefficient of friction on the process can be obtained by considering that

$$\mu(t) = \mu(\|\dot{\mathbf{u}}_\tau(t)\|, \alpha(t)), \quad \dot{\alpha}(t) = G(\alpha(t), \|\dot{\mathbf{u}}_\tau(t)\|) \tag{1.3}$$

where G is an appropriate function and α represents an internal state surface variable. Note that in such laws, the coefficient of friction depends both the rate of the slip, denoted $\|\dot{\mathbf{u}}_\tau\|$, and on the state variable α. For this reason, the literature refers to friction laws of the form (1.1)–(1.3) as rate-and-state friction laws. References in the field are [15, 16, 17, 20].

Contact models constructed by using equalities of the form (1.3) have been used in most geophysical publications dealing with earthquakes. A first example is the so-called Dieterich-Ruina model (see, e.g., [14]) in which

$$\mu = \mu_0 - A\ln\big(1 + \frac{\|\dot{\mathbf{u}}_\tau(t)\|}{v_\infty}\big) + B\ln\big(1 + \frac{\alpha(t)}{\alpha_0}\big). \tag{1.4}$$

Here μ_0 is the static friction coefficient, v_∞ is the maximal slip velocity in the system, and α is an internal state variable describing the surface, and whose equation of evolution is given by

$$\dot{\alpha}(t) = 1 - \frac{\|\dot{\mathbf{u}}_\tau(t)\|}{L^*}\alpha(t) \tag{1.5}$$

where L^*, A, B are adjusted system parameters. More elaborate expressions can be found in [6, 14, 15, 16], and we refer the reader there and the references therein. A second example is obtained by taking

$$\mu = \mu(\alpha), \quad \dot{\alpha}(t) = \|\dot{\mathbf{u}}_\tau(t)\|. \tag{1.6}$$

In this case state variable is the total slip rate, i.e., $\alpha(t) = \int_0^t ||\dot{\mathbf{u}}_\tau(s)||\,ds$. The dependence on the process history via this parameter takes into account the morphological changes undergone by the contacting surfaces as the process goes on. Finally, the slip rate dependence $\mu = \mu(\|\dot{\mathbf{u}}_\tau\|)$ is also an example of (1.3), in which α is a constant and G vanishes.

A friction coefficient which depends on the slip rate has been employed in dynamic cases in [8, 12, 13] where the non-uniqueness of the solution and possible solutions with shocks were investigated in a special setting. A result on quasistatic contact with slip rate or total slip rate dependent friction coefficient can be found in [1]. The modelling of dynamic contact problems with rate-and-state friction of the form (1.3) have been considered recently in [15, 16], associated to Kelvin-Voingt viscoelastic materials. An algorithm for the numerical simulation of these problems was considered in [17]. There, numerical simulations were provided and compared with experimental results made to a laboratory scale. However, the well-posedness of models with such friction conditions is, as yet, an unsolved problem. The reason arises in the coupling between the rate and the state variables in the friction law.

The aim of this paper is to present a rigorous analysis of a contact model with rate-and-state friction. In contrast with the models considered in [15, 16], in this paper we consider only quasistatic process of contact but we assume a more general viscoelastic constitutive law. Considering a dependence of the form (1.3) for the coefficient of friction leads to a new and nonstandard mathematical model which couples a variational inequality for the displacement field with an ordinary differential equation for the surface state variable. The analysis of this model represents the main trait of novelty of this paper.

The rest of the manuscript is structured as follows. In Section 2 we present the notation we shall use as well as some preliminary material. In Section 3 we describe the model of the contact process and list the assumptions on the data. Then, in Section 4 we derive the variational formulation of the problem and state our main existence and uniqueness result, Theorem 4.1. The proof of the theorem is provided in Section 5, based on arguments on history-dependent variational inequalities and nonlinear implicit integral equations in Banach spaces.

2. Notation and preliminaries

As already mentioned in the previous section, we start by introducing the notion we use everywhere in this paper together with some preliminary results.

General notation. Everywhere in this paper $d \in \{1,2,3\}$ and $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by $\mathbf{0}$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\mathbf{u}\cdot\mathbf{v} = u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v}\cdot\mathbf{v})^{1/2} \quad \forall \mathbf{u} = (u_i),\ \mathbf{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}\cdot\tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\| = (\tau\cdot\tau)^{1/2} \quad \forall \boldsymbol{\sigma} = (\sigma_{ij}),\ \tau = (\tau_{ij}) \in \mathbb{S}^d,$$

where the indices i, j run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used.

The norm on the space X will be denote by $\|\cdot\|_X$, and 0_X will represent the zero element of X. Moreover, we denote by $X = X_1 \times X_2 \times \ldots \times X_m$ the product of the normed spaces $X_1,\ X_2, \ldots, X_m$, endowed with the canonical product norm

$$\|\mathbf{u}\|_X = \sqrt{\|u_1\|_{X_1}^2 + \ldots + \|u_m\|_{X_m}^2}, \tag{2.1}$$

for all $\mathbf{u} = (u_1, \ldots, u_m) \in X$. For a Hilbert space X we denote by $(\cdot,\cdot)_X$ its inner product. In addition, if X_i are real Hilbert spaces with the inner products $(\cdot,\cdot)_{X_i}$ and associated norms $\|\cdot\|_{X_i}$, $i = 1, \ldots, m$, then the product space $X = X_1 \times X_2 \times \ldots \times X_m$ will be endowed with with the canonical inner product $(\cdot,\cdot)_X$ defined by

$$(\mathbf{u},\mathbf{v})_X = (u_1, v_1)_{X_1} + \ldots + (u_m, v_m)_{X_m}, \tag{2.2}$$

for all $\mathbf{u} = (u_1, \ldots, u_m)$, $\mathbf{v} = (v_1, \ldots, v_m) \in X$.

Below in this paper I will represent either a bounded interval of the form $[0,T]$ with $T > 0$, or the unbounded interval $\mathbb{R}_+ = [0, +\infty)$. We denote by $C(I;X)$ the space of continuous functions on I with values in X. In the case $I = [0,T]$, the space $C(I;X)$ will be equipped with the norm

$$\|v\|_{C([0,T];X)} = \max_{t\in[0,T]} \|v(t)\|_X. \tag{2.3}$$

It is well known that if X is a Banach space, then $C([0,T];X)$ is also a Banach space. Assume now that $I = \mathbb{R}_+$. It is well known that if X is a Banach space, then $C(I;X)$ can be organized in a canonical way as a Fréchet space, i.e., a complete metric space in which the corresponding topology is induced by a countable family of seminorms. The convergence of a sequence $\{v_k\}_k$ to the element v, in the space $C(\mathbb{R}_+;X)$, can be described as follows: $v_k \to v$ in $C(\mathbb{R}_+;X)$ as $k \to \infty$ if and only if

$$\max_{r\in[0,n]} \|v_k(r) - v(r)\|_X \to 0 \text{ as } k \to \infty \text{ for all } n \in \mathbb{N}. \tag{2.4}$$

In other words, the sequence $\{v_k\}_k$ converges to the element v in the space $C(\mathbb{R}_+;X)$ if and only if it converges to v in the space $C([0,n];X)$ for all $n \in \mathbb{N}$. In addition, we denote by $C^1(I;X)$ the space of continuously differentiable functions on I with values in X. Therefore, $v \in C^1(I;X)$ if and only if $v \in C(I;X)$ and $\dot{v} \in C(I;X)$ where, here and below, $\dot{v}$ represents the time derivative of the function v.

History-dependent variational inequalities. We proceed with an abstract existence and uniqueness result for a special class of time-dependent variational inequalities. To this end, we consider a real Hilbert space X and a normed space Y. Moreover, we consider the operators $A : X \to X$, $\mathcal{R} : C(I;X) \to C(I;Y)$, the functional $\varphi : Y \times X \times X \to \mathbb{R}$ and the function $f : I \to X$, and we assume that the following conditions hold.

(a) There exists $m_A > 0$ such that
$$(Au_1 - Au_2, u_1 - u_2)_X \geq m_A \|u_1 - u_2\|_X^2 \quad \forall u_1, u_2 \in X.$$
(b) There exists $M_A > 0$ such that
$$\|Au_1 - Au_2\|_X \leq M_A \|u_1 - u_2\|_X \quad \forall u_1,\, u_2 \in X. \tag{2.5}$$

For any compact $J \subset I$, there exists $L_J > 0$ such that
$$\|\mathcal{R}u_1(t) - \mathcal{R}u_2(t)\|_Y \leq L_J \int_0^t \|u_1(s) - u_2(s)\|_X \, ds \tag{2.6}$$
for all $u_1,\, u_2 \in C(I; X)$ and all $t \in J$.

(a) For all $y \in Y$ and $u \in X$, $\varphi(y, u, \cdot) : X \to \mathbb{R}$ is convex and lower semicontinuous on X.
(b) There exist $c_1 \geq 0$ and $c_2 \geq 0$ such that
$$\begin{aligned} &\varphi(y_1, u_1, v_2) - \varphi(y_1, u_1, v_1) + \varphi(y_2, u_2, v_1) - \varphi(y_2, u_2, v_2) \\ &\leq c_1 \|y_1 - y_2\|_Y \|v_1 - v_2\|_X + c_2 \|u_1 - u_2\|_X \|v_1 - v_2\|_X \end{aligned} \tag{2.7}$$
for all $y_1, y_2 \in Y$, $u_1, u_2, v_1, v_2 \in X$.
$$f \in C(I; X). \tag{2.8}$$

Note that assumption (2.5) shows that A is a Lipschitz continuous strongly monotone operator. Moreover, following the terminology introduced in [22], condition (2.6), shows that the operator $\mathcal{R}$ is a history-dependent operator. Such kind of operators arise both in Functional Analysis and Solid Mechanics, as explained in the recent book [23]. We have the following existence and uniqueness result for variational inequalities with history-dependent operators, the so-called history-dependent variational inequalities.

Theorem 2.1. *Assume that* (2.5)–(2.8) *hold. Moreover, assume that*
$$c_2 \geq m_A, \tag{2.9}$$
where m_A and c_2 are the constants in (2.5) *and* (2.7), *respectively. Then, there exists a unique function $u \in C(I; X)$ such that, for all $t \in I$, it holds*
$$\begin{aligned} &(Au(t), v - u(t))_X + \varphi(\mathcal{R}u(t), u(t), v) - \varphi(\mathcal{R}u(t), u(t), u(t)) \\ &\geq (f(t), v - u(t))_X \quad \forall v \in X. \end{aligned} \tag{2.10}$$

This theorem represents a particular case of a more general result presented in [23, pag 58]. Its proof is based on arguments of time-dependent quasivariational inequalities and a fixed point result for history-dependent operators defined on the Fréchet space $C(I; X)$. A version of Theorem 2.1 could be found in [25].

A nonlinear implicit equation. Assume in what follows that $(X, \|\cdot\|_X)$ is a normed space and $(Y, \|\cdot\|_Y)$ is a Banach space. Moreover, assume that the operators $A : X \to Y$ and $\mathcal{G} : I \times X \times Y \to Y$ satisfy the following conditions.

There exists $L_A > 0$ such that
$$\|Ax_1 - Ax_2\|_Y \leq L_A \|x_1 - x_2\|_X \quad \forall x_1,\, x_2 \in X. \tag{2.11}$$

(a) There exists $L_G > 0$ such that

$$\|\mathcal{G}(t, x_1, y_1) - \mathcal{G}(t, x_2, y_2)\|_Y \le L_{\mathcal{G}}(\|x_1 - x_2\|_X + \|y_1 - y_2\|_Y) \tag{2.12}$$

for all $x_1,\, x_2 \in X$, $y_1, y_2 \in Y$, $t \in I$.
(b) The mapping $t \mapsto \mathcal{G}(t, x, y) : I \to Y$ is continuous for all $x \in X$, $y \in Y$.

The following result will be used in the proof of Lemma 5.1 below.

Theorem 2.2. *Assume that* (2.11)–(2.12) *hold. Then:*

(1) For each function $x \in C(I; X)$, there exists a unique function $y \in C(I; Y)$ such that

$$y(t) = Ax(t) + \int_0^t \mathcal{G}(s, x(s), y(s))\, ds \quad \forall t \in I\,. \tag{2.13}$$

(2) There exists a history-dependent operator $\mathcal{R} : C(I; X) \to C(I; Y)$ (i.e., an operator which satisfies condition (2.6)*) such that for all functions $x \in C(I; X)$ and $y \in C(I; Y)$, equality* (2.13) *holds if and only if*

$$y(t) = Ax(t) + \mathcal{R}x(t) \quad \forall t \in I. \tag{2.14}$$

Note that this theorem describes the history-dependence feature of the solution of the implicit integral equation (2.13). Its proof can be found in [23, pag 52]. A versions of this theorem was previously obtained in [24], in the case when $I = [0, T]$ with $T > 0$.

Function spaces. Everywhere in this paper Ω denotes a bounded domain of $\mathbb{R}^d$ with a Lipschitz continuous boundary Γ and $\Gamma_1, \Gamma_2, \Gamma_3$ will represent a partition of Γ into three measurable parts such that $\mathrm{meas}(\Gamma_1) > 0$. We use $\mathbf{x} = (x_i)$ for the generic point in $\Omega \cup \Gamma$. An index that follows a comma will represent the partial derivative with respect to the corresponding component of the spatial variable $\mathbf{x} \in \Omega \cup \Gamma$, e.g. $f_{,i} = \partial f/\partial x_i$. Moreover, $\boldsymbol{\nu} = (\nu_i)$ denotes the outward unit normal at Γ.

We use standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. In particular, we use the spaces $L^2(\Omega)^d$, $L^2(\Gamma_2)^d$, $L^2(\Gamma_3)$ and $H^1(\Omega)^d$, endowed with their canonical inner products and associated norms. Moreover, we recall that for an element $\mathbf{v} \in H^1(\Omega)^d$ we sometimes write $\mathbf{v}$ for the trace $\gamma\mathbf{v} \in L^2(\Gamma)^d$ of $\mathbf{v}$ to Γ. In addition, we consider the following spaces:

$$V = \{\mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\},$$
$$Q = \{\boldsymbol{\sigma} = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}.$$

The spaces V and Q are real Hilbert spaces endowed with the canonical inner products

$$(\mathbf{u}, \mathbf{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx, \quad (\boldsymbol{\sigma}, \tau)_Q = \int_\Omega \boldsymbol{\sigma} \cdot \tau\, dx. \tag{2.15}$$

Here and below $\boldsymbol{\varepsilon}$ and Div represent the deformation and the divergence operators, respectively, i.e.,

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\, \boldsymbol{\sigma} = (\sigma_{ij,j}). \tag{2.16}$$

The associated norms on these spaces are denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Also, recall that the completeness of the space V follows from the assumption $\mathrm{meas}(\Gamma_1) > 0$ which allows the use of Korn's inequality.

For any element $\mathbf{v} \in V$ we denote by v_ν and $\mathbf{v}_\tau$ its normal and tangential components on Γ given by $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$ and $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$, respectively. For a regular function $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ we denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and tangential stress on Γ, that is $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$, and we recall that the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\mathbf{v})\, dx + \int_\Omega \operatorname{Div} \boldsymbol{\sigma} \cdot \mathbf{v}\, dx = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \mathbf{v}\, da \quad \text{for all } \mathbf{v} \in H^1(\Omega)^d. \tag{2.17}$$

We also recall that there exists $c_0 > 0$ which depends on Ω, Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \le c_0 \|\mathbf{v}\|_V \quad \text{for all } \mathbf{v} \in V. \tag{2.18}$$

Inequality (2.18) represents a consequence of the Sobolev trace theorem.

Finally, we denote by $\mathbf{Q}_\infty$ the space of fourth order tensor fields given by

$$\mathbf{Q}_\infty = \{\ \mathcal{E} = (\mathcal{E}_{ijkl}) : \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1 \le i,j,k,l \le d\ \}.$$

The space $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{1 \le i,j,k,l \le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Moreover, a simple calculation shows that

$$\|\mathcal{E}\tau\|_Q \le d \|\mathcal{E}\|_{\mathbf{Q}_\infty} \|\tau\|_Q \quad \forall \mathcal{E} \in \mathbf{Q}_\infty,\ \tau \in Q. \tag{2.19}$$

In addition to the spaces V, Q, $\mathbf{Q}_\infty$, whose properties will be used in various places in the next section, we shall use the space of vectorial functions $C(I;X)$ and $C^1(I;X)$ where X denotes one of the spaces V, Q, $\mathbf{Q}_\infty$ and, recall, I represents the time interval of interest.

3. The model

The classical formulation of the rate-and-state frictional contact problem we consider in this paper is the following.

Problem $\mathcal{P}$. Find a displacement field $\mathbf{u} : \Omega \times I \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times I \to \mathbb{S}^d$ and a surface state variable $\alpha : \Gamma_3 \times I \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{K}(t-s)\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s))\, ds \quad \text{in } \Omega, \tag{3.1}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \Omega, \tag{3.2}$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \mathbf{f}_2(t) \quad \text{on } \Gamma_2, \tag{3.4}$$

$$-\sigma_\nu(t) = p(u_\nu(t)) \quad \text{on } \Gamma_3, \tag{3.5}$$

$$\left.\begin{aligned} &\|\boldsymbol{\sigma}_\tau(t)\| \le \mu(\|\dot{\mathbf{u}}_\tau(t)\|; \alpha(t))|\sigma_\nu(t)| \\ &-\boldsymbol{\sigma}_\tau(t) = \mu(\|\dot{\mathbf{u}}_\tau(t)\|; \alpha(t))|\sigma_\nu(t)| \frac{\dot{\mathbf{u}}_\tau(t)}{\|\dot{\mathbf{u}}_\tau(t)\|} \quad \text{if } \dot{\mathbf{u}}_\tau(t) \ne \mathbf{0} \end{aligned}\right\} \quad \text{on } \Gamma_3, \tag{3.6}$$

$$\dot{\alpha}(t) = G(\alpha(t), \|\dot{\mathbf{u}}_\tau(t)\|) \quad \text{on } \Gamma_3, \tag{3.7}$$

for all $t \in I$ and, in addition,

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \alpha(0) = \alpha_0 \quad \text{on } \Gamma_3. \tag{3.8}$$

Problem $\mathcal{P}$ describes the evolution of a viscoelastic body under the action of body forces and surface tractions. In the reference configuration the body occupies the domain Ω and is in contact with a foundation on the part Γ_3 of its boundary.

For more details on the physical setting and the mathematical modeling of contact phenomena we send the reader to the monographs [7, 19, 23].

We now provide a description of the equations and the conditions (3.1)–(3.8) and introduce the assumptions on the data. Note that, here and below, to simplify the notation, we do not mention explicitly the dependence of various functions on the spatial variable $\mathbf{x} \in \Omega \cup \Gamma$.

First, equation (3.1) represents the viscoelastic constitutive law, in which $\mathcal{A}$ is the viscosity operator, $\mathcal{B}$ is the elasticity operator, $\mathcal{K}$ represents the relaxation tensor and $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the linearized strain tensor, see (2.16). Various results, examples and mechanical interpretations in the study of viscoelastic materials of the form (3.1), can be found in [5] and the references therein. Such kind of constitutive laws were used in the literature in order to model the behavior of real materials like rubbers, rocks, metals, pastes and polymers. In particular, equation (3.1) was employed in [3, 4] in order to model the hysteresis damping in elastomers. Moreover, incorporating it into equation of motion results in integro-partial differential equation which is computationally challenging both in simulation and control design balance, as mentioned in [5]. Note that when $\mathcal{K}$ vanishes (3.1) becomes the well-known Kelvin-Voigt constitutive law, used in [15, 16], for instance. The analysis of various mathematical models of contact problems with viscoelastic materials of the form (3.1) was provided in [21, 23, 25], for instance. Below in this paper we assume that the viscosity operator, the elasticity operator and the relaxation tensor in the constitutive law (3.1) satisfy the following conditions.

(a) $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$.
(b) There exists $L_{\mathcal{A}} > 0$ such that
$$\|\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|$$
for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.
(c) There exists $m_{\mathcal{A}} > 0$ such that
$$(\mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2$$
for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.
(d) The mapping $\mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\varepsilon})$ is measurable on Ω, for any $\boldsymbol{\varepsilon} \in \mathbb{S}^d$.
(e) The mapping $\mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0})$ belongs to Q.

(3.9)

(a) $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$.
(b) There exists $L_{\mathcal{B}} > 0$ such that
$$\|\mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{B}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|$$
for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.
(c) The mapping $\mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\varepsilon})$ is measurable on Ω, for any $\boldsymbol{\varepsilon} \in \mathbb{S}^d$.
(d) The mapping $\mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0})$ belongs to Q.

(3.10)

$$\mathcal{K} \in C(I; \mathbf{Q}_\infty). \tag{3.11}$$

Next, equation (3.2) represents the equation of equilibrium in which $\mathbf{f}_0$ represents the density of body forces, assumed to have the regularity
$$\mathbf{f}_0 \in C(I; L^2(\Omega)^d). \tag{3.12}$$

We use this equation in the statement of Problem $\mathcal{P}$ since we assume that the mechanical process is quasistatic and, therefore, the inertial terms in the equation of motion are neglected.

Conditions (3.3) and (3.4) are the displacement and the traction boundary condition, respectively, in which $\mathbf{f}_2$ represents the density of surface tractions, assumed to have the regularity

$$\mathbf{f}_2 \in C(I; L^2(\Gamma_2)^d). \tag{3.13}$$

These conditions show that the body is held fixed on the part Γ_1 on his boundary and is acted upon by time-dependent forces on the part Γ_2.

Condition (3.5) is the normal compliance contact condition on Γ_3 in which σ_ν denotes the normal stress, u_ν is the normal displacement and p is a given normal compliance function. This condition models the contact with a deformable foundation. It was first introduced in [11] and used in may publications see, e.g., [7, 19, 23] and the references therein. Moreover, the term normal compliance was first used in [9, 10]. Below in this paper we assume that the function p satisfies the following condition

(3.14)
(a) $p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$.
(b) There exists $L_p > 0$ such that
$$|p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2)| \le L_p\, |r_1 - r_2|$$
for all $r_1,\ r_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_3$.
(c) The mapping $\mathbf{x} \mapsto p(\mathbf{x}, r)$ is measurable on Γ_3 for all $r \in \mathbb{R}$.
(d) $p(\mathbf{x}, r) = 0$ for all $r \le 0$, a.e. $\mathbf{x} \in \Gamma_3$.
(e) There exists $p^* > 0$ such that $p(\mathbf{x}, r) \le p^*$ for all $r \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_3$.

A typical example of such function is

$$p(\mathbf{x}, r) = \begin{cases} \eta r^+ & \text{if } r < r_0 \\ \eta r_0 & \text{if } r \ge r_0 \end{cases} \tag{3.15}$$

for all $\mathbf{x} \in \Gamma_3$, where r^+ denotes the positive part of r, $r_0 > 0$ is a given bound and $\eta > 0$ represents the stiffness coefficient of the foundation.

Condition (3.6) represents the rate-and-state friction law, introduced in Section 1. It is obtained by using the Coulomb law of dry friction (1.1), with the friction bound (1.2) in which the coefficient of friction depends on the relative slip rate $\|\dot{\mathbf{u}}_\tau\|$ and the internal state variable α, as shown in (1.3). For the coefficient of friction we assume that

(3.16)
(a) $\mu : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$.
(b) There exists $L_\mu > 0$ such that
$$|\mu(\cdot, r_1, a_1) - \mu(\cdot, r_2, a_2)| \le L_\mu\, (|r_1 - r_2| + |a_1 - a_2|)$$
for all $r_1,\ r_2,\ a_1,\ a_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_3$.
(c) The mapping $\mathbf{x} \mapsto \mu(\mathbf{x}, r, a)$ is measurable on Γ_3, for all $r, a \in \mathbb{R}$.
(d) There exists $\mu^* > 0$ such that $\mu(\mathbf{x}, r, a) \le \mu^*$ for all $r, a \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_3$.

This assumption shows that μ is a Lipschitz continuous function of its arguments, which seems very reasonable in many applications. However, there are cases when the transition from the static to the dynamic value is rather sharp, and a graph may better describe the situation.

Next, (3.7) represents the differential equation which describes the evolution of the surface state variable. Here G is a given function assumed to satisfy

(a) $G : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.
(b) There exists $L_G > 0$ such that
$$|G(\mathbf{x}, \alpha_1, r_1) - G(\mathbf{x}, \alpha_2, r_2)| \le L_G(|\alpha_1 - \alpha_2| + |r_1 - r_2|)$$
for all $\alpha_1, \alpha_1, r_1,\, r_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Omega$.
(c) The mapping $\mathbf{x} \mapsto G(\mathbf{x}, \alpha, r)$ is measurable on Ω, for all $\alpha, r \in \mathbb{R}$.
(d) The mapping $\mathbf{x} \mapsto G(\mathbf{x}, 0, 0)$ belongs to $L^2(\Gamma_3)$.

(3.17)

Note that condition (3.17) is satisfied in the case of the total slip rate friction law (1.6) but is not satisfied for the Dietrich-Ruina model, see (1.5). Nevertheless, several regularized version of the differential equations (1.5) can be considered, in which the corresponding function G satisfies assumption (3.17). These regularizations are obtained by truncation, as explained in [15].

Finally, (3.8) represents the initial conditions in which $\mathbf{u}_0$ and α_0 denote the initial displacement and the initial surface state variable, respectively, supposed to have the regularity
$$\mathbf{u}_0 \in V, \quad \alpha_0 \in L^2(\Gamma_3). \tag{3.18}$$

We end this section with the remark that Problem $\mathcal{P}$ represents the classical formulation of the rate-and-state friction problem we consider in this paper. In general, this problem does not have classical solution, i.e., solution which have all the necessary classical derivatives. For this reason, as usual in the analysis of frictional contact problems, there is a need to associate to Problem $\mathcal{P}$ a new problem, the so called variational formulation.

4. Variational Formulation

In this section we derive the variational formulation of Problem $\mathcal{P}$ and state our main existence and uniqueness result, Theorem 4.1. To this end, we start by using use the Riesz representation theorem to define the function $\mathbf{f} : I \to V$ by equality
$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t) \cdot \mathbf{v}\, dx + \int_\Gamma \mathbf{f}_2(t) \cdot \mathbf{v}\, da, \tag{4.1}$$
for all $\mathbf{v} \in V$ and $t \in I$. The regularities (3.12), (3.13) imply that
$$\mathbf{f} \in C(I; V). \tag{4.2}$$

Next, we assume $(\mathbf{u}, \boldsymbol{\sigma}, \alpha)$ are sufficiently regular functions which satisfies (3.1)–(3.8). Let $\mathbf{v} \in V$ and $t \in I$ be given. We use the Green formula (2.17) and the equilibrium equation (3.2) to deduce that
$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma}(t) \cdot (\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))\, dx \\ &= \int_\Omega \mathbf{f}_0(t) \cdot (\mathbf{v} - \dot{\mathbf{u}}(t))\, dx + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\mathbf{v} - \dot{\mathbf{u}}(t))\, da\,. \end{aligned}$$

Then, we split the surface integral over Γ_1, Γ_2 and Γ_3, use equalities $\mathbf{v} - \dot{\mathbf{u}}(t) = \mathbf{0}$ on Γ_1 and $\boldsymbol{\sigma}(t)\nu = \mathbf{f}_2(t)$ on Γ_2 and definition (4.1) to deduce that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q = (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\mathbf{v} - \dot{\mathbf{u}}(t))\, da. \tag{4.3}$$

On the other hand, the boundary conditions (3.5), (3.6) combined with the positivity of the function p yield

$$\begin{gathered}\sigma_\nu(t)(v_\nu - \dot{u}_\nu(t)) = -p(u_\nu(t))(v_\nu - \dot{u}_\nu(t)),\\ \boldsymbol{\sigma}_\tau(t) \cdot (\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau(t)) \geq \mu(\|\dot{\mathbf{u}}_\tau(t)\|; \alpha(t))p(u_\nu(t))(\|\dot{\mathbf{u}}_\tau(t)\| - \|\mathbf{v}_\tau\|)\end{gathered}$$

on Γ_3. Therefore, since

$$\boldsymbol{\sigma}(t)\,\boldsymbol{\nu} \cdot (\mathbf{v} - \dot{\mathbf{u}}(t)) = \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t)) + \boldsymbol{\sigma}_\tau(t) \cdot (\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau(t)) \quad \text{on } \Gamma_3,$$

we deduce that

$$\begin{aligned}\int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\mathbf{v} - \dot{\mathbf{u}}(t))\, da &\geq -\Big(p(u_\nu(t)), v_\nu - \dot{u}_\nu(t)\Big)_{L^2(\Gamma_3)}\\ &\quad + \Big(\mu(\|\dot{\mathbf{u}}_\tau(t)\|; \alpha(t))p(u_\nu(t)), \|\dot{\mathbf{u}}_\tau(t)\| - \|\mathbf{v}_\tau\|\Big)_{L^2(\Gamma_3)}.\end{aligned}$$

We now combine this inequality with (4.3) to obtain

$$\begin{aligned}&(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q + \Big(p(u_\nu(t)), v_\nu - \dot{u}_\nu(t)\Big)_{L^2(\Gamma_3)}\\ &+ \Big(\mu(\|\dot{\mathbf{u}}_\tau(t)\|; \alpha(t))p(u_\nu(t)), \|\mathbf{v}_\tau\| - \|\dot{\mathbf{u}}_\tau(t)\|\Big)_{L^2(\Gamma_3)}\\ &\geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V.\end{aligned}$$

Finally, we substitute the constitutive law (3.1) in the previous inequality and gather the resulting inequality with the differential equation (3.7) and the initial conditions (3.8) to obtain the following variational formulation of Problem $\mathcal{P}$.

Problem $\mathcal{P}^V$. Find a displacement field $\mathbf{u} : I \to V$ and an surface state variable $\alpha : I \to L^2(\Gamma_3)$ such that $\mathbf{u}(0) = \mathbf{u}_0$, $\alpha(0) = \alpha_0$ and, for any $t \in I$, the following hold:

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{K}(t-s)\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s))\, ds, \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q\\ &+ \Big(p(u_\nu(t)), v_\nu - \dot{u}_\nu(t)\Big)_{L^2(\Gamma_3)}\\ &+ \Big(\mu(\|\dot{\mathbf{u}}_\tau(t)\|; \alpha(t))p(u_\nu(t)), \|\mathbf{v}_\tau\| - \|\dot{\mathbf{u}}_\tau(t)\|\Big)_{L^2(\Gamma_3)}\\ &\geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \quad \forall v \in V,\\ &\qquad\qquad \dot{\alpha}(t) = G(\alpha(t), \|\dot{\mathbf{u}}_\tau(t)\|).\end{aligned}$$

Note that Problem $\mathcal{P}^V$ represents a system which couples a differential equation for the surface state variable with a variational inequality for displacement field. Therefore, following the notion introduced in [2], it represents a differential variational inequality. In the study of this problem we have the following existence and uniqueness result.

Theorem 4.1. *Assume that* (3.9)–(3.18) *hold and, moreover, assume that*

$$c_0^2\, p^* L_\mu \le m_{\mathcal{A}}. \tag{4.4}$$

Then, Problem $\mathcal{P}^V$ has a unique solution with regularity

$$\mathbf{u} \in C^1(I;V), \quad \alpha \in C^1(I;L^2(\Gamma_3)). \tag{4.5}$$

A solution $(\mathbf{u},\alpha)$ of Problem $\mathcal{P}^V$ is called a weak solution to the contact problem $\mathcal{P}$. We conclude that Theorem 4.1 states the unique weak solvability of Problem $\mathcal{P}$, under the smallness assumption (4.4) on the normal compliance function and the coefficient of friction.

5. Proof of Theorem 4.1

The proof of Theorem 4.1 is carried out in several steps. Everywhere below we assume that (3.9)–(3.18) hold and we consider the operator $\mathcal{S} : C(I;V) \to C(I;V)$ defined by

$$\mathcal{S}\mathbf{w}(t) = \int_0^t \mathbf{w}(s)\, ds + \mathbf{u}_0, \tag{5.1}$$

for all $\mathbf{w} \in C(I;V)$ and $t \in I$. Note that

$$\|\mathcal{S}\mathbf{w}_1(t) - \mathcal{S}\mathbf{w}_2(t)\|_V \le \int_0^t \|\mathbf{w}_1(s) - \mathbf{w}_2(s)\|_V\, ds, \tag{5.2}$$

for all $\mathbf{w}_1$, $\mathbf{w}_2 \in C(I;V)$ and $t \in I$, and, therefore the operator $\mathcal{S}$ is a history-dependent operator. The first step in the proof of Theorem 4.1 is the following.

Lemma 5.1. *(1) For each function $\mathbf{w} \in C(I;V)$, there exists a unique function $\alpha \in C^1(I;L^2(\Gamma_3))$ such that*

$$\dot\alpha(t) = G(\alpha(t), \|\mathbf{w}_\tau(t)\|) \quad \forall t \in I, \tag{5.3}$$

$$\alpha(0) = \alpha_0. \tag{5.4}$$

(2) There exists a history-dependent operator $\mathcal{R}_1 : C(I;V) \to C(I;L^2(\Gamma_3))$ such that for all functions $\mathbf{w} \in C(I;V)$ and $\alpha \in C(I;L^2(\Gamma_3))$, the following statements are equivalent:

(a) $\alpha \in C^1(I;L^2(\Gamma_3))$ *and equalities* (5.3)–(5.4) *hold;*
(b) $\alpha(t) = \alpha_0 + \mathcal{R}_1\mathbf{w}(t)$ *for all* $t \in I$.

Proof. Let $\mathbf{w} \in C(I;V)$. Then, using assumptions (3.17), (3.18) it is easy to see that the function α is a solution to the Cauchy problem (5.3)–(5.4) with regularity $\alpha \in C^1(I, L^2(\Gamma_3))$ if and only if $\alpha \in C(I, L^2(\Gamma_3))$ and

$$\alpha(t) = \alpha_0 + \int_0^t G(\alpha(s); \|\mathbf{w}_\tau(s)\|)\, ds. \tag{5.5}$$

Then Lemma 5.1 is a direct consequence of Theorem 2.2 applied with $X = V$, $Y = L^2(\Gamma_3)$ and

$$A\mathbf{w} \equiv \alpha_0, \quad \mathcal{G}(t,\mathbf{w},\alpha) = G(\alpha; \|\mathbf{w}_\tau\|), \tag{5.6}$$

for all $\mathbf{w} \in V$, $\alpha \in L^2(\Gamma_3)$ and $t \in I$. □

We now state the following equivalence result whose proof is a direct consequence of Lemma 5.1 and definition (5.1).

Lemma 5.2. *The couple* $(\mathbf{u},\alpha)$ *is a solution of Problem* $\mathcal{P}^V$ *with regularity* (4.5) *if and only if there exists a function* $\mathbf{w}\in C(I;V)$ *such that*

$$\mathbf{u}(t)=\mathcal{S}\mathbf{w}(t), \tag{5.7}$$

$$\alpha(t)=\alpha_0+\mathcal{R}_1\mathbf{w}(t) \tag{5.8}$$

and, moreover, for all $t\in I$*, the inequality below holds:*

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{w}(t))+\mathcal{B}\boldsymbol{\varepsilon}((\mathcal{S}\mathbf{w})(t))+\int_0^t\mathcal{K}(t-s)\boldsymbol{\varepsilon}(\mathbf{w}(s))\,ds,\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\mathbf{w}(t)))_Q\\&+\Big(p((\mathcal{S}\mathbf{w})_\nu(t)),v_\nu-w_\nu(t)\Big)_{L^2(\Gamma_3)}\\&+\Big(\mu(\|\mathbf{w}_\tau(t)\|;\alpha_0+\mathcal{R}_1\mathbf{w}(t))p((\mathcal{S}\mathbf{w})_\nu(t)),\|\mathbf{v}_\tau\|-\|\mathbf{w}_\tau(t)\|\Big)_{L^2(\Gamma_3)}\\&\geq(\mathbf{f}(t),\mathbf{v}-\mathbf{w}(t))_V\quad\forall v\in V.\end{aligned} \tag{5.9}$$

Note that in (5.9) and below, $(\mathcal{S}\mathbf{w})_\nu(t)$ represents the normal component of the element $(\mathcal{S}\mathbf{w})(t)\in V$. The next step in the proof of Theorem 4.1 consists to obtain the unique solvability of the variational inequality (5.9) for the velocity field $\mathbf{w}=\dot{\mathbf{u}}$. We have the following existence and uniqueness result.

Lemma 5.3. *There exists a unique solution* $\mathbf{w}$ *of* (5.9)*. Moreover, the solution satisfies*

$$\mathbf{w}\in C(I;V). \tag{5.10}$$

Proof. We consider the product Hilbert space $\Lambda=L^2(\Gamma_3)\times Q\times L^2(\Gamma_3)$ and the set K defined by

$$K=\{z\in L^2(\Gamma_3):0\leq z\leq p^*\text{ a.e. on }\Gamma_3\}. \tag{5.11}$$

We note that K is a nonempty closed subset of the space $L^2(\Gamma_3)$ and we denote by $P_K:L^2(\Gamma_3)\to K$ the projection map on K. Next, we define the operators $A:V\to V$, $\mathcal{R}_2:C(I;V)\to C(I;Q)$, $\mathcal{R}_3:C(I;V)\to C(I;L^2(\Gamma_3))$ and $\mathcal{R}:C(I;V)\to C(I;\Lambda)$ by equalities

$$(A\mathbf{u},\mathbf{v})_V=(\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_Q, \tag{5.12}$$

$$\mathcal{R}_2\mathbf{w}(t)=\mathcal{B}\boldsymbol{\varepsilon}(\mathcal{S}\mathbf{w}(t))+\int_0^t\mathcal{K}(t-s)\boldsymbol{\varepsilon}(\mathbf{w}(s))\,ds, \tag{5.13}$$

$$\mathcal{R}_3\mathbf{w}(t)=p((\mathcal{S}\mathbf{w})_\nu(t)), \tag{5.14}$$

$$\mathcal{R}\mathbf{w}(t)=(\alpha_0+\mathcal{R}_1\mathbf{w}(t),\mathcal{R}_2\mathbf{w}(t),\mathcal{R}_3\mathbf{w}(t)) \tag{5.15}$$

for all $\mathbf{u},\mathbf{v}\in V$, $\mathbf{w}\in C(I;V)$ where, recall, $\mathcal{R}_1$ is the operator defined in Lemma 5.1. We also define the functional $\varphi:\Lambda\times V\times V\to\mathbb{R}$ by equality

$$\varphi(\boldsymbol{\lambda},\mathbf{w},\mathbf{v})=(\mathbf{y},\boldsymbol{\varepsilon}(\mathbf{v}))_Q+(z,v_\nu)_{L^2(\Gamma_3)}+(\mu(\|\mathbf{w}_\tau\|;x)P_Kz,\|\mathbf{v}_\tau\|)_{L^2(\Gamma_3)} \tag{5.16}$$

for all $\boldsymbol{\lambda}=(x,\mathbf{y},z)\in\Lambda$ and $\mathbf{w},\mathbf{v}\in V$. With these data we consider the problem of finding a function $\mathbf{w}:I\to V$ such that, for all $t\in I$, the following inequality holds:

$$\begin{aligned}&(A\mathbf{w}(t),\mathbf{v}-\mathbf{w}(t))_V+\varphi(\mathcal{R}\mathbf{w}(t),\mathbf{w}(t),\mathbf{v})-\varphi(\mathcal{R}\mathbf{w}(t),\mathbf{w}(t),\mathbf{w}(t))\\&\geq(\mathbf{f}(t),\mathbf{v}-\mathbf{w}(t))_V\quad\forall v\in V.\end{aligned} \tag{5.17}$$

We use the bound (3.14) (e) to see that for any function $\mathbf{w}\in C(I;V)$ we have $0\leq p((\mathcal{S}\mathbf{w})_\nu(t))\leq p^*$ a.e. on Γ_3 for all $t\in I$. Therefore, using definition (5.11) of the set K it follows that $P_Kp((\mathcal{S}\mathbf{w})_\nu(t))=p((\mathcal{S}\mathbf{w})_\nu(t))$ for all $t\in I$. Using this equality and the definitions (5.12)–(5.16) it is easy to see that a function $\mathbf{w} \in C(I;V)$ is a solution of (5.9) if and only if $\mathbf{w}$ is a solution of the inequality (5.17). For this reason, our aim in what follows is to prove the unique solvability of this problem and, to this end, we check the assumptions of Theorem 2.1 with $X = V$ and $Y = \Lambda$.

First, we use assumptions (3.9) to deduce that A satisfies (2.5) with

$$m_A = m_{\mathcal{A}} \quad \text{and} \quad M_A = L_{\mathcal{A}}. \tag{5.18}$$

Let $J \subset I$, $t \in J$ and let $\mathbf{u}, \mathbf{v} \in C(I;V)$. Lemma 5.1 (2) guarantees that $\mathcal{R}_1$ is a history dependent operator and, therefore, there exists $L_J^1 > 0$ such that

$$\|\mathcal{R}_1\mathbf{u}(t) - \mathcal{R}_1\mathbf{v}(t)\|_{L^2(\Gamma_3)} \le L_J^1 \int_0^t \|\mathbf{u}(s) - \mathbf{v}(s)\|_V \, ds. \tag{5.19}$$

On the other hand, definition (5.13), assumptions (3.10), (3.11) and inequalities (5.2), (2.19) imply that

$$\|\mathcal{R}_2\mathbf{u}(t) - \mathcal{R}_2\mathbf{v}(t)\|_Q \le \big(L_{\mathcal{B}} + d \max_{r\in J} \|\mathcal{K}(r)\|_{\mathbf{Q}_\infty}\big) \int_0^t \|\mathbf{u}(s) - \mathbf{v}(s)\|_V \, ds. \tag{5.20}$$

Finally, we use again inequality (5.2), assumption (3.14) and inequality (2.18) to deduce that

$$\|\mathcal{R}_3\mathbf{u}(t) - \mathcal{R}_3\mathbf{v}(t)\|_{L^2(\Gamma_3)} \le c_0 L_p \int_0^t \|\mathbf{u}(s) - \mathbf{v}(s)\|_V \, ds. \tag{5.21}$$

We now combine inequalities (5.19)–(5.21) to obtain that

$$\begin{aligned} &\|\mathcal{R}\mathbf{u}(t) - \mathcal{R}\mathbf{v}(t)\|_\Lambda \\ &\le \big(L_J^1 + L_{\mathcal{B}} + d \max_{r\in J} \|\mathcal{K}(r)\|_{\mathbf{Q}_\infty} + c_0 L_p\big) \int_0^t \|\mathbf{u}(s) - \mathbf{v}(s)\|_V \, ds \end{aligned} \tag{5.22}$$

which shows that the operator $\mathcal{R}$ satisfies condition (2.6) with

$$L_J = L_J^1 + L_{\mathcal{B}} + d \max_{r\in J} \|\mathcal{K}(r)\|_{\mathbf{Q}_\infty} + c_0 L_p.$$

On the other hand, it is easy to see that that the functional φ satisfies condition (2.7)(a). To satisfy condition (2.7)(b) let $\boldsymbol{\lambda}_1 = (x_1, \mathbf{y}_1, z_1), \boldsymbol{\lambda}_2 = (x_2, \mathbf{y}_2, z_2) \in \Lambda$ and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$. We use definition (5.16) to deduce that

$$\begin{aligned} &\varphi(\boldsymbol{\lambda}_1, \mathbf{w}_1, \mathbf{v}_2) - \varphi(\boldsymbol{\lambda}_1, \mathbf{w}_1, \mathbf{v}_1) + \varphi(\boldsymbol{\lambda}_2, \mathbf{w}_2, \mathbf{v}_1) - \varphi(\boldsymbol{\lambda}_2, \mathbf{w}_2, \mathbf{v}_2) \\ &= (\mathbf{y}_1 - \mathbf{y}_2, \boldsymbol{\varepsilon}(\mathbf{v}_2) - \boldsymbol{\varepsilon}(\mathbf{v}_1))_Q + (z_1 - z_2, v_{2\nu} - v_{1\nu})_{L^2(\Gamma_3)} \\ &\quad + \Big(\mu(\|\mathbf{w}_{1\tau}\|; x_1) P_K z_1 - \mu(\|\mathbf{w}_{2\tau}\|; x_2) P_K z_2, \|\mathbf{v}_{2\tau}\| - \|\mathbf{v}_{1\tau}\|\Big)_{L^2(\Gamma_3)}. \end{aligned} \tag{5.23}$$

Next, using the definition of the norm in the product space Λ and the trace inequality (2.18), it is easy to see that

$$(\mathbf{y}_1 - \mathbf{y}_2, \boldsymbol{\varepsilon}(\mathbf{v}_2) - \boldsymbol{\varepsilon}(\mathbf{v}_1))_Q \le \|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_\Lambda \|\mathbf{v}_1 - \mathbf{v}_2\|_V, \tag{5.24}$$

$$(z_1 - z_2, v_{2\nu} - v_{1\nu})_{L^2(\Gamma_3)} \le c_0 \|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_\Lambda \|\mathbf{v}_1 - \mathbf{v}_2\|_V. \tag{5.25}$$

We denote

$$\mu(\|\mathbf{w}_{1\tau}\|; x_1) = \mu_1, \quad \mu(\|\mathbf{w}_{2\tau}\|; x_2) = \mu_2.$$

Then, using inequalities $|\mu_1| \le \mu^*$, $0 \le P_K z_2 \le p^*$ a.e. on Γ_3, guaranteed by (3.16)(d) and (5.11), respectively, combined with the nonexpansivity of the projection map and assumption (3.16)(b), it is easy to see that

$$\begin{aligned}
&\Big(\mu(\|\mathbf{w}_{1\tau}\|;x_1)P_K z_1 - \mu(\|\mathbf{w}_{2\tau}\|;x_2)P_K z_2, \|\mathbf{v}_{2\tau}\| - \|\mathbf{v}_{1\tau}\|\Big)_{L^2(\Gamma_3)}\\
&= (\mu_1(P_K z_1 - P_K z_2), \|\mathbf{v}_{2\tau}\| - \|\mathbf{v}_{1\tau}\|)_{L^2(\Gamma_3)}\\
&\quad + ((\mu_1-\mu_2)P_K z_2, \|\mathbf{v}_{2\tau}\| - \|\mathbf{v}_{1\tau}\|)_{L^2(\Gamma_3)}\\
&\le \mu^*(|P_K z_1 - P_K z_2|, \|\mathbf{v}_1 - \mathbf{v}_2\|)_{L^2(\Gamma_3)}\\
&\quad + p^*(|\mu_1 - \mu_2|, \|\mathbf{v}_1 - \mathbf{v}_2\|)_{L^2(\Gamma_3)}\\
&\le \mu^*\|P_K z_1 - P_K z_2\|_{L^2(\Gamma_3)}\|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(\Gamma_3)^d}\\
&\quad + p^* L_\mu(\|\mathbf{w}_1 - \mathbf{w}_2\| + |x_1 - x_2|, \|\mathbf{v}_1 - \mathbf{v}_2\|)_{L^2(\Gamma_3)}\\
&\le \mu^*\|z_1 - z_2\|_{L^2(\Gamma_3)}\|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(\Gamma_3)^d}\\
&\quad + p^* L_\mu(\|\mathbf{w}_1 - \mathbf{w}_2\|_{L^2(\Gamma_3)^d} + \|x_1 - x_2\|_{L^2(\Gamma_3)})\|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(\Gamma_3)^d}.
\end{aligned}$$

Therefore, using again the definition of the norm in the product space Λ and the trace inequality (2.18) yields

$$\begin{aligned}
&\Big(\mu(\|\mathbf{w}_{1\tau}\|;x_1)P_K z_1 - \mu(\|\mathbf{w}_{2\tau}\|;x_2)P_K z_2, \|\mathbf{v}_{2\tau}\| - \|\mathbf{v}_{1\tau}\|\Big)_{L^2(\Gamma_3)}\\
&\le c_0\mu^*\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_\Lambda \|\mathbf{v}_1 - \mathbf{v}_2\|_V\\
&\quad + c_0^2 p^* L_\mu \|\mathbf{w}_1 - \mathbf{w}_2\|_V\|\mathbf{v}_1 - \mathbf{v}_2\|_V + c_0 p^* L_\mu\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_\Lambda\|\mathbf{v}_1 - \mathbf{v}_2\|_V.
\end{aligned} \tag{5.26}$$

We now combine equality (5.23) with inequalities (5.24)–(5.26) to find that

$$\begin{aligned}
&\varphi(\boldsymbol{\lambda}_1, \mathbf{w}_1, \mathbf{v}_2) - \varphi(\boldsymbol{\lambda}_1, \mathbf{w}_1, \mathbf{v}_1) + \varphi(\boldsymbol{\lambda}_2, \mathbf{w}_2, \mathbf{v}_1) - \varphi(\boldsymbol{\lambda}_2, \mathbf{w}_2, \mathbf{v}_2)\\
&\le (1 + c_0 + c_0\mu^* + c_0 p^* L_\mu)\|\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\|_\Lambda\|\mathbf{v}_1 - \mathbf{v}_2\|_V\\
&\quad + c_0^2 p^* L_\mu\|\mathbf{w}_1 - \mathbf{w}_2\|_V\|\mathbf{v}_1 - \mathbf{v}_2\|_V.
\end{aligned} \tag{5.27}$$

This inequality shows that the functional φ satisfies condition (2.7)(b) with

$$c_1 = 1 + c_0 + c_0\mu^* + c_0 p^* L_\mu \quad \text{and} \quad c_2 = c_0^2 p^* L_\mu. \tag{5.28}$$

Therefore, it follows from (5.18), (5.28) and (4.4) that the smallness condition (2.9) holds. Finally, taking into account the regularity (4.2) we find that (2.8) holds, too. We are now in a position to apply Theorem 2.1 and we deduce in this way that inequality (5.17) has a unique solution $\mathbf{w} \in C(I;V)$, which completes the proof. ☐

We now have all the ingredients to provide the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\mathbf{w}$ denote the unique solution of inequality (5.9) obtained in Lemma 5.3 and let $\mathbf{u} = \mathcal{S}\mathbf{w}$, $\alpha = \alpha_0 + \mathcal{R}_1\mathbf{w}$. Then, Lemma 5.2 implies that $(\mathbf{u}, \alpha)$ is a solution of Problem $\mathcal{P}^V$. This proves the existence part of the theorem. The uniqueness of the solution is now a consequence of the unique solvability of the variational inequality (5.9), guaranteed by Lemma 5.3, combined with the equivalence result in Lemma 5.2. ☐

References

[1] A. Amassad, M. Shillor, M. Sofonea; A quasistatic contact problem with slip dependent coefficient of friction, *Math. Meth. Appl. Sci.*, **22** (1999), pp. 267–284.

[2] J. P. Aubin, A. Cellina; *Differential Inclusions*, Springer-Verlag, New York, 1984.

[3] H. T. Banks, G. A. Pinter, L. K. Potter, B. C. Munoz, L. C. Yanyo; Estimation and control related issues in smart material structure and fluids, *Optimization Techniques and Applications*, L. Caccetta et al., Eds., Curtain University Press 1998, pp. 19–34.

[4] H. T. Banks, G. A. Pinter, L. K. Potter, J. M. Gaitens, L.C. Yanyo; Modeling of quasistatic and dynamic load responses of filled viesoelastic materials, Chapter 11 in *Mathematical Modeling : Case Studies from Industry*, E. Cumberbatch and A. Fitt, Eds., Cambridge University Press, 2011, pp. 229–252.

[5] H. T. Banks, S. Hu, Z. R. Kenz; A brief review of elasticity and viscoelasticity for solids, *Adv. Appl. Math. Mech.*, **3** (2011), pp. 1–51.

[6] W. R. Bielski, J. J. Telega; Modelling contact problems with friction in fault mechanics, *J. Theor. Appl. Mech.*, **39** (2001), pp. 475–505.

[7] W. Han, M. Sofonea; *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics **30**, American Mathematical Society, Providence, RI–International Press, Somerville, MA, 2002.

[8] I. R. Ionescu, J.-C. Paumier; On the contact problem with slip dependent friction in elastodynamics, *Eur. J. Mech. A Solids*, **13** (1994), pp. 555–568.

[9] A. Klarbring, A. Mikelič, M. Shillor; Frictional contact problems with normal compliance, *Int. J. Engrg. Sci.*, **26** (1988), pp. 811–832.

[10] A. Klarbring, A. Mikelič, M. Shillor; On friction problems with normal compliance, *Nonlinear Anal.*, **13** (1989), pp. 935–955.

[11] J. T. Oden, J. A. C. Martins; Models and computational methods for dynamic friction phenomena, *Comput. Methods Appl. Mech. Engrg.*, **52** (1985), pp. 527–634.

[12] J.-C. Paumier, Y. Renard; Existence et unicité pour le frottement élastodynamique avec perturbation par 'inertie de surface'; *C. R. Acad. Sci. Paris*, **330** (2000) Serie I, 1-4.

[13] J.-C. Paumier, Y. Renard; Surface perturbation of an elastodynamic contact problem with friction, *European J. Appl. Math.*, **14** (2003), pp. 465–483.

[14] G. J. Perrin, J. R. Rice, G. Zheng; Self-healing slip pulse on a frictional surface, *J. Mech. Phys. Solids*, **43** (1995), pp. 1461–1495.

[15] E. Pipping; Dynamic problems of rate-and-state friction in viscoelasticity, Ph.D. Thesis, 2014, Institute of Mathematics, Departament of Mathematics and Computer Science, Freie Universität Berlin.

[16] E. Pipping; Existence of long-time solutions to dynamic problems of viscoelasticity with rate-and-state friction problems, arXiv:1703.04289, 2017.

[17] E. Pipping, R. Kornhuber, M. Roseanu, O. Onken; On the efficient and relaible numerical solution of rate-and-state friction problems, *Geophysical Journal International*, **204** (2016), pp. 1858-1866.

[18] E. Rabinowiz; *Friction and Wear of Materials*, (2nd ed.) Wiley, New York, 1995.

[19] M. Shillor, M. Sofonea, J.J. Telega; *Models and Analysis of Quasistatic Contact*, Lecture Notes in Physics **655**, Springer, Berlin, 2004.

[20] C. H. Scholtz; Earthquakes and friction law, *Nature*, **391** (1998), pp. 37–48.

[21] M. Sofonea, A. Farcaş; Analysis of a history-dependent frictional contact problem, *Appl. Anal.*, **93** (2014), pp. 428–444.

[22] M. Sofonea, A. Matei; History-dependent quasivariational inequalities arising in Contact Mechanics, *European J. Appl. Math.*, **22** (2011), pp. 471–491.

[23] M. Sofonea, S. Migórski; *Variational-Hemivariational Inequalities with Applications*, Pure and Applied Mathematics, Chapman & Hall/CRC Press, Boca Raton-London, 2018.

[24] M. Sofonea, F. Pătrulescu, A. Ramadan; A mixed variational formulation of a contact problem with wear, *Acta. Appl. Math.* (2017), https://doi.org/10.1007/s10440-017-0123-4, to appear.

[25] M. Sofonea, Y. Xiao; Fully history-dependent quasivariational inequalities in contact mechanics, *Appl. Anal.*, **95** (2016), pp. 2464–2484.

[26] J. J. Telega; Topics on unilateral contact problems of elasticity and inelasticity, in *Nonsmooth Mechanics and Applications*, J. J. Moreau and P. D. Panagiotoupolos (Eds.), Springer, Vienna, 1988, pp. 340–461.

[27] W. R. D. Wilson, T.-C. Hsu, X.-B. Huang; A realistic friction model of sheet metal forming processes, *J. Eng. Ind.*, **117** (1995), pp. 202–209.

[28] W. R. D. Wilson; Modeling friction in sheet-metal forming simulation, in *The Integration of Materials, Processes and Product Design*, Zabaras et al. (Eds.), Balkema, Rotterdam, 1999, pp. 139–147.

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