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STABILITY OF SOLUTIONS FOR A HEAT EQUATION WITH MEMORY

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Abstract. This article concerns the heat equation with a memory term in the form of a time-convolution of a kernel with the time-derivative of the state. This problem appears in oil recovery simulation in fractured rock reservoir. It models the fluid flow in a fissured media where the history of the flow must be taken into account. Most of the existing papers on related works treat only (in addition to the well-posedness which is by now well understood in various spaces) the convergence of solutions to the equilibrium state without establishing any decay rate. In the present work we shall improve and extend the existing results. In addition to weakening the conditions on the kernel leading to exponential decay, we extend the decay rate to a general one.

1. Introduction

In this article we consider the problem

$$\begin{gathered} u_t(x,t)+\int_0^t k(t-s)u_t(x,s)ds=\Delta u(x,t), \quad (x,t)\in\Omega\times I \\ u(x,t)=0, \quad (x,t)\in\partial\Omega\times I, \\ u(x,0)=u_0(x), \quad x\in\Omega, \end{gathered} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^d$ ($d\geq 1$), with smooth boundary $\partial\Omega$ and $I=(0,T)$, $T>0$. This problem models the flow of a fluid in a fissured media when the history of the flow is taken into account. This is the case of some oil reservoirs where the media is formed by a matrix of porous blocks isolated by a well-developed system of fissures. Problem (1.1) has been derived by Hornung and Showalter [21]. This problem appears also in the heat conduction theory with memory term according to the theory of Gurtin-Pipkin. It is also known as the Basset problem when $k(t)=\frac{t^{-1/2}}{\Gamma(1/2)}$ (see Basset [7]). In this case the convolution term represents a fractional derivative of order $1/2$.

The literature is very rich in results on well-possedness for similar problems (see the section below). In fact, there are numerous works on existence and uniqueness as well as regularity in different spaces like: L^p space, Hölder spaces, Sobolev spaces

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and Besov spaces. There are also several generalizations of this problem to other linear as well as nonlinear cases encompassing other applications in other fields. The assumptions on the kernels in the memory terms are now reasonable (for the well-posedness). For this reason we shall not work on this issue and assume existence, uniqueness and enough regularity for our solutions to justify the computation. In contrast, we could not find many papers on the asymptotic behavior of solutions (see Section 3 below). Most of the existing papers treat rather the convergence of solutions (to the equilibrium state) without specifying the decay rate. It is exactly this last issue which we want to address here. We intend to shed some light on this matter of speed of convergence.

The next section contains a reminder of some results related to well-posedness. We recall the few results, we are aware of, on the asymptotic behavior of solutions in Section 3. In Section 4 we present some useful inequalities we need in our proof. Section 5 is devoted to our main result on the explicit decay rate of solutions.

2. Well-possedness

In the previous fifteen years a fairly large number of papers appeared in the literature with a large number of results on well-posedness for general problems. The main tools are several kinds of fixed point theorems and the semi-group theory. We shall not discuss these works here. We will restrict ourselves to the following ones just to give the reader a flavor of these generalizations.

In 1990, Hornung and Showalter [21] proved that problem (1.1) has a unique solution in the space of absolutely continuous functions with square summable derivatives provided that: $k \in L^1(0,T) \cap C^1(0,T)$, $k \geq 0$, $k' \leq 0$, k' is nondecreasing and not equal to a constant. In the same year, Clément and Da Prato [10] proved the existence and uniqueness of a mild solution in the space of continuous functions for a similar problem, namely

$$\frac{d}{dt}\left(u(t) + (k * u)(t)\right) = Au(t). \tag{2.1}$$

They assumed that the Laplace transform $\hat{k}(\sigma)$ of $k(t)$ admits an analytic extension in

$$S_{\nu,\theta} = \{\sigma \in \mathbb{C} \setminus \{0\} : |\arg(\sigma - \nu)| < \theta\}$$

and there exists $C > 0$, $\alpha \in (0,1)$, $\nu \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$ such that $\|\hat{k}(\sigma)\| \leq \frac{C}{|\sigma-\nu|^\alpha}$, for all $\sigma \in S_{\nu,\theta}$. These conditions are satisfied by $k(t) = e^{-\beta t} t^{\alpha-1}$, $\beta > 0$. They also considered the existence of a nonlinear source in the equation.

In 1995, Sforza [32] proved global existence and Hölder regularity of the solution when the kernel is nonnegative nondecreasing and summable for a little more general problem than (2.1)(with a nonlinear source).

For the same problem (with an external source term $f(x,t)$), Peszynska [31] presented a convergent method for fully-discrete approximation of solutions. They assumed that the kernel in (1.1) is nonnegative, monotone increasing and in $L^1(I) \cap C(I)$. The well-posedness being shown already in Peszynska [31]. The work of Hornung and Showalter [21] and Peszynska [31] was extended the following year by Slodicka [33]. The author established well-posedness in $C\left((0,\pi); L_2(\Omega)\right) \cap L_\infty\left((0,T); H_0^1(\Omega)\right)$ (with square summable time derivative) for weakly singular kernels ($k(t) \le ct^{-\alpha}$, $\alpha \in (0,1)$). The problem

$$\frac{d}{dt}[u(t) - a\int_{-\infty}^{t} k(t-s)u(s)ds] = Au(t) + a\int_{-\infty}^{t} l(t-s)u(s)ds - p(t) + q(t), \quad (2.2)$$

which arises in the study of dynamics of income and employment, has been treated in Dos Santos and Hernandez [17]. Existence and uniqueness of (continuous) mild solutions is established for continuous (matrices) k under a condition on the Laplace transform of k satisfied by $t^{\alpha}e^{-\beta t}$, $\beta > 0$, $\alpha \in (0,1)$. For continuous kernels the existence and uniqueness of Hölder continuous solutions has been discussed in Hernandez, Preto and O'Regan [20].

For more results one has to look into the abstract problem

$$u(t) + \int_{0}^{t} b(t-s)Au(s)ds \ni f(t), \quad u(0) = u_0. \quad (2.3)$$

This problem is shown to be equivalent to

$$\begin{gathered} a\frac{du}{dt} + \frac{d}{dt}\int_{0}^{t} k(t-s)u(s)ds + Au(t) \ni u_0k(t) + g(t), \\ u(0) = f(0) = u_0. \end{gathered} \quad (2.4)$$

where $g = af + k * f$. As an application, one may consider

$$\begin{gathered} \frac{\partial}{\partial t}[\alpha u + \int_{-\infty}^{t} b(t-s)u(s)ds] - \beta\sigma(u_x)_x = h(t,x), \quad x \in (0,1), \\ u(t,0) = u(t,1) = 0, \quad t > 0. \end{gathered} \quad (2.5)$$

This well-posedness (existence of generalized and strong solutions) of these problems is established in Clément and Nohel [9] for completely positive kernels b; a general definition satisfied, for instance, by

(i) $b \in L^1(0,T)$ nonnegative, non-increasing and log convex, or
(ii) (Special case of (i)), $b \in L^1(0,T)$ and is completely monotone on $(0,T)$.

The nonlinear case is treated in Crandall and Nohel [15] for $b \in AC[0,T] \cap BV[0,T]$, $b(0) > 0$ in addition to (i). Baillon and Clément [4] considered the same (abstract and application) problem and established existence and uniqueness under the assumption $b \ge 0$, b nonincreasing and $b \in BV_{\rm loc}[0,\infty)$. This work has been extended from Hilbert spaces to Banach spaces by Kato, Kobayasi and Miyadera [23].

Clément and Nohel [9] also considered problem (2.3) with $f(t) = u_0 + (b * g)(t)$ and completely positive kernels.

A nonlinear version of (2.5) is investigated in Jakubowski and Wittbold [22], namely

$$\begin{aligned} &\frac{\partial}{\partial t}[\alpha(\psi(u(t,x)) - \psi(u_0(x)) + \int_{0}^{t} k(t-s)(\psi(u(s,x)) - \psi(u_0(x)))ds)] \\ &= \operatorname{div}\ \sigma(x, \nabla u(t,,x)) + f(t,x)\,. \end{aligned}$$

Entropy solutions are sought in $L^1(\Omega)$ (a space which does not enjoy the Radon-Nikodyn property) and continuity of generalized solutions is proved when $k \in L^1_{\rm loc}(0,\infty)$ and $\alpha + \int_0^t k(s)ds > 0$, for all $t \ge 0$. Digging deeper we are lead to the theory of rigid heat-conductors with memory. Indeed, MacCamy [28], Nunziato

[30], Coleman and Gurtin [13], developed a theory for heat flow in materials with fading memory based on the balance of heat law

$$e_t = -q_x + h, \tag{2.6}$$

where

$$e(t,x) = \alpha u(t,x) + \int_0^t k(t-s)u(s,x)dx, t \geq 0, \quad 0 \leq x \leq 1. \tag{2.7}$$

is the internal energy,

$$q(t,x) = -\beta u_x(t,x) + \int_0^t l(t-s)u_x(s,x)ds, t \geq 0, \quad 0 \leq x \leq 1, \tag{2.8}$$

is the heat flux and $h(t,x)$ is the extended heat supply. For the remaining parameters and kernels, we note that α is the heat capacity, β is the thermal conductivity, k is the internal energy relation function, and l is the heat flux relaxation function. Taking into account (2.7) and (2.8) in (2.6) we find

$$\begin{gathered}
\frac{\partial}{\partial t}[\alpha u(t,x) + \int_0^t k(t-s)u(s,x)dx] \\
= \beta u_{xx}(t,x) - \int_0^t l(t-s)u_{xx}(s,x)ds + h(t,x), \quad t \geq 0, 0 \leq x \leq 1 \\
u(t,0) = u(t,1) = 0, \quad u(0,x) = u_0.
\end{gathered} \tag{2.9}$$

Many existing results in the literature, apply to this problem directly or indirectly through some transformations. Barbu and Malik [6] discussed the problem

$$\begin{gathered}
u'(t) + Bu(t) + \int_0^t l(t-s)Au(s)ds + \int_0^t k(t-s)u(s)ds \ni f(t), \\
u(0) = u_0.
\end{gathered}$$

They proved existence and uniqueness in the space of (weakly) continuous functions with the assumption $k, k' \in L^1_{\rm loc}([0,\infty];\mathbb{R})$.

A couple of years later Clément and Nohel [9] gave problem (2.9) as an application of the abstract equation (2.3) after transforming it into the Volterra equation

$$u(t) + (k * u)(t) + (\psi * Au)(t) = F(t),$$

for some $F(t)$, and then into the simple form

$$u(t) + (\psi * Au)(t) = G(t) = F(t) - (r(k) * F)(t).$$

In 1982, Londen and Nohel [26] investigated the problem

$$\begin{gathered}
\frac{du}{dt}(t) + Bu(t) + (l * Au)(t) + \frac{d}{dt}(k * u(t)) \ni f(t) \\
u(0) = u_0 \quad \text{a.e. in } \mathbb{R}^+
\end{gathered}$$

generalizing the work of Crandall, Lunardi and Nohel [14] where $k = 0$. They assumed that k is a locally absolutely continuous function on $[0,\infty)$ to prove existence (without uniqueness) in the space of continuous functions. A few years later, Da Prato and Lunardi [16] established the existence, uniqueness and regularity of solutions in some spaces of continuous functions under some assumptions on the kernel satisfied by $e^{-\beta t}t^{\alpha-1}$, $\beta > 0$, $\alpha \in (0,1)$.

In the same year, Clément and Da Prato published the paper [10] where they proved regularity in Hölder spaces, Sobolev spaces and spaces of bounded uniformly continuous functions. The kernel is assumed to be summable, nonnegative and nonincreasing. See also Keyantuo and Lizama [24] for regularity in $\mathbf{L^p}$ spaces, Hölder spaces and Besov space.

The same authors examined regularity in Hölder spaces for locally summable and 2-regular kernels on $\mathbb{R}$ in Keyantuo and Lizama [25]. For the same type of kernels we note that existence and uniqueness has been established in $\mathbf{L^p}$ space in Clément and Prüss [11] as well.

A slightly more general problem is treated in Grasselli and Lorenzi [18]. It is proved that a solution $u \in \mathbf{L}^\infty((0,T);\mathbf{L^2}(\Omega)) \cap \mathbf{L^2}((0,T);\mathbf{H_0^1}(\Omega))$ such that $u_t \in \mathbf{L^2}((0,T);\mathbf{H^{-1}}(\Omega))$ in case $k \in L^1(0,\infty)$.

The well-posedness in the space of continuous functions is shown also for summable kernels satisfying $\lambda\tilde{k}(\lambda) \geq 0$, for all $\lambda \in \mathbb{R}$ where $\tilde{k}(\lambda)$ is the Fourier sine transform of k. This condition is satisfied by summable nonincreasing functions. Before going to the more recent works, we pause to note that problem (1.1) with

$$k(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \in (0,1)$$

becomes the fractionally damped heat equation

$$u_t + D^\alpha u = \Delta u$$

where D^α is the Caputo fractional derivative operator. The well-posedness of parabolic fractional equations is established in Ashyralyev [3].

We refer the reader to the work of Yin [34] for the general problem

$$u_t = a(t,x,u,u_x)u_{xx} + b(t,x,u,u_x) + \int_0^t k(s,x,u,u_x)ds$$

and to the book [19] for more details.

In the context of neutral differential equations, equations of the form

$$\frac{d}{dt}[u(t) - f(t,u_t,\int_0^t k(t,s,u_t)ds)] = Au(t) + \int_0^t l(t,s,u_s)ds + g(t,u_t,\int_0^t m(t,s,u_s)ds)$$

have been investigated by many authors: Balachandran, Annapoorani and Kim [5], Akiladevi, Balachandran and Kim [2].

3. Asymptotic behavior

Regarding the long time behavior of solutions to problem (2.1) we could not find results on this precise form, so we moved to similar problems, namely problem (2.9). Barbu and Malik in [6] proved the convergence of solutions to zero when k, $k' \in L^1_{\rm loc}([0,\infty);\mathbb{R})$ and k is completely positive ($k \in \mathbf{C^2}(0,\infty) \cap C[0,\infty)$, $k(0) > 0$, $(-1)^n k^n(t) \geq 0$, $n = 0,1,2$), see also Clement, MacCamy and Nohel [12]. The same result is found in Kato, Kobayasi and Miyadera in [23] with $k \in \mathbf{BV}_{\rm loc}[0,\infty)$ and without the convexity assumption.

Londen and Nohel in [26] proved the convergence in case $k \in \mathbf{LAC}(\mathbb{R}^+)$, $k \geq 0, k' \leq 0$ on $\mathbb{R}^+$, and $|k'(t)| \leq ct^{-\alpha}$, $t \in [1,\infty)$, $c > 0$, $\alpha > 3/2$. A similar result is achieved in Aizicovici [1], but without this l ast condition on the growth of $k'(t)$. In these works (and many others which appeared in the same period and after that) the limit u_∞ is the equilibrium of the system. For instance, for the problem (2.9), we have $u_\infty(x) = \alpha\left(\beta - \int_0^\infty h(s)ds\right)^{-1} v(x)$, where v is the unique solution of $-v_{xx} = h_\infty$ with $v(0) = v(1) = 0$ and $h_\infty(x) = \lim_{t\to\infty} h(t,x)$ (assumed to exist).

Further, $u_\infty = 0$ if $\lim \int_{t-1}^{t} |h(s)|^2 ds = 0$, see Londen and Nohel [26]. These results hold for higher dimensions as well. In this case, for $h \neq 0$, another condition on k is added, namely $k'(t) + \frac{h(0)}{\beta} k(t) \leq 0$, $t \geq 0$. This condition has been removed later by Lunardi [27].

In our presentation above, while surveying some results, we focussed only on the assumptions on the kernel k and somewhat on the underlying spaces. This is intentional as we are concerned by problem (2.1) which corresponds to $h = 0$. We shall seek conditions on k which will ensure some specific decay rates of the solutions.

Of concern to us is the work of Nachlinger and Nunziato [29], where a similar problem to (2.9) is studied (with $-h(t)$ instead of $h(t)$ and an infinite history)

$$\frac{\partial}{\partial t}[\alpha u(t,x) + \int_0^\infty k(s)u(s,x)ds] = \beta \Delta u(t,x) + \int_0^\infty l(s)\Delta u(t-s,x)ds. \tag{3.1}$$

It is proved there that solutions decay exponentially to zero in the $\mathbf{L^2}$-norm provided that $k(0) \geq 0$, $k \geq 0$, $k(t) \to 0$ as $t \to \infty$, and $\sup_{t\in[0,\infty)} |\int_0^t e^{\delta\mu s} k'(s)ds| < \alpha\mu(1-\delta)\delta$ for some $0 < \delta < 2/3$, $\mu = \frac{1}{\alpha}[k(0) + \lambda\beta]$ where λ is the smallest positive eigenvalue of the problem

$$-\Delta v = \lambda v,$$
$$v|_{\partial\Omega} = 0$$

and same condition on l. It is our intention here to improve this work.

4. Preliminaries

In this section we shall present some material we will need in our paper later.

Lemma 4.1 (Young inequality). *For all $a, b \in \mathbb{R}$, we have $ab \leq \delta a^2 + \frac{b^2}{4\delta}$, $\delta > 0$.*

In the next lemma, $\|\cdot\|_p$ denotes the L^p-norm (where L^p is the usual Lebesgue space). The norm $\|\cdot\|$ will stand for $\|\cdot\|_2$.

Lemma 4.2 (Young inequality for convolution, see [8]). *If $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ with $1 \leq p, q, r \leq \infty$, then*

$$\|f \star g\|_r \leq \|f\|_p \, \|g\|_q \, ,$$

where $(f \star g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy$.

We will also use the well-known Poincaré inequality given in the following lemma.

Lemma 4.3. *Let Ω be a sufficiently regular domain in $\mathbb{R}^d$. Then, there exists a positive constant C_p such that, for every $u \in H_0^1(\Omega)$*

$$C_p\|u\|_2 \ \leq \ \|\nabla u\|_2$$

where $H_0^1(\Omega)$ is the Sobolev space of all functions $u \in H^1(\Omega)$ which vanish along the boundary of Ω.

5. Main results

Here we shall be concerned by weak and strong solutions.

Definition 5.1. A function $u : [0,T] \to H_0^1(\Omega)$ is called weak solution of (1.1) if $u \in L^2(0,T;H_0^1(\Omega))$, $u_t \in L^2(0,T;H^{-1}(\Omega))$ and for every $v \in H_0^1(\Omega)$ we have

$$\langle u_t(t), v\rangle + \Big\langle \int_0^t k(t-s)u_t(s)ds, v\Big\rangle + \langle \nabla u(t), \nabla v)\rangle = 0$$

a.e. in $[0,T]$. Moreover $u(0) = u_0$. Here, $\langle \cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

By the above considerations, if $u_0 \in H_0^1(\Omega)$, then there exists a unique weak solution to problem (1.1). In case $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ (which is the domain of our operator), there exists a unique strong solution of problem (1.1) which is a more regular function that satisfies the equation pointwise. These definitions justify our computation below.

The following functionals will be useful in order to cancel out some undesirable terms which will appear in the estimations:

$$\begin{gathered}
\phi(t) = \int_0^t \Big(\int_t^\infty |k'(\sigma - s)|d\sigma\Big)\|u\|^2 ds, \quad t \geq 0,\\
\psi(t) = \int_0^t \Big(\int_t^\infty k(\sigma - s)d\sigma\Big)\|u_t\|^2 ds, \quad t \geq 0.
\end{gathered}$$

Our main assumption on the kernel k is

- (H1) $k \in C^1[0,\infty) \cap L^1(0,\infty)$, $k \geq 0$ and there exists a continuous function $\mu(t)$ such that $\lim_{t\to\infty}\mu(t)$ exists and $|k'|(t-s) \geq \mu(t)\int_t^\infty |k'(\sigma - s)|d\sigma$, $t \geq s \geq 0$.

Theorem 5.2. *Assume that* (H1) *holds,* $u_0 \in H_0^1(\Omega)$, $k' \in L^1(0,\infty)$ *and* $\|k'\|_1 < C_p^2 + k(0)$. *We have*

- (a) *If* $\lim_{t\to\infty}\mu(t) = 0$, *then* $\|u\|^2 \leq Me^{-\alpha\int_0^t \mu(s)ds}$ *for some* $M,\alpha > 0$ *and* $t \geq 0$ *provided that* $\int_0^\infty k^2(s)e^{C_2\int_0^s \mu(\sigma)d\sigma}ds$ *is bounded.*
- (b) *If* $\lim_{t\to\infty}\mu(t) \neq 0$, *then* $\|u\|^2 \leq Ne^{-\beta t}$ *f or some* $N,\beta > 0$ *and* $t \geq 0$ *provided that* $\int_0^\infty k^2(s)e^{C_4 s}ds$ *is bounded*

where C_4 *and* C_2 *are constants, determined in the proof.*

Proof. Let us multiply the equation in (1.1) by u and integrate over Ω,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2dx + \int_\Omega u\int_0^t k(t-s)u_t(s)\,ds\,dx = -\int_\Omega |\nabla u|^2 dx, \quad t > 0. \tag{5.1}$$

Note that we have used Green's formula in the right-hand side and the homogeneous Dirichlet boundary condition. The second term in the left-hand side of (5.1) may be written as

$$\begin{aligned}
&\int_\Omega u\{k(0)u(t) - k(t)u(0) + \int_0^t k'(t-s)u(s)ds\}dx\\
&= k(0)\int_\Omega u^2dx - k(t)\int_\Omega u_0(x)udx + \int_\Omega u\int_0^t k'(t-s)u(s)\,ds\,dx,
\end{aligned}\tag{5.2}$$

for $t>0$. The last two terms in the right-hand side of (5.2) are estimated as follows:

$$k(t)\int_\Omega u_0 u dx \le \delta_1 \int_\Omega |u|^2 dx + \frac{k^2(t)}{4\delta_1}\int_\Omega |u_0|^2 dx, \quad \delta_1>0;$$

and using Young and Cauchy-Schwarz inequalities

$$\begin{aligned}
&\int_\Omega u \int_0^t k'(t-s)u(s)\,ds\,dx \\
&\le \delta_2 \int_\Omega |u|^2 dx + \frac{1}{4\delta_2}\Big(\int_0^t |k'|ds\Big)\int_0^t |k'|(t-s)\int_\Omega |u^2(s)|\,dx\,ds, \quad \delta_2>0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\phi'(t) &= \Big(\int_t^\infty |k'|(\sigma-t)d\sigma\Big)\int_\Omega |u|^2 dx - \int_0^t |k'|(t-s)\int_\Omega |u|^2(s)\,dx\,ds \\
&= \Big(\int_0^\infty |k'|ds\Big)\int_\Omega |u|^2 dx - \int_0^t |k'|(t-s)\int_\Omega |u|^2(s)\,ds\,dx, \quad t\ge 0.
\end{aligned}$$

By the assumption (H1), we see that for $0<\delta_3<1$,

$$\begin{aligned}
\phi'(t) &\le \Big(\int_0^\infty |k'|ds\Big)\|u\|^2 - \delta_3\int_0^t |k'|(t-s)\|u\|^2 ds \\
&\quad - (1-\delta_3)\int_0^t |k'|(t-s)\|u\|^2 ds \\
&\le \Big(\int_0^\infty |k'|ds\Big)\|u\|^2 - \delta_3\int_0^t |k'|(t-s)\|u\|^2 ds - (1-\delta_3)\mu(t)\phi(t),
\end{aligned} \tag{5.3}$$

for $t>0$. Therefore, the derivative of

$$\mathfrak{L}(t) := \frac{1}{2}\|u\|^2 + \lambda\phi(t), \quad \lambda>0,\ t\ge 0$$

along solutions of (1.1), is estimated, using (5.2) and (5.3) as follows

$$\begin{aligned}
\mathfrak{L}'(t) &= \frac{1}{2}\frac{d}{dt}\|u\|^2 + \lambda\phi'(t) \\
&\le -\|\nabla u\|^2 - k(0)\|u\|^2 + \delta_1\|u\|^2 + \frac{k^2(t)}{4\delta_1}\|u_0\|^2 \\
&\quad + \delta_2\|u\|^2 + \frac{1}{4\delta_2}\Big(\int_0^\infty |k'|ds\Big)\int_0^t |k'(t-s)|\|u(s)\|^2 ds \\
&\quad + \lambda\Big(\int_0^\infty |k'|ds\Big)\|u\|^2 - \lambda\delta_3\int_0^t |k'|(t-s)\|u(s)\|^2 ds \\
&\quad - \lambda(1-\delta_3)\mu(t)\phi(t), t\ge 0
\end{aligned}$$

or simply, using Poincaré inequality with constant C_p,

$$\begin{aligned}
\mathfrak{L}'(t) &\le -\big(C_p^2 + k(0) - \delta_1 - \delta_2 - \lambda\|k'\|_1\big)\|u\|^2 \\
&\quad - \Big(\lambda\delta_3 - \frac{1}{4\delta_2}\|k'\|_1\Big)\int_0^t |k'|(t-s)\|u\|^2 ds \\
&\quad - \lambda(1-\delta_3)\mu(t)\phi(t) + \frac{k^2(t)}{4\delta_1}\|u_0\|^2, \quad t\ge 0.
\end{aligned} \tag{5.4}$$

If

$$C_p^2 + k(0) - \delta_1 - \delta_2 - \lambda\|k'\|_1 > 0 \quad \text{and} \quad \lambda\delta_3 - \frac{\|k'\|_1}{4\delta_2} \geq 0 \tag{5.5}$$

then

$$\mathfrak{L}'(t) \leq -C_1\|u\|^2 - \lambda(1-\delta_3)\mu(t)\phi(t) + \frac{k^2(t)}{4\delta_1}\|u_0\|^2, \quad t \geq 0. \tag{5.6}$$

Let us first ignore δ_1 in (5.5) as it may be very small and will not affect the decay. Combining both relations in (5.5) shows that we may find $\lambda > 0$ provided that

$$\delta_2 + \frac{\|k'\|_1^2}{4\delta_2\delta_3} < C_p^2 + k(0) \quad \text{or} \quad \delta_2^2 - [C_p^2 + k(0)]\delta_2 + \frac{\|k'\|_1^2}{4\delta_3} < 0.$$

Solving this quadratic inequality shows that δ_2 exists if $\|k'\|_1^2 < \delta_3[C_p^2 + k(0)]^2$. In turn, under our assumption $\|k'\|_1 < C_p^2 + k(0)$ we may pick δ_3 close enough to (but smaller than) one. Now back to (5.6), we discuss two cases:

case 1: If $\lim_{t\to\infty}\mu(t) = 0$, then for any $M > 0$, there exits a $t_M > 0$ such that $\mu(t) \leq M$, $\forall t \geq t_M$. Therefore, this applies in particular to C_1 and we get a first order linear differential inequality in $\mathfrak{L}$

$$\mathfrak{L}'(t) \leq -C_2\mu(t)\mathfrak{L}(t) + \frac{k^2(t)}{4\delta_1}\|u_0\|^2$$

for some $C_2 > 0$ and $t \geq t_{C_1}$. Clearly

$$\mathfrak{L}(t) \leq \mathfrak{L}(0)e^{-C_2\int_0^t \mu(s)ds} + \frac{\|u_0\|^2}{4\delta_1}e^{-C_2\int_0^t \mu(s)ds}\int_0^t k^2(s)e^{C_2\int_0^s \mu(\sigma)d\sigma}ds,$$

for $t \geq t_{C_1}$. If

$$\int_0^\infty k^2(s)e^{C_2\int_0^s \mu(\sigma)d\sigma}ds \leq A \quad \text{for some } A > 0,$$

then

$$\mathfrak{L}(t) \leq \Big(\mathfrak{L}(0) + \frac{\|u_0\|^2 A}{4\delta_1}\Big)e^{-C_2\int_0^t \mu(s)ds}, \quad t \geq t_{C_1}. \tag{5.7}$$

case 2: If $\lim_{t\to\infty}\mu(t) \neq 0$, then there exists $t^\star > 0$ and $C_3 > 0$ such that $\mu(t) \geq C_3$, for all $t \geq t^\star$. We deduce that

$$\mathfrak{L}'(t) \leq -C_4\mathfrak{L}(t) + \frac{k^2(t)}{4\delta_1}\|u_0\|^2, \quad t \geq t^\star$$

for some $C_4 > 0$. We obtain

$$\frac{d}{dt}[\mathfrak{L}(t)e^{C_4 t}] \leq \frac{k^2(t)\|u_0\|^2}{4\delta_1}e^{C_4 t}, \quad t \geq t^\star$$

or

$$\mathfrak{L}(t)e^{C_4 t} \leq \mathfrak{L}(t^\star)e^{C_4 t^\star} + \int_0^{t^\star} \frac{k^2(s)\|u_0\|^2}{4\delta_1}e^{C_4 s}ds, \quad t \geq t^\star.$$

Hence,

$$\mathfrak{L}(t) \leq \mathfrak{L}(t^\star)e^{-C_4(t-t^\star)} + e^{-C_4 t}\frac{\|u_0\|^2}{4\delta_1}\int_0^{t^\star} k^2(s)e^{C_4 s}ds, \quad t \geq t^\star.$$

If $\int_0^\infty k^2(s)e^{C_4 s}ds \leq B$ for some finite positive number B, we find

$$\mathfrak{L}(t) \leq C_5 e^{-C_4 t}, \quad t \geq t^\star,\ c_5 \geq 0. \tag{5.8}$$

By continuity we may extend the relation (5.7) and (5.8) to $[0, t_{C_1}]$ and $[0, t^\star]$ (with different coefficients). $\square$

Remark 5.3. It is important to emphasize the following observations:

(1) Note that the assumption (H1) is satisfied by many functions and in particular by exponential (with negative powers) functions. Polynomially decaying functions are also there but do not satisfy the assumptions in Nachlinger and Nunziato [29]. Therefore, we have different kinds of decay corresponding to different classes of kernels including as special kernels those which are exponentially decaying functions.

(2) Our assumption $\|k'\|_1 < C_p^2 + k(0)$ is not a very restrictive condition. If for instance, k is a non-increasing function then

$$\int_0^\infty |k'(s)|ds = -\int_0^\infty k'(s)ds = -k(\infty) + k(0) \leq k(0).$$

This means that $\|k'\|_1 < C_p^2 + k(0)$ is trivially satisfied.

(3) Note also that this condition on k' is not tested against an exponential function as in Nachlinger and Nunziato [29].

(4) It is worth noting also that in the conditions

$$\int_0^\infty k^2(s)e^{c_2\int_0^s \mu(\sigma)d\sigma}ds \leq A, \quad \int_0^\infty k^2(s)e^{C_4 s}ds \leq B$$

A and B do not have to take specific values and therefore are arbitrary (we need them only to be finite). Again, these conditions are on k.

(5) In fact, we do not really need the boundedness of the expressions in the previous remark. We just need to ensure that they do not grow (with integrals from 0 to t) faster than the expressions: $e^{C_2\int_0^t \mu(s)ds}$ and $e^{C_4 t}$, respectively.

In the next theorem, we drop the conditions on the derivative of the kernel. We need, however, the initial data and the solution to be smoother. We shall assume that the initial data is in the domain of the operator and the solution to be a strong one. The multiplication of the equation in (1.1) by u_t and integration over Ω, taking into account the boundary conditions, yields

$$\int_\Omega u_t^2 dx + \int_\Omega u_t \int_0^t k(t-s)u_t(s)\,ds\,dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx,$$

or

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx = -\int_\Omega u_t^2 dx - \int_\Omega u_t \int_0^t k(t-s)u_t\,ds\,dx, t \geq 0. \tag{5.9}$$

Clearly, this gives rise to a nice term, namely $-\int_\Omega u_t^2 dx$ and suggests considering $\|\nabla u\|^2$ together with the first energy functional. That is, we let

$$E(t) = \frac{1}{2}(\|u\|^2 + \|\nabla u\|^2),\ t \geq 0.$$

For our kernel we shall assume that k satisfies the condition

(H2) $k \in C(0,\infty) \cap L^1(0,\infty)$, $k \geq 0$ and there exists a continuous function η such that $\lim_{t\to\infty} \eta(t)$ exists and $k(t-s) \geq \eta(t)\int_t^\infty k(\sigma - s)d\sigma$ for $t \geq s \geq 0$.

Theorem 5.4. *Assume* (H2) *holds,* $\|k\|_1^2 < \frac{2C_p^2}{1+2C_p^2}$, *and* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. *Then*

(a) *If* $\lim_{t\to\infty} \eta(t) = 0$, *then* $E(t) \leq E(0)e^{-\gamma\int_0^t \eta(s)ds}$, *for some* $\gamma > 0$, $t \geq 0$

(b) *If* $\lim_{t\to\infty}\eta(t)\neq 0$, *then* $E(t)\le E(0)e^{-\xi t}$, *for some* $\xi>0$, $t\ge 0$.

Proof. In view of (5.1) and (5.9), we have

$$\begin{aligned} E'(t) = &-\int_\Omega u\int_0^t k(t-s)u_t(s)\,ds\,dx - \int_\Omega |\nabla u|^2dx \\ &-\int_\Omega u_t^2dx - \int_\Omega u_t\int_0^t k(t-s)u_t(s)\,ds\,dx, \quad t\ge 0. \end{aligned} \tag{5.10}$$

Here, unlike in the first proof, we do not integrate by parts in the first term appearing in the right hand side of (5.10). We rather estimate it as follows

$$\begin{aligned} &\int_\Omega u\int_0^t k(t-s)u_t(s)\,ds\,dx \\ &\le \delta_1\|u\|^2 + \frac{1}{4\delta_1}\Big(\int_0^t kds\Big)\int_\Omega\int_0^t k(t-s)u_t^2(s)\,ds\,dx, \quad \delta_1>0, t\ge 0. \end{aligned} \tag{5.11}$$

Similarly,

$$\begin{aligned} &\int_\Omega u_t\int_0^t k(t-s)u_t(s)\,ds\,dx \\ &\le \delta_2\|u_t\|^2 + \frac{1}{4\delta_2}\Big(\int_0^t kds\Big)\int_\Omega\int_0^t k(t-s)u_t^2(s)\,ds\,dx, \quad \delta_2>0,\ t\ge 0. \end{aligned} \tag{5.12}$$

To deal with the last two terms in (5.11) and (5.12), we introduce the functional

$$\psi(t) = \int_0^t \Big(\int_t^\infty k(\sigma-s)d\sigma\Big)\|u_t(s)\|^2ds, \quad t\ge 0. \tag{5.13}$$

Its derivative is given by

$$\psi'(t) = \Big(\int_0^\infty k(s)ds\Big)\|u_t(s)\|^2 - \int_0^t k(t-s)\|u_t\|^2ds, \quad t\ge 0. \tag{5.14}$$

Now, we differentiate the expression

$$V(t) = E(t) + \gamma\psi(t)\ t\ge 0, \tag{5.15}$$

for some $\gamma>0$ to be determined, along solutions of (1.1). We find

$$\begin{aligned} V'(t) \le &-\|\nabla u\|^2 - \|u_t\|^2 + \delta_1\|u\|^2 + \delta_2\|u_t\|^2 \\ &+ \frac{\|k\|_1}{4}(\frac{1}{\delta_1}+\frac{1}{\delta_2})\int_0^t k(t-s)\|u_t(s)\|^2ds \\ &+ \gamma\|k\|_1\|u_t\|^2 - \gamma\int_0^t k(t-s)\|u_t(s)\|^2ds, \quad t\ge 0. \end{aligned} \tag{5.16}$$

or

$$\begin{aligned} V'(t) \le &-(\delta_3C_p^2-\delta_1)\|u\|^2 - (1-\delta_3)\|\nabla u\|^2 - [1-(\delta_2+\gamma\|k\|_1)]\|u_t\|^2 \\ &- \big[\delta_4\gamma - \frac{\|k\|_1}{4}(\frac{1}{\delta_1}+\frac{1}{\delta_2})\big]\int_0^t k(t-s)\|u_t(s)\|^2ds \\ &- (1-\delta_4)\gamma\int_0^t k(t-s)\|u_t(s)\|^2ds, \quad t\ge 0, \end{aligned} \tag{5.17}$$

for some δ_3 and δ_4 satisfying $0 < \delta_3 < 1$ and $0 < \delta_4 < 1$. We shall select the different parameters as follows:

$$\begin{gathered}\delta_3 C_p^2 - \delta_1 > 0,\\ \delta_2 + \gamma\|k\|_1 < 1,\\ \delta_4\gamma - \frac{\|k\|_1}{4}\big(\frac{1}{\delta_1} + \frac{1}{\delta_2}\big) \geq 0.\end{gathered} \tag{5.18}$$

Note that δ_3 and δ_4 may be selected if $\delta_1 < C_p^2$ and $\frac{\|k\|_1}{4}(\frac{1}{\delta_1} + \frac{1}{\delta_2}) < \gamma$.

To fix ideas, we pick $\delta_2 = 1/2$. Then, it is possible to choose γ so that the last two relations are fulfilled if

$$\frac{\|k\|_1}{4}\Big(\frac{1}{\delta_1} + 2\Big) < \frac{1}{2\|k\|_1}.$$

This necessitates

$$\|k\|_1^2 < \frac{2C_p^2}{2C_p^2 + 1}.$$

We are lead to

$$V'(t) \leq -C_1\|u\|^2 - C_2\|\nabla u\|^2 - (1-\delta_4)\gamma\int_0^t k(t-s)\|u_t(s)\|^2 ds, \quad t \geq 0.$$

By Assumption (H2) on the kernel k, we obtain

$$V'(t) \leq -C_1\|u\|^2 - C_2\|\nabla u\|^2 - (1-\delta_4)\gamma\eta(t)\psi(t), \quad t \geq 0.$$

At this stage we may proceed as in the proof of Theorem 5.2 and discuss two cases:

(a) If $\lim_{t\to\infty}\eta(t) = 0$, then there exists $t_1 > 0$ such that

$$V'(t) \leq -C_3\eta(t)V(t), \quad t \geq t_1$$

for some $C_3 > 0$. This implies that

$$V(t) \leq V(0)e^{-C_3\int_0^t \eta(s)ds}, \quad t \geq t_1.$$

(b) If $\lim_{t\to\infty}\eta(t) \neq 0$, then there exists $t_2 > 0$ and $C_4 > 0$ such that

$$V(t) \leq V(0)e^{-C_4 t}, \quad t \geq t_2.$$

By continuity, this estimation (as well as the previous one in (a)) may be extended to the interval $[0, t_2]$. This completes the proof. $\square$

The result in Theorem 5.4 may be improved further if we assume

$$\|e^{\alpha t}k\|_1 \leq \frac{2C_p^2}{1 + 2C_p^2}$$

for some $\alpha > 0$ instead of

$$\|k\|_1^2 < \frac{2C_p^2}{1 + 2C_p^2}.$$

Theorem 5.5. *If k is a nonnegative continuous function such that with $\|e^{\alpha t}k\|_1 \leq \frac{2C_p^2}{1+2C_p^2}$ for some $0 < \alpha \leq \frac{C_p^2}{2(1+C_p^2)}$, then $E(t) \leq E(0)e^{-2\alpha t}$, $t \geq 0$.*

Proof. Let us consider the functional

$$U(t) := \frac{e^{2\alpha t}}{2}\int_\Omega \big(|u|^2 + |\nabla u|^2\big)\,dx, \quad t \ge 0$$

for some $0 < \alpha < 1$. Differentiating this expression along solution of (1.1), we obtain

$$\begin{aligned}
U'(t) &= 2\alpha U(t) + e^{2\alpha t}\int_\Omega (u_t u + \nabla u_t \cdot \nabla u)dx \\
&= 2\alpha U(t) - e^{2\alpha t}\int_\Omega u \int_0^t k(t-s)u_t(s)\,ds\,dx \\
&\quad - e^{2\alpha t}\int_\Omega |\nabla u|^2 dx - e^{2\alpha t}\int_\Omega u_t^2 dx \\
&\quad - e^{2\alpha t}\int_\Omega u_t \int_0^t k(t-s)u_t(s)\,ds\,dx, \quad t \ge 0.
\end{aligned}$$

An integration over $(0,t)$, gives

$$\begin{aligned}
U(t) &= U(0) + 2\alpha\int_0^t U(s)ds - \int_\Omega\int_0^t e^{\alpha s}u(s)\int_0^s e^{\alpha(s-\sigma)}\cdot k(s-\sigma)e^{\alpha\sigma}u_t(\sigma)d\sigma\,ds\,dx \\
&\quad - \int_0^t e^{2\alpha s}\int_\Omega |\nabla u|^2\,dx\,ds - \int_0^t e^{2\alpha s}\int_\Omega u_t^2\,dx\,ds \\
&\quad - \int_\Omega\int_0^t e^{\alpha s}u_t(s)\int_0^s k(s-\sigma)e^{\alpha(s-\sigma)}\cdot e^{\alpha\sigma}u_t(\sigma)d\sigma\,ds\,dx, \quad t \ge 0.
\end{aligned} \tag{5.19}$$

By Young inequality, we can estimate

$$\begin{aligned}
&\int_\Omega\int_0^t e^{\alpha s}u(s)\int_0^s e^{\alpha(s-\sigma)}k(s-\sigma)e^{\alpha\sigma}u_t(\sigma)d\sigma\,ds\,dx \\
&\le \int_\Omega\Big(\int_0^t e^{2\alpha s}u^2(s)ds\Big)^{1/2}\Big[\int_0^t\Big(\int_0^s e^{\alpha(s-\sigma)}k(s-\sigma)e^{\alpha\sigma}u_t(\sigma)d\sigma\Big)^2 ds\Big]^{1/2}dx \\
&\le \int_\Omega\Big(\int_0^t e^{2\alpha s}u^2(s)ds\Big)^{1/2}\Big(\int_0^t e^{\alpha s}k(s)ds\Big)\Big(\int_0^t e^{2\alpha\sigma}u_t^2(\sigma)d\sigma\Big)^{1/2}dx \\
&\le \delta_1\int_0^t e^{2\alpha s}\|u(s)\|^2 ds + \frac{\|e^{\alpha t}k\|_1^2}{4\delta_1}\int_0^t e^{2\alpha\sigma}\|u_t(\sigma)\|^2 d\sigma, \quad t \ge 0.
\end{aligned} \tag{5.20}$$

Similarly,

$$\begin{aligned}
&\int_\Omega\int_0^t e^{\alpha s}u_t(s)\int_0^s e^{\alpha(s-\sigma)}k(s-\sigma)e^{\alpha\sigma}u_t(\sigma)d\sigma\,ds\,dx \\
&\le \Big(\int_0^t e^{\alpha s}k(s)ds\Big)\int_\Omega\int_0^t e^{2\alpha s}u_t^2(s)\,ds\,dx, \quad t \ge 0.
\end{aligned} \tag{5.21}$$

Taking into account (5.20) and (5.21) in (5.19), we find

$$\begin{aligned}
U(t) &\le U(0) + 2\alpha\int_0^t U(s)ds - \int_0^t e^{2\alpha s}\|\nabla u\|^2 ds - \int_0^t e^{2\alpha s}\|u_t(s)\|^2 ds \\
&\quad + \delta_1\int_0^t e^{2\alpha s}\|u(s)\|^2 ds + \frac{\|e^{\alpha t}k\|_1^2}{4\delta_1}\int_0^t e^{2\alpha s}\|u_t(s)\|^2 ds
\end{aligned}$$

$$+\Big(\int_0^t e^{\alpha s}k(s)ds\Big)\int_0^t e^{2\alpha s}\|u_t(s)\|^2 ds,\quad t\geq 0,$$

or

$$\begin{aligned}U(t)&\leq U(0)+(\alpha+\delta_1)\int_0^t e^{2\alpha s}\|u(s)\|^2ds+(\alpha-1)\int_0^t e^{2\alpha s}\|\nabla u\|^2ds\\&\quad+\Big(\|e^{\alpha t}k\|_1+\frac{\|e^{\alpha t}k\|_1^2}{4\delta_1}-1\Big)\int_0^t e^{2\alpha s}\|u_t(s)\|^2ds,\quad t\geq 0.\end{aligned}$$

As $0<\alpha<1$, we get

$$\begin{aligned}U(t)&\leq U(0)+[\alpha+\delta_1+(\alpha-1)C_p^2]\int_0^t e^{2\alpha s}\|u(s)\|^2ds\\&\quad+\big[(1+\frac{1}{4\delta_1})\|e^{\alpha t}k\|_1-1\big]\int_0^t e^{2\alpha s}\|u_t(s)\|^2ds.\end{aligned}$$

We need to select different parameters in such a manner that

$$\begin{gathered}\alpha+\delta_1+(\alpha-1)C_p^2\leq 0,\\(1+\frac{1}{4\delta_1})\|e^{\alpha t}k\|_1-1\leq 0.\end{gathered}$$

Let $\delta_1=C_p^2/2$ and $\alpha\leq\frac{C_p^2}{2(1+C_p^2)}$, then these relations are satisfied if

$$\|e^{\alpha t}k\|_1\leq\frac{2C_p^2}{2C_p^2+1}.$$

Under these conditions, we obtain

$$U(t)\leq U(0),\quad t\geq 0$$

and hence from the expression of the functional $U(t)$, we have

$$E(t)\leq e^{-2\alpha t}E(0),\quad t\geq 0.$$

This completes the proof. □

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