Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 305, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

APPROXIMATION OF THE LEADING SINGULAR COEFFICIENT OF AN ELLIPTIC FOURTH-ORDER EQUATION

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Communicated by Giovanni Molica Bisci

Abstract. The solution of the biharmonic equation with an homogeneous boundary conditions is decomposed into a regular part and a singular one. The later is written as a coefficient multiplied by the first singular function associated to the bilaplacian operator. In this paper, we consider the dual singular method for finding the value of the leading singular coefficient, and we use the mortar domain decomposition technique with the spectral discretization for its approximation. The numerical analysis leads to optimal error estimates. We present some numerical results which are in perfect coherence with the analysis developed in this paper.

1. Introduction

In a polygonal domain and when the data are smooth, the solution of an elliptic differential equation is irregular. For a homogeneous problem of the bilaplacian operator, we define some singular functions contingent to the geometry of the domain. The solution is the sum of two components: a regular part and singular functions ([16, 18, 19, 20]). The later are multiplied by appropriate coefficients called singular coefficients. To approximate the leading singularity coefficients two algorithms are deployed. We first refer to the Strang and Fix algorithm [21], permitting to add the leading singularity function to the discrete space [15]. Secondly, the singular dual method algorithm [5, 4]. In physics and particularly in solid mechanics (crack propagation), the leading singularity coefficient is very significant. The calculation of this coefficient was obtained by use of finite elements (Amara and Moussaoui [5, 6]). In this paper, we propose to use the mortar spectral element method combined with the method based on the singular dual function. We decompose the domain in an union of finite number of disjoint rectangles. On each rectangle, the discrete functions are polynomials of high degree. We enforce the discrete solution to satisfy a matching condition on the interfaces. Due to the non continuity of the discrete functions, mortar spectral element technique is nonconforming. For more details on the mortar spectral element method, we relate to Bernardi et al

2010 *Mathematics Subject Classification.* 35J15, 78M22.
Key words and phrases. Bilaplacian equation; singularity coefficient;
dual singular method; mortar spectral element method.

Submitted October 11, 2017. Published December 14, 2017.

[10, 11, 13, 14]. In this work, we prove that the order of the error estimation between the continuous leading singularity coefficient end the discrete one is optimal. This order is better than that obtained by the Strang and Fix algorithm.

The paper is outlined as follows. In the section 2, we present the geometry of the domain, the continuous problem and the dual singular method which allows us to calculate the leading coefficient of the singularity. This later depends only on the solution. The approximation of the leading singularity coefficient by the mortar element spectral method and the optimality estimation of the error is described in Section 3. Finally, the results of a numerical test are given in Section 4.

2. Geometry of the domain and the continuous problem

We denote by Ω a polygonal domain in $\mathbb{R}^2$ such that there exists a finite number of open rectangles $\Omega_i, 1 \le i \le I$, satisfying

$$\bar{\Omega} = \cup_{i=1}^{I} \bar{\Omega}_i, \quad \Omega_i \cap \Omega_l = \emptyset \quad \text{for } i \neq l. \tag{2.1}$$

and such that the intersection of each $\bar{\Omega}_i, 1 \le i \le K$, with the boundary $\partial\Omega$ is either empty or a corner or one or several entire edges of sub-domain Ω_i. We choose the coordinate axes parallel to the edge of the Ω_i. We denote by $\Gamma^{i,j}$, $1 \le j \le 4$ the edges of Ω^i and by $\gamma^{i,l}$, $1 \le i \neq l \le I$, the open segment such that

$$\bar{\gamma}^{i,l} = \partial\Omega^i \cap \partial\Omega^l.$$

The set $\mathcal{V}$ will be the set of all vertices of the Ω^i, $1 \le i \le I$ and $\mathcal{S} = \cup_{i=1}^{I} \cup_{j=1}^{4} \bar{\Gamma}^{i,j}$ the skeleton of the decomposition. We choose finite set of disjoint open segments γ^k, where k belongs to a finite set K such that $\mathcal{S} = \cup_{k\in K} \bar{\gamma}^k$, each γ^k, $k \in K$ is called mortar and its being a edge $\Gamma^{i(k),j(k)}$.

We are interested to non-convex domains, we assume that there exists an angle equal either to $\frac{3\pi}{2}$ or to 2π (case of the crack). Handling the singular function is local process, so that there is no restriction to suppose that the non-convex corner is unique.

2.1. **Notation.**

Let ω be the value of the non-convex angle equal either to $\frac{3\pi}{2}$ or to 2π, $\mathbf{a}$ be the corresponding corner of Ω and Δ be the open domain in Ω such that $\bar{\Delta}$ is the union of the $\bar{\Omega}_i$ which contain $\mathbf{a}$. We choose the origin of the coordinate axes at the point $\mathbf{a}$. We introduce a system of polar coordinates (r, θ) where r stands for the distance from $\mathbf{a}$ and θ is such that the line $\theta = 0$ contains an edge of $\partial\Omega$. For reasons which will appear later, we are lead to make the following conformity assumption: We suppose that the decomposition of the domain Δ is conforming (see figure 1). If $\mathbf{a}$ is a vertex of the mortar $\Gamma^{i(k),j(k)}$ which coincides with Γ^l a side of a sub-domain Ω^l, $l \neq i(k)$ then $N_{i(k)} \le N_l$, such that the restriction of a function to Δ is in $H^2(\Delta)$.

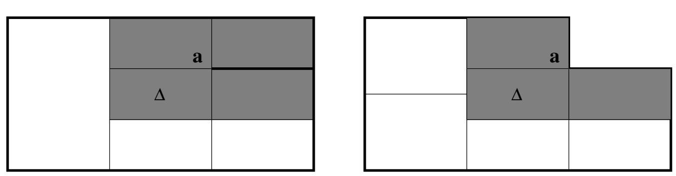


Figure 1. Domain Ω

The following biharmonic problem with homogenous boundary conditions is the model under consideration,

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

This problem admits the following variational formulation: Find $u \in H_0^2(\Omega)$ such that

$$\forall v \in H_0^2(\Omega) \quad \int_\Omega \Delta u . \Delta v \, dx\, dy = \int_\Omega f\, v \, dx\, dy \tag{2.3}$$

where f is in $L^2(\Omega)$.

Formulation (2.3) can be written in the equivalent form: Find $u \in H_0^2(\Omega)$ such that

$$\forall v \in H_0^2(\Omega) \quad \sum_{i=1}^{I} \int_{\Omega_i} (\Delta u|_{\Omega_i})\,(\Delta v|_{\Omega_i})\, dx\, dy = \sum_{i=1}^{I} \int_{\Omega_i} f|_{\Omega_i}\, v|_{\Omega_i}\, dx\, dy. \tag{2.4}$$

The function v is in $H_0^2(\Omega)$ if and only if $v|_{\Omega_i} \in H^2(\Omega_i)$, $1 \le i \le I$, v and $\partial_n v$ vanishes on $\partial\Omega$, $v|_{\Omega_i} = v|_{\Omega_l}$ and $\partial_n v|_{\Omega_i} = \partial_n v|_{\Omega_l}$ on $\gamma^{i,l}$, $1 \le i \ne l \le I$.

Using Lax-Milgram theorem we show that the problem is well posed. However, in a polygonal domain the global regularity of the solution depends on the angle ω. For a non negative real s, if f in $H^{s-2}(\Omega)$ then the solution u of problem (2.2) belongs to $H^{s+2}(\Omega)$ and there exists a positive constant C such that

$$\|u\|_{H^{s+2}(\Omega)} \le C\|f\|_{H^{s-2}(\Omega)}.$$

If $\omega = \frac{3\pi}{2}$, $s \le 0.544844$, and if $\omega = 2\pi$, $s \le 0.5$.

To enhance the regularity we decompose the solution as follows:

$$u = u_R + \mu\, \tau_1 \quad \text{such that } u_R \in H^{s+2}(\Omega) \tag{2.5}$$

where μ is the leading singular coefficient, τ_1 is the first singular function and there exists a positive constant C such that

$$\|u_R\|_{H^{s+2}(\Omega)} + |\mu| \le C\|f\|_{H^{s-2}(\Omega)}.$$

- When $\omega = \frac{3\pi}{2}$: $s < 1.544$ and $\tau_1(r,\theta) = \chi(r,\theta) r^{1.544}\psi(\theta)$,

$$\begin{aligned} \psi(\theta) =& 4.302\big(\cos(0.092\theta) - \cos(1.908\theta)\big) \\ &- 1.1815\big(10.869\sin(0.092\theta) - 0.524\sin(1.908\theta)\big). \end{aligned} \tag{2.6}$$

- In the case where $\omega = 2\pi$: $s < 1.5$ and $\tau_1(r,\theta) = \chi(r,\theta) r^{1.5}\psi(\theta)$,

$$\psi(\theta) = \big(\sin(1.5\theta) - 3\sin(0.5\theta) + \cos(1.5\theta) - \cos(0.5\theta)\big). \tag{2.7}$$

where χ is the C^∞ cut off function with support in $\overline{\Delta}$ which is equal to 1 in the neighborhood of $\mathbf{a}$.

To compute the leading singularity coefficient μ, we use the dual singularity functions [5, 17]. We consider the characteristic equation of the bilaplacian

$$F(z,\omega) = \sin^2(\omega z) - z^2 \sin(\omega^2) = 0. \tag{2.8}$$

The function F is even with respect to z, then if $z_0 = \xi + i\eta$ is the solution of (2.8) then it is the same for $-z_0 = -\xi - i\eta$.

The singular functions depend on the positive parameter $\xi = \mathrm{Re}(z)$ where z is solution of the equation (2.8). These singular functions are solutions of the following homogeneous problem:

$$\begin{gathered}\Delta^2\tau_\xi = 0 \quad \text{in } \Omega\\ \tau_\xi = 0 \quad \text{on } \Gamma\\ \frac{\partial\tau_\xi}{\partial n} = 0 \quad \text{on } \Gamma.\end{gathered} \tag{2.9}$$

To each τ_ξ corresponds a function $\tau_{-\xi}$ solution of problem (2.9).

Remark 2.1. We remark that the function τ_ξ is at least in $H^2(\Omega)$ while $\tau_{-\xi}$ is in $L^2(\Omega)$ in the neighborhood of $\mathbf{a}$. Let $\tau_1^*(r,\theta) = \chi(r,\theta)\tau_{-\xi}$ be the dual singular function and $\tau_1^*(r,\theta) = r^{1-\xi}\psi(\theta)$ in the neighborhood of $\mathbf{a}$, where ψ is defined in (2.6) and (2.7).

Since $\Delta^2(\tau_1^*) \in H^{-2}(\Omega)$, let φ^* the solution of the variational problem: find $\varphi^* \in H_0^2(\Omega)$ such that

$$\int_\Omega \Delta\varphi^*\Delta v\,dx = \langle\Delta^2(\tau_1^*), v\rangle, \quad \forall v \in H_0^2(\Omega). \tag{2.10}$$

Then the singularity coefficient is

$$\mu = c\int_\Omega f(\tau_1^* - \varphi^*)dx\,dy. \tag{2.11}$$

where c is given by [16, 17]

$$c = 8\xi(\omega)(\xi(\omega)+1)\int_0^\omega \exp(-(z+i)\theta)\psi(\theta)\,d\theta \tag{2.12}$$

where $\xi(\omega) = \sup\Big\{\mathrm{Re}(z) : z \text{ solution of (2.8)},\ z \neq \pm 1\Big\}$, ψ is defined in (2.6) and (2.7).

3. Discrete problem

The discretization is based on the Galerkin method. The goal is to construct the discrete space which is not included in the continuous one because the method is not conform. Since our problem is of order 4, we have to deal with two boundary conditions. Then we need two mortar functions; one for the trace and the other for the normal derivative. Let $\delta = (N_i)_{1\le i\le I}$ the discretization parameter where N_i are the degrees of the approximation polynomials in each sub-domain Ω_i, $1 \le i \le I$ ($\mathbb{P}_N(\Omega)$ is the space of polynomial functions of degree less or equal to N). We introduce the space $\mathcal{M}_\delta$ called the space of mortar functions made of couples (φ_0, φ_1) such that their restriction to each $\Gamma^{i(k),j(k)}$ is a polynomial of degree less than $N_i(k)$ and such that the following properties hold: at each vertex e of a sub-domain, each couple (φ_0, φ_1) defines a unique value φ^e, a unique derivative φ_x^e with respect to x, a unique derivative φ_y^e with respect to y and a unique mixed φ_{xy}^e with respect to x and y.

We define the discrete space X_δ as the space of functions v_δ such that:

(i) for each i, $1 \le i \le I$, the restriction of v_δ to Ω_i belongs to $\mathbb{P}_{N_i}(\Omega_i)$;
(ii) v_δ and its normal derivative vanish on $\partial\Omega$;

(iii) there exists a mortar couple (φ_0, φ_1) in $\mathcal{M}_\delta$ such that for $1 \le i \le I$,

$$\forall e \in \mathcal{V}, \quad v_\delta|_{\Omega_i}(e) = \varphi^e, \quad \frac{\partial}{\partial x}(v_\delta|_{\Omega_i})(e) = \varphi^e_x,$$
$$\frac{\partial}{\partial y}(v_\delta|_{\Omega_i})(e) = \varphi^e_y, \quad \frac{\partial^2}{\partial x \partial y}(v_\delta|_{\Omega_i})(e) = \varphi^e_{xy},$$

$$\forall \psi \in \mathbb{P}_{N_i-4}(\Gamma^{ij}), \quad \int_{\Gamma^{ij}} (v_\delta|_{\Omega_i} - \varphi_0)(\eta)\psi(\eta)\, d\eta = 0, \tag{3.1}$$

$$\int_{\Gamma^{ij}} (\frac{\partial v_\delta}{\partial n}|_{\Omega_i} - \varphi_1)(\eta)\psi(\eta)\, d\eta = 0, \quad 1 \le j \le 4. \tag{3.2}$$

The choice of $\mathbb{P}_{N_i-4}(\Gamma^{ij})$ is justified by the fact that four conditions are enforced on the vertices of Γ^{ij}, one on the function and one on its normal derivative at each vertex.

In the case of the problem of order four and to take into account the boundary conditions, it is more appropriate to use a quadrature formula which uses the function values on the extremities as well as the values of its normal derivative. The following lemma defines this quadrature formula (see [12, 22] for the proof)

Lemma 3.1. *Let $N \ge 2$ an integer, there exists a unique set of points ξ_j, $1 \le j \le N-1$, a unique set of positif reals ρ_j, $1 \le j \le N-1$, ρ_+, ρ_- such that for all polynomials φ in $\mathbb{P}_{2N-1}(]-1,1[)$*

$$\int_{-1}^{1} \varphi(x)\, dx = \sum_{j=1}^{N-1} \varphi(\xi_j)\rho_j + \varphi(-1)\rho_- + \varphi(1)\rho_+ \tag{3.3}$$

Remark 3.2. The nodes ξ_j; $1 \le j \le N-1$, are the zeros of the derivative of the Legendre polynomial L_N. We refer to [12] for the calculus of ξ_j and ρ_j, $1 \le j \le N-1$.

Given two functions u, v continuous on $\overline{\Omega} = [-1,1] \times [-1,1]$ and vanishes on its boundary, we define the following discrete scalar product

$$(u, v)_N = \sum_{j=1}^{N-1} \sum_{l=1}^{N-1} u(\xi_j, \xi_l) v(\xi_j, \xi_l)\rho_j\rho_l.$$

If T^i is the bijection from $]-1,1[^2$ in Ω_i, we define

$$(u, v)_{N_i} = \frac{|\Omega_i|}{4} \sum_{j=1}^{N_i-1} \sum_{l=1}^{N_i-1} (u \circ T^i)(\xi_j, \xi_l)(v \circ T^i)(\xi_j, \xi_l)\rho_j\rho_l.$$

Hence, for each value of δ, the discrete problem is written: Find u_δ in X_δ such that for all $v_\delta \in X_\delta$,

$$\sum_{i=1}^{I} (\Delta u_\delta|_{\Omega_i}, \Delta v_\delta|_{\Omega_i})_{N_i} = \sum_{i=1}^{I} (f, v_\delta|_{\Omega_i})_{N_i}. \tag{3.4}$$

See [8] for the numerical analysis and the implementation of problem (3.4) using the mortar spectral element method.

We present in this work a method based on dual singular function which will allow us to approximate the leading singularity coefficient of the bilaplacian operator. This method has high precision compared to the Strang and Fix algorithm [21].

Let $X_\delta^* = X_\delta + \mathbb{R}\tau_1$ be the augmented discrete space, which is a Banach space by the following discrete norm, for all $u_\delta^* = u_\delta + \mu\tau_1 \in X_\delta^*$,

$$\|u_\delta^*\|_{1*} = \sum_{i=1}^{I} \Big(\|u_\delta/_{\Omega_i}\|^2_{H^2(\Omega_i)} + |\mu|^2 \|\tau_1/_{\Omega_i}\|^2_{H^2(\Omega_i)} \Big)^{1/2}.$$

We ask the following two discrete problems:

(1) find the function $u_\delta^* = u_\delta + \mu\tau_1 \in X_\delta^*$ such that for all $v_\delta^* = v_\delta + \xi\tau_1 \in X_\delta^*$ we have

$$a_\delta^*(u_\delta^*, v_\delta^*) = \sum_{i=1}^{I} \int_{\Omega_i} f|_{\Omega_i} v_\delta^*|_{\Omega_i}\, dx\, dy; \tag{3.5}$$

(2) find φ_δ^* in X_δ^* such that for all $\psi_\delta^* \in X_\delta^*$,

$$a_\delta^*(\varphi_\delta^*, \psi_\delta^*) = \sum_{i=1}^{I} \int_{\Omega_i} \Delta^2 \tau_1^*|_{\Omega_i} \psi_\delta^*|_{\Omega_i}\, dx\, dy. \tag{3.6}$$

The bilinear form a_δ^* is defined by

$$\begin{aligned} a_\delta^*(u_\delta^*, v_\delta^*) = \sum_{i=1}^{I} \Big(&(\Delta u_\delta|_{\Omega_i}, \Delta v_\delta|_{\Omega_i})_{N_i} + \mu \int_{\Omega_i} \Delta\tau_1|_{\Omega_i} \Delta v_\delta|_{\Omega_i}\, dx\, dy \\ &+ \xi \int_{\Omega_i} \Delta\tau_1|_{\Omega_i} \Delta u_\delta|_{\Omega_i}\, dx\, dy + \mu\xi \int_{\Omega_i} (\Delta\tau_1|_{\Omega_i})^2\, dx\, dy \Big). \end{aligned} \tag{3.7}$$

We refer to [1] for the numerical analysis of this problem and to [3] for its implementation. The following proposition gives us the expression of the discrete leading singularity coefficient.

Proposition 3.3. *Let u, φ^*, u_δ^* and φ_δ^* be respectively the solutions of* (2.2), (2.10), (3.5) *and* (3.6), *we have (i)*

$$(\frac{1}{c})\mu_\delta = \int_\Omega f\tau_1^*\, dx\, dy + \int_\Omega u_\delta^* \Delta^2 \tau_1^*\, dx\, dy = \int_\Omega f(\tau_1^* - \varphi_\delta^*)\, dx\, dy \tag{3.8}$$

(ii)

$$\begin{aligned} &(\frac{1}{c})(\mu - \mu_\delta) \\ &= \sum_{i=1}^{I} \int_{\Omega_i} \Delta(u - u_\delta^*)|_{\Omega_i} \Delta(\varphi^* - \varphi_\delta^*)|_{\Omega_i}\, dx\, dy \\ &\quad + \sum_{1\le i\ne l\le I} \int_{\Gamma^{il}} \Big[\frac{\partial(\Delta u)}{\partial n_i} (\varphi_\delta^*|_{\Omega_i} - \varphi_\delta^*|_{\Omega_l}) - \frac{\partial(\Delta\varphi^*)}{\partial n_i} (u_\delta^*|_{\Omega_i} - u_\delta^*|_{\Omega_l}) \Big] \end{aligned} \tag{3.9}$$

where c is defined in (2.12)

Proof. We consider D the intersection of the domain Ω and the ball of center $\mathbf{a}$ and radius R. We consider that the cut-off function χ is equal 1 in D then we choose R such that $\Delta^2\tau_1 = \Delta^2\tau_1^* = 0$ then from (2.2) and (2.5) we have

$$\int_\Omega f\tau_1^*\, dx\, dy = \int_D -\Delta^2 u_\delta \tau_1^*\, dx\, dy + \int_{\Omega\setminus D} -\Delta^2 u_\delta^* \tau_1^*\, dx\, dy.$$

By double integration by parts we conclude

$$\begin{aligned}&\int_\Omega f\tau_1^*\,dx\,dy+\int_D u_\delta^*\Delta^2\tau_1^*\,dx\,dy\\&=\int_0^\omega\big(\partial_r(\Delta(u_\delta^*-u_\delta))\tau_1^*-(u_\delta^*-u_\delta)\partial_r(\Delta\tau_1^*)\big)(R,\theta)r\,d\theta\end{aligned}$$

Since $u_\delta^*-u_\delta=\mu_\delta\tau_1$, we have

$$c\Big(\int_\Omega f\tau_1^*\,dx\,dy+\int_\Omega u_\delta^*\Delta^2\tau_1^*\,dx\,dy\Big)=\mu_\delta.$$

To obtain the second equality we replace v by u in problem (2.10).

To show (3.9) we use (2.11) and (3.6). We obtain

$$(\frac{1}{c})(\mu-\mu_\delta)=\int_\Omega f(\varphi_\delta^*-\varphi^*)\,dx\,dy=\sum_{i=1}^I\int_{\Omega_i}\Delta^2u|_{\Omega_i}(\varphi^*-\varphi_\delta^*)|_{\Omega_i}\,dx\,dy. \tag{3.10}$$

By double integration by parts we have

$$\begin{aligned}&(\frac{1}{c})(\mu-\mu_\delta)\\&=\sum_{i=1}^I\int_{\Omega_i}\Delta u\Delta(\varphi^*-\varphi_\delta^*)\,dx\,dy+\sum_{1\le i<l\le I}\int_{\Gamma^{il}}\frac{\partial(\Delta u)}{\partial n_i}(\varphi^*-\varphi_\delta^*)\,d\tau\\&\quad-\sum_{1\le i<l\le I}\int_{\Gamma^{il}}(\Delta u)\frac{\partial}{\partial n_i}(\varphi^*-\varphi_\delta^*)\,d\tau.\end{aligned}\tag{3.11}$$

Let $\varphi_\delta^*=\varphi_\delta+\xi\tau_1$ and $u_\delta^*=u_\delta+\mu\tau_1$ in X_δ^*. Since

$$\begin{aligned}a_\delta^*(\varphi_\delta^*,u_\delta^*)=\sum_{i=1}^I(\Delta\varphi_\delta|_{\Omega_i},\Delta u_\delta|_{\Omega_i})_{N_i}+\mu\int_{\Omega_i}\Delta\tau_1\Delta u_\delta\,dx\,dy\\+\xi\int_{\Omega_i}\Delta\varphi_\delta\Delta\tau_1\,dx\,dy+\mu\xi\int_{\Omega_i}\Delta\tau_1^2\,dx\,dy,\end{aligned}$$

and if $u_\delta\in X_\delta^-=\{v_\delta\in X_\delta: v_{\delta/\Omega_i}\in\mathbb{P}_{N_i-1}(\Omega_i),\ 1\le i\le I\}$, we obtain

$$\sum_{i=1}^I(\Delta\varphi_\delta|_{\Omega_i},\Delta u_\delta|\Omega_i)_{N_i}=\sum_{i=1}^I\int_{\Omega_i}\Delta\varphi_\delta|_{\Omega_i}\Delta u_\delta|_{\Omega_i}\,dx\,dy.$$

Then using (2.10) we have

$$a_\delta^*(\varphi_\delta^*,u_\delta^*)=\sum_{i=1}^I\int_{\Omega_i}\Delta\varphi_\delta^*\Delta u_\delta^*\,dx\,dy=\sum_{i=1}^I\int_{\Omega_i}\Delta^2\tau_1^*u_\delta^*\,dx\,dy. \tag{3.12}$$

Following (3.12) we deduce that $\Delta^2\varphi^*=\Delta^2\tau_1^*$ in the sense of distribution and that $\varphi^*=\frac{\partial\varphi^*}{\partial n}=0$ on $\partial\Omega$ then

$$a_\delta^*(\varphi_\delta^*,u_\delta^*)=\sum_{i=1}^I\int_{\Omega_i}\Delta^2\varphi^*|_{\Omega_i}u_\delta^*|_{\Omega_i}\,dx\,dy.$$

Then by double integration by parts,

$$\begin{aligned} &a_\delta^*(\varphi_\delta^*, u_\delta^*)\\ &= \sum_{i=1}^{I} \int_{\Omega_i} \Delta\varphi^*|_{\Omega_i} \Delta u_\delta^*|_{\Omega_i} \, dx\, dy + \sum_{1\le i<l\le I} \int_{\Gamma^{il}} \frac{\partial(\Delta\varphi^*)}{\partial n_i} (u_\delta^*|_{\Omega_i} - u_\delta^*|_{\Omega_l})\, d\tau \\ &\quad - \sum_{1\le i<l\le I} \int_{\Gamma^{il}} \frac{\partial(\Delta u_\delta^*)}{\partial n_i} (\varphi^*|_{\Omega_i} - \varphi^*|_{\Omega_l})\, d\tau. \end{aligned} \tag{3.13}$$

Following (3.12) and (3.13) we conclude that

$$\begin{aligned} &\sum_{i=1}^{I} \int_{\Omega_i} \Delta(\varphi^* - \varphi_\delta^*)|_{\Omega_i} \Delta u_\delta^*|_{\Omega_i} \, dx \\ &= \sum_{1\le i<l\le I} \int_{\Gamma^{il}} \frac{\partial\Delta\varphi^*}{\partial n_i} (u_\delta^*|_{\Omega_i} - u_\delta^*|_{\Omega_l})\, d\tau \\ &- \sum_{1\le i<l\le I} \int_{\Gamma^{il}} \frac{\partial\Delta u_\delta^*}{\partial n_i} (\varphi_\delta^*|_{\Omega_i} - \varphi_\delta^*|_{\Omega_l})\, d\tau \end{aligned}$$

By adding this equality with (3.12), we obtain the desired result. □

We interested in the following the error estimate between μ and μ_δ.

Theorem 3.4. *Assume that f belongs to $H^{s-2}(\Omega)$ with $s>0$. The error between μ and μ_δ satisfies the following estimate, for $\varepsilon>0$,*

$$|\mu - \mu_\delta| \le CN^{-2}\Big(\sum_{1\le i\le I} N_i^{-\sigma_i}\Big)\|f\|_{H^{s-2}(\Omega)},$$

$N = \inf_{1\le i\le I} N_i$ and

$$\sigma_i = \begin{cases} s-2 & \text{if } \Omega_i \text{ does not contain any vertices of } \Omega,\\ \inf(s-2, 2\eta_1(\frac{\pi}{2}) - \varepsilon) & \text{if } \overline{\Omega_i} \text{ contains one vertex of } \Omega \text{ other than } \mathbf{a},\\ \inf(s-2, 2\eta_1(\omega) - \varepsilon) & \text{if } \overline{\Omega_i} \text{ contains } \mathbf{a}, \end{cases}$$

where $\eta_1(\omega)$ is the second real solution of equation (2.8) in the band $0<\mathrm{Re}(z)<s$.

Proof. Following (3.9) we have

$$\mu - \mu_\delta = c\int_\Omega f(\varphi_\delta^* - \varphi^*)\, dx\, dy = c\int_\Omega \Delta^2 u(\varphi_\delta^* - \varphi^*) dx\, dy.$$

By double integration by parts we obtain

$$\begin{aligned} \mu - \mu_\delta = c\Big(&\sum_{i=1}^{I} \int_{\Omega} \Delta u|_{\Omega_i} \Delta(\varphi - \mu_\varphi^*)\, dx\, dy + \sum_{1\le i\ne l\le I} \int_{\Gamma^{il}} \frac{\partial\Delta u}{\partial n_i} (\varphi_\delta^*|_{\Omega_i} - \varphi^*|_{\Omega_i})\, d\tau \\ &- \sum_{1\le i\ne l\le I} \int_{\Gamma^{il}} (\Delta u) \frac{\partial(\varphi_\delta^* - \varphi^*)}{\partial n_i}\, d\tau\Big). \end{aligned}$$

Otherwise taking $v_\delta^* \in X_\delta^*$ such that $v_\delta \in X_\delta^-$ we obtain

$$\begin{aligned} &a_\delta^*(\varphi_\delta^*, v_\delta^*) = \langle \Delta^2\varphi^*, v_\delta^*\rangle \\ &= \sum_{i=1}^{I} \int_{\Omega_i} \Delta\varphi \Delta v_\delta^*\, dx\, dy + \sum_{1\le i\ne l\le I} \int_{\Gamma^{il}} \frac{\partial\Delta\varphi^*}{\partial n} (v_\delta|_{\Omega_i} - v_\delta|_{\Omega_l})\, d\tau \end{aligned}$$

$$
- \sum_{1\le i\neq l\le I} \int_{\Gamma^{il}} (\Delta\varphi^*) \frac{\partial}{\partial n_i} (v_\delta^*|_{\Omega_i} - v_\delta^*|_{\Omega_l})\, d\tau.
$$

Then by adding and subtracting the same quantity, we write

$$
\begin{aligned}
\mu - \mu_\delta = c\Big(&\sum_{i=1}^{I} \int_{\Omega_i} (\Delta u|_{\Omega_i} - \Delta v_\delta^*|_{\Omega_i}) \Delta(\varphi_\delta^*|_{\Omega_i} - \varphi^*|_{\Omega_i})\, dx\, dy \\
&+ \sum_{1\le i\neq l\le I} \int_{\Gamma^{il}} \frac{\partial \Delta u}{\partial n_i} (\varphi_\delta^*|_{\Omega_i} - \varphi^*|_{\Omega_i})\, d\tau \\
&- \sum_{1\le i\neq l\le I} \int_{\Gamma^{il}} (\Delta u) \frac{\partial}{\partial n_i} (\varphi_\delta^*|_{\Omega_i} - \varphi^*|_{\Omega_i})\, d\tau \\
&+ \sum_{1\le i\neq l\le I} \int_{\Gamma^{il}} \frac{\partial \Delta\varphi^*}{\partial n_i} (v_\delta^*|_{\Omega_i} - v^*|_{\Omega_l})\, d\tau \\
&- \sum_{1\le i\neq l\le I} \int_{\Gamma^{il}} (\Delta\varphi^*) \frac{\partial}{\partial n_i} (v_\delta^*|_{\Omega_i} - v^*|_{\Omega_l})\, d\tau \Big).
\end{aligned}
$$

Proceeding as in [7, Chapter 4.2] we obtain

$$
\begin{aligned}
&\sum_{1\le i\neq l\le I} \int_{\Gamma^{il}} \frac{\partial \Delta u}{\partial n_i} (\varphi_\delta^*|_{\Omega_i} - \varphi^*|_{\Omega_i})\, d\tau - \sum_{1\le i\neq l\le I} \int_{\Gamma^{il}} (\Delta u) \frac{\partial}{\partial n_i} (\varphi_\delta^*|_{\Omega_i} - \varphi^*|_{\Omega_i})\, d\tau \\
&\le C\Big(\sum_{i=1}^{I} \sum_{j=1}^{4} \inf_{\psi^{ij}\in \mathbb{P}_{N_i-4}(\Gamma^{ij})} |\frac{\partial \Delta u_R}{\partial n} - \psi^{ij}\|_{[H^{3/2}(\Gamma^{ij})]'} \\
&\quad + \inf_{\psi^{ij}\in \mathbb{P}_{N_i-4}(\Gamma^{ij})} |\Delta u_R - \psi^{ij}\|_{[H^{1/2}(\Gamma^{ij})]'} \Big) \|\varphi^* - \varphi_\delta^*\|_{H^2(\Omega)}.
\end{aligned}
$$

On the other hand, by construction, $\Delta^2\tau_1^*$ is in $L^2(\Omega)$, therefore φ^* is the sum of τ_1 and a function $\widetilde{\varphi}$ in $H^4(\Omega)\cap H_0^2(\Omega)$ see [1]

$$
\begin{aligned}
&\sum_{1\le i\neq l\le I} \int_{\Gamma^{il}} \frac{\partial \Delta\varphi^*}{\partial n_i} (\varphi_\delta^*|_{\Omega_i} - \varphi_\delta^*|_{\Omega_l})\, d\tau - \sum_{1\le i\neq l\le I} \int_{\Gamma^{il}} (\Delta\varphi^*) \frac{\partial}{\partial n_i} (v_\delta^*|_{\Omega_i} - v_\delta^*|_{\Omega_l})\, d\tau \\
&\quad \le C\Big(\sum_{i=1}^{I} \sum_{j=1}^{4} \Big(\inf_{\psi^{ij}\in \mathbb{P}_{N_i-4}(\Gamma^{ij})} \|\frac{\partial \Delta\widetilde{\varphi}}{\partial n} - \psi^{ij}\|_{[H^{3/2}(\Gamma^{ij})]'} \\
&\qquad + \inf_{\psi^{ij}\in \mathbb{P}_{N_i-4}(\Gamma^{ij})} \|\Delta\widetilde{\varphi} - \psi^{ij}\|_{[H^{1/2}(\Gamma^{ij})]'} \Big)\Big) \|u_R - v_\delta^*\|_{1*}.
\end{aligned}
$$

Having φ^*, respectively φ_δ^* the solution of the continuous, respectively discrete, problem with second member in $L^2(\Omega)$, then we conclude from [1, Theorem 5.7]. □

Remark 3.5. We notice that the convergence order is $N^{\epsilon-4}$ and $N^{\epsilon-\frac{14}{3}}$ in the case of the crack and in the case of $\omega = \frac{3\pi}{2}$ respectively. This proves the high accuracy of the method.

4. Implementation and numerical results

To write the matrix system associated to the discrete problem (3.5) we have to choose a basis for the space X_δ^*. Let h_i be the Hermite interpolating polynomials on the interval $[-1,1]$ defined by

$$\begin{gathered}
h_j(\xi_l)=\delta_{jl} \quad 0\le j\le N,\\
h_j(-1)=h_j(1)=0 \quad 2\le l\le N-2,\\
h_j'(-1)=h_j'(1)=0.
\end{gathered}$$

It follows that any polynomial v_δ of X_δ is written as

$$v_\delta(x,y)|_{\Omega_i}=\sum_{j=0}^{N}\sum_{l=0}^{N} v_{N_i}^{jl} h_j^{N_i}(x) h_l^{N_i}(y),$$

where $v_{N_i}^{jl}=v_\delta(\xi_j^{N_i},\xi_l^{N_i})$, $\xi_j^{N_i}$ and $h_j^{N_j}$ are deduced respectively form ξ_j and h_j by translation and dilation. Therefore for v_δ^* in X_δ^* there exists $v_\delta\in X_\delta$ and $\mu\in\mathbb{R}$ such that $v_\delta^*=v_\delta+\mu\tau_1$ then

$$v_\delta^*(x,y)|_{\Omega_i}=\sum_{j=0}^{N_i}\sum_{l=0}^{N_i} v_{N_i}^{jl} h_j^{N_i}(x) h_l^{N_i}(y)+\mu\tau_1|_{\Omega_i}.$$

The two integral matching conditions (3.1) and (3.2) can be written in the matrix form

$$\begin{bmatrix} v_{jl}^i|_{\Omega_i} \\ v_{jl}|_{\text{edges}} \\ (\frac{\partial v^i}{\partial n})_{jl}|_{\text{edges}} \\ \mu_\delta \end{bmatrix} = Q \begin{bmatrix} v_{jl}^i|_{\Omega_i} \\ \varphi_0^i|_{\text{edges}} \\ \varphi_1^i|_{\text{edges}} \\ \mu_\delta \end{bmatrix}$$

where

$$Q=\begin{bmatrix} I & 0 & 0 & 0\\ 0 & Q_0 & 0 & 0\\ 0 & 0 & Q_1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The couple $[Q_0,Q_1]$ is called "rectangular transformation matrix". This matrix ensures the descendance of the mortar to the elements. While its transpose Q^T purges the unknown vectors from the false degree of freedom. It is clear that we evaluate $v_\delta/_{\Omega_i}$ without explicitly forming the global matrix projection. The calculation of this matrix is local for each edge-mortar. We observe that the discrete problem (3.5) is written equivalently in the form

$$AU_\delta^*=F \tag{4.1}$$

where A takes the form

$$\begin{pmatrix}
(\Delta(h_jh_l);\Delta(h_ph_q))_{N_1} & 0 & . & . & 0 & \int_{\Omega_1}\Delta\tau_1\Delta(h_ph_q)\\
0 & & & & & .\\
. & & & & & .\\
. & & & & & \int_{\Omega_{N_\Delta}}\Delta\tau_1\Delta(h_ph_q)\\
. & & & & & 0\\
. & & & & & .\\
. & & & & & .\\
. & & & & & \\
0 & . & . & 0 & (\Delta(h_ih_j);\Delta(h_ph_q))_{N_k} & 0\\
\int_{\Omega_1}\Delta\tau_1\Delta(h_jh_l) & . & \int_{\Omega_{N_\Delta}}\Delta\tau_1\Delta(h_jh_l) & 0 & . & \int_\Omega(\Delta\tau_1)^2
\end{pmatrix}$$

The i-th block $(\Delta(h_jh_l);\Delta(h_ph_q))_{N_i}$ for $1\le j,l\le N_i-1$ and $1\le p,q\le N_i-1$, represents the Bilaplacian operator on the sub-domain Ω_i, and F is the second member given by

$$F = \begin{bmatrix} (h_p h_q, f)_{N_1} \\ \dots \\ (h_p h_q, f)_{N_i} \\ \int_\Omega f \tau_1 \, dx \, dy \end{bmatrix}.$$

The vector U_δ^* is constituted by the values of the solution both at the interior nodes of each sub-domains and on the boundary interfaces.

Note that we do not solve the system (4.1) because it has false degrees of freedom. However, the global system that we solve is the following

$$Q^T A Q \tilde{U} = Q^T F \tag{4.2}$$

where $\tilde{U}$ is the vector composed from the unknown on the internal nodes and the value of the mortar functions (φ_0, φ_1) on the skeleton $\mathcal{S}$. The matrix $\widetilde{A} = Q^T A Q$ is symmetric and positive defined. Then, we will use the gradient conjugate method for solving the problem (4.2).

For this, let U_0 be arbitrary, $R_0 = Q^T F - \widetilde{A} U_0$, $T_0 = R_0$ and

$$\alpha_n = \frac{(R_n, R_n)}{(T_n, AT_n)}, \quad U_{n+1} = U_n + \alpha_n T_n, \quad R_{n+1} = R_n - \alpha_n A T_n,$$
$$\beta_n = \frac{(R_{n+1}, R_{n+1})}{(R_n, R_n)}, \quad T_{n+1} = R_{n+1} + \beta_n T_n \,.$$

The calculation is processed locally, Even though the resolution by the gradient algorithm is processed globally. We notice that the product matrix-vector is the most expensive. The local matrices $(\Delta(h_j h_l); \Delta(h_p h_q))_{N_i}$ are full, consequently, the calculation cost is high. It is as $O(N^4)$ operations and $O(N^4)$ memory space. This cost of operations is reduced to $O(N^3)$ and the memory space to $O(N^2)$ by sub-domain by the tensorisation.

Below, we present some numerical results to approximate the solution of problem 4.2 and the singularity coefficient by applying the dual method. In the following, we vary the parameter of discretization N and the data function of the biharminic problem.

The test cases are implemented in a neighborhood of the singular corner **a**, i.e. four sub-domains in the case of the crack and three sub-domains in the case of $\omega = 3\pi/2$ (see figure 2).

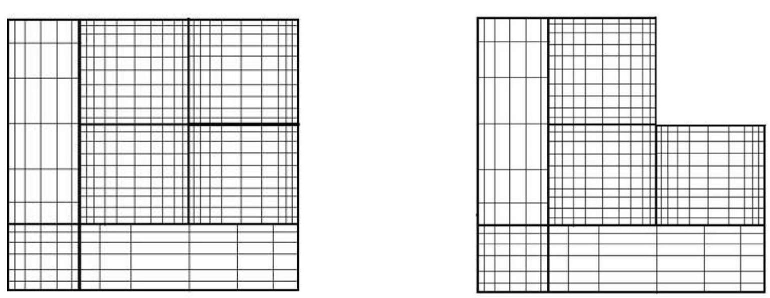


FIGURE 2. Spectral mesh of domain when $\omega = 2\pi$ and $\omega = \frac{3\pi}{2}$.

Below some numerical results are presented related to the calculation of the discrete solution of the problem 3.5 and the leading singularity coefficient by the dual method. In the following examples, μ_δ denotes the discrete leading singularity coefficient.

Example 4.1. $u(x,y) = \sin^2 \pi x^2 \sin^2 \pi y^2$ and $\omega = \frac{3\pi}{2}$.

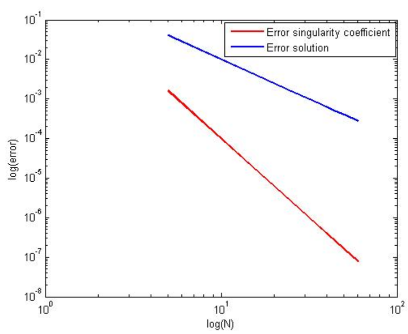
N	7	15	22	30	37
μ_δ	$4.0\,10^{-2}$	$2.463\,10^{-6}$	$-0.951\,10^{-12}$	$-3.041\,10^{-14}$	$1.382\,10^{-14}$

Example 4.2. $u(r,\theta) = r^{1.5}(\sin(1.5\theta) - 3\sin(0.5\theta) + \cos(1.5\theta) - \cos(0.5\theta))$, and $\omega = 2\pi$.

N	5	15	20	30	40
μ_δ	0.8995.	0.9599.	0.9993	0.9999	1.

Example 4.3. $u(r,\theta) = r^{1.544}\big(4.302(\cos(0.092\theta) - \cos(1.908\theta)) - 1.1815(10.869\sin(0.092\theta) - 0.524\sin(1.908))\big)$, and $\omega = \frac{3\pi}{2}$.

N	5	10	15	20	35
μ_δ	0.9017.	0.9896.	0.9991.	0.9998	1.



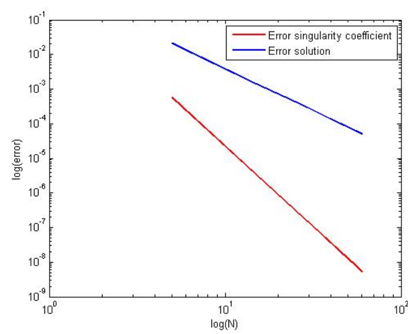


FIGURE 3. Error on the solution and the leading singularity coefficient.

Figure 3 shows the error curves on the solution of problem 4.2 (curves in blue) and the curves of error on the leading singularity coefficient (curves in red) in both $\omega = 2\pi$ and $\omega = \frac{3\pi}{2}$. The continuous solutions are equal to the first singular function which corresponds to a singularity coefficient equal to 1 (Examples 4.2 and 4.3). Error curves are calculated in logarithmic scale permitting the computing of the convergence order corresponding to the slope of the curve. We notice that the convergence order on the leading singularity coefficient is better than the convergence order of the solution. It is equal to 3.9986 for the crack and to 4.6519 for the L-domain. However in the case of the solution, this order is equal to 1.9997 for the crack and to 2.4131 for the L-domain.

Let $\Gamma_0 = \{(r,\theta)$ such that $\theta = 0$ and $\theta = \omega\}$. Figure 4 shows the iso-values of the discrete solution in the case of $\omega = \frac{3\pi}{2}$ for the below biharmonic problem

$$-\Delta^2 u = 0 \quad \text{in } \Omega$$
$$u = xy \quad \text{on } \partial\Omega/\bar{\Gamma}_0$$

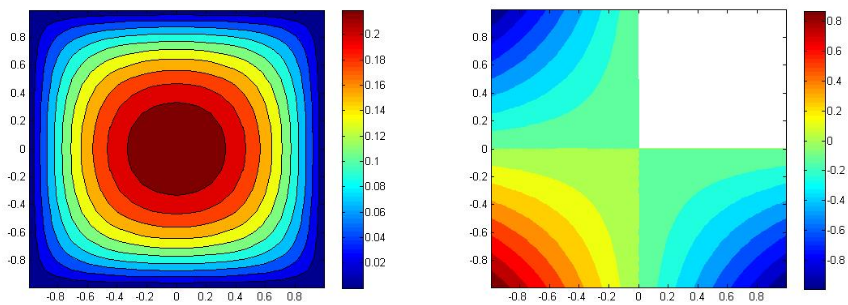


FIGURE 4. Discrete solution $\omega = 2\pi$ and $\omega = \frac{3\pi}{2}$.

$$
\begin{aligned}
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega/\bar{\Gamma}_0 \\
u &= 0 \quad \text{on } \Gamma_0 \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_0
\end{aligned}
$$

and in the case when $\omega = 2\pi$ the discrete solution corresponding to the problem:

$$
\begin{aligned}
-\Delta^2 u &= 1 \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega\,.
\end{aligned}
$$

The next example is related to the calculation of the leading singularity coefficient in the case of the crack for the biharmonic problem

$$
\begin{aligned}
-\Delta^2 u &= f \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega/\bar{\Gamma}_0 \\
\frac{\partial u}{\partial n} &= g \quad \text{on } \partial\Omega/\bar{\Gamma}_0 \\
u &= 0 \quad \text{on } \Gamma_0 \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_0.
\end{aligned}
$$

Example 4.4. $f = 0,\ g = x$ and $\omega = 2\pi$.

N	10	15	20	30	40
μ_δ	0.1580.	0.1559.	0.1561.	0.1562.	0.1562.

4.1. **Conclusion.** In this paper, we studied the approximation of the leading singularity coefficient by mortar spectral element method. This coefficient has a great importance in the solid mechanics domain. It informs on the crack propagation. The dual method permitted to improve the results. Indeed, the obtained results are better than those obtained by Strang and Fix algorithm (see [1]). Our conclusion is twofold: first, the theory is confirmed since the dual method gives us an optimal error estimate. Second, using the spectral discretization for such type of problem is more efficient.

Acknowledgments. The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this Research group No (RG-1435-026).

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