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STABILITY AND PHASE PORTRAITS OF SUSCEPTIBLE-INFECTIVE-REMOVED EPIDEMIC MODELS WITH VERTICAL TRANSMISSIONS AND LINEAR TREATMENT RATES

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Abstract. We study stability and phase portraits of susceptible-infective-removed (SIR) epidemic models with horizontal and vertical transmission rates and linear treatment rates by studying the reduced dynamical planar systems under the assumption that the total population keeps unchanged. We find out all the ranges of the parameters involved in the models for the infection-free equilibrium and the epidemic equilibrium to be positive. The novelty of this paper lies in the demonstration and justification of the parameter conditions under which the positive equilibria are stable focuses or nodes. These phase portraits provide more detailed descriptions of behaviors and extra biological understandings of the epidemic diseases than local or global stability of the models. Previous results only discussed the stability of the SIR models with horizontal or vertical transmission rates and without treatment rates. Our results involving vertical transmission and treatment rates will exhibit the effect of the vertical transmissions and the linear treatment rates on the epidemic models.

1. Introduction

The classic susceptible infectious recovered (SIR) models with vital dynamics (birth and death) (see Hethcote [7]) only deal with the horizontal transmission of epidemic diseases, that is, diseases are transmitted through contact between the infectives and the susceptibles. There are infectious diseases such as rubella, herpes simplex, hepatitis B, chagas disease, and AIDS which can be transmitted by horizontal transmission or by vertical transmission, i.e., the diseases are transmitted from infective parents to unborn or newly born offsprings[2, 3, 14]. Treatments, including isolation, quarantine and hospitalization, are important control measures to prevent epidemic diseases. A variety of treatment rates including linear, constant or saturated treatment rates have been incorporated into some epidemic disease models [9, 11, 20, 23, 24].

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Recently, Luo, Zhu and Lan [17] generalized the SIR models with horizontal and vertical transmissions in [18] by incorporating constant treatment rates on the infectives. They provided the conditions on the parameters involved and justified that under these conditions, the equilibria are stable focuses, stable nodes, saddle-nodes or cusps with dimension 2, and studied Bogdanov-Takens bifurcations containing saddle-node bifurcations, Hopf bifurcations and homoclinic bifurcations.

In this article, we consider the SIR models with horizontal and vertical transmissions by incorporating linear treatment rates on the infectives, and study the stability and phase portraits of the models. Some of models like the classic ratio-dependent predator-prey models [8, 10] have been incorporated with linear harvesting rates on predator, see for example, the second equation of [6, (1.1) p. 349], the second equation of (3) in [13, p.1866] and the second equation of [4, (2.2) p. 4046] and [15]. Similar to such predator-prey models, we shall exhibit the impact of the linear treatment rates on the SIR models by seeking the ranges of the treatment rates which show the changes of the dynamics of the models.

As in [17, 18], we assume that the birth rate equals the death rate and the recovered population never becomes susceptible. Under these assumptions, the SIR model we study in this paper is governed by the following system of three first-order ordinary differential equations

$$\begin{aligned} \dot{S} &= b - \beta SI - bS - qbI, \\ \dot{I} &= \beta SI - bI - rI + qbI - hI, \\ \dot{R} &= rI - bR + hI, \end{aligned} \tag{1.1}$$

where $S(t) \geq 0$, $I(t) \geq 0$ and $R(t) \geq 0$ denote the densities of the populations of the susceptible, infective and removed, respectively at time $t \geq 0$; the constant $b > 0$ in the first and last term of the right-side of the first equation denotes the birth rate of susceptible population and the coefficient $b > 0$ in the terms bS, bI and bR denotes the death rate of the corresponding population, respectively (note that the birth rate equals the death rate). The parameter $\beta > 0$ denotes the effective per capita transmission rate of infective individuals, and the term βSI is the incidence rate, which is bilinear, and $r > 0$ is the recovery rate of the infective individuals. The parameter $q \in [0, 1]$ is the fraction of unborn or newly born offsprings of the infective parents, and $h \geq 0$ is the proportionality of infective receiving treatments.

We emphasize that the three coefficients b, r and h in the terms bI, rI and hI in the second equation of (1.1) can not be simply reduced to one term $(b + r + h)I$ with $b + r + h > 0$ as one parameter. We must consider the ranges of (b, r, h) in the first quadrant of $\mathbb{R}^3$ instead of the ranges of one parameter $b + r + h$ in $\mathbb{R}^1$. We refer to [4, 6, 15, 13] for the study of ratio-dependent predator-prey models with linear harvesting rates, where the death rate and harvesting rate of predator cannot be combined into one parameter. We refer to [5, 26] and the references therein for the study of epidemic diseases with nonlinear incidence rates.

The stability of the model (1.1) with $q \in (0, 1)$ and $h = 0$ was studied in [18]. It was shown that when the basic reproductive rate $R_0 > 1$ the model has a unique positive infection-free unstable equilibrium, and one positive interior (epidemic) locally stable equilibrium; and when $R_0 < 1$, the infection-free equilibrium is locally stable and the interior equilibrium is unstable. But neither the case $R_0 = 1$ nor the phase portraits near the positive equilibria of (1.1) with $h = 0$ was studied in [18]. Our results will fill in the gap. It is well known that the phase portraits near the positive equilibria provide detailed descriptions of behaviors and extra biological understandings of the epidemic diseases We prove that when the basic reproduction number $\mathcal{R}_0 \le 1$, the model (1.1) with $q \in (0,1)$ and $h > 0$ has a unique disease-free equilibrium $(1,0)$, and when $\mathcal{R}_0 > 1$, the model has both a disease-free equilibrium $(1,0)$ and an interior (epidemic) equilibrium $(\bar{x}, \bar{y})$. (The symbols used in the Introduction will be given later). We show that when $\mathcal{R}_0 > 1$, $(1,0)$ is a saddle, when $\mathcal{R}_0 < 1$, $(1,0)$ is a stable node and when $\mathcal{R}_0 = 1$, it is a saddle-node and is stable in the triangle region $\{(u,v) \in \mathbb{R}_+ : u + v \le 1\}$. For the equilibrium $(\bar{x}, \bar{y})$, we provide sufficient conditions on the parameters involved and prove that under these conditions, the equilibria are stable focuses or stable nodes. The latter results are new and their proofs follow from several lemmas. Some simulations on our results will be provided to understand the phase portraits of the infection-free equilibrium and the positive equilibrium.

2. Positive equilibria

Since the birth rate equals the death rate, the total population keeps unchanged and can be normalized to 1. Hence, we have $S(t) + I(t) + R(t) = 1$ for $t \ge 0$ and (1.1) is equivalent to the following system

$$\begin{aligned} \dot{S} &= b - \beta SI - bS - qbI, \\ \dot{I} &= \beta SI - bI - rI + qbI - hI. \end{aligned} \tag{2.1}$$

As mentioned in the Introduction, the three parameters b, r, h in the second equation of (2.1) can not be combined into one parameter since they have their own biological meanings. For simplification of symbols, we let $x(t) = S(t)$ and $y(t) = I(t)$ for $t \ge 0$. Using $x(t)$ and $y(t)$, we rewrite (2.1) as follows

$$\begin{aligned} \dot{x} &= b - \beta xy - bx - qby := f(x,y), \\ \dot{y} &= \beta xy - by - ry + qby - hy := g(x,y), \end{aligned} \tag{2.2}$$

where $x(t)$ and $y(t)$ denote the densities of the populations of the susceptible and infective, respectively at time $t \ge 0$.

Recall that $(x,y) \in \mathbb{R}^2$ is an equilibrium of (2.2) if $f(x,y) = 0$ and $g(x,y) = 0$. An equilibrium (x,y) of (2.2) is said to be positive if $x \ge 0$ and $y \ge 0$, and a positive interior (endemic) equilibrium if $x > 0$ and $y > 0$.

Notation. Let

$$\eta := \eta(r,b,q) = r + (1-q)b \quad \text{and } q_1 = (b + r - \beta)/b.$$

We denote by $\mathcal{R}_0$ the basic reproduction number, representing the average number of cases that are induced by one infective individual, of the model (2.2) as

$$\mathcal{R}_0 = \beta/(\eta + h), \tag{2.3}$$

where $1/(\eta+h)$ is the average time for an infected individual staying in the infectious class.

We shall use the conditions: $\mathcal{R}_0 > 1, = 1$ or < 1, so the following two lemmas provide the ranges of the parameters b, r, β, q under which the above conditions hold and give better understanding on $\mathcal{R}_0$. The proofs are straightforward and we omit them.

Lemma 2.1. (1) *$\beta < \eta$ if and only if either $r \le \beta < b + r$ and $0 \le q < q_1$ or $\beta < r$ and $0 \le q \le 1$.*

(2) *$\eta < \beta$ if and only if $b + r < \beta$ and $0 \le q \le 1$ or $r < \beta \le b + r$ and $q_1 < q \le 1$.*
(3) *$\beta \le \eta$ if and only if either $b + r < \beta$ and $0 \le q \le 1$ or $r \le \beta \le b + r$ and $q_1 \le q \le 1$.*
(4) *$\beta \le \eta < \eta + b$ if and only if either $b + r < \beta < 2b + r - qb$ and $0 \le q \le 1$ or $r \le \beta \le b + r$ and $q_1 \le q \le 1$.*
(5) *$\beta = \eta$ if and only if $r \le \beta \le b + r$ and $q = q_1$.*

Lemma 2.2. (1) *$\mathcal{R}_0 > 1$ if and only if $\eta < \beta$ and $0 \le h < \beta - \eta$.*
(2) *$\mathcal{R}_0 < 1$ if and only if either $0 < \beta < \eta$ and $h \ge 0$ or $\eta \le \beta$ and $h > \beta - \eta$.*
(3) *$\mathcal{R}_0 = 1$ if and only if $\eta \le \beta$ and $h = \beta - \eta$.*

We prove the following main result on the number of equilibria of (2.2).

Theorem 2.3. (1) *If $\mathcal{R}_0 \le 1$, then $(1, 0)$ is the unique positive equilibrium of (2.2).*
(2) *If $\mathcal{R}_0 > 1$, then (2.2) has only two positive equilibria: $(1, 0)$ and $(\bar{x}, \bar{y})$, where*

$$\bar{x} = \frac{\eta + h}{\beta} \quad \text{and} \quad \bar{y} = \frac{b(\beta - \eta - h)}{\beta(b + r + h)}. \tag{2.4}$$

Proof. It is clear that (x, y) is an equilibrium of (2.2) if and only if (x, y) satisfies the system

$$\begin{aligned} b - \beta xy - bx - qby = 0, \\ \beta xy - by - ry + qby - hy = 0. \end{aligned} \tag{2.5}$$

For $b, r, \beta > 0$, $q \in [0, 1]$ and $h \ge 0$, it is clear that $(1, 0)$ is a solution of (2.5) and is a positive equilibrium of (2.2). It is easy to see that (2.5) with $y \ne 0$ is equivalent to the system

$$\begin{aligned} b - \beta xy - bx - qby = 0, \\ \beta x - r - b - h + qb = 0. \end{aligned} \tag{2.6}$$

Solving the second equation of (2.6) we obtain

$$x = \frac{r + (1 - q)b + h}{\beta} = \frac{\eta + h}{\beta}. \tag{2.7}$$

This, together with the first equation of (2.6), implies

$$y = \frac{b[\beta + qb - (r + b + h)]}{\beta(r + b + h)} = \frac{b(\beta - \eta - h)}{\beta(r + b + h)}. \tag{2.8}$$

If $\mathcal{R}_0 > 1$, then by Lemma 2.2 (1) we have $\eta < \beta$ and $0 \le h < \beta - \eta$. This, together with (2.8), implies $y > 0$. Hence, $(\bar{x}, \bar{y})$ given in (2.4) is a positive interior equilibrium of (2.2). If $\mathcal{R}_0 < 1$, then by Lemma 2.2 (2) we have either $0 < \beta < \eta$ and $h \ge 0$ or $\eta \le \beta$ and $h > \beta - \eta$. This implies that $\beta - \eta - h < 0$ and $y < 0$. Hence, (2.2) has no positive interior equilibria. If $\mathcal{R}_0 = 1$, then by Lemma 2.2 (3) we have $\eta \le \beta$ and $h = \beta - \eta$. This, together with (2.7) and (2.8), implies $y = 0$ and $x = 1$. The results follow. $\square$

Theorem 2.3 improves [7, Theorem 6.1] and the result on the number of positive equilibria obtained in [18, section 2].

3. Stability and phase portraits of the model

In this section, we study the stability and phase portraits of each positive equilibrium of (2.2). We recall some results on stability and phase portraits of planar systems near equilibria in the qualitative theory [1, 19, 21]. We consider the following planar system

$$\begin{aligned} \dot{x}(t) &= f(x(t), y(t)), \\ \dot{y}(t) &= g(x(t), y(t)) \end{aligned} \tag{3.1}$$

subject to the initial value condition:

$$(x(0), y(0)) = (x_0, y_0) \tag{3.2}$$

where $f, g \in C^1(\mathbb{R}^2)$.

Recall that (x, y) is said to be a solution of (3.1)-(3.2) if $x, y \in C^1(\mathbb{R}_+)$ and satisfy (3.1)-(3.2). A solution (x, y) is said to be positive if $x, y \in P$, where

$$P = \{x \in C^1(R_+) : x(t) \geq 0 \quad \text{for } t \in R_+\}.$$

Since $f, g \in C^1(\mathbb{R}^2)$, it is well known that for each initial value $(x_0, y_0) \in \mathbb{R}^2$, (3.1)-(3.2) has a unique solution. Moreover, if f and g satisfy

$$f(0, y) \geq 0 \quad \text{and } g(x, 0) \geq 0 \text{ for } x, y \in \mathbb{R}_+,$$

then for each initial value $(x_0, y_0) \in \mathbb{R}^2_+$, the unique solution of (3.1)-(3.2) is positive (see [21, Proposition B.7]).

We denote by $A(x, y)$ the Jacobian matrix of f and g at (x, y), that is,

$$A(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \tag{3.3}$$

and by $|A(x, y)|$ and $\operatorname{tr}(A(x, y))$ the determinant and trace of $A(x, y)$, respectively.

The following results can be found in [19] and have been used in [6, 12, 16, 17, 25].

Lemma 3.1. *If (x^*, y^*) is an equilibrium of (3.1), then the following assertions hold.*

- (i) *If $|A(x^*, y^*)| < 0$, then (x^*, y^*) is a saddle of (3.1).*
- (ii) *If $|A(x^*, y^*)| > 0$, $\operatorname{tr}(A(x^*, y^*)) < 0$ and $(\operatorname{tr}(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| \geq 0$ then (x^*, y^*) is a stable node of (3.1).*
- (iii) *If $|A(x^*, y^*)| > 0$, $\operatorname{tr}(A(x^*, y^*)) < 0$ and $(\operatorname{tr}(A(x^*, y^*)))^2 - 4|A(x^*, y^*)| < 0$ then (x^*, y^*) is a stable focus of (3.1).*
- (iv) *If $|A(x^*, y^*)| > 0$ and $\operatorname{tr}(A(x^*, y^*)) < 0$, then (x^*, y^*) is locally asymptotically stable.*

Recall that an equilibrium (x^*, y^*) is said to be globally asymptotically stable if it is locally asymptotically stable and each solution (x, y) of (3.1)-(3.2) with $(x_0, y_0) \in \mathbb{R}^2_+$ converges to (x^*, y^*) in $\mathbb{R}^2$; that is, $\lim_{t\to\infty}(x(t), y(t)) = (x^*, y^*)$.

The following result is a special case of the well-known Poincaré-Bendixson Trichotomy theorem, see [22, Theorem 8.8 and Lemma 8.9].

Lemma 3.2. *Assume that each positive solution of (3.1)-(3.2) with $(x_0, y_0) \in \mathbb{R}^2_+$ is contained in a bounded closed subset B of $\mathbb{R}^2$. Assume that B contains only one equilibrium (x^*, y^*) of (3.1) and (x^*, y^*) belongs to the boundary of B. Then each positive solution of (3.1)-(3.2) converges to (x^*, y^*).*

Recall that a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (f(x, y), g(x, y))$ is said to be regular if T is one to one and onto, T and T^{-1} are continuous and $|A(x, y)| \neq 0$ on $\mathbb{R}^2$. If T is regular, then the following transformation

$$\begin{aligned} u &= f(x, y),\\ v &= g(x, y) \end{aligned} \tag{3.4}$$

is said to be a regular transformation. If (3.1) is changed into another system under suitable regular transformations, then the two systems are said to be equivalent. It is known that under regular transformations, the topological structures of solutions of a planar system near equilibria including a variety of dynamics like saddles, topological saddles, nodes, saddle-nodes, foci, centers, or cusps remain unchanged.

Lemma 3.3. [12] *Let (x^*, y^*) be an equilibrium of (3.1). Assume that $|A(x^*, y^*)| = 0$, $\operatorname{tr}(A(x^*, y^*)) \neq 0$ and (3.1) is equivalent to the following system*

$$\begin{aligned} \dot{u} &= p(u, v),\\ \dot{v} &= \varrho v + q(u, v) \end{aligned} \tag{3.5}$$

with an isolated equilibrium point $(0, 0)$, where $\varrho \neq 0$, $p(u, v) = \sum_{i+j=2,i,j\geq 0}^{\infty} a_{ij} u^i v^j$ and $q(u, v) = \sum_{i+j=2,i,j\geq 0}^{\infty} b_{ij} u^i v^j$ are convergent power series. If $a_{20} \neq 0$, then (x^*, y^*) is a saddle-node of (3.1).

Now, we begin to study the stability and phase portraits near each of the positive equilibrium of (2.2).

Let $A(x, y)$ be the Jacobian matrix of f and g defined in (2.2). By (2.2) and (3.3), we have

$$A(x, y) = \begin{pmatrix} -\beta y - b & -\beta x - qb \\ \beta y & \beta x - \eta - h \end{pmatrix}.$$

Note that $\eta = r + (1 - q)b$, we have

$$|A(x, y)| = \beta y(b + r + h) - b\beta x + b(\eta + h) \tag{3.6}$$

and

$$\operatorname{tr}(A(x, y)) = -\beta(y - x) - (b + \eta + h). \tag{3.7}$$

We first prove the following result on the global stability and phase portraits near the infection-free equilibrium $(1, 0)$ of (2.2).

Theorem 3.4. (1) *If $\mathcal{R}_0 > 1$, then $(1, 0)$ is a saddle of (2.2).*
(2) *If $\mathcal{R}_0 < 1$, then $(1, 0)$ is a stable node of (2.2). Moreover, the infection-free equilibrium $(1, 0)$ of (2.2) is globally asymptotically stable.*
(3) *If $\mathcal{R}_0 = 1$, then $(1, 0)$ is a saddle-node of (2.2).*

Proof. By (3.6) and (3.7) with $(x, y) = (1, 0)$, we have

$$|A(1, 0)| = b(h - \beta + \eta), \tag{3.8}$$

$$\operatorname{tr}(A(1, 0)) = \beta - \eta - h - b. \tag{3.9}$$

(1) Since $\mathcal{R}_0 > 1$, by (3.8) and Lemma 2.2 (1), we have $|A(1, 0)| < 0$. The result follows from Lemma 3.1 (i).

(2) Since $\mathcal{R}_0 < 1$, by Lemma 2.2 (2), (3.6) and (3.7) we obtain $|A(1, 0)| > 0$ and $\operatorname{tr}(A(1, 0)) < 0$. Moreover, we have

$$\begin{aligned} \operatorname{tr}(A(1, 0))^2 - 4|A(1, 0)| &= (\beta - \eta - h - b)^2 - 4b(h - \beta + \eta)\\ &= (\beta - \eta - h)^2 - 2b(\beta - \eta - h) + b^2 + 4b(\beta - h - \eta) \end{aligned}$$

$$\begin{aligned}&= (\beta-\eta-h)^2 + 2b(\beta-\eta-h) + b^2\\&= (\beta-\eta-h+b)^2 \geq 0.\end{aligned}$$

The first result follows from Lemma 3.1 (ii).

Let $B = \{(u,v) \in \mathbb{R}^2_+ : u+v \leq 1\}$, which is a positive invariant set of (2.2). B is a bounded closed subset of $\mathbb{R}^2$ and contains only the equilibrium $(1,0)$ of (2.2). Since $(1,0)$ is on the boundary of B, it follows from Lemma 3.2 that every positive solution of (2.2) converges to $(1,0)$ as $t \to \infty$. Hence, $(1,0)$ is globally asymptotically stable.

(3) Since $\mathcal{R}_0 = 1$, by Lemma 2.2 (3), (3.6) and (3.7), we have $|A(1,0)| = 0$ and $\operatorname{tr}(A(1,0)) < 0$. We change the equilibrium $(1,0)$ to the origin $(0,0)$ by the change of variables $u_1 = x-1$ and $v_1 = y$. Note that $h = \beta - \eta$. Then system (2.2) becomes

$$\begin{gathered}\dot{u}_1 = \dot{x} = b - \beta(u_1+1)v_1 - b(u_1+1) - qbv_1 = -\beta u_1v_1 - (\beta+qb)v_1 - bu_1,\\ \dot{v}_1 = \dot{y} = \beta(u_1+1)v_1 - (\eta+h)v_1 = \beta u_1v_1 - (\eta+h-\beta)v_1 = \beta u_1 v_1.\end{gathered}$$

Let $\xi = (\beta+qb)b^{-1}$, $u_2 = u_1 + \xi v$ and $v_2 = v_1$. Then the last system becomes

$$\begin{aligned}\dot{u}_2 &= \dot{u}_1 + \xi\dot{v}_1 = -\beta u_1v_1 - (\beta+qb)v_1 - bu_1 + \xi\beta u_1v_1\\&= (\xi-1)\beta u_1v_1 - (\beta+qb)v_1 - bu_1\\&= (\xi-1)\beta\big[u_2v_2 - \xi v_2^2\big] - (\beta+qb)v_2 - b(u_2 - \xi v_2)\\&= (\xi-1)\beta u_2v_2 - \xi(\xi-1)\beta v_2^2 - bu_2\end{aligned}$$

and $\dot{v}_2 = \beta v_2[u_2 - \xi v_2] = -\xi b v_2^2 + \beta u_2 v_2$.

Let $u = v_2$ and $v = u_2$. Then the above last two equations become

$$\begin{gathered}\dot{u} = -\xi b u^2 + \beta uv,\\ \dot{v} = -bv + (\xi-1)\beta uv - \xi(\xi-1)\beta u^2.\end{gathered}$$

Since $\varrho := -b \neq 0$ and $a_{20} := -\xi b \neq 0$, it follows from Lemma 3.3 that $(1,0)$ is a saddle-node of (2.2). $\square$

Remark 3.5. When $\mathcal{R}_0 < 1$, Theorem 3.4 (2) shows that the infection- free equilibrium $(1,0)$ is a stable node and is globally asymptotically stable. By Lemma 2.2 (2), we see that the biological interpretation of Theorem 3.4 (2) is that if $0 < \beta < \eta$ with any treatment rate $h \geq 0$ or $\eta \leq \beta$ and the treatment rate $h > \beta - \eta$, then the epidemic disease will be eradicated and the epidemic can not maintain itself (see Figure 1 (a) as an example).

Theorem 3.4 (3) is new and from its proof, we see that $(1,0)$ is a saddle-node in its neighborhood, so it is unstable. But since we only consider the biologically meaningful solutions in the triangle $B = \{(u,v) \in \mathbb{R}^2_+ : u+v \leq 1\}$, from the Figure 1 ($b$) below, we see that all the positive solutions in B converge to $(1,0)$ and $(1,0)$ is stable in B. This conclusion is consistent with the result in [7, Theorem 6.1] which falls into the special case of $q = h = 0$. Results in Theorem 3.4 (1) and (2) were obtained in [18, section 2] which corresponds to the special condition of $h = 0$ in our model.

Now, we turn our attention to the positive endemic equilibrium $(\bar{x}, \bar{y})$ given in (2.4) of (2.2). We first prove that $(\bar{x}, \bar{y})$ is locally asymptotically stable under suitable conditions.

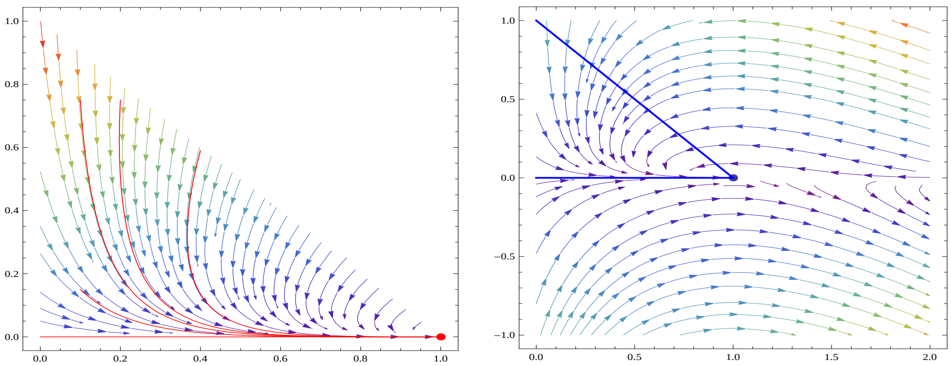

FIGURE 1. (a) shows that $(1,0)$ is globally asymptotically stable, where $b = 0.5$, $r = 2$, $\beta = 2$, $q = 0.5$, $h = 1$ and $\mathcal{R}_0 = \beta/(b+r-qr+h) = 2/3.25 < 1$. (b) shows that all the positive solutions in B converge to $(1,0)$, where b, r, q, h are same as in (a) and $\beta = 3.25$, so $\mathcal{R}_0 = 1$.

Theorem 3.6. *If $\mathcal{R}_0 > 1$, then $(\bar{x}, \bar{y})$ is locally asymptotically stable.*

Proof. Since $\mathcal{R}_0 > 1$, it follows from Theorem 2.3 (2) that $(\bar{x}, \bar{y})$ given in (2.4) is well defined. By (3.6) and (3.7) with $(x,y) = (\bar{x}, \bar{y})$, we have

$$|A(\bar{x}, \bar{y})| = \beta\bar{y}(b+r+h) + b(\eta+h) - \beta b\bar{x} = \beta\bar{y}(b+r+h), \tag{3.10}$$

$$\operatorname{tr}(A(\bar{x}, \bar{y})) = -\beta(\bar{y}-\bar{x}) - (b+h+\eta) = -(\beta\bar{y}+b). \tag{3.11}$$

Since $\bar{y} > 0$ and $b, r, \beta > 0$, by (3.10) and 3.11, we have $|A(\bar{x}, \bar{y})| > 0$ and $\operatorname{tr}(A(\bar{x}, \bar{y})) < 0$. The result follows from Lemma 3.1 (iv). □

Remark 3.7. By Lemma 2.2 (1), we see that $\mathcal{R}_0 > 1$ if and only if $\eta < \beta$ and $0 \le h < \beta - \eta$. Hence, Theorem 3.6 generalizes the result in [18, section 2] from $h = 0$ to $h \in [0, \beta - \eta)$. The biological interpretation of Theorem 3.6 is that if $\eta < \beta$, then the epidemic can not be eradicated if the treatment rate h is smaller than $\beta - \eta$.

It is not easy to determine if $(\bar{x}, \bar{y})$ is a stable node or stable focus. Our main goal in the rest of this paper is to find sufficient conditions on the parameters b, r, β, q, h under which $(\bar{x}, \bar{y})$ is a stable node or stable focus. All the results obtained below are new even when $q = 0$ or $h = 0$.

Our first result shows that $(\bar{x}, \bar{y})$ could be a stable node or stable focus for sufficiently small h. To do that, we first prove the following lemmas.

Let

$$\beta_1 := \frac{2(b+r)^{3/2}}{\sqrt{b+r}+\sqrt{r}} \quad \text{and} \quad \beta_2 := \frac{2(b+r)^{3/2}}{\sqrt{b+r}-\sqrt{r}}.$$

The following simple result will be useful in the proof of Lemma 3.9. Its proof is straightforward and we omit it.

Lemma 3.8. *Let $h(x) = 16x(1+x)^3 - 1$ for $x \in \mathbb{R}_+$. Then the following assertions hold.*

(i) *The equation $h(x) = 0$ has a unique solution $\gamma_1 \in (0.05, 0.055)$.*

(ii) $h(x) < 0$ *for* $x \in [0, \gamma_1)$ *and* $h(x) > 0$ *for* $x \in (\gamma_1, \infty)$.

Lemma 3.9. *(1) If* $b > 0$, $r > 0$, *then*

$$r < \beta_1 - b < b + r < \beta_2 - b \tag{3.12}$$

and

(i) *If* $r < \beta < \beta_1 - b$, *then* $0 < q_1 < 1 < \frac{\beta_1-\beta}{b} < \frac{\beta_2-\beta}{b}$;
(ii) *If* $\beta_1 - b < \beta \le b + r$, *then* $0 < q_1 < \frac{\beta_1-\beta}{b} < 1$ *and* $\frac{\beta_2-\beta}{b} > 1$.

(2) If $b > 0$ *and* $r > \gamma_1 b$, *then*

$$r < \beta_1 - b < b + r < \beta_1 < \beta_2 - b < \beta_2 \tag{3.13}$$

and the following assertions hold:

(i) *If* $b + r < \beta \le \beta_1$, *then* $0 \le \frac{\beta_1-\beta}{b} < 1$ *and* $\frac{\beta_2-\beta}{b} > 1$;
(ii) *If* $\beta_1 < \beta < \beta_2 - b$, *then* $\frac{\beta_1-\beta}{b} < 0$ *and* $\frac{\beta_2-\beta}{b} > 1$;
(iii) *If* $\beta_2 - b < \beta < \beta_2$, *then* $\frac{\beta_1-\beta}{b} < 0$ *and* $0 < \frac{\beta_2-\beta}{b} < 1$.

(3) If $0 < r < \gamma_1 b$, *then*

$$r < \beta_1 - b < b + r < \beta_2 - b < \beta_1 < \beta_2 \tag{3.14}$$

and the following assertions hold:

(i) *If* $b + r < \beta < \beta_2 - b$, *then* $0 < \frac{\beta_1-\beta}{b} < 1$ *and* $\frac{\beta_2-\beta}{b} > 1$;
(ii) *If* $\beta_2 - b < \beta \le \beta_1$, *then* $0 \le \frac{\beta_1-\beta}{b} < 1$ *and* $0 < \frac{\beta_2-\beta}{b} < 1$;
(iii) *If* $\beta_1 < \beta < \beta_2$, *then* $\frac{\beta_1-\beta}{b} < 0$ *and* $0 < \frac{\beta_2-\beta}{b} < 1$.

(4) If $r = \gamma_1 b$, *then*

$$r < \beta_1 - b < b + r < \beta_2 - b = \beta_1 < \beta_2 \tag{3.15}$$

and the following assertions hold:

(i) *If* $b + r < \beta < \beta_2 - b$, *then* $0 < \frac{\beta_1-\beta}{b} < 1$ *and* $\frac{\beta_2-\beta}{b} > 1$;
(ii) *If* $\beta_1 < \beta < \beta_2$, *then* $\frac{\beta_1-\beta}{b} < 0$ *and* $0 < \frac{\beta_2-\beta}{b} < 1$.

Proof. (1) Let $b > 0$ and $r > 0$. Since

$$\beta_1 = \frac{2(b+r)^{3/2}}{\sqrt{b+r} + \sqrt{r}} = \frac{2(b+r)}{1 + \sqrt{\frac{r}{b+r}}} > \frac{2(b+r)}{1+1} = b + r,$$

it follows that $r < \beta_1 - b$. Since

$$\begin{aligned}
\beta_1 - (2b+r) &= \frac{2(b+r)}{1 + \sqrt{\frac{r}{b+r}}} - (2b+r) = \frac{2(b+r) - (2b+r)\big[1 + \sqrt{\frac{r}{b+r}}\big]}{1 + \sqrt{\frac{r}{b+r}}} \\
&= \frac{r - (2b+r)\sqrt{\frac{r}{b+r}}}{1 + \sqrt{\frac{r}{b+r}}} = \frac{r^2 - (2b+r)^2 \frac{r}{b+r}}{\big[1 + \sqrt{\frac{r}{b+r}}\big]\big[r + (2b+r)\sqrt{\frac{r}{b+r}}\big]} \\
&= \frac{r^2(b+r) - r(2b+r)^2}{(b+r)\big[1 + \sqrt{\frac{r}{b+r}}\big]\big[r + (2b+r)\sqrt{\frac{r}{b+r}}\big]} \\
&= -\frac{rb(4b+3r)}{(b+r)\big[1 + \sqrt{\frac{r}{b+r}}\big]\big[r + (2b+r)\sqrt{\frac{r}{b+r}}\big]} < 0,
\end{aligned}$$

we have $\beta_1 - b < b + r$. Since

$$\beta_2 - (2b+r) = \frac{2(b+r)}{1-\sqrt{\frac{r}{b+r}}} - (2b+r) = \frac{2(b+r) - (2b+r)\big[1-\sqrt{\frac{r}{b+r}}\big]}{1-\sqrt{\frac{r}{b+r}}}$$
$$= \frac{r + (2b+r)\sqrt{\frac{r}{b+r}}}{1-\sqrt{\frac{r}{b+r}}} > 0,$$

we have $b + r < \beta_2 - b$. Hence, (3.12) holds. Note that $q_1 = (b + r - \beta)b^{-1}$. By (3.12), it is steadily verified that the results (i)-(ii) hold.

Let $b > 0$ and $r > 0$. By definition of β_1 and β_2 we have

$$\begin{aligned}
\beta_2 - b - \beta_1 &= \frac{2(b+r)^{3/2}}{\sqrt{b+r}-\sqrt{r}} - \frac{2(b+r)^{3/2}}{\sqrt{b+r}+\sqrt{r}} - b = \frac{4\sqrt{r}(b+r)^{3/2}}{b} - b \\
&= \frac{4\sqrt{r}(b+r)^{3/2} - b^2}{b} = \frac{16r(b+r)^3 - b^4}{b[4\sqrt{r}(b+r)^{3/2} + b^2]} = \frac{b^3\big[16\frac{r}{b}(1+\frac{r}{b})^3 - 1\big]}{4\sqrt{r}(b+r)^{3/2} + b^2} \\
&= \frac{b^3 h(\frac{r}{b})}{4\sqrt{r}(b+r)^{3/2} + b^2}
\end{aligned} \tag{3.16}$$

(2) If $b > 0$ and $r > \gamma_1 b$, then $\frac{r}{b} > \gamma_1$ and by Lemma 3.8, $h(\frac{r}{b}) > 0$. It follows from (3.16) that $\beta_1 < \beta_2 - b$. Since $\beta_1 > b + r$, we have $r < \beta_1 - b$. Hence, (3.13) holds. By (3.13), it is steadily verified that the results (i)-(iii) hold.

(3) If $0 < r < \gamma_1 b$, then $\frac{r}{b} < \gamma_1$ and by Lemma 3.8, $h(\frac{r}{b}) < 0$. It follows from (3.16) that $\beta_2 - b < \beta_1$. It is obvious that $\beta_1 < \beta_2$. Since

$$\beta_2 = \frac{2(b+r)^{3/2}}{\sqrt{b+r}-\sqrt{r}} = \frac{2(b+r)}{1-\sqrt{\frac{r}{b+r}}} > \frac{2(b+r)}{1-0} > 2b + r,$$

we obtain $b + r < \beta_2 - b$. It has been proved in (1) that $\beta_1 - b < b + r$. Hence, (3.14) holds. By (3.14), it is steadily verified that the results (i)-(iii) hold.

(4) If $r = \gamma_1 b$, then $\beta_1 = \beta_2 - b$. Hence, (3.15) holds and (i) and (ii) hold. □

Let

$$(\bar{x}, \bar{y}) = \Big(\frac{\eta + h}{\beta}, \frac{b(\beta - \eta - h)}{\beta(b + r + h)}\Big)$$

be same as in (2.4) and let

$$\Delta(q, h) = \operatorname{tr}(A(\bar{x}, \bar{y}))^2 - 4|A(\bar{x}, \bar{y})|. \tag{3.17}$$

Lemma 3.10. *If $\mathcal{R}_0 > 1$, then*

$$\Delta(q, h) = \frac{b}{(b + r + h)^2}\Big[b(\beta + qb)^2 - 4(b + r + h)^2(\beta + qb) + 4(b + r + h)^3\Big]. \tag{3.18}$$

Proof. Noting that $\eta = b + r - qb$, we have

$$\beta\bar{y} + b = \frac{b(\beta + qb)}{b + r + h} \quad \text{and} \quad 4\beta\bar{y}(b + r + h) = 4b[(\beta + qb) - (b + r + h)].$$

This, (3.10) and (3.11) imply

$$\Delta(q, h) := \operatorname{tr}(A(\bar{x}, \bar{y}))^2 - 4|A(\bar{x}, \bar{y})| = (\beta\bar{y} + b)^2 - 4\beta\bar{y}(b + r + h)$$

$$= \frac{b^2(\beta+qb)^2}{(b+r+h)^2} - 4b[(\beta+qb)-(b+r+h)]$$
$$= \frac{b}{(b+r+h)^2}\Big[b(\beta+qb)^2 - 4(b+r+h)^2(\beta+qb) + 4(b+r+h)^3\Big].$$

The result follows. $\square$

For the next theorem we use the following conditions

- (H1) $b>0$, $r>0$, $\beta_1-b<\beta\le b+r$ and $\frac{\beta_1-\beta}{b}<q\le 1$.
- (H2) $b>0$, $r>\gamma_1 b$ and one of the following conditions holds:
  - (i) $b+r<\beta\le\beta_1$ and $\frac{\beta_1-\beta}{b}<q\le 1$;
  - (ii) $\beta_1<\beta<\beta_2-b$ and $0\le q\le 1$;
  - (iii) $\beta=\beta_2-b$ and $0\le q<1$;
  - (iv) $\beta_2-b<\beta<\beta_2$ and $0\le q<\frac{\beta_2-\beta}{b}$.
- (H3) $b>0$, $0<r<\gamma_1 b$ and one of the following conditions holds:
  - (i) $b+r<\beta<\beta_2-b$ and $\frac{\beta_1-\beta}{b}<q\le 1$;
  - (ii) $\beta=\beta_2-b$ and $\frac{\beta_1-\beta}{b}<q<1$;
  - (iii) $\beta_2-b<\beta\le\beta_1$ and $\frac{\beta_1-\beta}{b}<q<\frac{\beta_2-\beta}{b}$;
  - (iv) $\beta_1<\beta<\beta_2$ and $0\le q<\frac{\beta_2-\beta}{b}$.
- (H4) $b>0$, $r=\gamma_1 b$ and one of the following conditions holds:
  - (i) $b+r<\beta<\beta_2-b$ and $\frac{\beta_1-\beta}{b}<q\le 1$;
  - (ii) $\beta=\beta_2-b$ and $\frac{\beta_1-\beta}{b}<q<1$.
  - (iii) $\beta_2-b<\beta<\beta_2$ and $0\le q<\frac{\beta_2-\beta}{b}$.

Theorem 3.11. *(1) Assume that one of the conditions* (H1)–(H4) *holds. Then there exists $h_0\in(0,\beta-\eta)$ such that $(\bar x,\bar y)$ is a stable focus of* (2.2) *for $h\in[0,h_0)$.*

(2) Assume that $b>0$, $r>0$ and one of the following conditions holds:

- (i) $r<\beta<\beta_1-b$ *and* $q_1<q\le 1$.
- (ii) $\beta=\beta_1-b$ *and* $q_1<q<1$.
- (iii) $\beta_2<\beta<\infty$ *and* $0\le q\le 1$.
- (iv) $\beta_2=\beta$ *and* $0<q\le 1$.

Then there exists $h_1\in(0,\beta-\eta)$ such that $(\bar x,\bar y)$ is a stable node of (2.2) *for $h\in[0,h_1)$.*

Proof. Under condition (H1), by (ii) of Lemma 3.9 (1), $r<\beta\le b+r$ and $q_1<q\le 1$. This, together with Lemma 2.1 (2), implies $\eta<\beta$. Similarly, by Lemma 3.9 (2), (3), (4), each of the hypotheses in (H2)–(H4) implies $b+r<\beta$ and $0\le q\le 1$. By Lemma 2.1 (2), we obtain $\eta<\beta$.

Let $0\le h<\beta-\eta$. By (3.18), we have

$$\Delta(q,h) = \frac{b^2\Gamma(h)}{(b+r+h)^2}, \tag{3.19}$$

where

$$\Gamma(q,h) = (\beta+qb)^2 - \frac{4(b+r+h)^2(\beta+qb)}{b} + \frac{4(b+r+h)^3}{b}. \tag{3.20}$$

We prove that

$$\Gamma(q,0) = b^2\Big(q-\frac{\beta_1-\beta}{b}\Big)\Big(q-\frac{\beta_2-\beta}{b}\Big). \tag{3.21}$$

Indeed, by (3.20) we have

$$
\begin{aligned}
&\Gamma(q,0)\\
&= (\beta+qb)^2 - \frac{4(b+r)^2(\beta+qb)}{b} + \frac{4(b+r)^3}{b}\\
&= \Big[\beta+qb-\frac{2(b+r)^2}{b}\Big]^2 - \frac{4r(b+r)^3}{b^2}\\
&= \Big[\beta+qb-\frac{2(b+r)^2}{b}+\frac{2\sqrt{r}(b+r)^{3/2}}{b}\Big]\Big[\beta+qb-\frac{2(b+r)^2}{b}-\frac{2\sqrt{r}(b+r)^{3/2}}{b}\Big]\\
&= \Big[\beta+qb-\frac{2(\sqrt{b+r}-\sqrt{r})(b+r)^{3/2}}{b}\Big]\Big[\beta+qb-\frac{2(\sqrt{b+r}+\sqrt{r})(b+r)^{3/2}}{b}\Big]\\
&= \Big[\beta+qb-\frac{2b(b+r)^{3/2}}{b(\sqrt{b+r}+\sqrt{r})}\Big]\Big[\beta+qb-\frac{2b(b+r)^{3/2}}{b(\sqrt{b+r}-\sqrt{r})}\Big]\\
&= (\beta+qb-\beta_1)(\beta+qb-\beta_2)
\end{aligned}
$$

and (3.21) holds.

We prove that under each of the conditions in (H1)–(H4),

$$\frac{\beta_1-\beta}{b} < q < \frac{\beta_2-\beta}{b}. \tag{3.22}$$

(H1) If $b>0$, $r>0$, $\beta_1-b<\beta<b+r$ and $\frac{\beta_1-\beta}{b}<q\le 1$. then by (ii) of Lemma 3.9 (1), we have

$$\frac{\beta_1-\beta}{b} < q \le 1 < \frac{\beta_2-\beta}{b}.$$

(H2) (i) If $b+r<\beta<\beta_1$ and $\frac{\beta_1-\beta}{b}\le q\le 1$, then by (i) of Lemma 3.9 (2),

$$\frac{\beta_1-\beta}{b} < q \le 1 < \frac{\beta_2-\beta}{b}.$$

(ii) If $\beta_1<\beta<\beta_2-b$ and $0\le q\le 1$, then by (ii) of Lemma 3.9 (2),

$$\frac{\beta_1-\beta}{b} < 0 \le q \le 1 < \frac{\beta_2-\beta}{b}.$$

(iii) If $\beta_2-b<\beta<\beta_2$ and $0\le q<\frac{\beta_2-\beta}{b}$, then by (iii) of Lemma 3.9 (2),

$$\frac{\beta_1-\beta}{b} < 0 \le q < \frac{\beta_2-\beta}{b}.$$

Hence, under each of the conditions (i), (ii) and (iii) in (H2), (3.22) holds. Similarly, (3.22) holds under each of the conditions in (H3) or (H4). By (3.21) and (3.22), we see that $\Gamma(q,0)<0$. It follows from the continuity of Γ that there exists $h_0\in(0,\beta-\eta)$ such that $\Gamma(h)<0$ for $h\in[0,h_0)$, and by (3.19), $\Delta(q,h)<0$ for $h\in[0,h_0)$. By Theorem 3.6 (1), (3.10) and (3.11) we see that for $h\in[0,h_0)$, $|A(\bar{x},\bar{y})|>0$ and $tr(\bar{x},\bar{y})<0$. The result follows from Lemma 3.1 (iii).

(2) (i) If $r<\beta<\beta_1-b$ and $q_1<q\le 1$, then by (i) of Lemma 3.9 (1), we have

$$1 < \frac{\beta_1-\beta}{b} < \frac{\beta_2-\beta}{b}. \tag{3.23}$$

(ii) If $\beta=\beta_1-b$ and $q_1<q<1$, then

$$q_1 < q < 1 = \frac{\beta_1-\beta}{b} < \frac{\beta_2-\beta}{b}. \tag{3.24}$$

(iii) If $\beta_2 < \beta < \infty$ and $0 \le q \le 1$, then

$$\frac{\beta_1 - \beta}{b} < \frac{\beta_2 - \beta}{b} < 0. \tag{3.25}$$

By (3.21) and each of (3.23), (3.24) and (3.25), we have $\Gamma(0) > 0$. It follows from the continuity of Γ that there exists $h_1 \in (0, \beta - \eta)$ such that $\Gamma(h) > 0$ for $h \in [0, h_1)$. It follows from (3.19) that $\Delta(h) > 0$ for $h \in [0, h_1)$. The result follows from Lemma 3.1 (ii). □

Simulation results for Theorem 3.11 (H2)(ii) and (2)(iii) are given in figure 2.

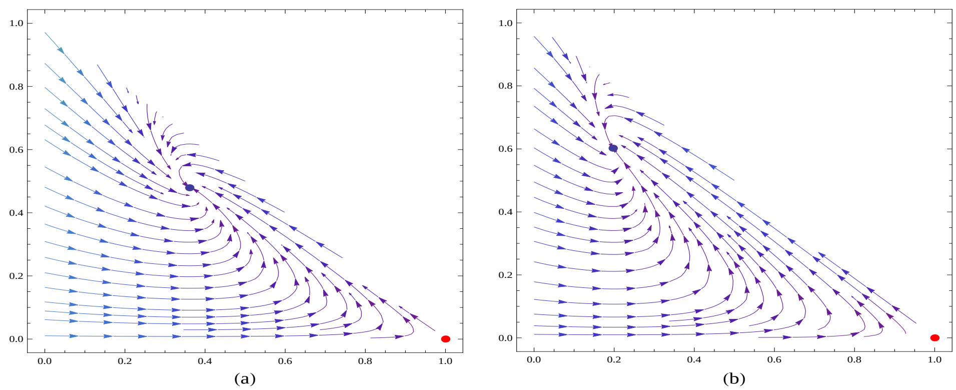

FIGURE 2. In (a) and (b), we use $b = 20$, $r = \frac{20}{3} > \gamma_1 b$ $(> \frac{1}{20}20 = 1)$, $q = 0.25$ and $h = 0$. Since $\beta_1 = \frac{320}{9}$, $\beta_2 - b = \frac{260}{3}$ and $\beta_2 = \frac{320}{3}$, conditions $(H_2)(ii)$ with $\beta = 60$ and $(2)(iii)$ with $\beta = 110$ hold.

By Theorem 3.11 with $h = q = 0$, we see that if $\beta_1 < \beta < \beta_2$, then $(\bar{x}, \bar{y})$ is a stable focus of (2.2) with $h = q = 0$ and if $\beta > \beta_2$, then $(\bar{x}, \bar{y})$ is a stable node of (2.2) with $h = q = 0$. Theorem 3.11 with $h = q = 0$ is inconclusive if $b + r < \beta \le \beta_1$ or $\beta = \beta_2$.

Using formulas (3.19) and (3.21), we can provide a direct proof to the following new result on the classic model (2.2) with $h = q = 0$.

Theorem 3.12. (1) *If $\beta_1 < \beta < \beta_2$, then $(\bar{x}, \bar{y})$ is a stable focus of* (2.2) *with* $h = q = 0$.

(2) *If either $b + r < \beta \le \beta_1$ or $\beta \ge \beta_2$, then $(\bar{x}, \bar{y})$ is a stable node of* (2.2) *with* $h = q = 0$.

Proof. By (3.19) and (3.21), we have

$$\Delta(0, 0) = \frac{b^2}{(b + r)^2}(\beta - \beta_1)(\beta - \beta_2).$$

(1) If $\beta_1 < \beta < \beta_2$, then $\Delta(0, 0) < 0$. By Lemma 3.1 (iii), the result (1) holds.

(2) If either $b + r < \beta \le \beta_1$ or $\beta \ge \beta_2$, then $\Delta(0, 0) \ge 0$ and the result (2) follows from Lemma 3.1 (ii). □

Theorem 3.11 shows that the interior equilibrium $(\bar{x}, \bar{y})$ of (2.2) can be a stable focus or a stable node for sufficiently small h. However, Theorem 3.11 does not provide any upper bounds for h. Hence, the question is that under which range of h, can the interior equilibrium $(\bar{x}, \bar{y})$ be a stable focus or a stable node?

In the following, we enhance the result (iii) of Theorem 3.11 (2) and partially answer the above question. We provide a range of h under which $(\bar{x}, \bar{y})$ is a stable node of (2.2). To do this, we first prove the following lemma. Let

$$\beta_0 = \frac{4(b+r)^2}{b} \quad \text{and} \quad h_1 = \frac{\sqrt{b(\beta+qb)}}{2} - b - r.$$

Lemma 3.13. *(1) $\beta_2 < \beta_0 - b$.*

(2) If $\beta_0 - b \le \beta \le \infty$ and $\max\{0, \frac{\beta_0-\beta}{b}\} \le q \le 1$, then $0 \le h_1 < \beta - \eta$.

Proof. (1) Since

$$\frac{2(b+r)}{b} = \frac{1}{1+\sqrt{\frac{r}{b+r}}} + \frac{1}{1-\sqrt{\frac{r}{b+r}}},$$

we have

$$\begin{aligned}
\beta_0 - \beta_2 - b &= \frac{4(b+r)^2}{b} - \frac{2(b+r)}{1-\sqrt{\frac{r}{b+r}}} - b = \frac{2(b+r)}{1+\sqrt{\frac{r}{b+r}}} - b \\
&= \frac{2(b+r) - b\big(1+\sqrt{\frac{r}{b+r}}\big)}{1+\sqrt{\frac{r}{b+r}}} = \frac{b\big(1-\sqrt{\frac{r}{b+r}}\big)+2r}{1+\sqrt{\frac{r}{b+r}}} > 0.
\end{aligned}$$

(2) We first prove that under the given hypotheses,

$$b - 4\beta - 4qb < 0. \tag{3.26}$$

In fact, if $\frac{\beta_0-\beta}{b} \ge 0$, then $q \ge \frac{\beta_0-\beta}{b}$ and

$$b - 4\beta - 4qb \le b - 4\beta - 4(\beta_0 - \beta) = b - 4\beta_0 = \frac{b^2 - 4(b+r)^2}{b} < 0.$$

If $\frac{\beta_0-\beta}{b} < 0$, then $q \ge 0$ and

$$b - 4\beta - 4qb \le b - 4\beta \le b - 4(\beta_0 - b) = \frac{5b^2 - 16(b+r)^2}{b} < -11b < 0.$$

Next, we prove that $h_1 < \beta - \eta$. Indeed, since

$$\begin{aligned}
h_1 - (\beta - \eta) &= \frac{\sqrt{b(\beta+qb)} - 2(b+r)}{2} - (\beta - b - r + qb) \\
&= \frac{\sqrt{b(\beta+qb)}}{2} - (\beta+qb) \\
&= \frac{(\beta+qb)[b - 4\beta - 4qb)]}{2\big[\sqrt{b(\beta+qb)} + 4(\beta+qb)\big]}.
\end{aligned}$$

This, together with (3.26), implies $h_1 < \beta - \eta$. Finally, we prove that $h_1 \ge 0$. In fact, since

$$\begin{aligned}
h_1 &= \frac{\sqrt{b(\beta+qb)} - 2(b+r)}{2} = \frac{b(\beta+qb) - 4(b+r)^2}{2[\sqrt{b(\beta+qb)} + 2(b+r)]} \\
&= \frac{b(\beta+qb) - 4(b+r)^2}{2[\sqrt{b(\beta+qb)} + 2(b+r)]} = \frac{b[(\beta+qb) - \beta_0]}{2[\sqrt{b(\beta+qb)} + 2(b+r)]}
\end{aligned}$$

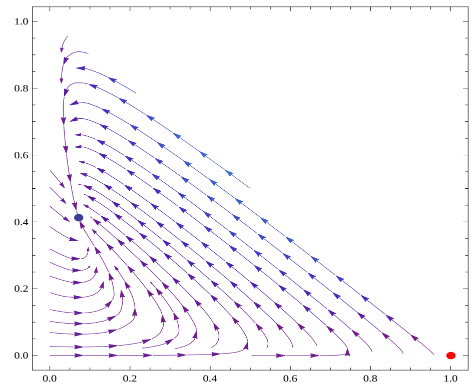

FIGURE 3. $b = r = 1$, $\beta = 24.5$, $q = 0.5$ and $h = 0.25$. Since $\beta_0 = 16$, $\beta_0 - b = 15$ and $h_1 = \frac{1}{2}$, the conditions of Theorem 3.14 are satisfied.

$$= \frac{b^2\left(q - \frac{\beta_0-\beta}{b}\right)}{2[\sqrt{b(\beta+qb)} + 2(b+r)]},$$

it follows from $q \geq \max\{0, \frac{\beta_0-\beta}{b}\}$ that $q - \frac{\beta_0-\beta}{b} \geq 0$ and $h_1 \geq 0$. □

Theorem 3.14. *If $\beta_0 - b \leq \beta \leq \infty$, $\max\{0, \frac{\beta_0-\beta}{b}\} \leq q \leq 1$ and $0 \leq h \leq h_1$, then $(\bar{x}, \bar{y})$ is a stable node of (2.2).*

Proof. By Lemma 3.9, we see that $b + r < \beta_2$ and by Lemma 3.13, $b + r < \beta_2 < \beta_0 - b \leq \beta$. Hence, by Lemma 2.1 (ii), Lemma 2.2 and Theorem 2.3 (2) we see that for $\max\{0, \frac{\beta_0-\beta}{b}\} \leq q \leq 1$, $(\bar{x}, \bar{y})$ given in (2.4) is well defined. It is easy to verify that if $h \leq h_1$, then $2(b + r + h) \leq \sqrt{b(\beta + qb)}$ and

$$b(\beta + qb) - 4(b + r + h)^2 \geq 0.$$

This, together with (3.18), implies

$$\Delta(q, h) = \frac{b}{(b+r+h)^2}\Big\{(\beta + qb)\big[b(\beta + qb) - 4(b + r + h)^2\big] + 4(b + r + h)^3\Big\} > 0.$$

The result follows from Lemma 3.1 (ii). □

Theorem 3.14 provides a range for h under which $(\bar{x}, \bar{y})$ is a stable node of (2.2), see Figure 3 below for a simulation result. By Lemma 3.13 (1), we see that under the hypothesis of Theorem 3.14: $\beta_0 - b \leq \beta \leq \infty$, we have $\beta_2 < \beta_0 - b \leq \beta$. Hence, Theorem 3.14 strengthens the result (iii) of Theorem 3.11 (2) which holds for sufficiently small h.

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