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LAYER POTENTIALS FOR GENERAL LINEAR ELLIPTIC SYSTEMS

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Abstract. In this article we construct layer potentials for elliptic differential operators using the Babuška-Lax-Milgram theorem, without recourse to the fundamental solution; this allows layer potentials to be constructed in very general settings. We then generalize several well known properties of layer potentials for harmonic and second order equations, in particular the Green's formula, jump relations, adjoint relations, and Verchota's equivalence between well-posedness of boundary value problems and invertibility of layer potentials.

1. Introduction

There is by now a very rich theory of boundary value problems for the Laplace operator, and more generally for second order divergence form operators $-\operatorname{div}\mathbf{A}\nabla$. The Dirichlet problem

$$-\operatorname{div}\mathbf{A}\nabla u = 0 \text{ in } \Omega, \quad u = f \text{ on } \partial\Omega, \quad \|u\|_{\mathfrak{X}} \leq C\|f\|_{\mathfrak{D}}$$

and the Neumann problem

$$-\operatorname{div}\mathbf{A}\nabla u = 0 \text{ in } \Omega, \quad \nu\cdot\mathbf{A}\nabla u = g \text{ on } \partial\Omega, \quad \|u\|_{\mathfrak{X}} \leq C\|g\|_{\mathfrak{N}}$$

are known to be well-posed for many classes of coefficients $\mathbf{A}$ and domains Ω, and with solutions in many spaces $\mathfrak{X}$ and boundary data in many boundary spaces $\mathfrak{D}$ and $\mathfrak{N}$.

A great deal of current research consists in extending these well posedness results to more general situations, such as operators of order $2m\geq 4$ (for example, [19, 25, 45, 47, 53, 54]; see also the survey paper [22]), operators with lower order terms (for example, [24, 30, 34, 55, 62]) and operators acting on functions defined on manifolds (for example, [46, 50, 51]).

Two very useful tools in the second order theory are the double and single layer potentials given by

$$\mathcal{D}^{\Omega}_{\mathbf{A}} f(x) = \int_{\partial\Omega} \overline{\nu\cdot\mathbf{A}^*(y)\nabla_y E^{L^*}(y,x)}\, f(y)\,d\sigma(y), \tag{1.1}$$

$$\mathcal{S}^{\Omega}_{L} g(x) = \int_{\partial\Omega} \overline{E^{L^*}(y,x)}\, g(y)\,d\sigma(y) \tag{1.2}$$

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where ν is the unit outward normal to Ω and where $E^L(y,x)$ is the fundamental solution for the operator $L = -\operatorname{div}\mathbf{A}\nabla$, that is, the formal solution to $LE^L(\,\cdot\,,x) = \delta_x$. These operators are inspired by a formal integration by parts

$$\begin{aligned}u(x) &= \int_\Omega \overline{L^* E^{L^*}(\,\cdot\,,x)}\,u \\ &= -\int_{\partial\Omega} \overline{\nu\cdot\mathbf{A}^*\nabla E^{L^*}(\cdot,x)}u\,d\sigma + \int_{\partial\Omega}\overline{E^{L^*}(\cdot,x)}\nu\cdot\mathbf{A}\nabla u\,d\sigma + \int_\Omega \overline{E^{L^*}(\cdot,x)}Lu\end{aligned}$$

which gives the Green's formula

$$u(x) = -\mathcal{D}^\Omega_{\mathbf{A}}(u\big|_{\partial\Omega})(x) + \mathcal{S}^\Omega_L(\nu\cdot\mathbf{A}\nabla u)(x) \quad \text{if } x\in\Omega \text{ and } Lu=0 \text{ in } \Omega$$

at least for relatively well-behaved solutions u.

Such potentials have many well known properties beyond the above Green's formula, including jump and adjoint relations. In particular, by a clever argument of Verchota [63] and some extensions in [21, 23], given certain boundedness and trace results, well posedness of the Dirichlet problem in both Ω and its complement is equivalent to invertibility of the operator $g\mapsto \mathcal{S}^\Omega_L g\big|_{\partial\Omega}$, and well posedness of the Neumann problem in both domains is equivalent to invertibility of the operator $f\mapsto \nu\cdot\mathbf{A}\nabla\mathcal{D}^\Omega_{\mathbf{A}}f$.

This equivalence has been used to solve boundary value problems in many papers, including [29, 32, 33, 63] in the case of harmonic functions (that is, the case $\mathbf{A}=\mathbf{I}$ and $L=-\Delta$) and [5, 14, 23, 35, 37, 38] in the case of more general second order operators under various assumptions on the coefficients $\mathbf{A}$. Layer potentials have been used in other ways in [4, 9, 21, 44, 48, 49, 56, 59, 65]. Boundary value problems were studied using a functional calculus approach in [6, 7, 8, 9, 10, 11, 12]; in [58] it was shown that certain operators arising in this theory coincided with layer potentials.

Thus, it is desirable to extend layer potentials to more general situations. It is possible to proceed as in the homogeneous second order case, by constructing the fundamental solution, formally integrating by parts, and showing that the resulting integral operators have appropriate properties. In the case of higher order operators with constant coefficients, this has been done in [2, 27, 28, 52, 53, 64]. All three steps are somewhat involved in the case of variable coefficient operators (although see [15, 30] for fundamental solutions, for higher order operators without lower order terms, and for second order operators with lower order terms, respectively).

An alternative, more abstract construction is possible. The fundamental solution for various operators was constructed in [15, 30, 36] as the kernel of the Newton potential, which may itself be constructed very simply using the Lax-Milgram theorem. It is possible to rewrite the formulas (1.1) and (1.2) for second order layer potentials directly in terms of the Newton potential, without mediating by the fundamental solution, and this construction generalizes very easily. It is this approach that was taken in [18, 20].

In this paper we will provide the details of this construction in a very general context. Roughly, this construction is valid for all differential operators L that may be inverted via the Babuška-Lax-Milgram theorem, and all domains Ω for which suitable boundary trace operators exist. We will also show that many properties of traditional layer potentials are valid in the general case.

The organization of this paper is as follows. The goal of this paper is to construct layer potentials associated to an operator L as bounded linear operators from a space $\mathfrak{D}_2$ or $\mathfrak{N}_2$ to a Hilbert space $\mathfrak{H}_2$ given certain conditions on $\mathfrak{D}_2$, $\mathfrak{N}_2$ and $\mathfrak{H}_2$. In Section 2 we will list these conditions and define our terminology. Because these properties are somewhat abstract, in Section 3 we will give an example of spaces $\mathfrak{H}_2$, $\mathfrak{D}_2$ and $\mathfrak{N}_2$ that satisfy these conditions in the case where L is a higher order differential operator in divergence form without lower order terms.

This is the context of the paper [19]; we intend to apply the results of the present paper therein to solve the Neumann problem with boundary data in L^2 for operators with transversally independent self-adjoint coefficients.

In Section 4 of this paper we will provide the details of the construction of layer potentials. We will prove the higher order analogues for the Green's formula, adjoint relations, and jump relations in Section 5. Finally, in Section 6 we will show that the equivalence between well posedness of boundary value problems and invertibility of layer potentials of [21, 23, 63] extends to the general case.

2. Terminology

We will construct layer potentials $\mathcal{D}^\Omega_{\mathfrak{B}}$ and $\mathcal{S}^\Omega_L$ using the following objects.

- Two Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$.
- Six (quasi)-normed vector spaces $\widehat{\mathfrak{H}}_1^\Omega$, $\widehat{\mathfrak{H}}_1^{\mathfrak{C}}$, $\widehat{\mathfrak{H}}_2^\Omega$, $\widehat{\mathfrak{H}}_2^{\mathfrak{C}}$, $\widehat{\mathfrak{D}}_1$ and $\widehat{\mathfrak{D}}_2$.
- Bounded sesquilinear forms $\mathfrak{B} : \mathfrak{H}_1 \times \mathfrak{H}_2 \to \mathbb{C}$, $\mathfrak{B}^\Omega : \mathfrak{H}_1^\Omega \times \mathfrak{H}_2^\Omega \to \mathbb{C}$, and $\mathfrak{B}^{\mathfrak{C}} : \mathfrak{H}_1^{\mathfrak{C}} \times \mathfrak{H}_2^{\mathfrak{C}} \to \mathbb{C}$. (We will define the spaces $\mathfrak{H}_j^\Omega$, $\mathfrak{H}_j^{\mathfrak{C}}$ momentarily.)
- Bounded linear operators $\dot{\mathbf{Tr}}_1 : \mathfrak{H}_1 \to \widehat{\mathfrak{D}}_1$ and $\dot{\mathbf{Tr}}_2 : \mathfrak{H}_2 \to \widehat{\mathfrak{D}}_2$.
- Bounded linear operators $(\cdot)\big|^1_\Omega : \mathfrak{H}_1 \to \widehat{\mathfrak{H}}_1^\Omega$ and $(\cdot)\big|^2_\Omega : \mathfrak{H}_2 \to \widehat{\mathfrak{H}}_2^\Omega$. When no ambiguity will arise we will suppress the superscript and refer to both operators as $\big|_\Omega$.
- Bounded linear operators $(\cdot)\big|^j_{\mathfrak{C}} : \mathfrak{H}_j \to \widehat{\mathfrak{H}}_j^{\mathfrak{C}}$ for $j = 1, 2$; we again often refer to both operators as $\big|_{\mathfrak{C}}$.

We will work not with the spaces $\widehat{\mathfrak{H}}_j^\Omega$, $\widehat{\mathfrak{H}}_j^{\mathfrak{C}}$ and $\widehat{\mathfrak{D}}_j$, but with the (normed) vector spaces $\mathfrak{H}_j^\Omega$, $\mathfrak{H}_j^{\mathfrak{C}}$ and $\mathfrak{D}_j$ defined as follows.

$$\mathfrak{H}_j^\Omega = \{F\big|_\Omega : F \in \mathfrak{H}_j\}/\sim \text{ with norm } \|f\|_{\mathfrak{H}_j^\Omega} = \inf\{\|F\|_{\mathfrak{H}_j} : F\big|_\Omega = f\}, \tag{2.1}$$

$$\mathfrak{H}_j^{\mathfrak{C}} = \{F\big|_{\mathfrak{C}} : F \in \mathfrak{H}_j\}/\sim \text{ with norm } \|f\|_{\mathfrak{H}_j^{\mathfrak{C}}} = \inf\{\|F\|_{\mathfrak{H}_j} : F\big|_{\mathfrak{C}} = f\}, \tag{2.2}$$

$$\mathfrak{D}_j = \{\dot{\mathbf{Tr}}_j F : F \in \mathfrak{H}_j\}/\sim \text{ with norm } \|\dot{\mathbf{f}}\|_{\mathfrak{D}_j} = \inf\{\|F\|_{\mathfrak{H}_j} : \dot{\mathbf{Tr}}_j F = \dot{\mathbf{f}}\} \tag{2.3}$$

where $\sim$ denotes the equivalence relation $f \sim g$ if $\|f - g\| = 0$.

Throughout we will impose the following conditions on the given function spaces and operators.

Condition 2.1. $\mathfrak{B}$ *is coercive; that is, there is some* $\lambda > 0$ *such that for every* $u \in \mathfrak{H}_1$ *and* $v \in \mathfrak{H}_2$ *we have that*

$$\sup_{w\in\mathfrak{H}_1\setminus\{0\}} \frac{|\mathfrak{B}(w,v)|}{\|w\|_{\mathfrak{H}_1}} \geq \lambda\|v\|_{\mathfrak{H}_2}, \quad \sup_{w\in\mathfrak{H}_2\setminus\{0\}} \frac{|\mathfrak{B}(u,w)|}{\|w\|_{\mathfrak{H}_2}} \geq \lambda\|u\|_{\mathfrak{H}_1}.$$

Condition 2.2. *If* $u \in \mathfrak{H}_1$ *and* $v \in \mathfrak{H}_2$, *then*

$$\mathfrak{B}(u,v) = \mathfrak{B}^\Omega(u\big|_\Omega, v\big|_\Omega) + \mathfrak{B}^{\mathfrak{C}}(u\big|_{\mathfrak{C}}, v\big|_{\mathfrak{C}}).$$

Condition 2.3. *If φ, $\psi \in \mathfrak{H}_j$ for $j = 1$ or $j = 2$, and if $\dot{\mathbf{Tr}}_j\,\varphi = \dot{\mathbf{Tr}}_j\,\psi$, then there is a $w \in \mathfrak{H}_j$ such that*

$$w\big|_\Omega = \varphi\big|_\Omega, \quad w\big|_{\mathfrak{C}} = \psi\big|_{\mathfrak{C}}, \quad \text{and} \quad \dot{\mathbf{Tr}}_j\,w = \dot{\mathbf{Tr}}_j\,\varphi = \dot{\mathbf{Tr}}_j\,\psi.$$

We now introduce some further terminology.

If $\mathfrak{X}$ is a quasi-Banach space, we will let $\mathfrak{X}^*$ be the space of conjugate linear functionals on $\mathfrak{X}$.

We define the conjugate linear operator L as follows. If $u \in \mathfrak{H}_2$, let Lu be the element of $\mathfrak{H}_1^*$ given by

$$\langle \varphi, Lu\rangle = \mathfrak{B}(\varphi, u). \tag{2.4}$$

Notice that L is bounded $\mathfrak{H}_2 \to \mathfrak{H}_1^*$.

If $u \in \mathfrak{H}_2^\Omega$, we let $(Lu)\big|_\Omega$ be the element of $\{\varphi \in \mathfrak{H}_1 : \dot{\mathbf{Tr}}_1\,\varphi = 0\}^*$ given by

$$\langle \varphi, (Lu)\big|_\Omega\rangle = \mathfrak{B}^\Omega(\varphi\big|_\Omega, u) \quad \text{for all } \varphi \in \mathfrak{H}_1 \text{ with } \dot{\mathbf{Tr}}_1\,\varphi = 0. \tag{2.5}$$

If $u \in \mathfrak{H}_2$, we will often use $(Lu)\big|_\Omega$ as shorthand for $(L(u\big|_\Omega))\big|_\Omega$. We will primarily be concerned with the case $(Lu)\big|_\Omega = 0$.

We will let

$$\mathfrak{N}_2 = \mathfrak{D}_1^*, \quad \mathfrak{N}_1 = \mathfrak{D}_2^* \tag{2.6}$$

denote the spaces of conjugate linear functionals on $\mathfrak{D}_1$ and $\mathfrak{D}_2$. We will now define the Neumann boundary values of an element u of $\mathfrak{H}_2^\Omega$ that satisfies $(Lu)\big|_\Omega = 0$. If $\dot{\mathbf{Tr}}_1\,\varphi = \dot{\mathbf{Tr}}_1\,\psi$ and $(Lu)\big|_\Omega = 0$, then $\mathfrak{B}^\Omega(\varphi\big|_\Omega - \psi\big|_\Omega, u) = 0$ by definition of $(Lu)\big|_\Omega$. Thus, $\mathfrak{B}^\Omega(\varphi\big|_\Omega, u)$ depends only on $\dot{\mathbf{Tr}}_1\,\varphi$, not on φ. Thus, $\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u$ defined as follows is a well defined element of $\mathfrak{N}_2$.

$$\langle \dot{\mathbf{Tr}}_1\,\varphi, \dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u\rangle = \mathfrak{B}^\Omega(\varphi\big|_\Omega, u) \quad \text{for all } \varphi \in \mathfrak{H}_1. \tag{2.7}$$

We can compute

$$|\langle \dot{\mathbf{f}}, \dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u\rangle| \leq \|\mathfrak{B}^\Omega\| \inf\{\|\varphi\|_{\mathfrak{H}_1} : \dot{\mathbf{Tr}}_1\,\varphi = \dot{\mathbf{f}}\}\|u\|_{\mathfrak{H}_2^\Omega} = \|\mathfrak{B}^\Omega\|\|\dot{\mathbf{f}}\|_{\mathfrak{D}_1}\|u\|_{\mathfrak{H}_2^\Omega}$$

and so we have the bound $\|\,\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u\|_{\mathfrak{N}_2} \leq \|\mathfrak{B}^\Omega\|\,\|u\|_{\mathfrak{H}_2^\Omega}$.

If $(Lu)\big|_\Omega \neq 0$, then the conjugate linear operator given by $\varphi \mapsto \mathfrak{B}^\Omega(\varphi\big|_\Omega, u)$ is still of interest. We will denote this operator $L_{\mathfrak{B}^\Omega}\,u$; that is, if $u \in \mathfrak{H}_2^\Omega$, then $L_{\mathfrak{B}^\Omega}\,u \in \mathfrak{H}_1^*$ is defined by

$$\langle \varphi, L_{\mathfrak{B}^\Omega}\,u\rangle = \mathfrak{B}^\Omega(\varphi\big|_\Omega, u) \quad \text{for all } \varphi \in \mathfrak{H}_1. \tag{2.8}$$

If $u \in \mathfrak{H}_2$ then as before we will use $L_{\mathfrak{B}^\Omega}\,u$ as a shorthand for $L_{\mathfrak{B}^\Omega}(u\big|_\Omega)$.

Remark 2.4. We observe that, for a given sesquilinear form $\mathfrak{B}$ defined on $\mathfrak{H}_1 \times \mathfrak{H}_2$, there are often many choices of forms $\mathfrak{B}^\Omega$ and $\mathfrak{B}^{\mathfrak{C}}$ that satisfy Condition 2.2. Conversely, for a given form $\mathfrak{B}^\Omega$ there may be many forms $\mathfrak{B}^{\mathfrak{C}}$ such that the operator $\mathfrak{B}$ given by Condition 2.2 satisfies Condition 2.1. See Remark 3.1 for an example.

The operator L depends only on $\mathfrak{B}$, and not on a particular choice of $\mathfrak{B}^\Omega$ and $\mathfrak{B}^{\mathfrak{C}}$. By contrast, the quantities $\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u$ and $L_{\mathfrak{B}^\Omega}\,u$ depend on $\mathfrak{B}^\Omega$ and not on $\mathfrak{B}$ (that is, not on $\mathfrak{B}^{\mathfrak{C}}$).

We also comment on the quantity $(Lu)\big|_\Omega$. If $u \in \mathfrak{H}_2^\Omega$, then by definition of $\mathfrak{H}_2^\Omega$ there is some $U \in \mathfrak{H}_2$ with $u = U\big|_\Omega$. If $\dot{\mathbf{Tr}}_1\,\varphi = 0 = \dot{\mathbf{Tr}}_1\,0$, then by Condition 2.3

there is some $w \in \mathfrak{H}_1$ with $w\big|_\Omega = \varphi\big|_\Omega$ and $w\big|_{\mathfrak{C}} = 0\big|_{\mathfrak{C}} = 0$. Thus, by the definition (2.5) of $(Lu)\big|_\Omega$ and Condition 2.2,

$$\langle \varphi, (Lu)\big|_\Omega \rangle = \mathfrak{B}^\Omega(\varphi\big|_\Omega, u) = \mathfrak{B}(w, U) - \mathfrak{B}^{\mathfrak{C}}(w\big|_{\mathfrak{C}}, U\big|_{\mathfrak{C}}) = \mathfrak{B}(w, U)$$

and so $(Lu)\big|_\Omega$ may be viewed as depending either on $\mathfrak{B}$ or on $\mathfrak{B}^\Omega$.

3. An example: higher order differential equations

In this section, we provide an example of a situation in which the terminology of Section 2 and the construction and properties of layer potentials of Sections 4 and 5 may be applied. We remark that this is the situation of [19], and that we will therein apply the results of this paper.

Let $m \geq 1$ be an integer, and let L be an elliptic differential operator of the form

$$Lu = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta} \partial^\beta u) \tag{3.1}$$

for some (possibly complex) bounded measurable coefficients $\mathbf{A}$ defined on $\mathbb{R}^d$. Here α and β are multiindices in $\mathbb{N}_0^d$, where $\mathbb{N}_0$ denotes the nonnegative integers. As is standard in the theory, we say that $Lu = 0$ in an open set Ω in the weak sense if

$$\int_\Omega \sum_{|\alpha|=|\beta|=m} \partial^\alpha \varphi A_{\alpha\beta} \partial^\beta u = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{3.2}$$

We impose the following ellipticity condition: we require that for some $\lambda > 0$,

$$\Re \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^d} \overline{\partial^\alpha \varphi}\, A_{\alpha\beta} \partial^\beta \varphi \geq \lambda \|\nabla^m \varphi\|^2_{L^2(\mathbb{R}^d)} \quad \text{for all } \varphi \in \dot{W}^2_m(\mathbb{R}^d). \tag{3.3}$$

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, and let $\mathfrak{C} = \mathbb{R}^d \setminus \overline{\Omega}$ denote the interior of its complement. Observe that $\partial\Omega = \partial\mathfrak{C}$.

The following function spaces and linear operators satisfy the conditions of Section 2.

- $\mathfrak{H}_1 = \mathfrak{H}_2 = \mathfrak{H}$ is the homogeneous Sobolev space $\dot{W}^2_m(\mathbb{R}^d)$ of locally integrable functions φ (or rather, of equivalence classes of functions modulo polynomials of degree $m-1$) with weak derivatives of order m, and such that the $\mathfrak{H}$-norm given by $\|\varphi\|_{\mathfrak{H}} = \|\nabla^m \varphi\|_{L^2(\mathbb{R}^d)}$ is finite. This space is a Hilbert space with inner product $\langle \varphi, \psi \rangle = \sum_{|\alpha|=m} \int_{\mathbb{R}^d} \overline{\partial^\alpha \varphi}\, \partial^\alpha \psi$.
- $\widehat{\mathfrak{H}}^\Omega$ and $\widehat{\mathfrak{H}}^{\mathfrak{C}}$ are the Sobolev spaces $\widehat{\mathfrak{H}}^\Omega = \dot{W}^2_m(\Omega) = \{\varphi : \nabla^m \varphi \in L^2(\Omega)\}$ and $\widehat{\mathfrak{H}}^{\mathfrak{C}} = \dot{W}^2_m(\mathfrak{C}) = \{\varphi : \nabla^m \varphi \in L^2(\mathfrak{C})\}$ with the expected norms.
- $\widehat{\mathfrak{D}}$ denotes the (vector-valued) Besov space $\dot{B}^{2,2}_{1/2}(\partial\Omega)$ of locally integrable functions modulo constants with norm

$$\|f\|_{\dot{B}^{2,2}_{1/2}(\partial\Omega)} = \Big(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^2}{|x-y|^d}\, d\sigma(x)\, d\sigma(y)\Big)^{1/2}.$$

- In [17, 18, 19], Ω is assumed to have connected boundary, and $\dot{\mathbf{Tr}}$ is the linear operator defined on $\mathfrak{H}$ by

$$\dot{\mathbf{Tr}}\, u = \mathrm{Tr}^\Omega\, \nabla^{m-1} u\big|_\Omega = \{\mathrm{Tr}^\Omega\, \partial^\gamma u\}_{|\gamma|=m-1},$$

where Tr^Ω is the standard boundary trace operator of Sobolev spaces.

Given a suitable modification of the trace space $\widehat{\mathfrak{D}}$, it is also possible to choose

$$\dot{\mathbf{Tr}}\, u = \{\operatorname{Tr}^\Omega \partial^\gamma u\}_{|\gamma|\leq m-1} \quad \text{or} \quad \dot{\mathbf{Tr}}\, u = (\operatorname{Tr}^\Omega u, \partial_\nu u, \dots, \partial_\nu^{m-1} u),$$

where ν is the unit outward normal, so that the boundary derivatives of u of all orders are recorded. See, for example, [3, 47, 53, 57, 60]. In this case, $\partial\Omega$ need not be connected.

- $\mathfrak{B}$ is the sesquilinear form on $\mathfrak{H}\times\mathfrak{H}$ given by

$$\mathfrak{B}(\psi,\varphi) = \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^d} \overline{\partial^\alpha \psi} A_{\alpha\beta} \partial^\beta \varphi. \tag{3.4}$$

The sesquilinear forms $\mathfrak{B}^\Omega$ and $\mathfrak{B}^{\mathfrak{C}}$ are defined analogously to $\mathfrak{B}$, but with the integral over $\mathbb{R}^d$ replaced by an integral over Ω or $\mathfrak{C}$.

We now discuss the conditions imposed in Section 2. The forms $\mathfrak{B}$, $\mathfrak{B}^\Omega$ and $\mathfrak{B}^{\mathfrak{C}}$ are clearly bounded and sesquilinear, and the restriction operators $\big|_\Omega : \mathfrak{H} \to \widehat{\mathfrak{H}}^\Omega$, $\big|_{\mathfrak{C}} : \mathfrak{H} \to \widehat{\mathfrak{H}}^{\mathfrak{C}}$ are bounded and linear.

The trace operator $\dot{\mathbf{Tr}}$ is linear. If $\Omega = \mathbb{R}^d_+$ is the half-space, then boundedness of $\dot{\mathbf{Tr}} : \mathfrak{H} \to \mathfrak{D}$ was established in [39, Section 5]; this extends to the case where Ω is the domain above a Lipschitz graph via a change of variables. If Ω is a bounded Lipschitz domain, then boundedness of $\dot{\mathbf{Tr}} : W \to \widehat{\mathfrak{D}}$, where W is the inhomogeneous Sobolev space with norm $\sum_{k=0}^m \|\nabla^k \varphi\|_{L^2(\mathbb{R}^d)}$, was established in [42, Chapter V]. Then boundedness of $\dot{\mathbf{Tr}} : \mathfrak{H} \to \widehat{\mathfrak{D}}$ follows by the Poincaré inequality.

By assumption, Condition 2.1 is valid. Because Ω is a Lipschitz domain, we have that $\partial\Omega$ has Lebesgue measure zero, and so Condition 2.2 is valid. A straightforward density argument shows that if $\dot{\mathbf{Tr}}$ is bounded, then Condition 2.3 is valid.

Thus, the given spaces and operators satisfy the conditions imposed at the beginning of Section 2.

We now comment on a few of the other quantities defined in Section 2. If $u \in \mathfrak{H}$, and if $Lu = 0$ in Ω in the weak sense of formula (3.2), then by density $\mathfrak{B}^\Omega(\varphi\big|_\Omega, u) = 0$ for all $\varphi \in \mathfrak{H}$ with $\dot{\mathbf{Tr}}\,\varphi = 0$; that is, $(Lu)\big|_\Omega$ as defined in Section 2 satisfies $(Lu)\big|_\Omega = 0$.

If $u \in \mathfrak{H}^\Omega$, then formally

$$L_{\mathfrak{B}^\Omega} u = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta} \mathcal{E}^\Omega(\partial^\beta u))$$

where $\mathcal{E}^\Omega$ denotes extension from Ω to $\mathbb{R}^d$ by zero.

If $m = 1$, then by an integration by parts argument we have that $\dot{\mathbf{M}}_{\mathfrak{B}^\Omega} u = \nu\cdot\mathbf{A}\nabla u$, where ν is the unit outward normal to Ω, whenever u is sufficiently smooth. The weak formulation of Neumann boundary values of formula (2.7) coincides with the formulation of higher order Neumann boundary data of [17, 18, 19] if $\dot{\mathbf{Tr}} = \operatorname{Tr}^\Omega \nabla^{m-1}$, with that of [3, 64] if $\dot{\mathbf{Tr}}\, u = (\operatorname{Tr}^\Omega u, \partial_\nu u, \dots, \partial_\nu^{m-1} u)$, and with [28, 52, 53] if $\dot{\mathbf{Tr}}\, u = \{\operatorname{Tr}^\Omega \partial^\gamma u\}_{|\gamma|\leq m-1}$.

Remark 3.1. Each of the sesquilinear forms $\mathfrak{B}$ and $\mathfrak{B}^\Omega$ may be associated with more than one choice of coefficients $A_{\alpha\beta}$.

For example, let $\widehat{A}_{\alpha\beta}$ satisfy $\widehat{A}_{\alpha\beta}(x) = A_{\alpha\beta}(x)$ for all $x \in \Omega$. Then $\mathfrak{B}^\Omega$ is unchanged if $A_{\alpha\beta}$ is replaced by $\widehat{A}_{\alpha\beta}$, but $\mathfrak{B}$ is not.

Conversely, let $\widetilde{A}_{\alpha\beta} = A_{\alpha\beta} + M_{\alpha\beta}$, where $M_{\alpha\beta}$ is a constant that satisfies $M_{\alpha\beta} = -M_{\beta\alpha}$. A straightforward integration by parts argument shows that $\mathfrak{B}$ (and thus L) is unchanged if $A_{\alpha\beta}$ is replaced by $\widetilde{A}_{\alpha\beta}$. However, the operators $\mathfrak{B}^\Omega$ and $\mathfrak{B}^\mathfrak{C}$ do take different values if $A_{\alpha\beta}$ is replaced by $\widetilde{A}_{\alpha\beta}$.

Thus, as mentioned in Remark 2.4, $\mathfrak{B}$ may be associated with more than one form $\mathfrak{B}^\Omega$, and $\mathfrak{B}^\Omega$ may be associated with more than one form $\mathfrak{B}$, that satisfy Condition 2.2.

For many classes of domains there is a bounded extension operator from $\widehat{\mathfrak{H}}^\Omega$ to $\mathfrak{H}$, and so $\mathfrak{H}^\Omega = \widehat{\mathfrak{H}}^\Omega = \dot{W}^2_m(\Omega)$ with equivalent norms. (If Ω is a Lipschitz domain then this is a well known result of Calderón [26] and Stein [61, Theorem 5, p. 181]; the result is true for more general domains, see for example [41].)

As mentioned above, if $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain, then $\dot{\mathbf{Tr}}$ is a bounded operator $\mathfrak{H} \to \widehat{\mathfrak{D}}$.

If $\dot{\mathbf{Tr}}\, u = \operatorname{Tr}^\Omega \nabla^{m-1} u$, as in [17, 18, 19], then $\dot{\mathbf{Tr}}$ has a bounded right inverse. See [16]. If $\dot{\mathbf{Tr}}\, u = (\operatorname{Tr}^\Omega u, \partial_\nu u, \dots, \partial_\nu^{m-1} u)$ or $\dot{\mathbf{Tr}}\, u = \{\operatorname{Tr}^\Omega \partial^\gamma u\}_{|\gamma| \leq m-1}$, as in [3, 47, 53, 57, 60], and if Ω is bounded, then $\dot{\mathbf{Tr}}$ has a bounded right inverse even if $\partial\Omega$ is not connected; see [42] or [47, Proposition 7.3]. Thus, in either of these cases, the norm in $\mathfrak{D}$ is comparable to the Besov norm. Furthermore, $\{\nabla^{m-1}\varphi\big|_{\partial\Omega} : \varphi \in C^\infty_0(\mathbb{R}^d)\}$ or $\{(\operatorname{Tr}^\Omega \varphi, \partial_\nu \varphi, \dots, \partial_\nu^{m-1}\varphi) : \varphi \in C^\infty_0(\mathbb{R}^d)\}$ is dense in $\mathfrak{D}$. Thus, if $m = 1$ then $\mathfrak{D} = \widehat{\mathfrak{D}} = \dot{B}^{2,2}_{1/2}(\partial\Omega)$. If $m \geq 2$ then $\mathfrak{D}$ is a closed proper subspace of $\widehat{\mathfrak{D}}$, as the different partial derivatives of a common function must satisfy certain compatibility conditions. In this case $\mathfrak{D}$ is the Whitney-Sobolev space used in many papers, including [1, 17, 25, 47, 52, 53, 54].

4. Construction of layer potentials

We will now use the Babuška-Lax-Milgram theorem to construct layer potentials. This theorem may be stated as follows.

Theorem 4.1 ([13, Theorem 2.1]). *Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two Hilbert spaces, and let $\mathfrak{B}$ be a bounded sesquilinear form on $\mathfrak{H}_1 \times \mathfrak{H}_2$ that is coercive in the sense that Condition 2.1 is valid.*

Then for every linear functional T defined on $\mathfrak{H}_1$ there is a unique $u_T \in \mathfrak{H}_2$ such that $\mathfrak{B}(v, u_T) = \overline{T(v)}$. Furthermore, $\|u_T\|_{\mathfrak{H}_2} \leq \frac{1}{\lambda}\|T\|_{\mathfrak{H}_1 \to \mathbb{C}}$, where λ is as in Condition 2.1.

We construct layer potentials as follows. Let $\dot{\mathbf{g}} \in \mathfrak{N}_2$. Then the operator $T_{\dot{\mathbf{g}}}\varphi = \langle \dot{\mathbf{g}}, \dot{\mathbf{Tr}}_1\, \varphi\rangle = \overline{\langle \dot{\mathbf{Tr}}_1\, \varphi, \dot{\mathbf{g}}\rangle}$ is a bounded linear operator on $\mathfrak{H}_1$. By the Babuška-Lax-Milgram theorem, there is a unique $u_T = \mathcal{S}^\Omega_L \dot{\mathbf{g}} \in \mathfrak{H}_2$ such that

$$\mathfrak{B}(\varphi, \mathcal{S}^\Omega_L \dot{\mathbf{g}}) = \langle \dot{\mathbf{Tr}}_1\, \varphi, \dot{\mathbf{g}}\rangle \quad \text{for all } \varphi \in \mathfrak{H}_1. \tag{4.1}$$

We will let $\mathcal{S}^\Omega_L \dot{\mathbf{g}}$ denote the single layer potential of $\dot{\mathbf{g}}$. Observe that the dependence of $\mathcal{S}^\Omega_L$ on the parameter Ω consists entirely of the dependence of the trace operator on Ω, and the connection between $\dot{\mathbf{Tr}}_1$ and Ω is given by Condition 2.3. This condition is symmetric about an interchange of Ω and $\mathfrak{C}$, and so

$$\mathcal{S}^\Omega_L \dot{\mathbf{g}} = \mathcal{S}^\mathfrak{C}_L \dot{\mathbf{g}}. \tag{4.2}$$

The double layer potential is somewhat more involved. We begin by defining the Newton potential.

Let H be an element of $\mathfrak{H}_1^*$. By the Babuška-Lax-Milgram theorem, there is a unique element $\mathcal{N}^L H$ of $\mathfrak{H}_2$ that satisfies

$$\mathfrak{B}(\varphi, \mathcal{N}^L H) = \langle \varphi, H\rangle \quad \text{for all } \varphi \in \mathfrak{H}_1. \tag{4.3}$$

We refer to $\mathcal{N}^L$ as the Newton potential. In some applications, it is easier to work with the Newton potential rather than the single layer potential directly; we remark that

$$\mathcal{S}_L^\Omega \dot{\mathbf{g}} = \mathcal{N}^L(T_{\dot{\mathbf{g}}}) \quad \text{where} \quad \langle \varphi, T_{\dot{\mathbf{g}}}\rangle = \langle \dot{\mathbf{Tr}}_1\, \varphi, \dot{\mathbf{g}}\rangle. \tag{4.4}$$

We now return to the double layer potential. Let $\dot{\mathbf{f}} \in \mathfrak{D}_2$. Then there is some $F \in \mathfrak{H}_2$ such that $\dot{\mathbf{Tr}}_2\, F = \dot{\mathbf{f}}$. Let

$$\mathcal{D}_{\mathfrak{B}}^\Omega \dot{\mathbf{f}} = \mathcal{D}_{L,\mathfrak{B}^\Omega}^\Omega \dot{\mathbf{f}} = -F\big|_\Omega + (\mathcal{N}^L(L_{\mathfrak{B}^\Omega}\, F))\big|_\Omega \quad \text{if } \dot{\mathbf{Tr}}_2\, F = \dot{\mathbf{f}}. \tag{4.5}$$

Notice that $\mathcal{D}_{\mathfrak{B}}^\Omega \dot{\mathbf{f}}$ is an element of $\mathfrak{H}_2^\Omega$, not of $\mathfrak{H}_2$. Further observe that the single layer potential $\mathcal{S}_L^\Omega$ depends only on $\dot{\mathbf{Tr}}_1$ and $\mathfrak{B}$ (equivalently on $\dot{\mathbf{Tr}}_1$ and the operator L), and not on the particular choice of $\mathfrak{B}^\Omega$. The double layer potential $\mathcal{D}_{\mathfrak{B}}^\Omega = \mathcal{D}_{L,\mathfrak{B}^\Omega}^\Omega$, by contrast, depends on both L (or $\mathfrak{B}$) and $\mathfrak{B}^\Omega$.

We conclude this section by showing that $\mathcal{D}_{\mathfrak{B}}^\Omega \dot{\mathbf{f}}$ is well defined, that is, does not depend on the choice of F in formula (4.5). We also establish that layer potentials are bounded operators.

Lemma 4.2. *The double layer potential is well defined. Furthermore, we have the bounds*

$$\|\mathcal{D}_{\mathfrak{B}}^\Omega \dot{\mathbf{f}}\|_{\mathfrak{H}_2^\Omega} \le \frac{\|\mathfrak{B}^{\mathfrak{C}}\|}{\lambda}\|\dot{\mathbf{f}}\|_{\mathfrak{D}_2}, \quad \|\mathcal{D}_{\mathfrak{B}}^{\mathfrak{C}} \dot{\mathbf{f}}\|_{\mathfrak{H}_2^{\mathfrak{C}}} \le \frac{\|\mathfrak{B}^\Omega\|}{\lambda}\|\dot{\mathbf{f}}\|_{\mathfrak{D}_2}, \quad \|\mathcal{S}_L^\Omega \dot{\mathbf{g}}\|_{\mathfrak{H}_2} \le \frac{1}{\lambda}\|\dot{\mathbf{g}}\|_{\mathfrak{N}_2}.$$

Proof. By Theorem 4.1, we have

$$\|\mathcal{S}_L^\Omega \dot{\mathbf{g}}\|_{\mathfrak{H}_2} \le \frac{1}{\lambda}\|T_{\dot{\mathbf{g}}}\|_{\mathfrak{H}_1\to\mathbb{C}} \le \frac{1}{\lambda}\|\,\dot{\mathbf{Tr}}_1\,\|_{\mathfrak{H}_1\to\mathfrak{D}_1}\|\dot{\mathbf{g}}\|_{\mathfrak{D}_1\to\mathbb{C}}.$$

By definition of $\mathfrak{D}_1$ and $\mathfrak{N}_2$, $\|\,\dot{\mathbf{Tr}}_1\,\|_{\mathfrak{H}_1\to\mathfrak{D}_1} = 1$ and $\|\dot{\mathbf{g}}\|_{\mathfrak{D}_1\to\mathbb{C}} = \|\dot{\mathbf{g}}\|_{\mathfrak{N}_2}$, and so $\mathcal{S}_L^\Omega : \mathfrak{N}_2 \to \mathfrak{H}_2$ is bounded with operator norm at most $1/\lambda$.

We now turn to the double layer potential. We will begin with a few properties of the Newton potential. By definition of L, if $\varphi \in \mathfrak{H}_1$ then $\langle \varphi, LF\rangle = \mathfrak{B}(\varphi, F)$. By definition of $\mathcal{N}^L$, $\mathfrak{B}(\varphi, \mathcal{N}^L(LF)) = \langle \varphi, LF\rangle$. Thus, by coercivity of $\mathfrak{B}$,

$$F = \mathcal{N}^L(LF) \quad \text{for all } F \in \mathfrak{H}_2. \tag{4.6}$$

By definition of $\mathfrak{B}^\Omega$, $\mathfrak{B}^{\mathfrak{C}}$ and $L_{\mathfrak{B}^\Omega}\, F$,

$$\langle \varphi, LF\rangle = \mathfrak{B}(\varphi, F) = \mathfrak{B}^\Omega(\varphi\big|_\Omega, F\big|_\Omega) + \mathfrak{B}^{\mathfrak{C}}(\varphi\big|_{\mathfrak{C}}, F\big|_{\mathfrak{C}}) = \langle \varphi, L_{\mathfrak{B}^\Omega}\, F\rangle + \langle \varphi, L_{\mathfrak{B}^{\mathfrak{C}}} F\rangle$$

for all $\varphi \in \mathfrak{H}_1$. Thus, $LF = L_{\mathfrak{B}^\Omega}\, F + L_{\mathfrak{B}^{\mathfrak{C}}} F$ and so

$$-F + \mathcal{N}^L(L_{\mathfrak{B}^\Omega}\, F) = -F + \mathcal{N}^L(LF) - \mathcal{N}^L(L_{\mathfrak{B}^{\mathfrak{C}}} F) = -\mathcal{N}^L(L_{\mathfrak{B}^{\mathfrak{C}}} F). \tag{4.7}$$

In particular, suppose that $\dot{\mathbf{f}} = \dot{\mathbf{Tr}}_2\, F = \dot{\mathbf{Tr}}_2\, F'$. By Condition 2.3, there is some $w \in \mathfrak{H}_2$ such that $w\big|_\Omega = F\big|_\Omega$ and $w\big|_{\mathfrak{C}} = F'\big|_{\mathfrak{C}}$. Then

$$\begin{aligned}
-F\big|_\Omega + (\mathcal{N}^L(L_{\mathfrak{B}^\Omega}\, F))\big|_\Omega &= -w\big|_\Omega + (\mathcal{N}^L(L_{\mathfrak{B}^\Omega}\, w))\big|_\Omega \\
&= -(\mathcal{N}^L(L_{\mathfrak{B}^{\mathfrak{C}}} w))\big|_\Omega = -(\mathcal{N}^L(L_{\mathfrak{B}^{\mathfrak{C}}} F'))\big|_\Omega \\
&= -F'\big|_\Omega + (\mathcal{N}^L(L_{\mathfrak{B}^\Omega}\, F'))\big|_\Omega
\end{aligned}$$

and so $\mathcal{D}_{\mathfrak{B}}^\Omega \dot{\mathbf{f}}$ is well-defined, that is, depends only on $\dot{\mathbf{f}}$ and not the choice of function F with $\dot{\mathbf{Tr}}_2\, F = \dot{\mathbf{f}}$.

Furthermore, we have the alternative formula

$$\mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}} = -(\mathcal{N}^L(L_{\mathfrak{B}^\mathfrak{C}} F))\big|_\Omega \quad \text{if } \dot{\mathbf{Tr}}_2\, F = \dot{\mathbf{f}}. \tag{4.8}$$

Thus,

$$\|\mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}}\|_{\mathfrak{H}_2^\Omega} \leq \inf_{\dot{\mathbf{Tr}}_2\, F=\dot{\mathbf{f}}} \|(\mathcal{N}^L(L_{\mathfrak{B}^\mathfrak{C}} F))\big|_\Omega\|_{\mathfrak{H}_2^\Omega} \leq \inf_{\dot{\mathbf{Tr}}_2\, F=\dot{\mathbf{f}}} \|\mathcal{N}^L(L_{\mathfrak{B}^\mathfrak{C}} F)\|_{\mathfrak{H}_2}$$

by definition of the $\mathfrak{H}_2^\Omega$-norm.

By Theorem 4.1 and the definition of $\mathcal{N}^L$, we have that

$$\|\mathcal{N}^L(L_{\mathfrak{B}^\mathfrak{C}} F)\|_{\mathfrak{H}_2} \leq \frac{1}{\lambda}\|L_{\mathfrak{B}^\mathfrak{C}} F\|_{\mathfrak{H}_1\to\mathbb{C}}.$$

Since $L_{\mathfrak{B}^\mathfrak{C}} F(\varphi) = \mathfrak{B}^\mathfrak{C}(\varphi\big|_\mathfrak{C}, F\big|_\mathfrak{C})$, we have that

$$\|L_{\mathfrak{B}^\mathfrak{C}} F\|_{\mathfrak{H}_1\to\mathbb{C}} \leq \|\mathfrak{B}^\mathfrak{C}\|\|F\big|_\mathfrak{C}\|_{\mathfrak{H}_2^\mathfrak{C}} \leq \|\mathfrak{B}^\mathfrak{C}\|\|F\|_{\mathfrak{H}_2}$$

and so

$$\|\mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}}\|_{\mathfrak{H}_2^\Omega} \leq \inf_{\dot{\mathbf{Tr}}_2\, F=\dot{\mathbf{f}}} \frac{1}{\lambda}\|\mathfrak{B}^\mathfrak{C}\|\|F\|_{\mathfrak{H}_2} = \frac{1}{\lambda}\|\mathfrak{B}^\mathfrak{C}\|\|\dot{\mathbf{f}}\|_{\mathfrak{D}_2}$$

as desired. □

5. Properties of layer potentials

We will begin this section by showing that layer potentials are solutions to the equation $(Lu)\big|_\Omega = 0$ (Lemma 5.1). We will then prove the Green's formula (Lemma 5.2), the adjoint formulas for layer potentials (Lemma 5.3), and conclude this section by proving the jump relations for layer potentials (Lemma 5.4).

Lemma 5.1. *Let $\dot{\mathbf{f}} \in \mathfrak{D}_2$, $\dot{\mathbf{g}} \in \mathfrak{N}_2$, and let $u = \mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}}$ or $u = \mathcal{S}^\Omega_L \dot{\mathbf{g}}\big|_\Omega$. Then $(Lu)\big|_\Omega = 0$.*

Proof. Recall that $(Lu)\big|_\Omega = 0$ if $\mathfrak{B}^\Omega(\varphi_+\big|_\Omega, u) = 0$ for all $\varphi_+ \in \mathfrak{H}_1$ with $\dot{\mathbf{Tr}}_1\, \varphi_+ = 0$. If $\dot{\mathbf{Tr}}_1\, \varphi_+ = 0 = \dot{\mathbf{Tr}}_1\, 0$, then by Condition 2.3 there is some $\varphi \in \mathfrak{H}_1$ with $\varphi\big|_\Omega = \varphi_+$, $\varphi\big|_\mathfrak{C} = 0$ and $\dot{\mathbf{Tr}}_1\, \varphi = 0$.

By definition (4.1) of the single layer potential,

$$0 = \mathfrak{B}(\varphi, \mathcal{S}^L \dot{\mathbf{g}}) = \mathfrak{B}^\Omega(\varphi\big|_\Omega, \mathcal{S}^\Omega_L \dot{\mathbf{g}}\big|_\Omega) + \mathfrak{B}^\mathfrak{C}(\varphi\big|_\mathfrak{C}, \mathcal{S}^\Omega_L \dot{\mathbf{g}}\big|_\mathfrak{C}) = \mathfrak{B}^\Omega(\varphi_+\big|_\Omega, \mathcal{S}^\Omega_L \dot{\mathbf{g}}\big|_\Omega)$$

as desired.

Turning to the double layer potential, if $\varphi \in \mathfrak{H}_1$, then by definition (4.5) of $\mathcal{D}^\Omega_{\mathfrak{B}}$, formula (4.8) for $\mathcal{D}^\mathfrak{C}_{\mathfrak{B}}$ and linearity of $\mathfrak{B}^\Omega$,

$$\mathfrak{B}^\Omega(\varphi\big|_\Omega, \mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}}) = -\mathfrak{B}^\Omega\big(\varphi\big|_\Omega, F\big|_\Omega\big) + \mathfrak{B}^\Omega\big(\varphi\big|_\Omega, (\mathcal{N}^L(L_{\mathfrak{B}^\Omega} F))\big|_\Omega\big),$$
$$\mathfrak{B}^\mathfrak{C}(\varphi\big|_\mathfrak{C}, \mathcal{D}^\mathfrak{C}_{\mathfrak{B}} \dot{\mathbf{f}}) = -\mathfrak{B}^\mathfrak{C}\big(\varphi\big|_\mathfrak{C}, (\mathcal{N}^L(L_{\mathfrak{B}^\Omega} F))\big|_\mathfrak{C}\big).$$

Subtracting and applying Condition 2.2,

$$\mathfrak{B}^\Omega(\varphi\big|_\Omega, \mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}}) - \mathfrak{B}^\mathfrak{C}(\varphi\big|_\mathfrak{C}, \mathcal{D}^\mathfrak{C}_{\mathfrak{B}} \dot{\mathbf{f}}\big|_\mathfrak{C}) = -\mathfrak{B}^\Omega\big(\varphi\big|_\Omega, F\big|_\Omega\big) + \mathfrak{B}\big(\varphi, \mathcal{N}^L(L_{\mathfrak{B}^\Omega} F)\big).$$

By definition (4.3) of $\mathcal{N}^L$,

$$\mathfrak{B}\big(\varphi, \mathcal{N}^L(L_{\mathfrak{B}^\Omega} F)\big) = \langle \varphi, L_{\mathfrak{B}^\Omega} F\rangle$$

and by the definition (2.8) of $L_{\mathfrak{B}^\Omega} F$,

$$\mathfrak{B}\big(\varphi, \mathcal{N}^L(L_{\mathfrak{B}^\Omega} F)\big) = \mathfrak{B}^\Omega(\varphi\big|_\Omega, F\big|_\Omega).$$

Thus,
$$\mathfrak{B}^\Omega(\varphi\big|_\Omega, \mathcal{D}^\Omega_{\mathfrak{B}}\dot{\mathbf{f}}) - \mathfrak{B}^{\mathfrak{C}}(\varphi\big|_{\mathfrak{C}}, \mathcal{D}^{\mathfrak{C}}_{\mathfrak{B}}\dot{\mathbf{f}}) = 0 \quad \text{for all } \varphi \in \mathfrak{H}_1. \tag{5.1}$$
In particular, as before if $\dot{\mathbf{Tr}}_1\,\varphi_+ = 0$ then there is some φ with $\varphi\big|_\Omega = \varphi_+\big|_\Omega$, $\varphi\big|_{\mathfrak{C}} = 0$ and so $\mathfrak{B}^\Omega(\varphi\big|_\Omega, \mathcal{D}^\Omega_{\mathfrak{B}}\dot{\mathbf{f}}) = 0$. This completes the proof. □

Lemma 5.2. *If $u \in \mathfrak{H}_2^\Omega$ and $(Lu)\big|_\Omega = 0$, then*
$$u = -\mathcal{D}^\Omega_{\mathfrak{B}}(\dot{\mathbf{Tr}}_2\,U) + \mathcal{S}^\Omega_L(\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u)\big|_\Omega, \quad 0 = \mathcal{D}^{\mathfrak{C}}_{\mathfrak{B}}(\dot{\mathbf{Tr}}_2\,U) + \mathcal{S}^{\mathfrak{C}}_L(\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u)\big|_{\mathfrak{C}}$$
for any $U \in \mathfrak{H}_2$ with $U\big|_\Omega = u$.

Proof. By definition (4.5) of the double layer potential,
$$-\mathcal{D}^\Omega_{\mathfrak{B}}(\dot{\mathbf{Tr}}_2\,U) = U\big|_\Omega - (\mathcal{N}^L(L_{\mathfrak{B}^\Omega}\,U))\big|_\Omega = u - (\mathcal{N}^L(L_{\mathfrak{B}^\Omega}\,u))\big|_\Omega$$
and by formula (4.8)
$$\mathcal{D}^{\mathfrak{C}}_{\mathfrak{B}}(\dot{\mathbf{Tr}}_2\,U) = -(\mathcal{N}^L(L_{\mathfrak{B}^\Omega}\,u))\big|_{\mathfrak{C}}.$$
It suffices to show that $\mathcal{N}^L(L_{\mathfrak{B}^\Omega}\,u) = \mathcal{S}^\Omega_L(\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u)$.

Let $\varphi \in \mathfrak{H}_1$. By formulas (4.1) and (2.7),
$$\mathfrak{B}(\varphi, \mathcal{S}^\Omega_L(\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u)) = \langle \dot{\mathbf{Tr}}_1\,\varphi, \dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u\rangle = \mathfrak{B}^\Omega(\varphi\big|_\Omega, u).$$
By formula (4.3) for the Newton potential and by the definition (2.8) of $L_{\mathfrak{B}^\Omega}\,u$,
$$\mathfrak{B}(\varphi, \mathcal{N}^L(L_{\mathfrak{B}^\Omega}\,u)) = \langle \varphi, L_{\mathfrak{B}^\Omega}\,u\rangle = \mathfrak{B}^\Omega(\varphi\big|_\Omega, u).$$
Thus, $\mathfrak{B}(\varphi, \mathcal{N}^L(L_{\mathfrak{B}^\Omega}\,u)) = \mathfrak{B}(\varphi, \mathcal{S}^\Omega_L(\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u))$ for all $\varphi \in \mathfrak{H}_1$; by coercivity of $\mathfrak{B}$, we must have that $\mathcal{N}^L(L_{\mathfrak{B}^\Omega}\,u) = \mathcal{S}^\Omega_L(\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u)$. This completes the proof. □

Let $\mathfrak{B}^*(\varphi, \psi) = \overline{\mathfrak{B}(\psi, \varphi)}$ and define $\mathfrak{B}^\Omega_*$, $\mathfrak{B}^{\mathfrak{C}}_*$ analogously. Then $\mathfrak{B}^*$ is a bounded and coercive operator $\mathfrak{H}_2 \times \mathfrak{H}_1 \to \mathbb{C}$, and so we can define the double and single layer potentials $\mathcal{D}^\Omega_{\mathfrak{B}^*} : \mathfrak{D}_1 \to \mathfrak{H}^\Omega_1$, $\mathcal{S}^\Omega_{L^*} : \mathfrak{N}_1 \to \mathfrak{H}_1$.

We then have the following adjoint relations.

Lemma 5.3. *We have the adjoint relations*
$$\langle \dot{\boldsymbol{\varphi}}, \dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,\mathcal{D}^\Omega_{\mathfrak{B}}\dot{\mathbf{f}}\rangle = \langle \dot{\mathbf{M}}_{\mathfrak{B}^\Omega_*}\,\mathcal{D}^\Omega_{\mathfrak{B}^*}\dot{\boldsymbol{\varphi}}, \dot{\mathbf{f}}\rangle, \tag{5.2}$$
$$\langle \dot{\boldsymbol{\gamma}}, \dot{\mathbf{Tr}}_2\,\mathcal{S}^\Omega_L\dot{\mathbf{g}}\rangle = \langle \dot{\mathbf{Tr}}_1\,\mathcal{S}^\Omega_{L^*}\dot{\boldsymbol{\gamma}}, \dot{\mathbf{g}}\rangle \tag{5.3}$$
for all $\dot{\mathbf{f}} \in \mathfrak{D}_2$, $\dot{\boldsymbol{\varphi}} \in \mathfrak{D}_1$, $\dot{\mathbf{g}} \in \mathfrak{N}_2$ and $\dot{\boldsymbol{\gamma}} \in \mathfrak{N}_1$.

If we let $\dot{\mathbf{Tr}}^\Omega_2\,\mathcal{D}^\Omega_{\mathfrak{B}}\dot{\mathbf{f}} = -\dot{\mathbf{Tr}}_2\,F + \dot{\mathbf{Tr}}_2\,\mathcal{N}^L(L_{\mathfrak{B}^\Omega}\,F))$ for any $F \in \mathfrak{H}_2$ with $\dot{\mathbf{Tr}}_2\,F = \dot{\mathbf{f}}$, then $\dot{\mathbf{Tr}}^\Omega_2\,\mathcal{D}^\Omega_{\mathfrak{B}}\dot{\mathbf{f}}$ does not depend on the choice of F, and we have the duality relations
$$\langle \dot{\boldsymbol{\gamma}}, \dot{\mathbf{Tr}}^\Omega_2\,\mathcal{D}^\Omega_{\mathfrak{B}}\dot{\mathbf{f}}\rangle = \langle -\dot{\boldsymbol{\gamma}} + \dot{\mathbf{M}}_{\mathfrak{B}^\Omega_*}\,\mathcal{S}^\Omega_{L^*}\dot{\boldsymbol{\gamma}}, \dot{\mathbf{f}}\rangle. \tag{5.4}$$

Proof. By formula (4.1),
$$\langle \dot{\mathbf{Tr}}_1\,\mathcal{S}^\Omega_{L^*}\dot{\boldsymbol{\gamma}}, \dot{\mathbf{g}}\rangle = \mathfrak{B}(\mathcal{S}^\Omega_{L^*}\dot{\boldsymbol{\gamma}}, \mathcal{S}^\Omega_L\dot{\mathbf{g}}\rangle,$$
$$\langle \dot{\mathbf{Tr}}_2\,\mathcal{S}^\Omega_L\dot{\mathbf{g}}, \dot{\boldsymbol{\gamma}}\rangle = \mathfrak{B}^*(\mathcal{S}^\Omega_L\dot{\mathbf{g}}, \mathcal{S}^\Omega_{L^*}\dot{\boldsymbol{\gamma}}\rangle$$
and so formula (5.3) follows by definition of $\mathfrak{B}^*$.

Let $\Phi \in \mathfrak{H}_1$ and $F \in \mathfrak{H}_2$ with $\dot{\mathbf{Tr}}_1\,\Phi = \dot{\boldsymbol{\varphi}}$, $\dot{\mathbf{Tr}}_2\,F = \dot{\mathbf{f}}$. Then by formulas (2.7) and (4.5),
$$\langle \dot{\boldsymbol{\varphi}}, \dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,\mathcal{D}^\Omega_{\mathfrak{B}}\dot{\mathbf{f}}\rangle = \mathfrak{B}^\Omega(\Phi\big|_\Omega, \mathcal{D}^\Omega_{\mathfrak{B}}\dot{\mathbf{f}}) = -\mathfrak{B}^\Omega(\Phi\big|_\Omega, F\big|_\Omega) + \mathfrak{B}^\Omega(\Phi\big|_\Omega, (\mathcal{N}^L(L_{\mathfrak{B}^\Omega}\,F))\big|_\Omega)$$

and so

$$\langle \dot{\boldsymbol{\varphi}}, \dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\, \mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}}\rangle = -\overline{\mathfrak{B}^\Omega_*(F\big|_\Omega, \Phi\big|_\Omega)} + \overline{\mathfrak{B}^\Omega_*((\mathcal{N}^L(L_{\mathfrak{B}^\Omega} F))\big|_\Omega, \Phi\big|_\Omega)}.$$

By formula (2.8),

$$\mathfrak{B}^\Omega_*((\mathcal{N}^L(L_{\mathfrak{B}^\Omega} F))\big|_\Omega, \Phi\big|_\Omega) = \langle \mathcal{N}^L(L_{\mathfrak{B}^\Omega} F), L_{\mathfrak{B}^\Omega_*} \Phi\rangle.$$

By formula (4.3), if $H \in \mathfrak{H}_1^*$ and $\varphi \in \mathfrak{H}_2$ then $\mathfrak{B}^*(\varphi, \mathcal{N}^{L^*} H) = \langle \varphi, H\rangle$. Letting $\varphi = \mathcal{N}^L(L_{\mathfrak{B}^\Omega} F)$ and $H = L_{\mathfrak{B}^\Omega_*} \Phi$, we see that

$$\mathfrak{B}^\Omega_*((\mathcal{N}^L(L_{\mathfrak{B}^\Omega} F))\big|_\Omega, \Phi\big|_\Omega) = \mathfrak{B}^*\big(\mathcal{N}^L(L_{\mathfrak{B}^\Omega} F), \mathcal{N}^{L^*}(L_{\mathfrak{B}^\Omega_*} \Phi)\big).$$

Therefore,

$$\langle \dot{\boldsymbol{\varphi}}, \dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\, \mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}}\rangle = -\overline{\mathfrak{B}^\Omega_*(F\big|_\Omega, \Phi\big|_\Omega)} + \overline{\mathfrak{B}^*(\mathcal{N}^L(L_{\mathfrak{B}^\Omega} F), \mathcal{N}^{L^*}(L_{\mathfrak{B}^\Omega_*} \Phi))}.$$

By the same argument

$$\langle \dot{\mathbf{f}}, \dot{\mathbf{M}}_{\mathfrak{B}^\Omega_*}\, \mathcal{D}^\Omega_{\mathfrak{B}^*} \dot{\boldsymbol{\varphi}}\rangle = -\overline{\mathfrak{B}^\Omega(\Phi\big|_\Omega, F\big|_\Omega)} + \overline{\mathfrak{B}(\mathcal{N}^{L^*}(L_{\mathfrak{B}^\Omega_*} \Phi), \mathcal{N}^L(L_{\mathfrak{B}^\Omega} F))}$$

and by definition of $\mathfrak{B}^*$ and $\mathfrak{B}^\Omega_*$ formula (5.2) is proven.

Finally, by definition of $\dot{\mathbf{Tr}}{}^\Omega_2\, \mathcal{D}^\Omega_{\mathfrak{B}}$,

$$\langle \dot{\boldsymbol{\gamma}}, \dot{\mathbf{Tr}}{}^\Omega_2\, \mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}}\rangle = -\langle \dot{\boldsymbol{\gamma}}, \dot{\mathbf{Tr}}_2\, F\rangle + \langle \dot{\boldsymbol{\gamma}}, \dot{\mathbf{Tr}}_2\, \mathcal{N}^L(L_{\mathfrak{B}^\Omega} F))\rangle.$$

By the definition (4.1) of the single layer potential,

$$\langle \dot{\boldsymbol{\gamma}}, \dot{\mathbf{Tr}}_2\, \mathcal{N}^L(L_{\mathfrak{B}^\Omega} F)\rangle = \overline{\mathfrak{B}^*(\mathcal{N}^L(L_{\mathfrak{B}^\Omega} F), \mathcal{S}^\Omega_{L^*} \dot{\boldsymbol{\gamma}})}.$$

By definition of $\mathfrak{B}^*$ and the definition (4.3) of the Newton potential,

$$\overline{\mathfrak{B}^*(\mathcal{N}^L(L_{\mathfrak{B}^\Omega} F), \mathcal{S}^\Omega_{L^*} \dot{\boldsymbol{\gamma}})} = \langle \mathcal{S}^\Omega_{L^*} \dot{\boldsymbol{\gamma}}, L_{\mathfrak{B}^\Omega} F\rangle$$

and by the definition (2.8) of $L_{\mathfrak{B}^\Omega} F$,

$$\langle \mathcal{S}^\Omega_{L^*} \dot{\boldsymbol{\gamma}}, L_{\mathfrak{B}^\Omega} F\rangle = \mathfrak{B}^\Omega(\mathcal{S}^\Omega_{L^*} \dot{\boldsymbol{\gamma}}\big|_\Omega, F\big|_\Omega).$$

By the definition (2.7) of Neumann boundary values,

$$\overline{\mathfrak{B}^\Omega_*(F\big|_\Omega, \mathcal{S}^\Omega_{L^*} \dot{\boldsymbol{\gamma}}\big|_\Omega)} = \overline{\langle \dot{\mathbf{Tr}}_2\, F, \dot{\mathbf{M}}_{\mathfrak{B}^\Omega_*}(\mathcal{S}^\Omega_{L^*} \dot{\boldsymbol{\gamma}}\big|_\Omega)\rangle}$$

and so

$$\langle \dot{\boldsymbol{\gamma}}, \dot{\mathbf{Tr}}{}^\Omega_2\, \mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}}\rangle = -\langle \dot{\boldsymbol{\gamma}}, \dot{\mathbf{f}}\rangle + \langle \dot{\mathbf{M}}_{\mathfrak{B}^\Omega_*}(\mathcal{S}^\Omega_{L^*} \dot{\boldsymbol{\gamma}}\big|_\Omega), \dot{\mathbf{f}}\rangle$$

for any choice of F. Thus $\dot{\mathbf{Tr}}{}^\Omega_2\, \mathcal{D}^\Omega_{\mathfrak{B}}$ is well-defined and formula (5.4) is valid. $\square$

We conclude this section with the jump relations for layer potentials.

Lemma 5.4. *Let $\dot{\mathbf{Tr}}{}^\Omega_2\, \mathcal{D}^\Omega_{\mathfrak{B}}$ be as in Lemma 5.3. If $\dot{\mathbf{f}} \in \mathfrak{D}_2$ and $\dot{\mathbf{g}} \in \mathfrak{N}_2$, then we have the jump and continuity relations*

$$\dot{\mathbf{Tr}}{}^\Omega_2\, \mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}} + \dot{\mathbf{Tr}}{}^{\mathfrak{C}}_2\, \mathcal{D}^{\mathfrak{C}}_{\mathfrak{B}} \dot{\mathbf{f}} = -\dot{\mathbf{f}}, \tag{5.5}$$

$$\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}(\mathcal{S}^\Omega_L \dot{\mathbf{g}}\big|_\Omega) + \dot{\mathbf{M}}_{\mathfrak{B}^{\mathfrak{C}}}(\mathcal{S}^{\mathfrak{C}}_L \dot{\mathbf{g}}\big|_{\mathfrak{C}}) = \dot{\mathbf{g}}, \tag{5.6}$$

$$\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}(\mathcal{D}^\Omega_{\mathfrak{B}} \dot{\mathbf{f}}) - \dot{\mathbf{M}}_{\mathfrak{B}^{\mathfrak{C}}}(\mathcal{D}^{\mathfrak{C}}_{\mathfrak{B}} \dot{\mathbf{f}}) = 0. \tag{5.7}$$

If there are bounded operators $\dot{\mathbf{Tr}}{}^\Omega_2 : \mathfrak{H}^\Omega_2 \to \mathfrak{D}_2$ and $\dot{\mathbf{Tr}}{}^{\mathfrak{C}}_2 : \mathfrak{H}^{\mathfrak{C}}_2 \to \mathfrak{D}_2$ such that $\dot{\mathbf{Tr}}_2\, F = \dot{\mathbf{Tr}}{}^\Omega_2(F\big|_\Omega) = \dot{\mathbf{Tr}}{}^{\mathfrak{C}}_2(F\big|_{\mathfrak{C}})$ for all $F \in \mathfrak{H}_2$, then in addition

$$\dot{\mathbf{Tr}}{}^\Omega_2(\mathcal{S}^\Omega_L \dot{\mathbf{g}}\big|_\Omega) - \dot{\mathbf{Tr}}{}^{\mathfrak{C}}_2(\mathcal{S}^{\mathfrak{C}}_L \dot{\mathbf{g}}\big|_{\mathfrak{C}}) = 0. \tag{5.8}$$

In the absence of an operator $\dot{\mathbf{Tr}}_2^\Omega$, the continuity relation

$$\dot{\mathbf{Tr}}_2\,\mathcal{S}_L^\Omega\dot{\mathbf{g}} - \dot{\mathbf{Tr}}_2\,\mathcal{S}_L^{\mathfrak{C}}\dot{\mathbf{g}} = 0 \tag{5.9}$$

is valid (and follows immediately from formula (4.2)). Existence of the operator $\dot{\mathbf{Tr}}_2^\Omega$ is equivalent to the condition that $\dot{\mathbf{Tr}}_2\,u = 0$ whenever $u\big|_\Omega = 0$. This condition is natural if $\Omega \subset \mathbb{R}^d$ is an open set, $\mathfrak{C} = \mathbb{R}^d \setminus \overline{\Omega}$ and $\dot{\mathbf{Tr}}_2$ denotes a trace operator restricting functions to the boundary $\partial\Omega$. Observe that if such operators $\dot{\mathbf{Tr}}_2^\Omega$ and $\dot{\mathbf{Tr}}_2^{\mathfrak{C}}$ exist, then by the definition (4.5) of the double layer potential and by the definition of $\dot{\mathbf{Tr}}_2^\Omega\,\mathcal{D}_{\mathfrak{B}}^\Omega$ in Lemma 5.3, $\dot{\mathbf{Tr}}_2^\Omega(\mathcal{D}_{\mathfrak{B}}^\Omega\dot{\mathbf{f}}) = (\dot{\mathbf{Tr}}_2^\Omega\,\mathcal{D}_{\mathfrak{B}}^\Omega)\dot{\mathbf{f}}$ and so there is no ambiguity of notation.

Proof of Lemma 5.4. The continuity relation (5.8) follows from formula (4.2) because $\mathcal{S}_L^\Omega\dot{\mathbf{g}} \in \mathfrak{H}_2$ and by the definition of $\dot{\mathbf{Tr}}_2^\Omega$, $\dot{\mathbf{Tr}}_2^{\mathfrak{C}}$.

The jump relation (5.5) follows from the definition of $\dot{\mathbf{Tr}}_2^\Omega\,\mathcal{D}_{\mathfrak{B}}^\Omega$ and by using formula (4.7) to rewrite $\dot{\mathbf{Tr}}_2^{\mathfrak{C}}\,\mathcal{D}_{\mathfrak{B}}^{\mathfrak{C}}$.

We observe that by the definition (2.7) of Neumann boundary values and the definitions (2.3) and (2.6) of $\mathfrak{D}_1$ and $\mathfrak{N}_2$, if $u \in \mathfrak{H}_2^\Omega$ and $v \in \mathfrak{H}_2^{\mathfrak{C}}$ with $(Lu)\big|_\Omega = 0$ and $(Lv)\big|_{\mathfrak{C}} = 0$, then

$$\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u + \dot{\mathbf{M}}_{\mathfrak{B}^{\mathfrak{C}}}\,v = \dot{\boldsymbol{\psi}} \text{ if and only if } \langle\dot{\mathbf{Tr}}_1\,\varphi, \dot{\boldsymbol{\psi}}\rangle = \mathfrak{B}^\Omega(\varphi\big|_\Omega, u) + \mathfrak{B}^{\mathfrak{C}}(\varphi\big|_{\mathfrak{C}}, v)$$

for all $\varphi \in \mathfrak{H}_1$.

Therefore, the continuity relation (5.7) follows from formula (5.1), and the jump relation (5.6) follows from formula (4.2) and from the definition (4.1) of the single layer potential. $\square$

6. Layer potentials and boundary value problems

We now discuss boundary value problems. We routinely wish to establish existence and uniqueness of solutions to the Dirichlet problem

$$(\widehat{L}u)\big|_\Omega = 0, \quad \widehat{\mathbf{Tr}}_{\mathfrak{X}}^\Omega\,u = \dot{\mathbf{f}}, \quad \|u\|_{\mathfrak{X}^\Omega} \le C\|\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}},$$

and the Neumann problem

$$(\widehat{L}u)\big|_\Omega = 0, \quad \widehat{\mathbf{M}}_{\mathfrak{X}}^\Omega\,u = \dot{\mathbf{g}}, \quad \|u\|_{\mathfrak{X}^\Omega} \le C\|\dot{\mathbf{g}}\|_{\mathfrak{N}_{\mathfrak{X}}}$$

for some constant C and some solution space $\mathfrak{X}$ and spaces of Dirichlet and Neumann boundary data $\mathfrak{D}_{\mathfrak{X}}$ and $\mathfrak{N}_{\mathfrak{X}}$. For example, if $\widehat{L}$ is a second-order differential operator, then as in [31, 40, 43, 44] we might wish to establish well-posedness with $\mathfrak{D}_{\mathfrak{X}} = \dot{W}_1^p(\partial\Omega)$, $\mathfrak{N}_{\mathfrak{X}} = L^p(\partial\Omega)$ and $\mathfrak{X}^\Omega = \{u : \widetilde{N}(\nabla u) \in L^p(\partial\Omega)\}$, where $\widetilde{N}$ is the modified nontangential maximal function introduced in [43].

If $\mathfrak{X}^\Omega = \mathfrak{H}_2^\Omega$, $\mathfrak{D}_{\mathfrak{X}} = \mathfrak{D}_2$ and $\mathfrak{N}_{\mathfrak{X}} = \mathfrak{N}_2$, then under some modest additional assumptions, a brief and fairly standard argument involving the Babuška-Lax-Milgram theorem yields well posedness. We will provide these arguments in Section 6.1.

In more general spaces, the method of layer potentials states that if layer potentials, originally defined as bounded operators $\mathcal{D}_{\mathfrak{B}}^\Omega : \mathfrak{D}_2 \to \mathfrak{H}_2$ and $\mathcal{S}_L^\Omega : \mathfrak{N}_2 \to \mathfrak{H}_2$, may be extended to operators $\widehat{\mathcal{D}}^\Omega : \mathfrak{D}_{\mathfrak{X}} \to \mathfrak{X}$ and $\widehat{\mathcal{S}}^\Omega : \mathfrak{N}_{\mathfrak{X}} \to \mathfrak{X}$, and if certain of the properties of layer potentials of Section 5 are preserved by that extension, then well posedness of boundary value problems are equivalent to certain invertibility properties of layer potentials.

In Sections 6.2 and 6.3 we will make this notion precise.

As in Sections 2, 4 and 5, we will work with layer potentials and function spaces in a very abstract setting.

6.1. **Boundary value problems via the Babuška-Lax-Milgram theorem.** Consider the Dirichlet problem of finding a $u \in \mathfrak{H}_2$ that satisfies

$$(Lu)\big|_\Omega = 0, \quad \dot{\mathbf{Tr}}_2\, u = \dot{\mathbf{f}}, \quad \|u\|_{\mathfrak{H}_2} \le C\|\dot{\mathbf{f}}\|_{\mathfrak{D}_2} \tag{6.1}$$

or the Neumann problem of finding a $u \in \mathfrak{H}_2^\Omega$ that satisfies

$$(Lu)\big|_\Omega = 0, \quad \dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\, u = \dot{\mathbf{g}}, \quad \|u\|_{\mathfrak{H}_2^\Omega} \le C\|\dot{\mathbf{g}}\|_{\mathfrak{N}_2}. \tag{6.2}$$

Under some modest additional assumptions on the operators L and $\mathfrak{B}^\Omega$, a standard argument involving Theorem 4.1 yields unique solvability of these problems.

Lemma 6.1. *Let $\mathring{\mathfrak{H}}_j = \{\varphi \in \mathfrak{H}_j : \dot{\mathbf{Tr}}_j\, \varphi = 0\}$. Suppose that there is a $\lambda' > 0$ such that*

$$\sup_{w \in \mathring{\mathfrak{H}}_1 \setminus \{0\}} \frac{|\mathfrak{B}(w,v)|}{\|w\|_{\mathfrak{H}_1}} \ge \lambda' \|v\|_{\mathfrak{H}_2}, \quad \sup_{w \in \mathring{\mathfrak{H}}_2 \setminus \{0\}} \frac{|\mathfrak{B}(u,w)|}{\|w\|_{\mathfrak{H}_2}} \ge \lambda' \|u\|_{\mathfrak{H}_1} \tag{6.3}$$

for all $u \in \mathring{\mathfrak{H}}_1$ and $v \in \mathring{\mathfrak{H}}_2$. Then there is a C such that, for each $\dot{\mathbf{f}} \in \mathfrak{D}_2$, there is a function $u \in \mathfrak{H}_2$ such that the problem (6.1) *is valid.*

Furthermore, if u_1 and u_2 are two solutions to this problem then $u_1\big|_\Omega = u_2\big|_\Omega$. Thus, there is a unique $u \in \mathfrak{H}_2^\Omega$ such that

$$(Lu)\big|_\Omega = 0, \quad \dot{\mathbf{Tr}}_2\, U = \dot{\mathbf{f}} \text{ for some } U \in \mathfrak{H}_2 \text{ with } U\big|_\Omega = u, \quad \|u\|_{\mathfrak{H}_2} \le C\|\dot{\mathbf{f}}\|_{\mathfrak{D}_2}.$$

In particular, if operators $\dot{\mathbf{Tr}}_2^\Omega$ as in Lemma 5.4 exist, then there exists a unique solution $u \in \mathfrak{H}_2^\Omega$ to the problem

$$(Lu)\big|_\Omega = 0, \quad \dot{\mathbf{Tr}}_2^\Omega\, u = \dot{\mathbf{f}}, \quad \|u\|_{\mathfrak{H}_2^\Omega} \le C\|\dot{\mathbf{f}}\|_{\mathfrak{D}_2}.$$

If the condition (3.3) is valid, or more generally if $\mathfrak{H}_1 = \mathfrak{H}_2$ and Condition 2.1 is strengthened to the condition $|\mathfrak{B}(u,u)| \ge \lambda\|u\|^2$, then the condition (6.3) is valid.

Proof of Lemma 6.1. We will in fact produce a $u \in \mathfrak{H}_2$ that is a joint solution both to the problem (6.1) and to the problem

$$(Lu)\big|_{\mathfrak{C}} = 0, \quad \dot{\mathbf{Tr}}_2\, u = \dot{\mathbf{f}}, \quad \|u\|_{\mathfrak{H}_2} \le C\|\dot{\mathbf{f}}\|_{\mathfrak{D}_2}.$$

Because $\dot{\mathbf{f}} \in \mathfrak{D}_2$, there is some $F \in \mathfrak{H}_2$ such that $\dot{\mathbf{Tr}}_2\, F = \dot{\mathbf{f}}$. Observe that $\mathring{\mathfrak{H}}_j$ is a Hilbert space and that the operator T given by $T\varphi = \overline{\mathfrak{B}(\varphi, F)}$ is bounded. By Theorem 4.1, there is a unique $w \in \mathring{\mathfrak{H}}_2$ such that $\mathfrak{B}(\varphi, w) = \mathfrak{B}(\varphi, F)$ for each $\varphi \in \mathring{\mathfrak{H}}_1$. Let $u = F - w$. Then u is the unique element of $\mathfrak{H}_2$ that satisfies $\dot{\mathbf{Tr}}_2\, u = \dot{\mathbf{Tr}}_2\, F - \dot{\mathbf{Tr}}_2\, w = \dot{\mathbf{f}}$ and $\mathfrak{B}(\varphi, u) = 0$ for all $\varphi \in \mathring{\mathfrak{H}}_1$. By Conditions 2.2 and 2.3 and the definition (2.5) of $(Lu)\big|_\Omega$, $(Lu)\big|_\Omega = 0$ and $(Lu)\big|_{\mathfrak{C}} = 0$ if and only if $\mathfrak{B}(\varphi, u) = 0$ for all $\varphi \in \mathring{\mathfrak{H}}_1$. Thus, u is the the unique element of $\mathfrak{H}_2$ that satisfies $\dot{\mathbf{Tr}}_2\, u = \dot{\mathbf{f}}$ and $(Lu)\big|_\Omega = 0 = (Lu)\big|_{\mathfrak{C}}$.

We now turn to uniqueness. Let u be as before. Suppose that $(Lu_1)\big|_\Omega = 0$ and $\dot{\mathbf{Tr}}_2\, u_1 = \dot{\mathbf{f}}$. Then by Condition 2.3, there is some $w \in \mathfrak{H}_2$ such that $w\big|_\Omega = u_1\big|_\Omega$ and $w\big|_{\mathfrak{C}} = u\big|_{\mathfrak{C}}$. But then $(Lw)\big|_\Omega = (Lu_1)\big|_\Omega = 0$ and $(Lw)\big|_{\mathfrak{C}} = (Lu)\big|_{\mathfrak{C}} = 0$, and $\dot{\mathbf{Tr}}_2\, w = \dot{\mathbf{Tr}}_2\, u_1 = \dot{\mathbf{Tr}}_2\, u = \dot{\mathbf{f}}$, and so $w = u$. In particular $u_1\big|_\Omega = w\big|_\Omega = u\big|_\Omega$, as desired. $\square$

Lemma 6.2. *Suppose that there is a $\lambda' > 0$ such that*

$$\sup_{w\in\mathfrak{H}_1^\Omega\setminus\{0\}} \frac{|\mathfrak{B}^\Omega(w,v)|}{\|w\|_{\mathfrak{H}_1^\Omega}} \geq \lambda'\|v\|_{\mathfrak{H}_2^\Omega}, \qquad \sup_{w\in\mathfrak{H}_2^\Omega\setminus\{0\}} \frac{|\mathfrak{B}^\Omega(u,w)|}{\|w\|_{\mathfrak{H}_2^\Omega}} \geq \lambda'\|u\|_{\mathfrak{H}_1^\Omega} \tag{6.4}$$

for all $u \in \mathfrak{H}_1^\Omega$ and $v \in \mathfrak{H}_2^\Omega$.

Let $\mathring{\mathfrak{D}}_1 = \{\dot{\mathbf{Tr}}_1\,\varphi : \varphi \in \mathfrak{H}_1, \varphi\big|_\Omega = 0\}$. Suppose that $\dot{\mathbf{g}} \in \mathfrak{N}_2$ and that $\langle\dot{\mathbf{f}}, \dot{\mathbf{g}}\rangle = 0$ for all $\dot{\mathbf{f}} \in \mathring{\mathfrak{D}}_1$. Then there is a C independent of $\dot{\mathbf{g}}$ such that there is exactly one function $u \in \mathfrak{H}_2^\Omega$ such that the problem (6.2) *is valid.*

Recall that $\langle\dot{\mathbf{Tr}}_1\,\varphi, \dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u\rangle = \mathfrak{B}^\Omega(\varphi\big|_\Omega, u)$ for all $\varphi \in \mathfrak{H}_1$; thus, the given condition on $\dot{\mathbf{g}}$ is necessary. If operators $\dot{\mathbf{Tr}}_1^\Omega$ parallel to those in Lemma 5.4 exist, then $\mathring{\mathfrak{D}}_1 = \{0\}$ and so solutions to the Neumann problem exist for all $\dot{\mathbf{g}} \in \mathfrak{N}_2$. In the case of the operators of Section 3, the condition (6.4) does not follow from the condition (3.3); this condition must be replaced by the condition

$$\Re \sum_{|\alpha|=|\beta|=m} \int_\Omega \overline{\partial^\alpha\varphi}\, A_{\alpha\beta}\,\partial^\beta\varphi \geq \lambda\|\nabla^m\varphi\|^2_{L^2(\Omega)} \quad \text{for all } \varphi \in \dot{W}^2_m(\mathbb{R}^d).$$

Proof of Lemma 6.2. Let $\dot{\mathbf{g}} \in \mathfrak{N}_2$ with $\langle\dot{\mathbf{f}}, \dot{\mathbf{g}}\rangle = 0$ for all $\dot{\mathbf{f}} \in \mathring{\mathfrak{D}}_1$. Let $T_{\dot{\mathbf{g}}}$ be the operator on $\mathfrak{H}_1^\Omega$ given by $T_{\dot{\mathbf{g}}}\varphi = \overline{\langle\dot{\mathbf{Tr}}_1\,\Phi, \dot{\mathbf{g}}\rangle}$ for any $\Phi \in \mathfrak{H}_1$ with $\Phi\big|_\Omega = \varphi$. Then $T_{\dot{\mathbf{g}}}$ is bounded and well defined.

By Theorem 4.1, there is a unique $u \in \mathfrak{H}_2^\Omega$ such that $\mathfrak{B}^\Omega(\varphi, u) = \overline{T_{\dot{\mathbf{g}}}\varphi}$ for all $\varphi \in \mathfrak{H}_1^\Omega$. By definition of $T_{\dot{\mathbf{g}}}$, we have that $\mathfrak{B}^\Omega(\Phi\big|_\Omega, u) = \langle\dot{\mathbf{Tr}}_1\,\Phi, \dot{\mathbf{g}}\rangle$ for any $\Phi \in \mathfrak{H}_1$. By the definitions (2.5) and (2.7), we have that $(Lu)\big|_\Omega = 0$ and $\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u = \dot{\mathbf{g}}$. Conversely, if $(Lu_1)\big|_\Omega = 0$ and $\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}\,u_1 = \dot{\mathbf{g}}$, then $\mathfrak{B}^\Omega(\varphi, u_1) = \overline{T_{\dot{\mathbf{g}}}\varphi}$ for all $\varphi \in \mathfrak{H}_1^\Omega$, and so $u_1 = u$ and the solution is unique. $\square$

6.2. **From invertibility to well posedness.** In this section we will need the following objects.

- Quasi-Banach spaces $\mathfrak{Y}^\Omega$, $\mathfrak{D}_{\mathfrak{X}}$ and $\mathfrak{N}_{\mathfrak{X}}$.
- Linear operators $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^\Omega : \mathfrak{Y}^\Omega \to \mathfrak{D}_{\mathfrak{X}}$ and $\widehat{\mathbf{M}}_{\mathfrak{X}}^\Omega : \mathfrak{Y}^\Omega \to \mathfrak{N}_{\mathfrak{X}}$.
- Linear operators $\widehat{\mathcal{D}}^\Omega : \mathfrak{D}_{\mathfrak{X}} \to \mathfrak{Y}^\Omega$ and $\widehat{\mathcal{S}}^\Omega : \mathfrak{N}_{\mathfrak{X}} \to \mathfrak{Y}^\Omega$.

For the sake of the applications, we will introduce the following notation.

Definition 6.3. We will let $\mathfrak{X}^\Omega$ be any superspace of $\mathfrak{Y}^\Omega$, that is, any quasi-Banach space with $\mathfrak{X}^\Omega \supseteq \mathfrak{Y}^\Omega$ and with $\|u\|_{\mathfrak{X}^\Omega} = \|u\|_{\mathfrak{Y}^\Omega}$ for any $u \in \mathfrak{Y}^\Omega$.

We will let $(\widehat{L}\,\cdot)\big|_\Omega$ be any operator defined on $\mathfrak{X}^\Omega$ such that $(\widehat{L}u)\big|_\Omega = 0$ if and only if $u \in \mathfrak{Y}^\Omega$. Thus, $\mathfrak{Y}^\Omega = \{u \in \mathfrak{X}^\Omega : (\widehat{L}u)\big|_\Omega = 0\}$; we will routinely use this expression for $\mathfrak{Y}^\Omega$.

Such a superspace and operator must exist. For example, we could take $\mathfrak{X}^\Omega = \mathfrak{Y}^\Omega$, and given an $\mathfrak{X}^\Omega \supseteq \mathfrak{Y}^\Omega$ we could let $(\widehat{L}\,\cdot)\big|_\Omega$ be the (nonlinear) indicator function of $\mathfrak{X}^\Omega \setminus \mathfrak{Y}^\Omega$.

Remark 6.4. In the situation of Section 6.1, $\mathfrak{Y}^\Omega = \{u \in \mathfrak{H}_2^\Omega : (Lu)\big|_\Omega = 0\}$, and so the use of the space $\mathfrak{X}^\Omega = \mathfrak{H}_2^\Omega$ and operator $(\widehat{L}\,\cdot)\big|_\Omega = (L\,\cdot)\big|_\Omega$ is very natural. As discussed above, in the situation of [31, 40, 43, 44], the use of the space $\mathfrak{X}^\Omega = \{u : \widetilde{N}(\nabla u) \in L^p(\Omega)\}$ and the operator L given by formula (3.2) is equally natural.

This section could be written strictly in terms of the space $\mathfrak{Y}^\Omega$; however, we have chosen to use the auxiliary space $\mathfrak{X}^\Omega$ and operator $(\widehat{L}\,\cdot\,)\big|_\Omega$ because of their natural roles in the applications.

Remark 6.5. Inherent in the requirements that $\widehat{\mathcal{D}}^\Omega : \mathfrak{D}_\mathfrak{X} \to \mathfrak{Y}^\Omega$ and $\widehat{\mathcal{S}}^\Omega : \mathfrak{N}_\mathfrak{X} \to \mathfrak{Y}^\Omega$ is the requirement that if $\dot{\mathbf{g}} \in \mathfrak{N}_\mathfrak{X}$ then $(\widehat{L}(\widehat{\mathcal{S}}^\Omega \dot{\mathbf{g}}))\big|_\Omega = 0$, and if $\dot{\mathbf{f}} \in \mathfrak{D}_\mathfrak{X}$ then $(\widehat{L}(\widehat{\mathcal{D}}^\Omega \dot{\mathbf{f}}))\big|_\Omega = 0$.

Remark 6.6. Recall that $\mathcal{S}_L^\Omega = \mathcal{S}_L^\mathfrak{C}$ is defined in terms of a "global" Hilbert space $\mathfrak{H}_2$. If $\mathfrak{X}^\Omega = \mathfrak{H}_2^\Omega$, then we have in mind the example $\widehat{\mathcal{S}}^\Omega \dot{\mathbf{g}} = \mathcal{S}_L^\Omega \dot{\mathbf{g}}\big|_\Omega$. In the general case, we do not assume the existence of a global quasi-Banach space $\mathfrak{X}$ whose restrictions to Ω lie in $\mathfrak{X}^\Omega$, and thus we will let $\widehat{\mathcal{S}}^\Omega \dot{\mathbf{g}}$ be an element of $\mathfrak{X}^\Omega$ without assuming a global extension.

In applications it is often useful to define $\dot{\mathbf{Tr}}_2^\Omega$, $\dot{\mathbf{M}}_{\mathfrak{B}^\Omega}$, L, $\mathcal{D}_\mathfrak{B}^\Omega$ and $\mathcal{S}_L^\Omega$ in terms of some Hilbert spaces $\mathfrak{H}_j$, $\mathfrak{H}_j^\Omega$ and to extend these operators to operators with domain or range $\mathfrak{X}^\Omega$ by density or some other means. See, for example, [19]. We will not assume that the operators $\widehat{\mathbf{Tr}}_\mathfrak{X}^\Omega$, $\widehat{\mathbf{M}}_\mathfrak{X}^\Omega$, $\widehat{L}$, $\widehat{\mathcal{D}}^\Omega$ and $\widehat{\mathcal{S}}^\Omega$ arise by density; we will merely require that they satisfy certain properties similar to those established in Section 5.

Specifically, we will often use the following conditions; observe that if $\mathfrak{X}^\Omega = \mathfrak{H}_2^\Omega$ for some $\mathfrak{H}_2^\Omega$ as in Section 2, and if $\widehat{\mathbf{Tr}}_\mathfrak{X}^\Omega$ is the operator $\dot{\mathbf{Tr}}_2^\Omega$ of Lemma 5.4, then these properties are valid.

Condition 6.7. *$\widehat{\mathbf{Tr}}_\mathfrak{X}^\Omega$ is bounded $\mathfrak{Y}^\Omega \to \mathfrak{D}_\mathfrak{X}$; that is, $\widehat{\mathbf{Tr}}_\mathfrak{X}^\Omega$ is a bounded operator from $\{u \in \mathfrak{X}^\Omega : (\widehat{L}u)\big|_\Omega = 0\}$ to $\mathfrak{D}_\mathfrak{X}$.*

Condition 6.8. *$\widehat{\mathbf{M}}_\mathfrak{X}^\Omega$ is a bounded operator $\{u \in \mathfrak{X}^\Omega : (\widehat{L}u)\big|_\Omega = 0\} \to \mathfrak{N}_\mathfrak{X}$.*

Condition 6.9. *The single layer potential $\widehat{\mathcal{S}}^\Omega$ is bounded $\mathfrak{N}_\mathfrak{X} \to \mathfrak{Y}^\Omega$; equivalently, $\widehat{\mathcal{S}}^\Omega$ is bounded $\mathfrak{N}_\mathfrak{X} \to \mathfrak{X}^\Omega$.*

Condition 6.10. *The double layer potential $\widehat{\mathcal{D}}^\Omega$ is bounded $\mathfrak{D}_\mathfrak{X} \to \mathfrak{X}^\Omega$.*

Condition 6.11. *If $u \in \mathfrak{Y}^\Omega$, that is, if $u \in \mathfrak{X}^\Omega$ and $(\widehat{L}u)\big|_\Omega = 0$, then we have the Green's formula*
$$u = -\widehat{\mathcal{D}}^\Omega(\widehat{\mathbf{Tr}}_\mathfrak{X}^\Omega u) + \widehat{\mathcal{S}}^\Omega(\widehat{\mathbf{M}}_\mathfrak{X}^\Omega u).$$

The following theorem is straightforward to prove and is the core of the classic method of layer potentials.

Theorem 6.12. *Let $\mathfrak{X}^\Omega$, $(\widehat{L}\,\cdot\,)\big|_\Omega$, $\mathfrak{D}_\mathfrak{X}$, $\mathfrak{N}_\mathfrak{X}$, $\widehat{\mathbf{Tr}}_\mathfrak{X}^\Omega$, $\widehat{\mathbf{M}}_\mathfrak{X}^\Omega$, $\widehat{\mathcal{D}}^\Omega$, and $\widehat{\mathcal{S}}^\Omega$ be as given at the beginning of this section.*

Suppose that $\widehat{\mathbf{Tr}}_\mathfrak{X}^\Omega \widehat{\mathcal{S}}^\Omega : \mathfrak{N}_\mathfrak{X} \to \mathfrak{D}_\mathfrak{X}$ is surjective. Then for every $\dot{\mathbf{f}} \in \mathfrak{D}_\mathfrak{X}$, there is some u such that
$$(\widehat{L}u)\big|_\Omega = 0, \quad \widehat{\mathbf{Tr}}_\mathfrak{X}^\Omega u = \dot{\mathbf{f}}, \quad u \in \mathfrak{X}^\Omega. \tag{6.5}$$
Suppose in addition that Condition 6.9 is valid, and that $\widehat{\mathbf{Tr}}_\mathfrak{X}^\Omega \widehat{\mathcal{S}}^\Omega : \mathfrak{N}_\mathfrak{X} \to \mathfrak{D}_\mathfrak{X}$ has a bounded right inverse, that is, there is a constant C_0 such that if $\dot{\mathbf{f}} \in \mathfrak{D}_\mathfrak{X}$, then there is some pre-image $\dot{\mathbf{g}}$ of $\dot{\mathbf{f}}$ with $\|\dot{\mathbf{g}}\|_{\mathfrak{N}_\mathfrak{X}} \le C_0 \|\dot{\mathbf{f}}\|_{\mathfrak{D}_\mathfrak{X}}$. Then there is some constant C_1 depending on C_0 and the implicit constant in Condition 6.9 such that if $\dot{\mathbf{f}} \in \mathfrak{D}_\mathfrak{X}$, then there is some $u \in \mathfrak{X}^\Omega$ such that
$$(\widehat{L}u)\big|_\Omega = 0, \quad \widehat{\mathbf{Tr}}_\mathfrak{X}^\Omega u = \dot{\mathbf{f}}, \quad \|u\|_{\mathfrak{X}^\Omega} \le C_1 \|\dot{\mathbf{f}}\|_{\mathfrak{D}_\mathfrak{X}}. \tag{6.6}$$

Suppose that $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{D}}^{\Omega} : \mathfrak{D}_{\mathfrak{X}} \to \mathfrak{N}_{\mathfrak{X}}$ *is surjective. Then for every* $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$*, there is some* u *such that*

$$(\widehat{L}u)\big|_{\Omega} = 0, \quad \widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\, u = \dot{\mathbf{g}}, \quad u \in \mathfrak{X}^{\Omega}. \tag{6.7}$$

If Condition 6.10 is valid and $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{D}}^{\Omega} : \mathfrak{D}_{\mathfrak{X}} \to \mathfrak{N}_{\mathfrak{X}}$ *has a bounded right inverse, then there is some constant* C_1 *depending on the bound on that inverse and the implicit constant in Condition 6.10 such that if* $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$*, then there is some* $u \in \mathfrak{X}^{\Omega}$ *such that*

$$(\widehat{L}u)\big|_{\Omega} = 0, \quad \widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\, u = \dot{\mathbf{g}}, \quad \|u\|_{\mathfrak{X}^{\Omega}} \leq C_1 \|\dot{\mathbf{g}}\|_{\mathfrak{N}_{\mathfrak{X}}}. \tag{6.8}$$

Thus, surjectivity of layer potentials implies existence of solutions to boundary value problems.

We may also show that injectivity of layer potentials implies uniqueness of solutions to boundary value problems. This argument appeared first in [23] and is the converse to an argument of [63].

Theorem 6.13. *Let* $\mathfrak{X}^{\Omega}$, $(\widehat{L}\,\cdot)\big|_{\Omega}$, $\mathfrak{D}_{\mathfrak{X}}$, $\mathfrak{N}_{\mathfrak{X}}$, $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}$, $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}$, $\widehat{\mathcal{D}}^{\Omega}$, *and* $\widehat{\mathcal{S}}^{\Omega}$ *be as given at the beginning of this section.*

Suppose that Condition 6.11 is valid. Suppose furthermore that the operator $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{S}}^{\Omega} : \mathfrak{N}_{\mathfrak{X}} \to \mathfrak{D}_{\mathfrak{X}}$ *is one-to-one. Then for each* $\dot{\mathbf{f}} \in \mathfrak{D}_{\mathfrak{X}}$*, there is at most one solution* u *to the Dirichlet problem*

$$(\widehat{L}u)\big|_{\Omega} = 0, \quad \widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\, u = \dot{\mathbf{f}}, \quad u \in \mathfrak{X}^{\Omega}.$$

If Conditions 6.7, 6.9, 6.10 and 6.11 are all valid, and if $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{S}}^{\Omega} : \mathfrak{N}_{\mathfrak{X}} \to \mathfrak{D}_{\mathfrak{X}}$ *has a bounded left inverse, that is, there is a constant* C_0 *such that the estimate* $\|\dot{\mathbf{g}}\|_{\mathfrak{N}_{\mathfrak{X}}} \leq C_0 \|\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{S}}^{\Omega}\dot{\mathbf{g}}\|_{\mathfrak{D}_{\mathfrak{X}}}$ *is valid for all* $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$*, then there is some constant* C_1 *such that every* $u \in \mathfrak{X}^{\Omega}$ *with* $(\widehat{L}u)\big|_{\Omega} = 0$ *satisfies* $\|u\|_{\mathfrak{X}^{\Omega}} \leq C_1 \|\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\, u\|_{\mathfrak{D}_{\mathfrak{X}}}$ *(that is, if* u *satisfies the Dirichlet problem* (6.5) *then* u *must satisfy the Dirichlet problem* (6.6)*).*

Similarly, if Condition 6.11 is valid and the operator $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{D}}^{\Omega} : \mathfrak{D}_{\mathfrak{X}} \to \mathfrak{N}_{\mathfrak{X}}$ *is one-to-one, then for each* $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$*, there is at most one solution* u *to the Neumann problem*

$$(\widehat{L}u)\big|_{\Omega} = 0, \quad \widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\, u = \dot{\mathbf{g}}, \quad u \in \mathfrak{X}^{\Omega}.$$

If Conditions 6.8, 6.9, 6.10 and 6.11 are all valid, and if $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{D}}^{\Omega} : \mathfrak{D}_{\mathfrak{X}} \to \mathfrak{N}_{\mathfrak{X}}$ *has a bounded left inverse, then there is some constant* C_1 *such that every* $u \in \mathfrak{X}^{\Omega}$ *with* $(\widehat{L}u)\big|_{\Omega} = 0$ *satisfies* $\|u\|_{\mathfrak{X}^{\Omega}} \leq C_1 \|\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\, u\|_{\mathfrak{D}_{\mathfrak{X}}}$*.*

Proof. We present the proof only for the Neumann problem; the argument for the Dirichlet problem is similar.

Throughout the proof we will let C denote a constant whose value may change from line to line.

Suppose that u, $v \in \mathfrak{X}^{\Omega}$ with $(\widehat{L}u)\big|_{\Omega} = (\widehat{L}v)\big|_{\Omega} = 0$ in Ω and $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\, u = \dot{\mathbf{g}} = \widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\, v$. By Condition 6.11,

$$\begin{gathered}
u = -\widehat{\mathcal{D}}^{\Omega}(\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\, u) + \widehat{\mathcal{S}}^{\Omega}(\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\, u) = -\widehat{\mathcal{D}}^{\Omega}(\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\, u) + \widehat{\mathcal{S}}^{\Omega}\dot{\mathbf{g}}, \\
v = -\widehat{\mathcal{D}}^{\Omega}(\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\, v) + \widehat{\mathcal{S}}^{\Omega}(\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\, v) = -\widehat{\mathcal{D}}^{\Omega}(\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\, v) + \widehat{\mathcal{S}}^{\Omega}\dot{\mathbf{g}}.
\end{gathered}$$

In particular, $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{D}}^{\Omega}(\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\, u) = \widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{D}}^{\Omega}(\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\, v)$. If $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{D}}^{\Omega}$ is one-to-one, then $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\, u = \widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\, v$. Another application of Condition 6.11 yields that $u = v$.

Now, suppose that $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{D}}^{\Omega}$ has a bounded left inverse; this implies that for any $\dot{\mathbf{f}}\in\mathfrak{D}_{\mathfrak{X}}$ we have the estimate $\|\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}}\le C_0\|\,\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{D}}^{\Omega}\dot{\mathbf{f}}\|_{\mathfrak{N}_{\mathfrak{X}}}$. Let $u\in\mathfrak{X}^{\Omega}$ with $(\widehat{L}u)\big|_{\Omega}=0$; we want to show that $\|u\|_{\mathfrak{X}^{\Omega}}\le C\|\,\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,u\|_{\mathfrak{D}_{\mathfrak{X}}}$.

By Condition 6.11, and because $\mathfrak{X}^{\Omega}$ is a quasi-Banach space,

$$\|u\|_{\mathfrak{X}^{\Omega}}\le C\|\widehat{\mathcal{D}}^{\Omega}(\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\,u)\|_{\mathfrak{X}^{\Omega}}+C\|\widehat{\mathcal{S}}^{\Omega}(\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,u)\|_{\mathfrak{X}^{\Omega}}.$$

By Conditions 6.9 and 6.10,

$$\|u\|_{\mathfrak{X}}\le C\|\,\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\,u\|_{\mathfrak{D}_{\mathfrak{X}}}+C\|\,\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,u\|_{\mathfrak{N}_{\mathfrak{X}}}.$$

Applying our estimate on $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{D}}^{\Omega}$, we see that

$$\|u\|_{\mathfrak{X}}\le C\|\,\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{D}}^{\Omega}\,\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\,u\|_{\mathfrak{N}_{\mathfrak{X}}}+C\|\,\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,u\|_{\mathfrak{N}_{\mathfrak{X}}}.$$

By Condition 6.11, $\widehat{\mathcal{D}}^{\Omega}(\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\Omega}\,u)=\widehat{\mathcal{S}}^{\Omega}(\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,u)-u$, and so

$$\|u\|_{\mathfrak{X}}\le C\|\,\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,\widehat{\mathcal{S}}^{\Omega}\,\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,u\|_{\mathfrak{N}_{\mathfrak{X}}}+C\|\,\widehat{\mathbf{M}}_{\mathfrak{X}}^{\Omega}\,u\|_{\mathfrak{N}_{\mathfrak{X}}}.$$

An application of Conditions 6.8 and 6.9 completes the proof. □

6.3. **From well posedness to invertibility.** We are now interested in the converse results. That is, we have shown that results for layer potentials imply results for boundary value problems; we would like to show that results for boundary value problems imply results for layer potentials.

Notice that the above results were built on the Green's formula (that is, Condition 6.11). The converse results will be built on jump relations, as in Lemma 5.4. Recall that jump relations treat the interplay between layer potentials in a domain and in its complement; thus we will need to impose conditions in both domains.

In this section we will need the following spaces and operators.

- Quasi-Banach spaces $\mathfrak{Y}^{\mathfrak{U}}$, $\mathfrak{Y}^{\mathfrak{W}}$, $\mathfrak{D}_{\mathfrak{X}}$ and $\mathfrak{N}_{\mathfrak{X}}$. As in Section 6.2, we will let $\mathfrak{Y}^{\mathfrak{U}}=\{u\in\mathfrak{X}^{\mathfrak{U}}:(\widehat{L}u)\big|_{\mathfrak{U}}=0\}$ and $\mathfrak{Y}^{\mathfrak{W}}=\{v\in\mathfrak{X}^{\mathfrak{W}}:(\widehat{L}v)\big|_{\mathfrak{W}}=0\}$ for some superspaces $\mathfrak{X}^{\mathfrak{U}}$, $\mathfrak{X}^{\mathfrak{W}}$ and operators $(\widehat{L}\,\cdot)\big|_{\mathfrak{U}}$, $(\widehat{L}\,\cdot)\big|_{\mathfrak{W}}$.
- Linear operators $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}}:\mathfrak{Y}^{\mathfrak{U}}\to\mathfrak{D}_{\mathfrak{X}}$, $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}:\mathfrak{Y}^{\mathfrak{U}}\to\mathfrak{N}_{\mathfrak{X}}$, $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{W}}:\mathfrak{Y}^{\mathfrak{W}}\to\mathfrak{D}_{\mathfrak{X}}$, and $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{W}}:\mathfrak{Y}^{\mathfrak{W}}\to\mathfrak{N}_{\mathfrak{X}}$.
- Linear operators $\widehat{\mathcal{D}}^{\mathfrak{U}}:\mathfrak{D}_{\mathfrak{X}}\to\mathfrak{Y}^{\mathfrak{U}}$, $\widehat{\mathcal{D}}^{\mathfrak{W}}:\mathfrak{D}_{\mathfrak{X}}\to\mathfrak{Y}^{\mathfrak{W}}$, $\widehat{\mathcal{S}}^{\mathfrak{U}}:\mathfrak{N}_{\mathfrak{X}}\to\mathfrak{Y}^{\mathfrak{U}}$ and $\widehat{\mathcal{S}}^{\mathfrak{W}}:\mathfrak{N}_{\mathfrak{X}}\to\mathfrak{Y}^{\mathfrak{W}}$.

In the applications $\mathfrak{U}$ is an open set in $\mathbb{R}^d$ or in a smooth manifold, and $\mathfrak{W}=\mathbb{R}^d\setminus\overline{\mathfrak{U}}$ is the interior of its complement. The space $\mathfrak{X}^{\mathfrak{W}}$ is then a space of functions defined in $\mathfrak{W}$ and is thus a different space from $\mathfrak{X}^{\mathfrak{U}}$. However, we emphasize that we have defined only one space $\mathfrak{D}_{\mathfrak{X}}$ of Dirichlet boundary values and one space $\mathfrak{N}_{\mathfrak{X}}$ of Neumann boundary values; that is, the traces from both sides of the boundary must lie in the same spaces.

We will often use the following conditions. Note the similarity between Conditions 6.7–6.11 and Conditions 6.14–6.18; Conditions 6.14–6.18 state that Conditions 6.7–6.11 hold for both $\Omega=\mathfrak{U}$ and $\Omega=\mathfrak{W}$.

Condition 6.14. $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}}$ *is bounded* $\{u\in\mathfrak{X}^{\mathfrak{U}}:(\widehat{L}u)\big|_{\mathfrak{U}}=0\}\to\mathfrak{D}_{\mathfrak{X}}$, *and* $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{W}}$ *is bounded* $\{v\in\mathfrak{X}^{\mathfrak{W}}:(\widehat{L}v)\big|_{\mathfrak{W}}=0\}\to\mathfrak{D}_{\mathfrak{X}}$.

Condition 6.15. $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}$ *is bounded* $\{u\in\mathfrak{X}^{\mathfrak{U}}:(\widehat{L}u)\big|_{\mathfrak{U}}=0\}\to\mathfrak{N}_{\mathfrak{X}}$, *and* $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{W}}$ *is bounded* $\{v\in\mathfrak{X}^{\mathfrak{W}}:(\widehat{L}v)\big|_{\mathfrak{W}}=0\}\to\mathfrak{N}_{\mathfrak{X}}$.

Condition 6.16. *$\widehat{\mathcal{S}}^{\mathfrak{U}}$ is bounded $\mathfrak{N}_{\mathfrak{X}} \to \mathfrak{X}^{\mathfrak{U}}$, and $\widehat{\mathcal{S}}^{\mathfrak{W}}$ is bounded $\mathfrak{N}_{\mathfrak{X}} \to \mathfrak{X}^{\mathfrak{W}}$.*

Condition 6.17. *$\widehat{\mathcal{D}}^{\mathfrak{U}}$ is bounded $\mathfrak{D}_{\mathfrak{X}} \to \mathfrak{X}^{\mathfrak{U}}$, and $\widehat{\mathcal{D}}^{\mathfrak{W}}$ is bounded $\mathfrak{D}_{\mathfrak{X}} \to \mathfrak{X}^{\mathfrak{W}}$.*

Condition 6.18. *If $u \in \mathfrak{X}^{\mathfrak{U}}$ and $(\widehat{L}u)\big|_{\mathfrak{U}} = 0$, and if $v \in \mathfrak{X}^{\mathfrak{W}}$ and $(\widehat{L}v)\big|_{\mathfrak{W}} = 0$, then*

$$u = -\widehat{\mathcal{D}}^{\mathfrak{U}}(\widehat{\mathbf{Tr}}^{\mathfrak{U}}_{\mathfrak{X}}\, u) + \widehat{\mathcal{S}}^{\mathfrak{U}}(\widehat{\mathbf{M}}^{\mathfrak{U}}_{\mathfrak{X}}\, u) \quad \text{and} \quad v = -\widehat{\mathcal{D}}^{\mathfrak{W}}(\widehat{\mathbf{Tr}}^{\mathfrak{W}}_{\mathfrak{X}}\, v) + \widehat{\mathcal{S}}^{\mathfrak{W}}(\widehat{\mathbf{M}}^{\mathfrak{W}}_{\mathfrak{X}}\, v).$$

Condition 6.19. *If $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$, then we have the continuity relation*

$$\widehat{\mathbf{Tr}}^{\mathfrak{U}}_{\mathfrak{X}}(\widehat{\mathcal{S}}^{\mathfrak{U}}\dot{\mathbf{g}}) - \widehat{\mathbf{Tr}}^{\mathfrak{W}}_{\mathfrak{X}}(\widehat{\mathcal{S}}^{\mathfrak{W}}\dot{\mathbf{g}}) = 0.$$

Condition 6.20. *If $\dot{\mathbf{f}} \in \mathfrak{D}_{\mathfrak{X}}$, then we have the continuity relation*

$$\widehat{\mathbf{M}}^{\mathfrak{U}}_{\mathfrak{X}}(\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}) - \widehat{\mathbf{M}}^{\mathfrak{W}}_{\mathfrak{X}}(\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{f}}) = 0.$$

Condition 6.21. *If $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$, then we have the jump relation*

$$\widehat{\mathbf{M}}^{\mathfrak{U}}_{\mathfrak{X}}(\widehat{\mathcal{S}}^{\mathfrak{U}}\dot{\mathbf{g}}) + \widehat{\mathbf{M}}^{\mathfrak{W}}_{\mathfrak{X}}(\widehat{\mathcal{S}}^{\mathfrak{W}}\dot{\mathbf{g}}) = \dot{\mathbf{g}}.$$

Condition 6.22. *If $\dot{\mathbf{f}} \in \mathfrak{D}_{\mathfrak{X}}$, then we have the jump relation*

$$\widehat{\mathbf{Tr}}^{\mathfrak{U}}_{\mathfrak{X}}(\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}) + \widehat{\mathbf{Tr}}^{\mathfrak{W}}_{\mathfrak{X}}(\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{f}}) = -\dot{\mathbf{f}}.$$

We now move from well posedness of boundary value problems to invertibility of layer potentials. The following theorem uses an argument of Verchota from [63].

Theorem 6.23. *Assume that Conditions 6.19 and 6.21 are valid. Suppose that, for any $\dot{\mathbf{f}} \in \mathfrak{D}_{\mathfrak{X}}$, there is at most one solution u or v to each of the two Dirichlet problems*

$$(\widehat{L}u)\big|_{\mathfrak{U}} = 0, \quad \widehat{\mathbf{Tr}}^{\mathfrak{U}}_{\mathfrak{X}}\, u = \dot{\mathbf{f}}, \quad u \in \mathfrak{X}^{\mathfrak{U}},$$
$$(\widehat{L}v)\big|_{\mathfrak{W}} = 0, \quad \widehat{\mathbf{Tr}}^{\mathfrak{W}}_{\mathfrak{X}}\, v = \dot{\mathbf{f}}, \quad v \in \mathfrak{X}^{\mathfrak{W}}.$$

Then $\widehat{\mathbf{Tr}}^{\mathfrak{U}}_{\mathfrak{X}}\,\widehat{\mathcal{S}}^{\mathfrak{U}} : \mathfrak{N}_{\mathfrak{X}} \to \mathfrak{D}_{\mathfrak{X}}$ is one-to-one.

If in addition Condition 6.15 is valid and there is a constant C_0 such that every $u \in \mathfrak{X}^{\mathfrak{U}}$ and $v \in \mathfrak{X}^{\mathfrak{W}}$ with $(\widehat{L}u)\big|_{\mathfrak{U}} = 0$ and $(\widehat{L}v)\big|_{\mathfrak{W}} = 0$ satisfies

$$\|u\|_{\mathfrak{X}^{\mathfrak{U}}} \leq C_0 \| \widehat{\mathbf{Tr}}^{\mathfrak{U}}_{\mathfrak{X}}\, u\|_{\mathfrak{D}_{\mathfrak{X}}}, \quad \|v\|_{\mathfrak{X}^{\mathfrak{W}}} \leq C_0 \| \widehat{\mathbf{Tr}}^{\mathfrak{W}}_{\mathfrak{X}}\, v\|_{\mathfrak{D}_{\mathfrak{X}}},$$

then there is a constant C_1 such that the bound $\|\dot{\mathbf{g}}\|_{\mathfrak{N}_{\mathfrak{X}}} \leq C_1 \| \widehat{\mathbf{Tr}}^{\mathfrak{U}}_{\mathfrak{X}}\,\widehat{\mathcal{S}}^{\mathfrak{U}}\dot{\mathbf{g}}\|_{\mathfrak{D}_{\mathfrak{X}}}$ is valid for all $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$.

Similarly, assume that Conditions 6.20 and 6.22 are valid. Suppose that for any $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$, there is at most one solution u or v to each of the two Neumann problems

$$(\widehat{L}u)\big|_{\mathfrak{U}} = 0, \quad \widehat{\mathbf{M}}^{\mathfrak{U}}_{\mathfrak{X}}\, u = \dot{\mathbf{g}}, \quad u \in \mathfrak{X}^{\mathfrak{U}},$$
$$(\widehat{L}v)\big|_{\mathfrak{W}} = 0, \quad \widehat{\mathbf{M}}^{\mathfrak{W}}_{\mathfrak{X}}\, v = \dot{\mathbf{g}}, \quad v \in \mathfrak{X}^{\mathfrak{W}}.$$

Then $\widehat{\mathbf{M}}^{\mathfrak{U}}_{\mathfrak{X}}\,\widehat{\mathcal{D}}^{\mathfrak{U}} : \mathfrak{D}_{\mathfrak{X}} \to \mathfrak{N}_{\mathfrak{X}}$ is one-to-one.

If Condition 6.14 is valid and there is a constant C_0 such that every $u \in \mathfrak{X}^{\mathfrak{U}}$ and $v \in \mathfrak{X}^{\mathfrak{W}}$ with $(\widehat{L}u)\big|_{\mathfrak{U}} = 0$ and $(\widehat{L}v)\big|_{\mathfrak{W}} = 0$ satisfies

$$\|u\|_{\mathfrak{X}^{\mathfrak{U}}} \leq C_0 \| \widehat{\mathbf{M}}^{\mathfrak{U}}_{\mathfrak{X}}\, u\|_{\mathfrak{D}_{\mathfrak{X}}}, \quad \|v\|_{\mathfrak{X}^{\mathfrak{W}}} \leq C_0 \| \widehat{\mathbf{M}}^{\mathfrak{W}}_{\mathfrak{X}}\, v\|_{\mathfrak{D}_{\mathfrak{X}}},$$

then there is a constant C_1 such that the bound $\|\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}} \leq C_1 \| \widehat{\mathbf{M}}^{\mathfrak{U}}_{\mathfrak{X}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}\|_{\mathfrak{N}_{\mathfrak{X}}}$ is valid for all $\dot{\mathbf{f}} \in \mathfrak{D}_{\mathfrak{X}}$.

Proof. As in the proof of Theorem 6.13, we will consider only the relationship between the Neumann problem and the double layer potential.

Let $\dot{\mathbf{f}}$, $\dot{\mathbf{h}} \in \mathfrak{D}_{\mathfrak{X}}$ and let $u = \widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}$, $w = \widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{h}}$. Then $u \in \mathfrak{X}^{\mathfrak{U}}$ and $w \in \mathfrak{X}^{\mathfrak{U}}$. If $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}} = \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{h}}$, then $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}\,u = \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}\,w$. Because there is at most one solution to the $\mathfrak{U}$-Neumann problem, we must have that $u = w$, and in particular $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}} = \widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{h}}$.

By Condition 6.20, we have that $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{W}}\,\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{f}} = \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{W}}\,\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{h}}$. By uniqueness of solutions to the $\mathfrak{W}$-Neumann problem,

$$\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{W}}\,\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{f}} = \widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{W}}\,\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{h}}.$$

By Condition 6.22, we have that

$$\dot{\mathbf{f}} = -\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}} - \widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{W}}\,\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{f}} = -\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{h}} - \widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{W}}\,\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{h}} = \dot{\mathbf{h}}$$

and so $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}$ is one-to-one.

Now assume the stronger condition, that is, that $C_0 < \infty$. Because $\mathfrak{D}_{\mathfrak{X}}$ is a quasi-Banach space, if $\dot{\mathbf{f}} \in \mathfrak{D}_{\mathfrak{X}}$ then by Condition 6.22,

$$\|\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}} \le C\|\,\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}} + C\|\,\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{W}}\,\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}}.$$

By definition of $\widehat{\mathcal{D}}^{\mathfrak{U}}$, $\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}} \in \mathfrak{X}^{\mathfrak{U}}$ with $(\widehat{L}(\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}))\big|_{\mathfrak{U}} = 0$. By Condition 6.14,

$$\|\,\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}} \le C_2\|\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}\|_{\mathfrak{X}^{\mathfrak{U}}}$$

for some C_2. Thus,

$$\|\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}} \le CC_2\|\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}\|_{\mathfrak{X}^{\mathfrak{U}}} + CC_2\|\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{f}}\|_{\mathfrak{X}^{\mathfrak{W}}}.$$

By definition of C_0,

$$\|\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}\|_{\mathfrak{X}^{\mathfrak{U}}} \le C_0\|\,\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}\|_{\mathfrak{N}_{\mathfrak{X}}} \quad \text{and} \quad \|\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{f}}\|_{\mathfrak{X}^{\mathfrak{W}}} \le C_0\|\,\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{W}}\,\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{f}}\|_{\mathfrak{N}_{\mathfrak{X}}}.$$

By Condition 6.20, $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{W}}\,\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{f}} = \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}$ and so

$$\|\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}} \le 2CC_2C_0\|\,\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}}\|_{\mathfrak{N}_{\mathfrak{X}}}$$

as desired. □

Finally, we consider the relationship between existence and surjectivity. The following argument appeared first in [21].

Theorem 6.24. *Assume that Conditions 6.18, 6.19, and 6.22 are valid. Suppose that, for any $\dot{\mathbf{f}} \in \mathfrak{D}_{\mathfrak{X}}$, there is at least one pair of solutions u and v to the pair of Dirichlet problems*

$$(\widehat{L}u)\big|_{\mathfrak{U}} = (\widehat{L}v)\big|_{\mathfrak{W}} = 0, \quad \widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}}\,u = \widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{W}}\,v = \dot{\mathbf{f}}, \quad u \in \mathfrak{X}^{\mathfrak{U}}, \quad v \in \mathfrak{X}^{\mathfrak{W}}. \tag{6.9}$$

Then $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{S}}^{\mathfrak{U}} : \mathfrak{N}_{\mathfrak{X}} \to \mathfrak{D}_{\mathfrak{X}}$ is onto.

Suppose in addition that Condition 6.15 is valid, and that there is some $C_0 < \infty$ such that, if $\dot{\mathbf{f}} \in \mathfrak{D}_{\mathfrak{X}}$, then there is some pair of solutions u and v to the problem (6.9) *with*

$$\|u\|_{\mathfrak{X}^{\mathfrak{U}}} \le C_0\|\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}}, \quad \|v\|_{\mathfrak{X}^{\mathfrak{W}}} \le C_0\|\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}}.$$

Then there is a constant C_1 such that for any $\dot{\mathbf{f}} \in \mathfrak{D}_{\mathfrak{X}}$, there is a $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$ such that $\widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}}\,\widehat{\mathcal{S}}^{\mathfrak{U}}\dot{\mathbf{g}} = \dot{\mathbf{f}}$ and $\|\dot{\mathbf{g}}\|_{\mathfrak{N}_{\mathfrak{X}}} \le C_1\|\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}}$.

Similarly, assume that Conditions 6.18, 6.20, and 6.21 are valid. Suppose that for any $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$*, there is at least one pair of solutions* u *and* v *to the pair of Neumann problems*

$$(\widehat{L}u)\big|_{\mathfrak{U}} = (\widehat{L}v)\big|_{\mathfrak{W}} = 0, \quad \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}} u = \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{W}} v = \dot{\mathbf{g}}, \quad u \in \mathfrak{X}^{\mathfrak{U}}, \quad v \in \mathfrak{X}^{\mathfrak{W}}. \tag{6.10}$$

Then $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}} \widehat{\mathcal{D}}^{\mathfrak{U}} : \mathfrak{D}_{\mathfrak{X}} \to \mathfrak{N}_{\mathfrak{X}}$ *is onto.*

If in addition Condition 6.14 is valid, and if there is some $C_0 < \infty$ *such that, if* $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$*, then there is some pair of solutions* u *and* v *to the problem* (6.10) *with*

$$\|u\|_{\mathfrak{X}^{\mathfrak{U}}} \le C_0 \|\dot{\mathbf{g}}\|_{\mathfrak{N}_{\mathfrak{X}}}, \quad \|v\|_{\mathfrak{X}^{\mathfrak{W}}} \le C_0 \|\dot{\mathbf{g}}\|_{\mathfrak{N}_{\mathfrak{X}}}, \tag{6.11}$$

then there is a constant C_1 *such that for any* $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$*, there is an* $\dot{\mathbf{f}} \in \mathfrak{D}_{\mathfrak{X}}$ *such that* $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}} \widehat{\mathcal{D}}^{\mathfrak{U}} \dot{\mathbf{f}} = \dot{\mathbf{g}}$ *and* $\|\dot{\mathbf{f}}\|_{\mathfrak{D}_{\mathfrak{X}}} \le C_1 \|\dot{\mathbf{g}}\|_{\mathfrak{N}_{\mathfrak{X}}}$*.*

Proof. As usual we present the proof for the Neumann problem. Choose some $\dot{\mathbf{g}} \in \mathfrak{N}_{\mathfrak{X}}$ and let u and v be the solutions to the problem (6.10) assumed to exist. (If $C_0 < \infty$ we further require that the bound (6.11) be valid.)

By definition of $\widehat{\mathbf{Tr}}_{\mathfrak{X}}$, $\dot{\mathbf{f}} = \widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{U}} u$ and $\dot{\mathbf{h}} = \widehat{\mathbf{Tr}}_{\mathfrak{X}}^{\mathfrak{W}} v$ exist and lie in $\mathfrak{D}_{\mathfrak{X}}$. By Condition 6.18,

$$\begin{aligned} 2\dot{\mathbf{g}} &= \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}} u + \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{W}} v \\ &= \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}(-\widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}} + \widehat{\mathcal{S}}^{\mathfrak{U}}\dot{\mathbf{g}}) + \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{W}}(-\widehat{\mathcal{D}}^{\mathfrak{W}}\dot{\mathbf{h}} + \widehat{\mathcal{S}}^{\mathfrak{W}}\dot{\mathbf{g}}). \end{aligned}$$

By Conditions 6.20 and 6.21 and linearity of the operators $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}}$, $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{W}}$, we have that

$$\begin{aligned} 2\dot{\mathbf{g}} &= -\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}} \widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{f}} + \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}} \widehat{\mathcal{S}}^{\mathfrak{U}}\dot{\mathbf{g}} - \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}} \widehat{\mathcal{D}}^{\mathfrak{U}}\dot{\mathbf{h}} + \dot{\mathbf{g}} - \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}} \widehat{\mathcal{S}}^{\mathfrak{U}}\dot{\mathbf{g}} \\ &= \dot{\mathbf{g}} - \widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}} \widehat{\mathcal{D}}^{\mathfrak{U}}(\dot{\mathbf{f}} + \dot{\mathbf{h}}). \end{aligned}$$

Thus, $\widehat{\mathbf{M}}_{\mathfrak{X}}^{\mathfrak{U}} \widehat{\mathcal{D}}^{\mathfrak{U}}$ is surjective. If $C_0 < \infty$, then because $\mathfrak{D}_{\mathfrak{X}}$ is a quasi-Banach space and by Condition 6.14,

$$\|\dot{\mathbf{f}} + \dot{\mathbf{h}}\|_{\mathfrak{D}_{\mathfrak{X}}} \le C C_0 \|\dot{\mathbf{g}}\|_{\mathfrak{N}_{\mathfrak{X}}}$$

for some constant C, as desired. □

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