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ALMOST GLOBAL EXISTENCE FOR THE NEUMANN PROBLEM OF QUASILINEAR WAVE EQUATIONS OUTSIDE STAR-SHAPED DOMAINS IN 3D

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Abstract. In this article, we prove the almost global existence of solutions for quasilinear wave equations in the complement of star-shaped domains in three dimensions, with a Neumann boundary condition.

1. Introduction

Assume the obstacle $\mathcal{K} \subset \mathbb{R}^3$ be a smooth, closed and strictly star-shaped domain with respect to the origin. Then consider the Neumann problem for the quasilinear wave equation

$$\begin{gathered}\Box_c u = F(du, d^2u), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \backslash \mathcal{K}, \\ \partial_\nu u\big|_{\partial\mathcal{K}} = 0, \\ u(0,x) = f(x), \quad \partial_t u(0,x) = g(x).\end{gathered} \tag{1.1}$$

Here $\Box_c = (\Box_{c_1}, \Box_{c_2}, \cdots, \Box_{c_N})$ is a vector-value multiple-speed D'Alembertian with $\Box_{c_I} = \partial_t^2 - c_I^2\Delta$, and we suppose that all c_I's are positive but not necessarily distinct.

$$\partial_\nu u = \vec{n} \cdot \nabla_x u = \sum_{j=1}^{3} \frac{\partial u}{\partial x_j} n_j$$

denotes differentiation with respect to the outward normal to $\mathcal{K}$. If we set $\partial_t = \partial_0$, then

$$F^I(du, d^2u) = \sum^{3}_{0\le j,k,l\le 3,\, 0\le J,K\le N} C^{IJ,jk}_{K,l} \partial_l u^K \partial_j \partial_k u^J, \quad 1 \le I \le N,$$

where $C^{IJ,jk}_{K,l}$ are real constants satisfying the symmetry conditions

$$C^{IJ,jk}_{K,l} = C^{JI,jk}_{K,l} = C^{IJ,kj}_{K,l}.$$

Let $\partial = (\partial_t, \partial_1, \partial_2, \partial_3) = (\partial_0, \nabla)$ Denote the time-space gradient, and $\partial u = u'$. We write $\Omega = \{\Omega_{ij}\}$, where $\Omega_{ij} = x_i\partial_j - x_j\partial_i$, $1 \le i \le j \le 3$, are the Euclidean

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$\mathbb{R}^3$ rotation operators. Set $Z = \{\partial_t, \partial_j, \Omega_{ij}\}$, $S = t\partial_t + x \cdot \nabla_x = t\partial_t + r\partial_r$, $\langle x \rangle = (1 + |x|^2)^{1/2}$.

To simplify the notation, we let

$$\Box = \partial_t^2 - \Delta$$

be the scale unit-speed D'Alembertian. Since the estimates for $\Box$ yield ones for $\Box_c$, we will state most of our estimates in terms of $\Box$ instead of $\Box_c$.

We suppose that the Cauchy data satisfies the relevant compatibility conditions. Let $J_k u = \{\partial_x^\alpha u : 0 \le |\alpha| \le k\}$. If m is fixed and u is a formal H^m solution of (1.1), then we write $\partial_t^k u(0, \cdot) = \psi_k(J_k f, J_{k-1} g)(0 \le k \le m)$. The compatibility condition for (1.1) with $(f, g) \in H^m \times H^{m-1}$ is just the requirement that ψ_k vanish on $\partial\mathcal{K}$ for $0 \le k \le m$. Furthermore, $(f, g) \in C^\infty$ satisfies the compatibility conditions to infinite order if these conditions hold for all m.

There have been many results on the almost global existence of wave equations, mostly with Dirichlet boundary condition. The almost global existence for nonlinear wave equations was proved in [1] on Minkowski space by using the Lorentz invariance of the wave operator. In [3], the authors gave the same result without relying on Lorentz invariance. The exterior problem of nonlinear wave equation was considered in [2]. Mitsuru Ikawa [4] studied some mixed problems for hyperbolic system of second order. The almost global existence for the Dirichlet problem of quasilinear, semilinear wave equations in three space dimensions were proved in [5, 8] and [7], respectively. Also [12, 13, 14] give the global existence for Dirichlet problem of nonlinear wave equations in exterior domains. The nonexistence of global solutions for exterior problem to critical semilinear wave equations in high dimensions was obtained in [9].

There are also some results on the almost existence to Neumann problem for wave equations. The Neumann problem for the wave equation in wedge was considered in [15]. [16] considered the Neumann exterior problem for wave equation in 2D and studied the asymptotic behavior of the solutions for large times. Katayama et al [10] proved the almost global existence of solutions to exterior problem for semilinear wave equations with Neumann condition. Metcalfe et al [11] gave the almost global existence for quasilinear Neumann wave equations on infinite homogeneous waveguides.

To our acknowledge there are very few results on the almost global existence or lifespan estimate of exterior Neumann problem for quasilinear wave equations in 3D. In this paper, we study the almost global existence of solutions to the exterior problem for quasilinear wave equations with Neumann condition by using the estimates similar to Dirichlet problem in [5]. Compared with the Dirichlet problem, $u = 0$ changes into $\partial_\nu u = 0$ on $\partial\Omega$. So the estimates on the boundary, we decompose the estimated terms into the terms which contain $\partial_\nu u$. The key steps in this paper are the piontwise estimates and weighted L^2 estimates. At last, we proof the almost global existence to this problem and give a lower bound for the lifespan of the solutions. To study this problem conveniently, we need some known lemmas (see [5]).

Lemma 1.1. *Suppose that $u \in C^5$ solves the Cauchy problem*

$$\begin{gathered} \Box u = F(s, x), \quad (s, x) \in [0, t] \times \mathbb{R}^3 \\ u(0, x) = \partial_t u(0, x) = 0. \end{gathered} \tag{1.2}$$

Then

$$(1+t)|u(t,x)| \le C\int_0^t \int_{\mathbb{R}^3} \sum_{|\alpha|+j\le 3, j\le 1} |S^j Z^\alpha F(s,y)| \frac{1}{|y|}\, dy\, ds. \tag{1.3}$$

Lemma 1.2. *Let $u \in C^5$ solve (1.2), and fix $x \in \mathbb{R}^3$ with $|x| = r$. Then*

$$|x|\,|u(t,x)| \le \frac{1}{2}\int_0^t \int_{|r-(t-s)|}^{r+t-s} \sup_{|\theta|=1} |F(s,\rho\theta)|\rho d\rho ds. \tag{1.4}$$

Lemma 1.3. *Suppose that u solves the Cauchy problem*

$$\begin{gathered} \Box u = F, \\ u(0,x) = f, \quad \partial_t u(0,x) = g. \end{gathered} \tag{1.5}$$

Then

$$\begin{aligned} &(\ln(2+t))^{-1/2} \|\langle x\rangle^{-1/2} u'\|_{L^2(\mathbb{R}^3)} \\ &\le C\|u'(0,x)\|_{L^2(\mathbb{R}^3)} ds + C\int_0^t \|F(s,\cdot)\|_{L^2(\mathbb{R}^3)} ds, \end{aligned} \tag{1.6}$$

$$\|u'\|_{L^2([0,t]\times\{|x|<1\}} \le C\|u'(0,x)\|_{L^2(\mathbb{R}^3)} ds + C\int_0^t \|F(s,\cdot)\|_{L^2(\mathbb{R}^3)} ds. \tag{1.7}$$

Lemma 1.4. *Suppose that $h \in C^\infty(\mathbb{R}^3)$. Then for $R > 1$,*

$$\|h\|_{L^\infty(R/2<|x|<R)} \le CR^{-1} \sum_{|\alpha|+|\gamma|\le 2} \|\Omega^\alpha \partial_x^\gamma h\|_{L^2(R/4<|x|<2R)}.$$

2. Pointwise estimates outside of obstacles

In this section, we shall consider the exterior problem of Neumann wave equations

$$\begin{gathered} \Box u = F(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3\backslash\mathcal{K}, \\ \partial_\nu u(t,x) = 0, \quad x \in \partial\mathcal{K}, \\ u(t,x) = 0, \quad t \le 0. \end{gathered} \tag{2.1}$$

Any of the following estimates for $\Box$ extend to estimates for $\Box_c$ after applying straightforward scaling argument. We will prove the following pointwise estimate.

Theorem 2.1. *Suppose that $u = u(t,x) \in C^\infty$ is the solution of (2.1). Then for each $|\alpha| = N > 1$,*

$$\begin{aligned} t|Z^\alpha u(t,x)| &\le C\int_0^t \sum_{|\gamma|+j\le N+3,\, j\le 1} \|S^j \partial^\gamma F(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} ds \\ &\quad + C\int_0^t \int_{\mathbb{R}^3\backslash\mathcal{K}} \sum_{|\beta|+j\le N+6\ \ j\le 1} |S^j Z^\beta F(s,y)| \frac{1}{|y|}\, dy\, ds. \end{aligned} \tag{2.2}$$

We assume, without loss of generality, that $\mathcal{K} \subset \{x \in \mathbb{R}^3 : |x| < 1\}$. As a first step, we prove the following lemma.

Lemma 2.2. *Suppose that $u = u(t,x) \in C^\infty$ is the solution of (2.1). Then for each $|\alpha| = N > 1$,*

$$\begin{aligned} t|Z^\alpha u(t,x)| \le C \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} \sum_{|\gamma|+j\le 3, j\le 1} |S^j Z^{\alpha+\gamma} F(s,y)| \frac{1}{|y|}\,dy\,ds \\ + C \sup_{|y|\le 2, 0\le s\le t} (1+s)(|Z^\alpha u'(s,y)| + |Z^\alpha u(s,y)|). \end{aligned} \tag{2.3}$$

Proof. Inequality (2.3) obviously holds for $|x| < 2$. Let $\rho \in C^\infty(\mathbb{R})$ be a cut function satisfying

$$\rho(r) = \begin{cases} 1, & r \ge 2, \\ 0, & r \le 1. \end{cases}$$

Then $\omega(t,x) = \rho(|x|)\partial^\alpha u(t,x)$, solves the following problem in $\mathbb{R}^3$,

$$\begin{gathered} \Box\omega = \rho\partial^\alpha F + G, \\ \omega(t,x) = 0, \quad t \le 0, \end{gathered}$$

where

$$G = -2\nabla\rho(|x|)\cdot\nabla\partial^\alpha u - (\Delta\rho(|x|))u.$$

Split $\omega = \omega_1 + \omega_2$, where ω_1 and ω_2 solve the following problems:

$$\begin{gathered} \Box\omega_1 = \rho\partial^\alpha F, \\ \omega_1(t,x) = 0, \quad t \le 0, \end{gathered}$$

and

$$\begin{gathered} \Box\omega_2 = G, \\ \omega_2(t,x) = 0, \quad t \le 0, \end{gathered}$$

respectively. Applying Lemma 1.1, we conclude that

$$t|\omega_1(t,x)| \le C \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} \sum_{|\gamma|+j\le 3, j\le 1} |S^j Z^\gamma \partial^\alpha F(s,y)| \frac{1}{|y|}\,dy\,ds.$$

By Lemma 1.2,

$$|\omega_2(t,x)| \le C\frac{1}{|x|} \int_0^t \int_{||x|-(t-s)|}^{|x|+t-s} \sup_{|\theta|=1} |G(s,r\theta)| r\,dr\,ds. \tag{2.4}$$

For $|x| \le 1$ and $|x| \ge 2$, $G(t,x) = 0$. Hence the right-hand side of (2.4) is nonzero only when

$$-2 \le |x| - (t-s) \le 2,$$

namely,

$$(t-|x|) - 2 \le s \le (t-|x|) + 2.$$

We conclude that

$$\begin{aligned} &|\omega_2(t,x)| \\ &\le C\frac{1}{|x|}\frac{1}{1+|t-|x||} \sup_{\substack{(t-|x|)-2\le s\le(t-|x|)+2, \\ |y|\le 2}} (1+s)(|Z^\alpha u'(s,y)| + |Z^\alpha u(s,y)|). \end{aligned} \tag{2.5}$$

This implies that (2.3) still holds for $|x| \ge 2$. $\square$

Lemma 2.3. *Suppose that $u \in C^\infty$ solves (2.1) and $F(t,x)=0$ for $|x|>4$. Then there exists a constant $c>0$ such that*

$$\|u'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|<4)} \le C\int_0^t e^{-c(t-s)}\|F(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds. \tag{2.6}$$

Consequently, for any fixed nonnegative integer M, we have

$$\begin{aligned}
&\sum_{|\alpha|+j\le M,\, j\le 1} \|(t\partial_t)^j\partial^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|<4)} \\
&\le C\sum_{|\alpha|+j\le M-1,\, j\le 1}\|(t\partial_t)^j\partial^\alpha F(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \\
&\quad + C\int_0^t e^{-\frac{c}{2}(t-s)}\sum_{|\alpha|+j\le M,\, j\le 1}\|(s\partial_s)^j\partial^\alpha F(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
&\sum_{|\alpha|+j\le M,\, j\le 1} \|(t\partial_t)^j\partial^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|<4)} \\
&\le C\sum_{|\alpha|+j\le M-1,\, j\le 1}\|S^j\partial^\alpha f(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \\
&\quad + C\int_0^t e^{-\frac{c}{2}(t-s)}\sum_{|\alpha|+j\le M,\, j\le 1}\|S^j\partial^\alpha F(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds.
\end{aligned} \tag{2.8}$$

Proof. First, we provide the exponential energy decay [17, Theorem III, p. 480] and [18, (iii), p. 230]: Suppose that ω is the solution to the problem

$$\begin{gathered}
\Box\omega = 0, \\
\partial_\nu\omega = 0, \quad x\in\partial\mathcal{K}.
\end{gathered} \tag{2.9}$$

Let

$$E(\omega,D,t) = \frac{1}{2}\int_D \Big(|\partial_t\omega|^2 + |\nabla\omega|^2\Big)dx.$$

Then there exist positive constants C, c, such that

$$E(\omega,D,t) \le Ce^{-ct}E(\omega,D,0).$$

Next, homogenizing (2.1), we have

$$\begin{gathered}
\Box\omega = 0, \\
\partial_\nu\omega\big|_{\partial\mathcal{K}} = 0, \\
\omega|_{t=s} = 0, \quad \partial_t\omega|_{t=s} = F(s,x).
\end{gathered} \tag{2.10}$$

Suppose that ω solves problem (2.10), then $u=\int_0^t\omega(x,t,s)ds$ solves (2.1). Thus we derive

$$\begin{aligned}
\|u'\|^2_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|<4)} &\le \int_0^t \|\omega'(x,t,s)\|^2_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|<4)}ds \\
&\le C\int_0^t E(\omega,(\mathbb{R}^3\backslash\mathcal{K}:|x|<4),t-s)ds \\
&\le C\int_0^t e^{-c(t-s)}E(\omega,(\mathbb{R}^3\backslash\mathcal{K}:|x|<4),s)ds \\
&\le C\int_0^t e^{-c(t-s)}\|F(s,\cdot)\|^2_{L^2(\mathbb{R}^3\backslash\mathcal{K}:|x|<4)}ds,
\end{aligned}$$

which implies

$$\|u'\|_{L^2(\mathbb{R}^3\setminus\mathcal{K}:|x|<4)} \le C\int_0^t e^{-c(t-s)}\|F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds.$$

Therefore, estimate (2.6) holds.

Estimate (2.8) follows from (2.7). Using induction and elliptic regularity we can prove the estimate (2.7). □

Proof of Theorem 2.1. By Lemma 2.2, we need only to proof that the last term on the right-hand side of (2.3) can be dominated by the right-hand side of (2.2), namely prove

$$t\sup_{|x|<2}|\partial^\alpha u(t,x)| \le \text{ right-hand side of (2.3)},$$

holds for each $|\alpha| = N$. We have

$$|t\partial^\alpha u(t,x)| \le \int_0^t \sum_{j\le 1}|(s\partial_s)^j\partial^\alpha u(s,x)|ds. \tag{2.11}$$

First we discuss the case: $F(s,y)\equiv 0$ when $|y|>4$.

By Sobolev Lemma, from (2.8), we obtain that for $|\alpha| = N$,

$$\begin{aligned} &t\sup_{|x|<2}|\partial^\alpha u(t,x)| \\ &\le C\int_0^t \sum_{|\gamma|+j\le N+2,\, j\le 1}\|S^j\partial^\gamma F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K}:|x|\le 4)}ds \\ &\quad + C\int_0^t\int_0^s e^{-\frac{c}{2}(s-\tau)}\sum_{|\gamma|+j\le N+2,\, j\le 1}\|S^j\partial^\gamma F(\tau,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K}:|x|\le 4)}d\tau ds. \end{aligned}\tag{2.12}$$

Therefore,

$$t\sup_{|x|<2}|\partial^\alpha u(t,x)| \le \text{first term on the right-hand side of (2.3)}.$$

Now we deal with the second case: $F(s,y)\equiv 0$ when $|y|<3$. Suppose that u_0 solves the Cauchy problem

$$\begin{gathered} \Box u_0 = F(t,x), \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3, \\ u_0(t,x) = 0, \quad t\le 0. \end{gathered}\tag{2.13}$$

Let $\eta\in C_0^\infty(\mathbb{R}^3)$ be a cut function satisfying

$$\eta(x) = \begin{cases} 1, & |x|<2, \\ 0, & |x|\ge 3. \end{cases}$$

If we set $\tilde{u} = (\eta-1)u_0 + u$, then $\tilde{u}$ solves the problem

$$\begin{gathered} \Box\tilde{u} = G(t,x), \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3\setminus\mathcal{K}, \\ \partial_\nu\tilde{u}\big|_{\partial\mathcal{K}} = 0, \\ \tilde{u}(t,x) = 0, \quad t\le 0, \end{gathered}\tag{2.14}$$

where

$$G = -2\nabla\eta\cdot\nabla u_0 - (\Delta\eta)u_0$$

vanishes unless $2 \le |x| \le 4$. Hence by the first case,

$$\begin{aligned} t \sup_{|x|<2} |\partial^\alpha u(t,x)| &= t \sup_{|x|<2} |\partial^\alpha \tilde{u}(t,x)| \\ &\le C \int_0^t \sum_{|\gamma|\le N+2,\, j\le 1} \|S^j \partial^\gamma G(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\ &\le C \int_0^t \sum_{|\gamma|\le N+3,\, j\le 1} \|S^j \partial^\gamma u_0(s,\cdot)\|_{L^\infty(\mathbb{R}^3\setminus\mathcal{K}:2\le|x|\le 4)} ds. \end{aligned} \tag{2.15}$$

Set $\omega = S^j\partial^\gamma u_0$ with $j = 0, 1$. By (1.4), we obtain

$$\begin{aligned} &\|S^j\partial^\gamma u_0(s,\cdot)\|_{L^\infty(2\le|x|\le 4)} \\ &\le C\int_0^s \int_{|s-\tau-\rho|\le 4} \sup_{|\theta|=1} |S^j\partial^\gamma F(\tau,\rho\theta)|\rho d\rho d\tau \\ &\le C \sum_{|\mu|\le 2} \int_0^s \int_{|s-\tau-\rho|\le 4} |S^j\partial^\gamma \Omega^\mu F(\tau,\rho\theta)|\rho d\rho d\theta d\tau \\ &= C \sum_{|\mu|\le 2} \int_0^s \int_{|s-\tau-|y||\le 4} |S^j\partial^\gamma \Omega^\mu F(\tau,y)| \frac{dy d\tau}{|y|}. \end{aligned} \tag{2.16}$$

Set $\Lambda_s = \{(\tau,y) : 0 \le \tau \le s, |s-\tau-|y|| \le 4\}$ satisfying $\Lambda_s \cap \Lambda_{s'} = \emptyset$ if $|s-s'| > 20$. Therefore, by (2.15) and (2.16), we conclude that

$$t \sup_{|x|<2} |\partial^\alpha u(t,x)| \le C \sum_{\gamma\le N+3,\, |\mu|\le 2, j\le 1} \int_0^t \int_{\mathbb{R}^3\setminus\mathcal{K}} |S^j\Omega^\mu\partial^\gamma F(\tau,y)| \frac{dy d\tau}{|y|}\,.$$

The proof is complete. □

3. Weighted $L^2_{t,x}$ estimates for D'Alembertian outside of star-shaped obstacles

In this section, we prove the following theorem.

Theorem 3.1. *Suppose that $u = u(t,x)$ solves problem (2.1). Then if N is fixed, we have*

$$\begin{aligned} &(\ln(2+t))^{-1/2} \sum_{|\alpha|\le N} \|\langle x\rangle^{-1/2}\partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})} \\ &\le C\int_0^t \sum_{|\alpha|\le N} \|\Box\partial^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\ &\quad + C \sum_{|\alpha|\le N-1} \|\Box\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}, \quad \forall t\ge 0. \end{aligned} \tag{3.1}$$

Additionally,

$$\begin{aligned} &(\ln(2+t))^{-1/2} \sum_{|\alpha|+m\le N,\, m\le 1} \|\langle x\rangle^{-1/2} S^m \partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})} \\ &\le C\int_0^t \sum_{|\alpha|+m\le N,\, m\le 1} \|\Box S^m\partial^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\ &\quad + C \sum_{|\alpha|+m\le N-1,\, m\le 1} \|\Box S^m \partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}, \quad \forall t\ge 0, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} &(\ln(2+t))^{-1/2} \sum_{|\alpha|+m\le N,\, m\le 1} \|\langle x\rangle^{-1/2} S^m Z^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})} \\ &\le C\int_0^t \sum_{|\alpha|+m\le N,\, m\le 1} \|\Box S^m Z^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\ &\quad + C \sum_{|\alpha|+m\le N-1,\, m\le 1} \|\Box S^m Z^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}, \quad \forall t\ge 0. \end{aligned} \tag{3.3}$$

Proposition 3.2. *Suppose that u solves problem* (2.1)*. Then we have*

$$\|u'\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K}:|x|<2)} \le C\int_0^t \|\Box u(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds, \quad \forall t\ge 0 \tag{3.4}$$

and for any given positive integer N,

$$\begin{aligned} &\sum_{|\alpha\le N} \|\partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K}:|x|<2)} \\ &\le C\int_0^t \sum_{|m\le N} \|\Box\partial_s^m u(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds + C\sum_{|\alpha\le N-1} \|\Box\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}, \\ &\quad \forall t\ge 0. \end{aligned} \tag{3.5}$$

Proof. Using the elliptic regularity argument, we know that (3.5) is a consequence of (3.4). To prove (3.4), we discuss the first case: $F(s,y)\equiv 0$ for $|y|>6$.

By (2.6) and the Schwarz inequality, we have

$$\begin{aligned} &\|u'(\tau,\cdot)\|^2_{L^2(\mathbb{R}^3\setminus\mathcal{K}:|x|<2)} \\ &\le C\int_0^\tau e^{-c(\tau-s)}\|F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \int_0^\tau \|F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds, \end{aligned}$$

for all $\tau\ge 0$. Integrating τ from 0 to t on the above inequality,

$$\begin{aligned} &\int_0^t \|u'(\tau,\cdot)\|^2_{L^2(\mathbb{R}^3\setminus\mathcal{K}:|x|<2)} d\tau \\ &\le C\int_0^t\int_0^\tau e^{-c(\tau-s)}\|F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \int_0^\tau \|F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds d\tau \\ &\le C\int_0^t\int_0^\tau e^{-c(\tau-s)}\|F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds d\tau \int_0^t \|F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\ &= C\int_0^t\int_s^t e^{-c(\tau-s)}\|F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} d\tau ds \int_0^t \|F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\ &\le C\Big(\int_0^t \|F(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds\Big)^2, \ \forall t\ge 0 \end{aligned}$$

therefore, (3.4) holds.

Now we consider the second case: $F(s,y)\equiv 0$ for $|y|<4$. By (3.4), we have

$$\|u'\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K}:|x|<2)} \le C\|F\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K}:|x|<2)}, \text{ if } F(s,y)\equiv 0,\ |y|>4. \tag{3.6}$$

Let $\eta\in C^\infty(\mathbb{R}^3)$ be a cut function satisfying

$$\eta(x)=\begin{cases} 1, & |x|\le 2, \\ 0, & |x|\ge 4. \end{cases}$$

Suppose that u_0 solves the Cauchy problem (2.13). Set $\tilde{u} = (\eta - 1)u_0 + u$, then $\tilde{u}$ solves the following problem

$$\begin{gathered} \Box \tilde{u} = \tilde{F}, \\ \partial_\nu \tilde{u}\big|_{\partial\mathcal{K}} = 0, \\ \tilde{u}(0,x) = 0, \quad t \le 0, \end{gathered}$$

where

$$\tilde{F} = -2\nabla\eta \cdot \nabla u_0 - (\Delta\eta)u_0.$$

Note that $\tilde{u} = u$ for $|x| < 2$, and $\tilde{F}(s,y) = 0$ for $|y| > 4$. Then by (3.6) and (1.7), we obtain

$$\begin{aligned} \|u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<2)} &\le C\|u_0'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<4)} + C\|u_0\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<4)} \\ &\le C\int_0^t \|\Box u\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} ds, \quad \forall t \ge 0. \end{aligned}$$

□

Repeating the proof of Proposition 3.2 and using (2.8), we have the following proposition.

Proposition 3.3. *Suppose that u solves problem* (2.1). *Then*

$$\begin{aligned} &\sum_{|\alpha|+m\le N,\, m\le 1} \|S^m\partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<2)} \\ &\le C\int_0^t \sum_{|\alpha|+m\le N,\, m\le 1} \|\Box S^m\partial^m u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} ds \\ &\quad + C\sum_{|\alpha|+m\le N-1,\, m\le 1} \|\Box S^m\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}, \quad \forall t \ge 0. \end{aligned} \tag{3.7}$$

Additionally,

$$\begin{aligned} &\sum_{|\alpha|+|\gamma|+m\le N,\, m\le 1} \|S^m\Omega^\gamma\partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|<2)} \\ &\le C\int_0^t \sum_{|\alpha|+|\gamma|+m\le N,\, m\le 1} \|\Box S^m\Omega^\gamma\partial^m u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} ds \\ &\quad + C\sum_{|\alpha|+|\gamma|+m\le N-1,\, m\le 1} \|\Box S^m\Omega^\gamma\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}, \quad \forall t \ge 0. \end{aligned} \tag{3.8}$$

Proof of Theorem 3.1. Let us first proof estimate (3.1). By Proposition 3.2 it suffices to prove that

$$\begin{aligned} &(\ln(2+t))^{-1/2}\sum_{|\alpha|\le N} \|\langle x\rangle^{-1/2}\partial^\alpha u'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|>2)} \\ &\le C\int_0^t \sum_{|\alpha|\le N} \|\Box\partial^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} ds + C\sum_{|\alpha|\le N-1} \|\Box\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}. \end{aligned} \tag{3.9}$$

Let $\beta \in C^\infty(\mathbb{R}^3)$ be a cut function satisfying

$$\beta(x) = \begin{cases} 1, & |x| \ge 2, \\ 0, & |x| \le 1. \end{cases}$$

Then $\omega = \beta u$ solves the Cauchy problem

$$\Box\omega = \beta\Box u - 2\nabla\beta\cdot\nabla u - (\Delta\beta)u,$$
$$\omega(t,x) = 0, \quad t \le 0,$$

We split $\omega = \omega_1 + \omega_2$, where $\Box\omega_1 = \beta\Box u$ and $\Box\omega_2 = -2\nabla\beta\cdot\nabla u - (\Delta\beta)u$. By (1.6), we have

$$\begin{aligned}&(\ln(2+t))^{-1/2}\sum_{|\alpha|\le N}\|\langle x\rangle^{-1/2}\partial^\alpha\omega_1'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|>2)}\\ &\le C\sum_{|\alpha|\le N}\int_0^t\|\partial^\alpha(\beta\Box u)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds \le C\sum_{|\alpha|\le N}\int_0^t\|\Box\partial^\alpha u\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds\end{aligned}$$

To bound the left of (3.9) it suffices to proof

$$\begin{aligned}&(\ln(2+t))^{-1/2}\sum_{|\alpha|\le N}\|\langle x\rangle^{-1/2}\partial^\alpha\omega_2'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|>2)}\\ &\le C\int_0^t\sum_{|\alpha|\le N}\|\Box\partial^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds + C\sum_{|\alpha|\le N-1}\|\Box\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}.\end{aligned} \tag{3.10}$$

Note that $G = -2\nabla\beta\cdot\nabla u - (\Delta\beta)u = \Box\omega_2$ vanishes unless $1 < |x| < 2$. To use this, let $\chi \in C_0^\infty(\mathbb{R})$ satisfying $\chi(s) = 0,\ |s| > 2$, and $\sum_j \chi(s-j) = 1$. Then we split $G = \sum_j G_j$, where $G_j(s,x) = \chi(s-j)G(s,x)$, and let $\omega_{2,j}$ solves $\omega_{2,j} = G_j$ on Minkowski space with zero initial data. By the sharp Huygens principle, we have $|\partial^\alpha\omega_2(t,x)|^2 \le C\sum_j|\partial^\alpha\omega_{2,j}(t,x)|^2$. Therefore, by (1.6) it follows that

$$\begin{aligned}&\Big((\ln(2+t))^{-1/2}\sum_{|\alpha|\le N}\|\langle x\rangle^{-1/2}\partial^\alpha\omega_2'\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K}:|x|>2)}\Big)^2\\ &\le \sum_{|\alpha|\le N}\sum_j\Big(\int_0^t\|\partial^\alpha G_j(s,\cdot)\|_{L^2(\mathbb{R}^3)}ds\Big)^2\\ &\le C\sum_{|\alpha|\le N}\|\partial^\alpha G\|^2_{L^2([0,t]\times\mathbb{R}^3)}\\ &\le C\sum_{|\alpha|\le N}\|\partial^\alpha u'\|^2_{L^2([0,t]\times\{1<|x|<2\})} + C\sum_{|\alpha|\le N}\|\partial^\alpha u\|^2_{L^2([0,t]\times\{1<|x|<2\})}\\ &\le C\sum_{|\alpha|\le N}\|\partial^\alpha u'\|^2_{L^2([0,t]\times\{|x|<2\})}\\ &\le C\Big(\int_0^t\sum_{|\alpha|\le N}\|\Box\partial^\alpha u(s,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds + C\sum_{|\alpha|\le N-1}\|\Box\partial^\alpha u\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}\Big)^2,\end{aligned}$$

which completes the proof of (3.1). Estimates (3.2) and (3.3) follow by a similar argument. □

4. L^2_x estimates outside of obstacles

Suppose that v is a sufficiently smooth function such that

$$\|\nabla v\|_{L^\infty([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \le \delta, \tag{4.1}$$

$$\|\partial\nabla v\|_{L^1_tL^\infty_x([0,T]\times\mathbb{R}^3\backslash\mathcal{K})} \le C_0, \tag{4.2}$$

where $\delta > 0$ is a sufficiently small constant, C_0 is a positive constant. Let $\Box_\gamma$ denote a second order operator given by

$$\Box_\gamma = \Box_c - \sum_{l,m} C^{lm}(\nabla v)\partial_l\partial_m. \tag{4.3}$$

Consider the Neumann wave equations

$$\begin{gathered}\Box_\gamma \omega = G, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3\backslash\mathcal{K},\\ \partial_\nu \omega\big|_{\partial\mathcal{K}} = 0,\\ \omega(t,x) = 0, \quad t \le 0.\end{gathered} \tag{4.4}$$

Let

$$\begin{gathered}E_0 = |\partial_0\omega|^2 + c_I^2|\nabla\omega|^2 + \sum_{l,m=1}^{3}(\partial_l\omega)^T C^{lm}(\nabla v)\partial_m\omega,\\ E_j = -2c_I^2(\partial_0\omega)^T(\partial_j\omega) - 2\sum_{k=1}^{3}(\partial_0\omega)^T C^{jk}(\nabla v)\partial_k\omega, \quad j = 1,2,3,\\ e = \sum_{l,m=1}^{3}\left((\partial_l\omega)^T\partial_0 C^{lm}(\nabla v)\partial_m\omega - 2(\partial_l\omega)^T\partial_l C^{lm}(\nabla v)\partial_m\omega\right).\end{gathered}$$

Noting the symmetry condition of $C^{lm}(\nabla v)$, we have

$$\partial_0 E_0 + \sum_{j=1}^{3}\partial_j E_j = 2(\partial_0\omega)^T\Box_\gamma\omega + e. \tag{4.5}$$

By (4.1), there exist positive constants λ, μ depending only on c_1, c_2, δ, such that

$$\lambda|\omega'|^2 \le E_0 \le \mu|\omega'|^2. \tag{4.6}$$

Integrating (4.6) over $[0,t] \times \mathbb{R}^3\backslash\mathcal{K}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3\backslash\mathcal{K}} E_0(t,x)dx - \int_{\mathbb{R}^3\backslash\mathcal{K}} E_0(0,x)dx - \int_{[0,t]\times\partial\mathcal{K}}\sum_{j=1}^{3}E_j n_j d\sigma\, ds\\ &= 2\int_{[0,t]\times\mathbb{R}^3\backslash\mathcal{K}}(\partial_0\omega)^T\Box_\gamma\omega\, ds\, dx + \int_{[0,t]\times\mathbb{R}^3\backslash\mathcal{K}} e\, ds\, dx.\end{aligned} \tag{4.7}$$

Noticing the Neumann condition $\partial_\nu\omega = \sum_{j=1}^3 \partial_j\omega n_j = 0$ when $\omega \in \partial\mathcal{K}$, we have $\sum_{j=1}^3 E_j n_j = 0$ on $\partial\mathcal{K}$, and $E_0(0,x) = 0$. Therefore,

$$\int_{\mathbb{R}^3\backslash\mathcal{K}} E_0(t,x)dx = 2\int_{[0,t]\times\mathbb{R}^3\mathcal{K}}(\partial_0\omega)^T\Box_\gamma\omega\, ds\, dx + \int_{[0,t]\times\mathbb{R}^3\mathcal{K}} e\, ds\, dx. \tag{4.8}$$

Using (4.6) and (4.8), we have

$$\begin{aligned}\|\omega'\|^2_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \le & C\int_0^t \|\omega'\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\|G\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds\\ &+ C\int_0^t\sum_{l,m}\|\partial C^{lm}(\nabla v)\|_{L^\infty_x(\mathbb{R}^3\backslash\mathcal{K})}\|\omega'\|^2_{L^2(\mathbb{R}^3\backslash\mathcal{K})}ds.\end{aligned} \tag{4.9}$$

From assumption (4.2) and applying Gronwall inequality, we obtain

$$\begin{aligned}\|\omega'\|^2_{L^2(\mathbb{R}^3\setminus\mathcal{K})} &\le C\int_0^t \|\omega'\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\|G\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds\\ &\le C\Big(\sup_{0\le s\le t}\|\omega'\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\Big)\int_0^t \|G\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds.\end{aligned}$$

Therefore,

$$\|\omega'\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le C\int_0^t \|G\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds, \quad 0\le t\le T. \tag{4.10}$$

In general, we have the following theorem.

Theorem 4.1. *Assume that* (4.1) *and* (4.2) *hold, and* $\omega=\omega(t,x)\in C^\infty$ *solves problem* (4.4)*. Then for any nonnegative integer* N*, there is a positive constant* C*, such that*

$$\begin{aligned}&\sum_{|\alpha|\le N}\|\partial^\alpha\omega'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\ &\le C\int_0^t\sum_{|\alpha|\le N}\|\Box_\gamma\partial_s^m\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}ds\\ &\quad + C\sum_{|\alpha|\le N-1}\|\Box_c\partial^\alpha\omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}, \quad 0\le t\le T.\end{aligned} \tag{4.11}$$

The second term on the right-hand side of (4.11) *vanishes when* $N=0$.

Proof. Proof by induction. When $N=0$, (4.10) shows that (4.11) holds.

We suppose that (4.11) is valid if N is replaced by $N-1$, then we proof it is valid for N. We first notice that $\partial_t\omega$ satisfies (4.4), then by the assumption of induction,

$$\sum_{|\alpha|\le N-1}\|\partial^\alpha(\partial_t\omega)'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le \text{ right-hand side of (4.11)}.$$

Hence it suffices to show that, for $N\ge 1$

$$\sum_{|\alpha|\le N}\|\partial_x^\alpha\nabla_x\omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le \text{the right side of (4.11)}.$$

However,

$$\begin{aligned}&\sum_{|\alpha|\le N-1}\|\Delta\partial_x^\alpha\omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\ &\le C\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\partial_t^2\omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + C\sum_{|\alpha|\le N-1}\|\Box_c\partial_x^\alpha\omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})},\end{aligned} \tag{4.12}$$

where C depends only on the wave speeds c_I.

The first term on the right-hand side of (4.12) is bounded by the right-hand side of (4.11), thus the right-hand side of (4.12) is similarly bounded. By elliptic regularity, so is $\sum_{|\alpha|=N}\|\partial_x^\alpha\nabla_x\omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}$, which completes the proof. $\square$

5. L_x^2 ESTIMATES INVOLVING OPERATORS S^jZ^α OUTSIDE OF OBSTACLES

We suppose that ω solves problem (4.4). Let $P = P(t,x,D)$ be differential operator and $\partial_\nu P\omega$ not necessarily vanishes on $\partial\mathcal{K}$. We will give some rough L^2 estimates for $P\omega$. In this section, we assume that v satisfies (4.1) and (4.2).

Proposition 5.1. *Suppose that $P\omega(0,\cdot) = \partial_t P\omega(0,\cdot) = 0$ and there exist an integer M and a constant C_0 such that*

$$|(P\omega)'(t,x)| \le C_0 t \sum_{|\alpha|\le M-1} |\partial_t\partial^\alpha\omega'(t,x)| + C_0 \sum_{|\alpha|\le M} |\partial^\alpha\omega'(t,x)|, \ x \in \partial\mathcal{K}. \tag{5.1}$$

Then,

$$\begin{aligned}
\|(P\omega)'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le C\int_0^t \|\Box_\gamma P\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\
+ C\int_0^t \sum_{|\alpha|+j\le M+1,\, j\le 1} \|\Box_c S^j\partial^\alpha\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\
+ \sum_{|\alpha|+j\le M,\, j\le 1} \|\Box_c S^j\partial^\alpha\omega(s,\cdot)\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}.
\end{aligned} \tag{5.2}$$

Proof. We will use the analogue of (4.7) where ω is replaced by $P\omega$. Then we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3\setminus\mathcal{K}} E_0(t,x)dx - \int_{\mathbb{R}^3\setminus\mathcal{K}} E_0(0,x)dx - \int_{[0,t]\times\partial\mathcal{K}} \sum_{j=1}^3 E_j n_j d\sigma ds \\
&= 2\int_{[0,t]\times\mathbb{R}^3\setminus\mathcal{K}} (\partial_0 P\omega)^T \Box_\gamma P\omega \, ds\, dx + \int_{[0,t]\times\mathbb{R}^3\setminus\mathcal{K}} e\, ds\, dx,
\end{aligned} \tag{5.3}$$

where

$$E_0 = |\partial_0 P\omega|^2 + c_I^2|\nabla P\omega|^2 + \sum_{l,m=1}^3 (\partial_l P\omega)^T C^{lm}(\nabla v)\partial_m P\omega,$$

$$E_j = -2c_I^2(\partial_0 P\omega)^T(\partial_j P\omega) - 2\sum_{k=1}^3 (\partial_0 P\omega)^T C^{jk}(\nabla v)\partial_k P\omega, \quad j = 1,2,3,$$

$$e = \sum_{l,m=1}^3 \left((\partial_l P\omega)^T \partial_0 C^{lm}(\nabla v)\partial_m P\omega - 2(\partial_l P\omega)^T \partial_l C^{lm}(\nabla v)\partial_m P\omega\right).$$

It is obvious that $E_0(0,x) = 0$. Use (4.1) and (4.2) and apply Gronwall's inequality, we obtain that if $\delta > 0$ is small enough, then

$$\begin{aligned}
&\|(P\omega)'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\
&\le C\int_0^t \|\Box_\gamma P\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\
&\quad + C\Big(\int_{[0,t]\times\partial\mathcal{K}} (|\partial_t P\omega(s,x)|^2 + |\nabla_x P\omega(s,x)|^2) d\sigma\Big)^{1/2}.
\end{aligned} \tag{5.4}$$

Recall that $\mathcal{K} \subset \{|x| < 1\}$. By (5.1) and trace inequality, we have

$$\begin{aligned}
&\int_{[0,t]\times\partial\mathcal{K}} (|\partial_t P\omega(s,x)|^2 + |\nabla_x P\omega(s,x)|^2) d\sigma \\
&\le C \int_{[0,t]\times\partial\mathcal{K}} \sum_{|\alpha|+j\le M,\, j\le 1} |S^j \partial^\alpha \omega'|^2 d\sigma \\
&\le C \sum_{|\alpha|+j\le M+1,\, j\le 1} \|S^j \partial^\alpha \omega'\|^2_{L^2([0,t]\times\partial\mathcal{K}:|x|<2)}, \quad \forall t \ge 0.
\end{aligned} \tag{5.5}$$

Therefore, by (5.4), (5.5) and (3.7), we obtain

$$\begin{aligned}
&\|(P\omega)'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\
&\le C \int_0^t \|\Box_\gamma P\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds + C \sum_{|\alpha|+j\le M+1,\, j\le 1} \|S^j \partial^\alpha \omega'\|_{L^2([0,t]\times\partial\mathcal{K}:|x|<2)} \\
&\le C \int_0^t \|\Box_\gamma P\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds + C \int_0^t \sum_{|\alpha|+j\le M+1,\, j\le 1} \|\Box_c S^j \partial^\alpha \omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\
&\quad + C \sum_{|\alpha|+j\le M,\, j\le 1} \|\Box_c S^j \partial^\alpha \omega\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}, \quad \forall t \ge 0.
\end{aligned}$$

□

Obviously, $P = S^j Z^\alpha (j \le 1)$ satisfies (5.1), then we have the following theorem.

Theorem 5.2. *Suppose that $\omega = \omega(t,x) \in C^\infty$ solves (4.4). If $M = 1, 2, \dots,$ we have*

$$\begin{aligned}
&\sum_{|\alpha|+j\le M,\, j\le 1} \|(S^j Z^\alpha \omega)'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\
&\le C \int_0^t \sum_{|\alpha|+j\le M,\, j\le 1} \|\Box_\gamma S^j Z^\alpha \omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\
&\quad + C \int_0^t \sum_{|\alpha|+j\le M+1,\, j\le 1} \|\Box_c S^j \partial^\alpha \omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\
&\quad + \sum_{|\alpha|+j\le M,\, j\le 1} \|\Box_c S^j \partial^\alpha \omega(s,\cdot)\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}.
\end{aligned} \tag{5.6}$$

6. L^2_x estimates involving $S^m\partial^\alpha$ outside of star-shaped obstacles

In this section, we shall assume furthermore that

$$\|\nabla v\|_{L^\infty(\mathbb{R}^3\setminus\mathcal{K})} \le \frac{\delta}{1+t}, \tag{6.1}$$

with δ small enough. Assume that ω solves problem (4.4). Using that $\mathcal{K}$ is a star-shaped obstacle, we will obtain a better estimate for $S\omega$.

Proposition 6.1. *Suppose that* (6.1) *holds and* $\omega = \omega(t,x) \in C^\infty$ *solves problem* (4.4)*, then*

$$\begin{aligned} \|(S\omega)'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le C \int_0^t \|\Box_\gamma S\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\ + C\int_0^t \sum_{|\alpha|\le 2} \|\Box_c \partial^\alpha \omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\ + C\sum_{|\alpha|\le 1} \|\Box_c \partial^\alpha \omega\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}. \end{aligned} \tag{6.2}$$

Proof. Using the analogue of (4.7) where ω is replaced by $S\omega$, we have

$$\begin{aligned} &\int_{\mathbb{R}^3\setminus\mathcal{K}} E_0(t,x)dx - \int_{\mathbb{R}^3\setminus\mathcal{K}} E_0(0,x)dx - \int_{[0,t]\times\partial\mathcal{K}} \sum_{j=1}^3 E_j n_j d\sigma ds \\ &= 2\int_{[0,t]\times\mathbb{R}^3\setminus\mathcal{K}} (\partial_0 S\omega)^T \Box_\gamma S\omega \, ds\, dx + \int_{[0,t]\times\mathbb{R}^3\setminus\mathcal{K}} e\, ds\, dx, \end{aligned} \tag{6.3}$$

where

$$E_0 = |\partial_0 S\omega|^2 + c_I^2|\nabla S\omega|^2 + \sum_{l,m=1}^3 (\partial_l S\omega)^T C^{lm}(\nabla v)\partial_m S\omega,$$

$$E_j = -2c_I^2(\partial_0 S\omega)^T(\partial_j S\omega) - 2\sum_{k=1}^3 (\partial_0 S\omega)^T C^{jk}(\nabla v)\partial_k S\omega, \quad j=1,2,3,$$

$$e = \sum_{l,m=1}^3 \left((\partial_l S\omega)^T \partial_0 C^{lm}(\nabla v)\partial_m S\omega - 2(\partial_l S\omega)^T\partial_l C^{lm}(\nabla v)\partial_m S\omega\right).$$

First we consider the right most term on the left-hand side of (6.3). When $(s,x)\in \mathbb{R}_+\times\partial\mathcal{K}$, the Neumann condition $\partial_\nu\omega = \langle\vec{n},\nabla_x\rangle\omega = 0$ gives us

$$\partial_s S\omega = s\partial_s^2\omega + \partial_s\omega + \partial_s\langle x,\nabla_x\rangle\omega = s\partial_s^2\omega + \partial_s\omega + \langle x,\vec{n}\rangle\partial_s\partial_\nu\omega = s\partial_s^2\omega + \partial_s\omega.$$

Similarly,

$$\sum_{j=1}^3 n_j\partial_j S\omega = \sum_{j=1}^3 s n_j \partial_j\partial_s\omega + \sum_{j=1}^3 n_j\partial_j\langle x,\nabla_x\rangle\omega = s\partial_\nu\partial_s\omega + \partial\nu\langle x,\nabla_x\rangle\omega = 0$$

on $\mathbb{R}_+\times\partial\mathcal{K}$. Noticing the assumption (6.1), we have

$$\begin{aligned} &-\sum_{j=1}^3 E_j n_j = 2(s\partial_s^2\omega + \partial_s\omega)^T \sum_{j,k=1}^3 C^{jk}(\nabla v)(s\partial_k\partial_s\omega + \partial_k(\langle x,\nabla\rangle\omega))n_j \\ &\le C\sum_{1\le|\alpha|\le 2} |\partial^\alpha\omega|^2. \end{aligned}$$

Hence, identity (6.3) yields

$$\begin{aligned} \int_{\mathbb{R}^3\setminus\mathcal{K}} E_0(t,x)dx \le 2\int_{[0,t]\times\mathbb{R}^3\mathcal{K}} (\partial_0 S\omega)^T\Box_\gamma S\omega\, ds\, dx \\ + \int_{[0,t]\times\mathbb{R}^3\mathcal{K}} e\, ds\, dx + C\int_{[0,t]\times\partial\mathcal{K}} \sum_{1\le|\alpha|\le 2} |\partial^\alpha\omega|^2 d\sigma. \end{aligned} \tag{6.4}$$

Applying Gronwall's inequality, we obtain

$$\begin{aligned} &\|(S\omega)'(t,\cdot)\|_{\mathbb{R}^3\setminus\mathcal{K}} \\ &\le \int_0^t \|\Box_\gamma S\omega(t,\cdot)\|_{\mathbb{R}^3\setminus\mathcal{K}} ds + C\Big(\int_{[0,t]\times\partial\mathcal{K}} \sum_{1\le|\alpha|\le 2} |\partial^\alpha \omega|^2 d\sigma\Big)^{1/2}. \end{aligned} \tag{6.5}$$

By the trace inequality and (3.5), we obtain

$$\begin{aligned} &\Big(\int_{[0,t]\times\partial\mathcal{K}} \sum_{1\le|\alpha|\le 2} |\partial^\alpha \omega|^2 d\sigma\Big)^{1/2} \\ &\le \sum_{|\alpha|\le 2} \|\partial^\alpha \omega'(s,\cdot)\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K}:|x|<2)} \\ &\le C\int_0^t \sum_{|\alpha|\le 2} \|\Box_c \partial^\alpha \omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds + C\sum_{|\alpha|\le 1} \|\Box_c \partial^\alpha \omega\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}. \end{aligned} \tag{6.6}$$

Inequalities (6.5) and (6.6) complete the proof of (6.2). □

Applying Proposition 6.1 and repeating the procedure of Theorem 4.1, we have the following theorem.

Theorem 6.2. *Suppose that* (6.1) *holds and* $\omega = \omega(t,x) \in C^\infty$ *solves problem* (4.4). *Then for any nonnegative integer* N,

$$\begin{aligned} &\sum_{|\alpha|+m\le N,\, m\le 1} \|S^m\partial^\alpha \omega'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\ &\le C\int_0^t \sum_{|\alpha|+m\le N,\, m\le 1} \|\Box_\gamma S^m \partial^\alpha \omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\ &\quad + \sum_{|\alpha|+m\le N-1,\, m\le 1} \|\Box_c S^m \partial^\alpha \omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \\ &\quad + C\int_0^t \sum_{|\alpha|\le N+1} \|\Box_c \partial^\alpha \omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds \\ &\quad + C\sum_{|\alpha|\le N} \|\Box_c \partial^\alpha \omega\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}, \quad \forall t\ge 0\,. \end{aligned} \tag{6.7}$$

7. Main L^2 estimates outside of star-shaped obstacles

We assume that v satisfies (4.2) and (6.1), then we have the following result.

Proposition 7.1. *Suppose that $\omega = \omega(t,x) \in C^\infty$ solves problem (4.4). Then for any fixed nonnegative integer N, we have*

$$
\begin{aligned}
&\sum_{|\alpha|\le N+4} \|\partial^\alpha \omega'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \sum_{|\alpha|+m\le N+2,\, m\le 1} \|S^m\partial^\alpha \omega'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&+ \sum_{|\alpha|+m\le N,\, m\le 1} \|S^m Z^\alpha \omega'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\le C\int_0^t \Big(\sum_{|\alpha|\le N+4} \|\Box_\gamma \partial^\alpha \omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad + \sum_{|\alpha|+m\le N+2,\, m\le 1} \|\Box_\gamma S^m\partial^\alpha \omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad + \sum_{|\alpha|+m\le N,\, m\le 1} \|\Box_\gamma S^m Z^\alpha \omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\Big) ds \qquad (7.1)\\
&\quad + C\sum_{|\alpha|\le N+3} \|\Box_\gamma \partial^\alpha \omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad + C\sum_{|\alpha|+m\le N+1,\, m\le 1} \|\Box_\gamma S^m\partial^\alpha \omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad + C\sum_{|\alpha|\le N+2} \|\Box_c \partial^\alpha \omega\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad + C\sum_{|\alpha|+m\le N,\, m\le 1} \|\Box_c S^m\partial^\alpha \omega\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}.
\end{aligned}
$$

Proof. We denote the left side of (7.1) by $I + II + III$, and the right-hand side side of (7.1) by RHS. Noticing that $\Box_c = \Box_\gamma + \sum_{l,m=1}^3 C^{lm}(\nabla v)\partial_l\partial_m$, then by Theorem 4.1, we have

$$
I \le RHS + C\sum_{l,m=1}^3 \sum_{|\alpha|\le N+3} \|C^{lm}(\nabla v)\partial_l\partial_m\partial^\alpha\omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}. \qquad (7.2)
$$

Similarly, by Theorem 6.2, we obtain

$$
\begin{aligned}
II \le RHS &+ C\int_0^t \sum_{l,m=1}^3 \sum_{|\alpha|\le N+3} \|C^{lm}(\nabla v)\partial_l\partial_m\partial^\alpha\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds\\
&+ C\sum_{j,k=1}^3 \sum_{|\alpha|+m\le N+1,\, m\le 1} \|C^{jk}(\nabla v)\partial_j\partial_k S^m\partial^\alpha\omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}.
\end{aligned} \qquad (7.3)
$$

Similarly, by Theorem 5.2, we obtain

$$
III \le RHS + C\int_0^t \sum_{j,k=1}^3 \sum_{|\alpha|+m\le N+1,\, m\le 1} \|C^{jk}(\nabla v)\partial_j\partial_k S^m\partial^\alpha\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds.
$$

Applying assumption (6.1), the last term on the right-hand side of (7.2) is dominated by

$$
C\Big(\sup_{x\in\mathbb{R}^3,\, l,m} |C^{lm}(\nabla v)|\Big) \sum_{|\alpha|\le N+4} \|\partial^\alpha\omega'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} \le C\delta I.
$$

It can be counteracted by the left-hand side of (7.2), if δ is small enough. Similarly, the last term on the right-hand side of (7.3) can be counteracted by the left side of (7.3). Hence, we have

$$\begin{aligned}
&I+II+III\\
&\le RHS + C\int_0^t \Big(\sup_{x\in\mathbb{R}^3,\, l,m} |C^{lm}(\nabla v)|\Big) \sum_{|\alpha|\le N+4} \|\partial^\alpha \omega'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds\\
&\quad + C\int_0^t \Big(\sup_{x\in\mathbb{R}^3,\, l,m} |C^{lm}(\nabla v)|\Big) \sum_{|\alpha|+m\le N+2,\, m\le 1} \|S^m\partial^\alpha \omega'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} ds\\
&\le RHS + C\int_0^t \Big(\sup_{x\in\mathbb{R}^3,\, l,m} |C^{lm}(\nabla v)|\Big)(I+II) ds.
\end{aligned}$$

Applying Gronwall's inequality and assumption (6.1), we conclude that $I+II+III\le RHS$. $\square$

Using Theorem 3.1 and repeating above proof yields the following theorem.

Theorem 7.2. *Suppose that $\omega=\omega(t,x)\in C^\infty$ solves problem (4.4). Then for any fixed nonnegative integer N,*

$$\begin{aligned}
&\sum_{|\alpha|\le N+4} \|\partial^\alpha \omega'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \sum_{|\alpha|+m\le N+2,\, m\le 1} \|S^m\partial^\alpha \omega'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&+ \sum_{|\alpha|+m\le N,\, m\le 1} \|S^m Z^\alpha \omega'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&+ (\ln(2+t))^{-1/2}\Big(\sum_{|\alpha|\le N+3} \|\langle x\rangle^{-1/2}\partial^\alpha\omega'\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}\\
&+ \sum_{|\alpha|+m\le N+1,\, m\le 1} \|\langle x\rangle^{-1/2}S^m\partial^\alpha\omega'\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}\\
&+ \sum_{|\alpha|+m\le N-1,\, m\le 1} \|\langle x\rangle^{-1/2}S^m Z^\alpha\omega'\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}\Big)\\
&\le C\int_0^t \Big(\sum_{|\alpha|\le N+4} \|\Box_\gamma \partial^\alpha\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad + \sum_{|\alpha|+m\le N+2,\, m\le 1} \|\Box_\gamma S^m\partial^\alpha\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad + \sum_{|\alpha|+m\le N,\, m\le 1} \|\Box_\gamma S^m Z^\alpha\omega(s,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\Big) ds\\
&\quad + C\sum_{|\alpha|\le N+3} \|\Box_\gamma \partial^\alpha\omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad + C\sum_{|\alpha|+m\le N+1,\, m\le 1} \|\Box_\gamma S^m\partial^\alpha\omega(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\
&\quad + C\sum_{|\alpha|\le N+2} \|\Box_c \partial^\alpha\omega\|_{L^2([0,t]\times\mathbb{R}^3\setminus\mathcal{K})}
\end{aligned}$$

$$+ C \sum_{|\alpha|+m\le N,\, m\le 1} \|\Box_c S^m \partial^\alpha \omega(s,\cdot)\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}$$
$$+ C \sum_{|\alpha|+m\le N-2,\, m\le 1} \|\Box_c S^m Z^\alpha \omega(s,\cdot)\|_{L^2([0,t]\times\mathbb{R}^3\backslash\mathcal{K})}. \tag{7.4}$$

8. Almost global existence for quasilinear wave equations outside of star-sharped obstacles

In this section, we shall use above estimates to give the main result of this article, namely the following theorem.

Theorem 8.1. *Suppose that $f, g \in C^\infty(\mathbb{R}^3\backslash\mathcal{K})$ satisfies the compatibility conditions of infinite order. Then there exist constants $\kappa, \varepsilon_0 > 0$, and a positive integer N, such that for all $\varepsilon \le \varepsilon_0$, if*

$$\sum_{|\alpha|\le N} \|\langle x\rangle^{|\alpha|}\partial_x^\alpha f\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + \sum_{|\alpha|\le N-1} \|\langle x\rangle^{|\alpha|+1}\partial_x^\alpha g\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \le \varepsilon, \tag{8.1}$$

then (1.1) *has a unique solution $u \in C^\infty([0,T_\varepsilon]\times\mathbb{R}^3\backslash\mathcal{K})$, with*

$$T_\varepsilon = \exp\Big(\frac{\kappa}{\varepsilon}\Big). \tag{8.2}$$

Proof. Suppose that the integer $N > 14$ and we will take $N = 14$ in the following proof. By local existence we know that if ε is small enough, problem (1.1) has a local solution u in $0 < t < 1$ satisfying the estimate

$$\begin{aligned}
&\sup_{0\le t\le 1}\Big(\sum_{|\alpha|\le 14}\|\partial^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} + \sum_{|\alpha|+m\le 12,\, m\le 1}\|S^m\partial^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\\
&+ \sum_{|\alpha|+m\le 10,\, m\le 1}\|S^m Z^\alpha u'(t,\cdot)\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})}\Big)\\
&+ \sum_{|\alpha|\le 13}\|\langle x\rangle^{-1/2}\partial^\alpha u'(t,\cdot)\|_{L^2([0,1]\times\mathbb{R}^3\backslash\mathcal{K})}\\
&+ \sum_{|\alpha|+m\le 11,\, m\le 1}\|\langle x\rangle^{-1/2}S^m\partial^\alpha u'(t,\cdot)\|_{L^2([0,1]\times\mathbb{R}^3\backslash\mathcal{K})}\\
&+ \sum_{|\alpha|+m\le 9,\, m\le 1}\|\langle x\rangle^{-1/2}S^m Z^\alpha u'(t,\cdot)\|_{L^2([0,1]\times\mathbb{R}^3\backslash\mathcal{K})}\\
&\le C\varepsilon.
\end{aligned} \tag{8.3}$$

Let $\eta \in C^\infty(\mathbb{R})$ be a cut function satisfying

$$\eta(t) = \begin{cases} 1, & t \le \frac12,\\ 0, & t \ge 1.\end{cases}$$

Set $u_0 = \eta u$, $\omega = u - u_0 = (1-\eta)u$, where u is the local solution. Since $\omega = 0$ for $t \le \frac12$, we shall prove the almost global existence of ω by iteration instead of u. Also,

$$\Box_c u_0 = \eta F(\nabla u, \nabla^2 u) + [\Box_c, \eta]u.$$

Thus u solves problem (1.1) for $0 < t < T_\varepsilon$ if and only if ω solves

$$\begin{gathered}\Box_c\omega = (1-\eta)F(\nabla u_0 + \omega), \nabla^2(u_0+\omega) - [\Box_c, \eta](u_0+\omega),\\ \partial_\nu\omega\big|_{\partial\mathcal{K}} = 0,\\ \omega(t,x) = 0, \quad t \le 0,\end{gathered} \tag{8.4}$$

for $0 < t < T_\varepsilon$.

Set $\omega_0 = 0$, and define ω_k recursively for $k = 1, 2, \dots$ by requiring that

$$\begin{gathered}\Box_c\omega_k = (1-\eta)F(\nabla u_0 + \omega_{k-1}), \nabla^2(u_0+\omega_k) - [\Box_c, \eta](u_0+\omega_k),\\ \partial_\nu\omega_k\big|_{\partial\mathcal{K}} = 0,\\ \omega_k(t,x) = 0, \quad t \le 0.\end{gathered} \tag{8.5}$$

Let

$$\begin{aligned}&M_k(T)\\ &= \sup_{0\le t\le T}\Big(\sum_{|\alpha|\le 14}\|\partial^\alpha\omega_k'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + \sum_{|\alpha|+m\le 12,\, m\le 1}\|S^m\partial^\alpha\omega_k'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})}\\ &\quad + \sum_{|\alpha|+m\le 10,\, m\le 1}\|S^m Z^\alpha\omega_k'(t,\cdot)\|_{L^2(\mathbb{R}^3\setminus\mathcal{K})} + (1+t)\sum_{|\alpha|\le 2}\|Z^\alpha\omega_k'(t,\cdot)\|_{L^\infty(\mathbb{R}^3\setminus\mathcal{K})}\Big)\\ &\quad + \big(\ln(2+T)\big)^{-\frac12}\Big(\sum_{|\alpha|\le 13}\|\langle x\rangle^{-\frac12}\partial^\alpha\omega_k'\|_{L^2([0,T]\times\mathbb{R}^3\setminus\mathcal{K})}\\ &\quad + \sum_{|\alpha|+m\le 11,\, m\le 1}\|\langle x\rangle^{-\frac12}S^m\partial^\alpha\omega_k'\|_{L^2([0,T]\times\mathbb{R}^3\setminus\mathcal{K})}\\ &\quad + \sum_{|\alpha|+m\le 9,\, m\le 1}\|\langle x\rangle^{-\frac12}S^m Z^\alpha\omega_k'\|_{L^2([0,T]\times\mathbb{R}^3\setminus\mathcal{K})}\Big)\\ &= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7.\end{aligned}$$

Now we prove that there exists a constant C_1, such that

$$M_k(T_\varepsilon) \le C_1\varepsilon, \quad k = 0, 1, 2, \dots \tag{8.6}$$

if $\varepsilon > 0$ and constant κ in $T_\varepsilon = \exp^{\frac{\kappa}{\varepsilon}}$ are sufficiently small. It is obviously that $M_0(T_\varepsilon) \le C_1\varepsilon$. Providing $M_{k-1}(T_\varepsilon) \le C_1\varepsilon$, we shall proof $M_k(T_\varepsilon) \le C_1\varepsilon$. To do this, we first prove

$$M_k(T_\varepsilon) \le C\varepsilon + CC_1\kappa\big(M_{k-1}(T_\varepsilon) + M_k(T_\varepsilon)\big). \tag{8.7}$$

The bound (8.6) follows from (8.7). By Theorem 2.1 and (8.3), we know that A_4 can be controlled by the right-hand side of (8.7). The other terms of $M_k(T_\varepsilon)$ can be controlled by the right-hand side of (7.4), where $N = 10$, $\omega = \omega_k$. Denote the right-hand side of (7.4) by

$$B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8.$$

$B_1 + B_2 + B_3$ can be controlled by the right-hand side of (8.7) using the argument in [5]. It is easy to prove that $B_4 + B_5$ is estimated by the right-hand side of (8.7).

Now we deal with B_6. For $t > 1$, we have

$$\sum_{|\alpha|\le 12}|S\partial^\alpha\omega_k| \le C\sum_{|\alpha|\le 13, |\beta|\le 6}\big(|\partial^\alpha\omega_k'||\partial^\beta\omega_{k-1}'| + |\partial^\alpha\omega_{k-1}'||\partial^\beta\omega_k'|\big),$$

therefore,

$$
\begin{aligned}
&\sum_{|\alpha|\le 12} \|S\partial^\alpha \omega_k\|_{L^2([1,T_\varepsilon]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&\le C \sum_{|\alpha|\le 13,|\beta|\le 6} \big(\|\partial^\alpha\omega_k'\partial^\beta\omega_{k-1}'\|_{L^2([1,T_\varepsilon]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&\quad + \|\partial^\alpha\omega_{k-1}'\partial^\beta\omega_k'\|_{L^2([1,T_\varepsilon]\times\mathbb{R}^3\backslash\mathcal{K})}\big).
\end{aligned} \tag{8.8}
$$

Consider the first term on the right-hand side of (8.8). Applying Lemma 1.4, we have

$$
\begin{aligned}
&\sum_{|\alpha|\le 13,|\beta|\le 6} \|\partial^\alpha\omega_k'\partial^\beta\omega_{k-1}'\|_{L^2([1,T_\varepsilon]\times\mathbb{R}^3\backslash\mathcal{K})} \\
&\le C\sum_{|\beta|\le 8} \|\langle x\rangle^{-1} z^\beta\omega_{k-1}'\|_{L^2([1,T_\varepsilon]\times\mathbb{R}^3\backslash\mathcal{K})} \sup_{1<t<T_\varepsilon} \sum_{|\alpha|\le 13} \|\partial^\alpha\omega_k'\|_{L^2(\mathbb{R}^3\backslash\mathcal{K})} \\
&\le CC_1\varepsilon \ln(T_\varepsilon)^{1/2} M_k(T_\varepsilon) \\
&\le CC_1\kappa M_k(T_\varepsilon).
\end{aligned}
$$

In a similar way, we can prove the second term on right-hand side of (8.8) can be controlled by the right-hand side of (8.7). For $t<1$, noticing the estimate of local solution and the assumption of induction, we can get that B_6 is bounded by the right-hand side of (8.7). Similarly, we obtain that B_7+B_8 is also estimated by the right-hand side of (8.7). Hence, we complete the proof of (8.7).

Next, using the energy inequality, we can show that $\{\omega_k(t,x)\}$ converges in the energy norm. Suppose that its limit is $\omega(t,x)$, then $u=u_0+\omega$ solves problem (1.1). If $(f,g)\in C^\infty(\mathbb{R}^3\backslash\mathcal{K})$ satisfying the compatibility conditions to infinite order, then $u\in C^\infty([0,T_\varepsilon)\times\mathbb{R}^3\backslash\mathcal{K})$. □

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