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AN INVERSE STURM-LIOUVILLE PROBLEM WITH A GENERALIZED SYMMETRIC POTENTIAL

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Abstract. We consider the normal form of Sturm-Liouville differential equations with separable boundary conditions. For this problem we know that the potential function is determined uniquely by two spectra and that if the potential is symmetric, then it is determined uniquely by just one spectrum. In this paper firstly we generalize symmetric potential and then investigate change of needed data to determine potential function uniquely.

1. Introduction

The inverse Sturm-Liouville problem was firstly studied by Ambartsumyan in 1929. He considered the boundary value problem

$$-y''(x) + (\lambda + q(x))y(x) = 0,$$
$$y'(0) = y'(1) = 0$$

and showed that if eigenvalues of the problem are $\lambda_n = n^2\pi^2$, then the potential function $q \equiv 0$ [1]. This work naturally led to question whether the potential is determined uniquely by one spectrum or not. This problem is called the inverse Sturm- Liouville problem. Borg showed that it is not true generally and that two spectra are needed to determine the potential uniquely by changing one of boundary conditions. He also showed that if the potential q is symmetric about midpoint of the interval $[0,\pi]$ for his problem, then it is determined uniquely by one spectrum in 1949 [3]. Levinson shortened the proof of Borg by using complex analysis techniques[11]. From then on, various versions of Borg's work have been considered. Some of them can be seen in [6, 8, 9].

The study we conducted here is motivated by [11, 12, 13]. Firstly we need to give some theorems and definitions. The structure of BVP we considered is as follows:

$$L[y] = \mu y, \tag{1.1}$$
$$y'(0) - hy(0) = 0, \tag{1.2}$$
$$y'(a) + Hy(a) = 0, \tag{1.3}$$

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where $L[u] = -u'' + qu$ and h, H, a are real constants. The operator L is a self-adjoint operator defined on $L^2[0, a]$ provided that q satisfies suitable regularity conditions. Under these conditions L has a discrete spectrum consisting of the simple and real eigenvalues $\{\mu_i\}_{i=0}^{\infty}$ [2]. Let us consider (1.1) with the initial conditions

$$y(0) = 1, \quad y'(0) = h. \tag{1.4}$$

Theorem 1.1. *Suppose that $\phi(x, \lambda)$ is the solution of (1.1) satisfying the initial conditions (1.4) and that $q(x)$ has m locally integrable derivatives. Then there exists function $K(x, t)$ having $m + 1$ locally integrable derivatives with respect to each of variables such that*

$$\phi(x, \lambda) = \cos\sqrt{\lambda}x + \int_0^x K(x, t)\cos\sqrt{\lambda}t dt, \tag{1.5}$$

$$K(x, x) = h + \frac{1}{2}\int_0^x q(t)dt. \tag{1.6}$$

The above theorem and other properties of function $K(x, t)$ can be found in [12].

Lemma 1.2. *Let (a, b) be a finite interval and $f(x) \in L(a, b)$. Then*

$$\lim_{|\lambda|\mapsto\infty} \int_a^b f(x)\cos\sqrt{\lambda}x dx = 0.$$

The above lemma and its proof can be found in [7].

Lemma 1.3. *If a real sequence $\{\mu_n\}$ is the spectrum of BVP (1.1)-(1.3) with function $q(x)$ where $q \in L^1(0, a)$, then it satisfies the asymptotic formula*

$$\sqrt{\mu_n} = \frac{n\pi}{a} + \frac{aa_0}{n} + o(\frac{1}{n}) \tag{1.7}$$

where $\mu_n \neq \mu_m$ for $m \neq n$ and $a_0 = (K(a, a) + H(/(a\pi)$.

Proof. By using the boundary conditions directly we derive $\sqrt{\mu_n}$. From Theorem 1.1, we have solution (1.5). By considering (1.3) we obtain

$$\begin{aligned} &-\sqrt{\mu}\sin\sqrt{\mu}a + \int_0^a \frac{\partial K(x, t)}{\partial x}\cos\sqrt{\mu}t dt + K(a, a)\cos\sqrt{\mu}a \\ &+ H\big[\cos\sqrt{\mu}a + \int_0^a K(a, t)\cos\sqrt{\mu}t dt\big] = 0. \end{aligned} \tag{1.8}$$

The numbers $\{\mu_n\}$ are the roots of equation (1.8). Since $\lambda_n \mapsto \infty$ as $n \mapsto \infty$ the first approximation for μ_n's is obtained as follows:

$$\sin\sqrt{\mu_n}a + O\big(\frac{1}{\sqrt{\mu_n}}\big) = 0.$$

Therefore

$$\sqrt{\mu_n} = \frac{n\pi}{a} + O\big(\frac{1}{n}\big).$$

Now let us put

$$\sqrt{\mu_n} = \frac{n\pi}{a} + \frac{aa_0}{n} + \frac{\gamma_0}{n}.$$

Then

$$(-1)^n a\Big[\frac{aa_0 + \gamma_n}{n} + O\big(\frac{1}{n^3}\big)\Big] - \frac{(-1)^n[K(a, a) + H]}{\frac{n\pi}{a}\big(\frac{1+a^2a_0+\gamma_n a}{\pi n^2}\big)}\Big[1 + O\big(\frac{1}{n^2}\big)\Big]$$

$$-\frac{1}{\sqrt{\mu_n}}\int_0^a \big[HK(a,t)+\frac{\partial}{\partial x}K(x,t)\Big|_{x=a}\big]\cos\sqrt{\mu_n}t\,dt=0$$

from (1.8). Since $[HK(a,t)+\frac{\partial}{\partial x}K(x,t)\Big|_{x=a}]\in L^1(0,a)$, by using Lemma 1.2 it follows that as $n\to\infty$

$$\frac{1}{\sqrt{\mu_n}}\int_0^a \big[HK(a,t)+\frac{\partial}{\partial x}K(x,t)\Big|_{x=a}\big]\cos\sqrt{\mu_n}t\,dt\to 0$$

Therefore $a_0=\frac{K(a,a)+H}{a\pi}$ and $\gamma_n\to 0$ as $n\to\infty$. This completes the proof. $\square$

Remark 1.4. There is no $\tilde{a}$ different from a such that asymptotic formula (1.7) is admitted by the problem (1.1)-(1.3) with $\tilde{a}$ instead of a.

Let us consider a function $f:[0,1]\mapsto\mathbb{R}$. If for $\forall x\in[0,1]$ and $n=1,2,\dots,k$,

$$f(x)=f\Big(\frac{1}{2^{n-1}}-x\Big),$$

then we will call the function f as kth-order symmetric function.

Example 1.5. Define the function

$$f(x)=\begin{cases}x^2+x+0.01 & 0<x<0.25,\\ x^2-2x+0.76 & 0.25<x<0.5,\\ x^2-0.24 & 0.5<x<0.75,\\ x^2-3x+1.01 & 0.75<x<1,\end{cases}$$

Then f is a second-order symmetric function. It can be seen in Figure reffig1.

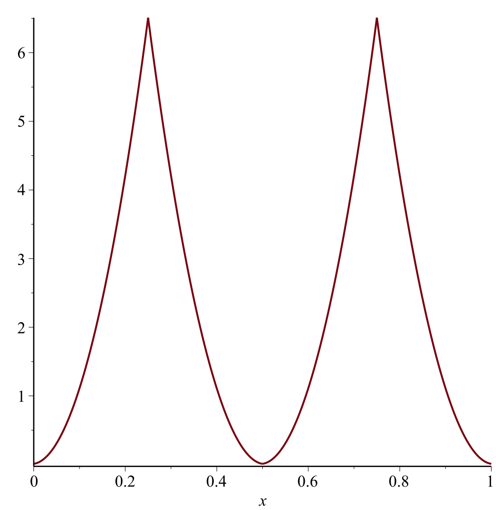

FIGURE 1. A second-order symmetric function.

2. Main results

In this part we consider the BVP

$$y''(x) + (\lambda - q(x))y(x) = 0, \tag{2.1}$$

$$y'(0) - hy(0) = 0, \quad y'(1) + Hy(1) = 0. \tag{2.2}$$

Theorem 2.1. *Let be $q \in L^1[0,1]$, q is a second-order symmetric function and $h = H$. Then for (2.1)-(2.2) the potential q is determined uniquely by its half spectrum except finite numbers of its eigenvalues.*

Proof. To prove this theorem let us consider the BVP

$$-y''(x) + q(x)y(x) = \tilde{\lambda} y(x), \tag{2.3}$$

$$y'(0) - \frac{h}{2}y(0) = 0, \quad y'(1/2) + \frac{H}{2}y(1/2) = 0. \tag{2.4}$$

From the theory about symmetric potential we know that for all $x \in [0,1]$ when q is symmetric about midpoint, one spectrum is sufficient to determine potential q uniquely. Furthermore since $q(x) = q(\frac{1}{2} - x)$ for all $x \in [0, 1/2]$, the same situation holds for (11)-(12). We intend to show that the spectrum of (11)-(12) represents half spectrum of the problem (2.1)-(2.2) except for a finite numbers of its eigenvalues. From Lemma 1.2 we see that spectrum of (11)-(12) has the asymptotic form

$$\sqrt{\tilde{\lambda}_k} = 2k\pi + \frac{\tilde{a}_0}{2k} + o\Big(\frac{1}{k}\Big)$$

and that the spectrum of (2.1)-(2.2) has also asymptotic form

$$\sqrt{\lambda_k} = n\pi + \frac{a_0}{n} + o\Big(\frac{1}{n}\Big).$$

Also we have

$$\begin{aligned}
\tilde{a}_0 &= \frac{\tilde{K}(1/2,1/2) + H/2}{\pi/2} = \frac{1}{\pi}\big[2\tilde{K}(1/2,1/2) + H\big] \\
&= \frac{1}{\pi}\Big[2\Big(\frac{h}{2} + \frac{1}{2}\int_0^{1/2} q(x)dx\Big) + H\Big] \\
&= \frac{1}{\pi}\Big[H + \Big(h + \frac{1}{2}\int_0^{1} q(x)dx\Big)\Big] \\
&= \frac{1}{\pi}\big[H + K(1,1)\big] = a_0
\end{aligned}$$

where K and $\tilde{K}$ are kernels of solutions which represent (2.1)-(2.2) and (11)-(12) respectively. Having the above results we obtain

$$\sqrt{\tilde{\lambda}_n} - \sqrt{\lambda_{2n}} = o\Big(\frac{1}{n}\Big) \to 0$$

for large n. This completes the proof. □

Corollary 2.2. *Let be $q \in L^1[0,1]$, q is a kth-order symmetric function and $h = H$. Then for (2.1)-(2.2) the potential q is determined uniquely by its $\frac{1}{2^{k-1}}$ spectrum except for a finite numbers of eigenvalues of spectrum.*

The above corollary can be shown easily by using induction with the above process. This result leads to a question: What is the situation when q is an infinite-order symmetric function? Our related theorem is as follows.

Theorem 2.3. *Let $q \in C[0,1]$, $H = -h$ and q be an infinite-order symmetric function. Then for problem (2.1)-(2.2) the potential function q is determined uniquely by just its one eigenvalue.*

Proof. Since $q \in C[0,1]$ from Weierstrass approximation theorem (see [10]) there exists polynomial P with degree m on $[a,b]$ which represents potential q. Then for $\forall k$ it follows that $P(x) = P\big(\frac{1}{2^{k-1}} - x\big)$. By rewriting this for $k = 1, 2, \dots, n$ and then adding those for $P(x) = c_0 + c_1 x + \dots + c_m x^m$ we obtain

$$\begin{aligned}
P(x) &= \frac{1}{n}\Big\{nc_0 + c_1\big[(1-x) + (\frac{1}{2} - x) + \dots + (\frac{1}{2^{n-1}})\big] + \dots \\
&\quad + c_m\big[(1-x)^m + \dots + (\frac{1}{2^{n-1}})^m\big]\Big\} \\
&= c_0 + \frac{1}{n}\sum_{k=1}^{m} c_k\big[(1-x)^k + \dots + \big(\frac{1}{2^{n-1}} - x\big)^k\big].
\end{aligned}$$

By taking limits for the general term of the sum as $n \mapsto \infty$ it follows that

$$\begin{aligned}
&\lim_{n\mapsto\infty} \frac{(1-x)^k + \dots + \big(\frac{1}{2^{n-1}} - x\big)^k}{n} \\
&= \lim_{n\mapsto\infty} \frac{1}{n}\Big\{\big(1 + \frac{1}{2^k} + \dots + \frac{1}{(2^k)^{n-1}}\big) \\
&\quad - x\big(1 + \frac{1}{2^{k-1}} + \dots + \frac{1}{(2^{k-1})^{n-1}}\big) + \dots + (-1)^k x^k n\Big\} \\
&= (-1)^k x^k
\end{aligned}$$

by using Stolz-Cesáro theorem for each term in parenthesis. We can assume that $m = 2k$ without loss of generality. Thus we see that $P(x) = c_0 + c_2 x^2 + \dots + c_{2k} x^{2k}$. Besides we have $P(0) = P\big(\frac{1}{2^{n-1}}\big)$ for $x = 0$. So for $n = 2, 3, \dots, k+1$ we obtain

$$\begin{pmatrix} 2^{-2} & 2^{-4} & \dots & 2^{-2k} \\ 2^{-4} & 2^{-8} & \dots & 2^{-4k} \\ \vdots & \vdots & \vdots & \vdots \\ 2^{-2k} & 2^{-4k} & \dots & 2^{-2k^2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_4 \\ \vdots \\ c_{2k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

By using the LU decomposition [15] for coefficient matrix, say A, it follows that

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 2^{-2k} & 2^{-4k} & 2^{-6k} & \dots & 2^{-2k^2} \\ 0 & 3\cdot 2^{-4k+2} & 2^{-6k+2} & \dots & \cdot \\ 0 & 0 & 45\cdot 2^{-6k+6} & \dots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M\cdot 2^{-k^2-k} \end{vmatrix} \\
&= (2^2-1)^{k-1}(2^4-1)^{k-2}\dots(2^{2(k-1)}-1)2^{-\frac{k(k+1)(2k+1)}{3}} \neq 0
\end{aligned}$$

where $M = (2^2-1)(2^4-1)\dots(2^{2(k-1)}-1)$. So

$$c_2 = c_4 = \dots = c_{2k} = 0.$$

Consequently $q(x) = c_0$. Finally by using (2.1)-(2.2) with the fact $h = -H$ we see that potential q can be determined by just its one eigenvalue. ☐

Conclusion. In this study, we considered the variation of needed data to be the spectrum of the operator to determine uniqueness of the potential function that has property $q(x) = q(\frac{1}{2^{n-1}} - x)$ by changing natural numbers n. In the recent times the studies which include reconstruction of the potential have been conducted. Some of them can be seen in [4, 5, 14]. For an application of this study, Theorem 2.1, Corollary 2.2, and Theorem 2.3 can be considered. We think that one can obtain an approximation to the potential by using the sets $\{\lambda_1\}$, $\{\lambda_1, \lambda_2\}$, $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}, \dots$. It is clear that obtaining a good approximation is difficult. However it can be considered to be an initial potential in numerical algorithms like the algorithm described in [14].

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