BOUNDEDNESS AND SQUARE INTEGRABILITY OF
SOLUTIONS OF NONLINEAR FOURTH-ORDER DIFFERENTIAL
EQUATIONS WITH BOUNDED DELAY

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Communicated by Mokhtar Kirane

ABSTRACT. In this article, we give sufficient conditions for the boundedness,
uniformly asymptotic stability and square integrability of the solutions to a
fourth-order non-autonomous differential equation with bounded delay by us-
ing Lyapunov’s second method.

1. INTRODUCTION

Ordinary differential equations have been studied for more than 300 years since
the seventeenth century after the concepts of differentiation and integration were
formulated by Newton and Leibniz. By means of ordinary differential equations,
researchers can explain many natural phenomena like gravity, projectiles, wave,
vibration, nuclear physics, and so on. In addition, in Newtonian mechanics, the
system’s state variable changes over time, and the law that governs the change
of the system’s state is normally described by an ordinary differential equation.
The question concerning the stability of ordinary differential equations has been
originally raised by the general problem of the stability of motion (Lyapunov [22]).

However, thereafter along with the development of technology, it is seen that
the ordinary differential equations cannot respond to the needs arising in sciences
and engineering. For example, in many applications, it can be seen that physical
or biological background of modeling system shows that the change rate of the
system’s current status often depends not only on the current state but also on the
history of the system. This usually leads to so-called retarded functional differential
equations (Smith [33]).

To the best of our knowledge, the study of qualitative properties of functional
differential equations of higher order has been developed at a high rate in the last
four decades. Functional differential equations of higher order can serve as ex-
cellent tools for description of mathematical modeling of systems and processes in
economy, stochastic processes, biomathematics, population dynamics, medicine, in-
formation theory, physics, chemistry, aerodynamics and many fields of engineering
like atomic energy, control theory, mechanics, etc., Therefore, the investigation of
the qualitative properties of solutions of functional differential equations of higher order, stability, boundedness, oscillation, integrability etc. of solutions play an important role in many real world phenomena related to the sciences and engineering technique fields. In fact, we would not like to give here the details of the applications related to functional differential equations of higher order here.

In particular, for more results on the stability, boundedness, convergence, etc. of ordinary or functional equations differential equations of fourth order, see the book of Reissig et al. [30] as a good survey for the works done by 1974 and the papers of Burton [6], Cartwright [7], Ezeilo [11, 12, 13, 14], Harrow [15, 16], Tunç book of Reissig et al. [30] as a good survey for the works done by 1974 and the papers of Reissig et al. [30, 31, 32, 33, 34, 35, 36, 37]. Remili et al. [25, 26, 27, 28, 29], Wu [44] and others and theirs references. These information indicate the importance of investigating the qualitative properties, of solutions of retarded functional differential equations of fourth order.

In this article, we study the uniformly asymptotic stability of the solutions for

\[ p(t, x, x', x'', x''') = 0 \]

and also square integrable and boundedness of solutions to the fourth order nonlinear differential equation with delay

\[
x^{(4)} + a(t)(g(x(t)) x''(t))' + b(t)(q(x(t)) x'(t))' + c(t)f(x(t))x'(t) + d(t)h(x(t - r(t))) = p(t, x, x', x'', x''').
\]

(1.1)

For convenience, we let

\[
\theta_1(t) = q'(x(t))x'(t), \quad \theta_2(t) = q'(x(t))x'(t), \quad \theta_3(t) = f'(x(t))x'(t).
\]

We write (1.1) in the system form

\[
\begin{align*}
x' & = y, \\
y' & = z, \\
z' & = w, \\
w' & = -a(t)g(x)w - (b(t)q(x) + a(t))\theta_1 z - (b(t)\theta_2 + c(t)f(x))y \\
& \quad - d(t)h(x) + d(t)\int_{t-r(t)}^{t} h'(x)yd\eta + p(t, x, y, z, w),
\end{align*}
\]

(1.2)

where \( r \) is a bounded delay, \( 0 \leq r(t) \leq \psi, r'(t) \leq \xi, 0 < \xi < 1, \xi \) and \( \psi \) some positive constants, \( \psi \) which will be determined later, the functions \( a, b, c, d \) are continuously differentiable and the functions \( f, h, q, p \) are continuous functions depending only on the arguments shown. Also derivatives \( g'(x), q'(x), f'(x) \) and \( h'(x) \) exist and are continuous. The continuity of the functions \( a, b, c, d, p, q, g, q, f, h \) guarantees the existence of the solutions of equation (1.1). If the right hand side of the system (1.2) satisfies a Lipchitz condition in \( x(t), y(t), z(t), w(t) \) and \( x(t - r) \), and exists of solutions of system (1.2), then it is unique solution of system (1.2).

Assume that there are positive constants \( a_0, b_0, c_0, d_0, f_0, g_0, a_1, b_1, c_1, d_1, f_1, g_1, q_1, m, M, \delta, \eta_1 \) such that the following assumptions hold:

\begin{enumerate}
  \item [(A1)] \( 0 < a_0 \leq a(t) \leq a_1, 0 < b_0 \leq b(t) \leq b_1, 0 < c_0 \leq c(t) \leq c_1, 0 < d_0 \leq d(t) \leq d_1 \) for \( t \geq 0; \)
  \item [(A2)] \( 0 < f_0 \leq f(x) \leq f_1, 0 < g_0 \leq g(x) \leq g_1, 0 < q_0 \leq q(x) \leq q_1 \) for \( x \in R \) and \( 0 < m < \min\{f_0, g_0, 1\}, M > \max\{f_1, g_1, 1\}; \)
  \item [(A3)] \( \frac{h(t)}{x} \geq \delta > 0 \) for \( x \neq 0, h(0) = 0; \)
  \item [(A4)] \( \int_{0}^{\infty} (|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|)dt < \eta_1; \)
  \item [(A5)] \( |p(t, x, y, z, w)| \leq |e(t)|. \)
\end{enumerate}
Motivated by the results of references, we obtain some new results on the uniformly asymptotic stability and boundedness of the solutions by means of the Lyapunov’s functional approach. Our results differ from that obtained in the literature (see, the references in this article and the references therein). By this way, we mean that this paper has a contribution to the subject in the literature, and it may be useful for researchers working on the qualitative behaviors of solutions of functional differential equations of higher order. In view of all the mentioned information, it can be checked the novelty and originality of the current paper.

2. Preliminaries

We also consider the functional differential equation
\[ \dot{x} = f(t, x), \quad x(t) = x(t + \theta), \quad -r \leq \theta \leq 0, \quad t \geq 0, \tag{2.1} \]
where \( f : I \times C_H \to \mathbb{R}^n \) is a continuous mapping, \( f(t, 0) = 0, \) \( C_H := \{ \phi \in (C[-r, 0], \mathbb{R}^n) : \| \phi \| t \leq H \} \), and for \( H_1 < H \), there exists \( L(H_1) > 0 \), with \( |f(t, \phi)| < L(H_1) \) when \( \| \phi \| t < H_1 \).

**Lemma 2.1** (19). Let \( V(t, \phi) : I \times C_H \to \mathbb{R} \) be a continuous functional satisfying a local Lipschitz condition, \( V(t, 0) = 0 \), and wedges \( W_i \) such that:

(i) \( W_1(\| \phi \| t) \leq V(t, \phi) \leq W_2(\| \phi \| t) \);

(ii) \( V'(t, \phi) \leq -W_3(\| \phi \| t) \).

Then, the zero solution of (2.1) is uniformly asymptotically stable.

3. Main results

**Lemma 3.1** (19). Let \( h(0) = 0, \ x_h(x) > 0 \ (x \neq 0) \) and \( \delta(t) - h'(x) \geq 0 \ (\delta(t) > 0) \), then \( 2\delta(t)H(x) \geq h^2(x), \) where \( H(x) = \int_0^x h(s)ds \).

**Theorem 3.2.** In addition to the basic assumptions imposed on the functions \( a, b, c, d, p, f, h, g, q \) suppose that there are positive constants \( h_0, h_1, \delta_0, \delta_1, \eta_2, \eta_3 \) such that the following conditions are satisfied:

(i) \( h_0 - \frac{a_0b_0}{d^2} \leq h'(x) \leq \frac{h_0}{2} \) for \( x \in R; \)

(ii) \( \delta_1 = \frac{d^3b_0a_1M}{c_0m} + \frac{c_1M + b_0}{a_0m} < b_0q_0; \)

(iii) \( \int_{-\infty}^\infty (|g'(s)| + |q'(s)| + |f'(s)|)ds < \eta_2; \)

(iv) \( \int_0^\infty |e(t)|dt < \eta_3. \)

Then any solution \( x(t) \) of (1.1) and its derivatives \( x'(t), x''(t), x'''(t) \) are bounded and satisfy
\[ \int_0^\infty (x'^2 + x''^2 + x'''^2)ds < \infty, \]
provided that
\[ \psi < \frac{1 - \xi}{d^2h_1} \min \left\{ \frac{\varepsilon \alpha + \varepsilon \beta (1 - \xi)}{2 + \xi}, \frac{2b_0q_0 - \delta_1 - \varepsilon M(a_1 + c_1)}{\alpha (1 - \xi)}, \frac{2\varepsilon}{\alpha (1 - \xi)} \right\}, \]

**Proof.** We define a Lyapunov functional
\[ W = W(t, x, y, z, w) = e^{\frac{\psi}{\varepsilon} t} \int_0^t \gamma(s)ds V, \tag{3.1} \]
where
\[ \gamma(t) = |a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| + |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|, \]
and

\[ 2V = 2\beta d(t)H(x) + c(t)f(x)y^2 + a_b(t)q(x)z^2 + a(t)g(x)z^2 + 2\beta a(t)g(x)yz \]
\[ + [\beta b(t)q(x) - \alpha h_0 d(t)]y^2 - \beta z^2 + \alpha w^2 + 2d(t)h(x)y + 2\alpha d(t)h(x)z \]
\[ + 2\alpha c(t)f(x)yz + 2\beta yw + 2zw + \sigma \int_{-r(t)}^{t} y^2(\gamma)d\gamma ds \]

with \( H(x) = \int_{0}^{x} h(s)ds \), \( \alpha = \frac{1}{a_0m} + \varepsilon, \beta = \frac{d_1h_0}{c_0m} + \varepsilon, \varepsilon \) and \( \eta \) are positive constants to be determined later in the proof. We can rearrange \( 2V \) as

\[ 2V = a(t)g(x)\left[ \frac{w}{a(t)g(x)} + z + \beta y \right]^2 + c(t)f(x)\left[ \frac{d(t)h(x)}{c(t)f(x)} + y + \alpha z \right]^2 \]
\[ + \frac{d^2(t)h^2(x)}{c(t)f(x)} + 2\varepsilon d(t)H(x) + \sigma \int_{-r(t)}^{t} y^2(\gamma)d\gamma ds + V_1 + V_2 + V_3, \]

where

\[ V_1 = 2d(t) \int_{0}^{x} h(s)\left[ \frac{d_1h_0}{c_0m} - \frac{2d(t)}{c(t)f(x)}h'(s) \right]ds, \]
\[ V_2 = [ab(t)q(x) - \beta - \alpha^2 c(t)f(x)]z^2, \]
\[ V_3 = [\beta b(t)q(x) - \alpha h_0 d(t) - \beta^2 a(t)g(x)]y^2 + [\alpha - \frac{1}{a(t)g(x)}]w^2. \]

Let

\[ \varepsilon < \min \left\{ \frac{1}{a_0m}, \frac{d_1h_0}{c_0m}, \frac{b_0q_0 - \delta_1}{M(a_1 + c_1)} \right\} \quad (3.2) \]

then

\[ \frac{1}{a_0m} < \alpha < \frac{2}{a_0m}, \quad \frac{d_1h_0}{c_0m} < \beta < \frac{2d_1h_0}{c_0m} \quad (3.3) \]

By using conditions (A1)–(A3), (i)–(ii) and inequalities (3.2), (3.3) we obtain

\[ V_1 \geq 4d(t)\frac{d_1}{c_0m} \int_{0}^{x} h(s)\frac{h_0}{2} - h'(s)]ds \geq 0, \]
\[ V_2 = (\alpha \beta b(t)q(x) - \beta a(t) - \alpha c(t)f(x)) + \beta(\alpha a(t) - 1)z^2 \]
\[ \geq \alpha \left( b_0q_0 - \frac{d_1h_0a_1}{c_0m} - \frac{c_1M}{a_0m} - \varepsilon(a_1 + a_1M) \right)z^2 + \beta(1 - 1)z^2 \]
\[ \geq \alpha (b_0q_0 - \delta_1 - \varepsilon M(a_1 + c_1))z^2 \geq 0, \]

and

\[ V_3 \geq \beta \left( b_0q_0 - \frac{1}{\beta}h_0d_1 - \beta a_1M \right)y^2 + (\alpha - \frac{1}{a_0m})w^2 \]
\[ \geq \beta \left( b_0q_0 - \frac{c_0}{a_0} - a_1 \frac{d_1h_0M}{c_0m} - \varepsilon(c_0m + a_1M) \right)y^2 + \varepsilon w^2 \]
\[ \geq \beta(b_0q_0 - \delta_1 - \varepsilon M(c_1 + a_1))y^2 + \varepsilon w^2 \geq 0. \]

Thus, it is clear from the above inequalities that there exists positive constant \( D_0 \) such that

\[ 2V \geq D_0(y^2 + z^2 + w^2 + H(x)). \quad (3.4) \]

From Lemma 3.1 (A3) and (i), it follows that there is a positive constant \( D_1 \) such that

\[ 2V \geq D_1(x^2 + y^2 + z^2 + w^2) \quad (3.5) \]
In this way $V$ is positive definite. From (A1)–(A3), it is clear that there is a positive constant $U_1$ such that
\[ V \leq U_1(x^2 + y^2 + z^2 + w^2). \] (3.6)

From (iii), we have
\[
\int_0^t (|\theta_1(s)| + |\theta_2(s)| + |\theta_3(s)|) ds \\
= \int_{\alpha_2(t)}^{\alpha_1(t)} (|g'(u)| + |q'(u)| + |f'(u)|) du \\
\leq \int_{-\infty}^{+\infty} (|g'(u)| + |q'(u)| + |f'(u)|) du < \eta_2 < \infty
\] (3.7)

where $\alpha_1(t) = \min\{x(0), x(t)\}$ and $\alpha_2(t) = \max\{x(0), x(t)\}$. From inequalities (3.2), (3.6) and (3.7), it follows that
\[ W \geq D_2(x^2 + y^2 + z^2 + w^2) \] (3.8)

where $D_2 = \frac{D_2}{2} e^{-\frac{\alpha_2}{4} \eta_2}$. Also, it is easy to see that there is a positive constant $U_2$ such that
\[ W \leq U_2(x^2 + y^2 + z^2 + w^2) \] (3.9)

for all $x, y, z, w$ and all $t \geq 0$.

Now, we show that $W$ is negative definite function. The derivative of the function $V$ along any solution $(x(t), y(t), z(t), w(t))$ of system (1.2), with respect to $t$ is after simplifying
\[ 2\dot{V}_{(1.2)} = -2\varepsilon c(t)f(x)y^2 + V_4 + V_5 + V_6 + V_7 + V_8 + V_9 + V_0 + 2(\beta y + z + \alpha w)p(t, x, y, z, w) \]

where
\[
V_4 = -2\left( \frac{d_1 h_0}{c(t)} f(x) - d(t) h'(x) \right) y^2 - 2\alpha d(t) (h_0 - h'(x)) yz,
\]
\[
V_5 = -2(b(t) q(x) - \alpha c(t) f(x) - \beta a(t) g(x)) z^2,
\]
\[
V_6 = -2(\alpha a(t) g(x) - 1) w^2,
\]
\[
V_7 = 2\alpha d(t) w \int_{t-r(t)}^{t} h'(x(\eta)) x'(\eta) d\eta + 2\beta d(t) y(t) \int_{t-r(t)}^{t} h'(x(\eta)) x'(\eta) d\eta
\]
\[
+ 2d(t) z(t) \int_{t-r(t)}^{t} h'(x(\eta)) x'(\eta) d\eta + \sigma r(t) y^2(t) - \sigma(1 - r'(t)) \int_{t-r(t)}^{t} y^2(\eta) d\eta,
\]
\[
V_8 = -a(t) \theta_1(z_2 + 2\alpha z w) - b(t) \theta_2(\alpha z^2 + 2\alpha z w + \beta y^2 + 2yz)
\]
\[
+ c(t) \theta_3(y^2 + 2\alpha yz),
\]
\[
V_9 = d'(2\beta H(x) - ah_0 y^2 + 2h(x) y + 2\alpha h(x) z)
\]
\[
+ c'(t) [f(x)y^2 + 2af(x)yz] + b'(t) [a q(x) z^2 + \beta q(x) y^2]
\]
\[
+ a'(t) [g(x) y^2 + 2\beta g(x) yz].
\]

By regarding conditions (A1), (A2), (i), (ii) and inequalities (3.3), (3.4), we have
\[
V_4 \leq -2d(t) h_0 - d(t) h'(x) y^2 - 2\alpha d(t) [h_0 - h'(x)] yz
\]
\[
\leq -2d(t) [h_0 - h'(x)] y^2 - 2\alpha d(t) [h_0 - h'(x)] yz
\]
\[
\leq 2d(t) [h_0 - h'(x)] [(y + \frac{\alpha}{2} z)^2 - (\frac{\alpha}{2} z)^2]
Lemma 3.1, we can write

\[
\frac{\alpha^2}{2} d(t) [h_0 - h'(x)] z^2.
\]

In this case,

\[
V_4 + V_5 \leq -2[b(t) q(x) - \alpha(t) f(x) - \beta a(t) g(x) - \frac{\alpha^2}{4} d(t) [h_0 - h'(x)]] z^2
\]

\[
\leq -2[b_0 q_0 - \frac{1}{a_0 m} + \varepsilon] M - \frac{d_1 h_0}{c_0 m} - \frac{d_1 (a_0 m \delta_0)}{2} \varepsilon M(a_1 + c_1)] z^2
\]

\[
\leq -2[b_0 q_0 - \delta_1 - \varepsilon M(a_1 + c_1)] z^2 \leq 0,
\]

and

\[
V_6 \leq -2[a a_0 m - 1] w^2 = -2 \varepsilon w^2 \leq 0.
\]

By taking \( h_1 = \max \{ |a a_0 m - 1|, |b_0 h_1| \} \), we have

\[
V_7 \leq d_1 h_1 r(t) [(\alpha w^2 + \beta y^2 + z^2) + \sigma r(t) y^2 + c_0 M a_1 (\alpha + \beta + 1) - \sigma (1 - \xi)] \int_{t-r(t)}^t y^2(s) ds
\]

If we choose \( \sigma = \frac{d_1 h_1 (\alpha + \beta + 1)}{1 - \xi} \), we obtain

\[
V_7 \leq \frac{d_1 h_1}{1 - \xi} r(t) [(1 - \xi) w^2 + (\alpha + \beta (2 - \xi) + 1) y^2 + (1 - \xi) z^2].
\]

Thus, there exists a positive constant \( D_3 \) such that

\[-\varepsilon c(t) f(x) y^2 + V_4 + V_5 + V_6 + V_7 \leq -2 D_3 (y^2 + z^2 + w^2).\]

From \([3.4]\), and the Cauchy Schwartz inequality, we obtain

\[
V_8 \leq a(t) \theta_1 [(z^2 + \alpha (z^2 + w^2)) + b(t) \theta_2 [(\alpha z^2 + \alpha (z^2 + w^2) + \beta y^2 + y^2 + z^2) + c(t) \theta_3 (y^2 + \alpha (y^2 + z^2))]
\]

\[
\leq \lambda_1 (\theta_1 + \theta_2 + \theta_3) (y^2 + z^2 + w^2 + H(x))
\]

\[
\leq \frac{2 \lambda_1}{D_0} (\theta_1 + \theta_2 + \theta_3) V,
\]

where \( \lambda_1 = \max \{ a_1 (1 + \alpha), b_1 (1 + 2 \alpha + \beta), c_1 (1 + \alpha) \} \). Using condition (iii) and Lemma \([3.1]\) we can write

\[
h^2(x) \leq h_0 H(x),
\]

hereby,

\[
|V_9| \leq |d^2(t)||2 \beta H(x) + a h_0 y^2 + h^2(x) + y^2 + \alpha (h^2(x) + z^2)|
\]

\[
+ \beta c^2(t)|y^2 + \alpha (y^2 + z^2)| + b(t)|\alpha z^2 + \beta y^2|
\]

\[
+ |a'(t)| |z^2 + 2 \beta (y^2 + z^2)|
\]

\[
\leq \lambda_2 (|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|) (y^2 + z^2 + w^2 + H(x))
\]

\[
\leq \frac{2 \lambda_2}{D_0} (|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|) V,
\]
such that \( \lambda_2 = \max\{2\beta + (\alpha + 1)h_0, \alpha h_0 + 1, \alpha + 1\} \). By taking \( \frac{1}{\eta} = \frac{1}{h_0} \max\{\lambda_1, \lambda_2\} \), we obtain

\[
\dot{V}_{1,2} \leq -D_3(y^2 + z^2 + w^2) + (\beta y + z + \alpha w)p(t, x, y, z, w) + \frac{1}{\eta}(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| + |\theta_1| + |\theta_2| + |\theta_3|)V.
\]

(3.10)

From (A4), (A5),(iii), (3.7), (3.8), (3.10) and the Cauchy Schwartz inequality, we obtain

\[
\dot{W}_{1,2} = \left(\dot{V}_{1,2} - \frac{1}{\eta}\gamma(t)V\right)e^{-\frac{1}{\eta}\int_0^t \gamma(s)ds}
\]

\[
\leq (-D_3(y^2 + z^2 + w^2) + (\beta y + z + \alpha w)p(t, x, y, z, w))e^{-\frac{1}{\eta}\int_0^t \gamma(s)ds}
\]

\[
\leq (\beta|y| + |z| + \alpha|w|)|p(t, x, y, z, w)|
\]

\[
\leq D_4(|y| + |z| + |w|)|e(t)|
\]

\[
\leq D_4(3 + y^2 + z^2 + w^2)|e(t)|
\]

\[
\leq D_4(3 + \frac{1}{D_2}W)|e(t)|
\]

\[
\leq 3D_4|e(t)| + \frac{D_4}{D_2}W|e(t)|,
\]

(3.12)

where \( D_4 = \max\{\alpha, \beta, 1\} \). Integrating (3.12) from 0 to \( t \) and using the condition (iv) and the Gronwall inequality, we have

\[
W \leq W(0, x(0), y(0), z(0), w(0)) + 3D_4\eta_3
\]

\[
+ \frac{D_4}{D_2} \int_0^t W(s, x(s), y(s), z(s), w(s))|e(s)|ds
\]

\[
\leq (W(0, x(0), y(0), z(0), w(0)) + 3D_4\eta_3)e^{\frac{D_4}{D_2}\int_0^t |e(s)|ds}
\]

\[
\leq (W(0, x(0), y(0), z(0), w(0)) + 3D_4\eta_3)e^{\frac{D_4}{D_2}\eta_1} = K_1 < \infty
\]

(3.13)

Because of inequalities (3.8) and (3.13), we write

\[
(x^2 + y^2 + z^2 + w^2) \leq \frac{1}{D_2} W \leq K_2,
\]

(3.14)

where \( K_2 = \frac{K_1}{D_2} \). Clearly (3.14) implies

\[
|x(t)| \leq \sqrt{K_2}, \quad |y(t)| \leq \sqrt{K_2}, \quad |z(t)| \leq \sqrt{K_2}, \quad |w(t)| \leq \sqrt{K_2} \quad \text{for } t \geq 0.
\]

Hence

\[
|x(t)| \leq \sqrt{K_2}, \quad |x'(t)| \leq \sqrt{K_2}, \quad |x''(t)| \leq \sqrt{K_2}, \quad |x'''(t)| \leq \sqrt{K_2}
\]

(3.15)

for \( t \geq 0 \). Now, we proof the square integrability of solutions and their derivatives. We define

\[
F_1 = F(t, x(t), y(t), z(t), w(t)) = W + \rho \int_0^t (y^2(s) + z^2(s) + w^2(s))ds,
\]

where \( \rho > 0 \). It is easy to see that \( F_1 \) is positive definite, since \( W = W(t, x, y, z, w) \) is already positive definite. Using the following estimate

\[
e^{-\frac{2\beta + (\alpha + 1)h_0}{\eta}} \leq e^{-\frac{1}{\eta}\int_0^t \gamma(s)ds} \leq 1
\]
by (3.12) we have
\[
\dot{\mathcal{F}}_{t}(1.2) \leq -D_{3}(y(t)^{2} + z(t)^{2} + w(t)^{2})e^{-\eta_{1} - \eta_{2} \frac{t}{\eta}} + D_{4}(|y(t)| + |z(t)| + |w(t)|)|e(t)|
+ \rho(y(t)^{2} + z(t)^{2} + w(t)^{2})
\]  
(3.16)

By choosing \(\rho = D_{3}e^{-\eta_{1} - \eta_{2} \frac{t}{\eta}}\) we obtain
\[
\dot{\mathcal{F}}_{t}(1.2) \leq D_{4}(3 + y(t)^{2} + z(t)^{2} + w(t)^{2})|e(t)|
\leq 3D_{4}|e(t)| + \frac{D_{4}}{D_{2}}\mathcal{F}_{t}|e(t)|.
\]  
(3.17)

Integrating from 0 to \(t\) and using again the Gronwall inequality and the condition (iv), we obtain
\[
\mathcal{F}_{t} \leq \mathcal{F}_{0} + 3D_{4}\eta_{3} + \frac{D_{4}}{D_{2}} \int_{0}^{t} \mathcal{F}_{s}|e(s)|ds
\leq (\mathcal{F}_{0} + 3D_{4}\eta_{3})e^{\frac{D_{4}}{D_{2}} \int_{0}^{t} |e(s)|ds}
\leq (\mathcal{F}_{0} + 3D_{4}\eta_{3})e^{\frac{D_{4}}{D_{2}} \mathcal{F}_{t}} = K_{3} < \infty
\]  
(3.18)

Therefore,
\[
\int_{0}^{\infty} y^{2}(s)ds < K_{3}, \quad \int_{0}^{\infty} z^{2}(s)ds < K_{3}, \quad \int_{0}^{\infty} w^{2}(s)ds < K_{3},
\]
which implies
\[
\int_{0}^{\infty} |x'(s)|^{2}ds < K_{3}, \quad \int_{0}^{\infty} |x''(s)|^{2}ds < K_{3}, \quad \int_{0}^{\infty} |x'''(s)|^{2}ds < K_{3}.
\]  
(3.19)

which completes the proof. \( \square \)

Remark 3.3. If \(p(t, x, y, z, w) \equiv 0\), similarly to the above proof, the inequality (3.11) becomes
\[
\dot{W}_{t}(1.2) = \left(\dot{\gamma}(t) \frac{1}{\eta} \gamma(t) V e^{-\eta \int_{0}^{t} \gamma(s)ds} - \frac{1}{\eta} \dot{f}_{0}^{t} \gamma(s)ds\right)
\leq -D_{3}(y^{2} + z^{2} + w^{2})e^{-\eta_{1} \frac{t}{\eta}} + \frac{D_{4}}{D_{2}}\mathcal{F}_{t}|e(t)|
\leq -\mu(y^{2} + z^{2} + w^{2}),
\]
where \(\mu = D_{3}e^{-\eta_{1} \frac{t}{\eta}}\). It can also be observed that the only solution of system (1.2) for which \(\dot{W}_{t}(1.2) = 0\) is the solution \(x = y = z = w = 0\). The above discussion guarantees that the trivial solution of equation (1.1) is uniformly asymptotically stable, and the same conclusion as in the proof of theorem can be drawn for square integrability of solutions of equation (1.1).
Example 3.4. We consider the fourth-order nonlinear differential equation with delay

\[ x^{(4)} + (e^{-2t}\sin 3t + 2) \left( \frac{5x + 2e^x + 2e^{-x}}{e^x + e^{-x}} \right)' + \frac{\sin 2t + 11t^2 + 11}{t^2 + 1} \left( \frac{\sin x + 9e^x + 9e^{-x}}{e^x + e^{-x}} \right)' + (e^{-t}\sin t + 3) \left( \frac{x\cos x + x^4 + 1}{x^4 + 1} \right)' + \frac{\sin^2 t + t^2 + 1}{5t^2 + 5} \left( \frac{x(t - \frac{1}{\sqrt{t+15}})}{x^2(t - \frac{1}{\sqrt{t+15}}) + 1} \right) \]

\[ = \frac{2\sin t}{t^2 + 1 + (xx'x'')^2 + (x''')^2} \]

by taking

\[ g(x) = \frac{5x + 2e^x + 2e^{-x}}{e^x + e^{-x}} \quad q(x) = \frac{\sin x + 9e^x + 9e^{-x}}{e^x + e^{-x}} \quad f(x) = \frac{x\cos x + x^4 + 1}{x^4 + 1} \]

\[ h(x) = \frac{x}{x^2 + 1}, \quad a(t) = e^{-2t}\sin 3t + 2, \quad b(t) = \frac{\sin 2t + 11t^2 + 11}{t^2 + 1}, \]

\[ c(t) = e^{-t}\sin t + 3, \quad d(t) = \frac{\sin^2 t + t^2 + 1}{5t^2 + 5}, \quad r(t) = \frac{1}{e^t + 15}, \]

\[ p(t, x, x', x'', x''') = \frac{2\sin t}{t^2 + 1 + (xx'x'')^2 + (x''')^2}. \]

We obtain \( g_0 = 0.33, \quad g_1 = 3.7, \quad f_0 = 0.5, \quad f_1 = 1.5, \quad q_0 = 8.5, \quad q_1 = 9.5, \quad a_0 = 1, \quad a_1 = 3, \quad b_0 = 10, \quad b_1 = 12, \quad c_0 = 2, \quad c_1 = 4, \quad d_0 = 0.2, \quad d_1 = 0.3, \quad m = 0.3, \quad M = 3.8, \quad h_0 = 2, \quad \alpha = \frac{23}{6}, \quad \beta = \frac{3}{2}, \quad \delta_0 = \frac{17}{8} \quad \text{and} \quad \delta_1 = 69.15. \]

Also we have

\[ \int_{-\infty}^{\infty} |g'(x)|dx = 5 \int_{-\infty}^{\infty} \left| \frac{1}{e^x + e^{-x}} + x \frac{e^{-x} - e^x}{(e^x + e^{-x})^2} \right| dx \]

\[ \leq 5 \int_{-\infty}^{0} \left| \frac{1}{e^x + e^{-x}} - x \frac{e^{-x} - e^x}{(e^x + e^{-x})^2} \right| dx + 5 \int_{0}^{\infty} \left| \frac{1}{e^x + e^{-x}} - x \frac{e^{-x} - e^x}{(e^x + e^{-x})^2} \right| dx = 5\pi, \]

\[ \int_{-\infty}^{\infty} |g'(x)|dx = \int_{-\infty}^{\infty} \left| \frac{(e^x + e^{-x}) \cos x - (e^x - e^{-x}) \sin x}{(e^x + e^{-x})^2} \right| dx \]

\[ \leq \int_{-\infty}^{\infty} \left| \frac{1}{e^x + e^{-x}} + x \frac{e^{-x} - e^x}{(e^x + e^{-x})^2} \right| dx = \pi, \]

\[ \int_{-\infty}^{\infty} |f'(x)|dx = \int_{-\infty}^{\infty} \left| \frac{\cos x}{x^4 + 1} - 4x^4 \frac{\cos x}{(x^4 + 1)^2} - x \frac{\sin x}{x^4 + 1} \right| dx \]

\[ \leq \int_{-\infty}^{\infty} \left| \frac{5}{x^4 + 1} + \frac{x^2}{x^4 + 1} \right| dx = 6\sqrt{2}\pi, \]

\[ \int_{0}^{\infty} \left| p(t, x, x', x'', x''') \right| dt = \int_{0}^{\infty} \frac{2\sin t}{t^2 + 1 + (xx'x'')^2 + (x''')^2} dt \]
\[
\int_0^\infty |a'(t)|dt = \int_0^\infty \left| -2e^{-2t} \sin 3t + 3e^{-2t} \cos 3t \right|dt \\
\leq \int_0^\infty 5e^{-2t}dt = \frac{5}{2},
\]

\[
\int_0^\infty |b'(t)|dt = \int_0^\infty \left| \frac{2 \cos 2t}{t^2 + 1} - 2t \frac{\sin 2t}{(t^2 + 1)^2} \right|dt \\
\leq \int_0^\infty 3 \frac{dt}{t^2 + 1} = \frac{3\pi}{2},
\]

\[
\int_0^\infty |c'(t)|dt = \int_0^\infty \left| -e^{-t} \sin t + e^{-t} \cos t \right|dt \\
\leq \int_0^\infty 2e^{-t}dt = 2,
\]

\[
\int_0^\infty |d'(t)|dt = \int_0^\infty \left| \frac{2 \sin t \cos t}{5t^2 + 5} - 2t \frac{\sin^2 t}{(5t^2 + 5)^2} \right|dt \\
\leq \frac{11}{25} \int_0^\infty \frac{1}{t^2 + 1}dt = \frac{11\pi}{50}.
\]

Consequently
\[
\int_{-\infty}^{+\infty} \left( |g'(s)| + |q'(s)| + |f'(s)| \right)ds < \infty,
\]
\[
\int_0^\infty \left( |a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| \right)dt < \infty.
\]

Thus all the assumptions of Theorem 3.2 hold, this shows that every solutions of equation \((3.20)\) are bounded and derivatives of solutions are square integrable.

**Conclusion.** A class of nonlinear retarded functional differential equations of fourth order is considered. Sufficient conditions are established guaranteeing the uniformly asymptotic stability of the solutions for \( p(t, x, x', x'', x''') = 0 \) and also square integrable and boundedness of solutions of equation \((1.1)\) with delay. In the proofs of the main results, we benefit from the Lyapunov functional approach. The results obtained essentially improve, include and complement the results in the literature. An example is furnished to illustrate the hypotheses by MATLAB-Simulink.
The asymptotically stability of the null solution for the mentioned differential equation is shown by the following graph.

![Graph showing trajectory of x(t) for Example.](image)

**Figure 1. Trajectory of x(t) for Example.**

The boundedness of all the solutions for the mentioned differential equation is shown by the following graph.

![Graph showing boundedness of solutions for Example.](image)

**Figure 2. Trajectory of x(t) for Example.**

**REFERENCES**


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