EXISTENCE OF GLOBAL SOLUTIONS AND DECAY ESTIMATES FOR A VISCOELASTIC PETROVSKY EQUATION WITH A DELAY TERM IN THE NON-LINEAR INTERNAL FEEDBACK

NADIA MEZOUAR, MAMA ABDELLI, AMIRA RACHAH

Abstract. In this article we consider a nonlinear viscoelastic Petrovsky equation in a bounded domain with a delay term in the weakly nonlinear internal feedback:

\[ |u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\,ds + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0. \]

We prove the existence of global solutions in suitable Sobolev spaces by using the energy method combined with Faedo-Galarkin method under condition on the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we study general stability estimates by using some properties of convex functions.

1. Introduction

1.1. The model. In this article we consider the existence and decay properties of global solutions for the initial boundary value problem of viscoelastic Petrovsky equation

\[ |u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s)\Delta^2 u(s)\,ds + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau)) = 0 \quad \text{in } \Omega \times ]0,\infty[. \]

\[ u(x,t) = 0 \quad \text{on } \partial \Omega \times ]0,\infty[, \]

\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \]

\[ u_t(x,t-\tau) = f_0(x,t-\tau) \quad \text{in } \Omega \times ]0,\tau[, \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), \( n \in \mathbb{N}^* \), \( \partial \Omega \) is a smooth boundary, \( t > 0 \), \( \mu_1 \) and \( \mu_2 \) are positive real numbers, \( h \) is a positive non-increasing function defined on \( \mathbb{R}^+ \), \( g_1 \) and \( g_2 \) are two functions, \( \tau > 0 \) is a time delay and \( (u_0, u_1, f_0) \) are the initial data in a suitable function space. Cavalcanti et al. [10] studied the following

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nonlinear viscoelastic problem with strong damping
\[ |u_t|^l u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds - \gamma \Delta u_t = 0, \quad x \in \Omega, \quad t > 0. \tag{1.2} \]

Under the assumptions \(0 < l \leq \frac{2}{n-2}\) if \(n \geq 3\) or \(l > 0\) if \(n = 1, 2\) and \(h\) decays exponentially, they obtained the global existence of weak solutions for \(\gamma \geq 0\) and the uniform exponential decay rates of the energy for \(\gamma > 0\). In the case of \(\gamma = 0\) when a source term competes with the dissipation induced by the viscoelastic term, Messaoudi and Tatar [22] studied the equation
\[ |u_t|^l u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds + b|u|^{p-2}u = 0, \quad x \in \Omega, \quad t > 0. \]

They used the potential well method to show that the damping induced by the viscoelastic term is enough to ensure global existence and uniform decay of solutions provided that the initial data are in some stable set. Han and Wang [15], investigated a related problem with linear damping
\[ |u_t|^l u_{tt} - \Delta u - \Delta u_{tt} - \int_0^t h(t-s)\Delta u(s)\,ds + u_t = 0, \quad x \in \Omega, \quad t > 0. \]
Using the Faedo-Galerkin method, they showed the global existence of weak solutions and obtained uniform exponential decay of solutions by introducing a perturbed energy functional. Recently, these results have been extended by Wu [32] to a general case where a source term and a nonlinear damping term are present.

In the presence of the source term, problem (1.2) has been discussed by many authors, and related results concerning local or global existence, asymptotic behavior and blow-up of solution have been recently established (see [4, 20, 23]).

Park and Kang [26] studied the following nonlinear viscoelastic problem with damping
\[ |u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - M(\|\nabla u\|^2_2)\Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + u_t = 0, \quad x \in \Omega, \quad t > 0. \]

Santos et al. [27] considered the existence and uniform decay for the following nonlinear beam equation in a non-cylindrical domain:
\[ u_{tt} + \Delta^2 u - M(\|\nabla u\|^2_2)\Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + \alpha u_t = 0, \quad \text{in} \ \hat{Q}, \]
where \(\hat{Q} = \cup_{0 \leq t \leq \infty} \Omega_t \times \{t\}\). Benaissa, Benguessoum and Messaoudi [6] proved the existence of global solution, as well as, a general stability result for the equation
\[ u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + \mu_1 g_1(u_t(x, t)) + \mu_2 g_2(u(t, x, t - \tau)) = 0, \tag{1.3} \]
for \(x \in \Omega\) and \(t > 0\), when \(h\) is decays at a certain rate.

In the absence of the viscoelastic term (i.e. if \(h \equiv 0\)), problem (1.3) has been studied by many authors. It is well known that in the further absence of a damping mechanism, the delay term causes instability of the system (see, for instance, Datko et al. [11]). On the contrary, in the absence of the delay term, the damping term assures global existence for arbitrary initial data and energy decay is estimated depending on the rate of growth of \(g_1\) (see Alabau-Boussouira, [3], Benaissa and Guesmia [8], Haraux [13], Komornik [16], Lasiecka and Tataru [18]).
Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see Shinskey [26]). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. Time delays so often arise in many physical, chemical, biological, and economic phenomena. In recent years, the control of PDEs with time delay effects has become an active area of research (see Abdallah et al. [2], Suh and Bien [29] and Zhong [31]). To stabilize a hyperbolic system involving delay terms, additional control terms are necessary (see Nicaise and Pignotti [24], Nicaise and Pignotti [25], Xu et al. [11]). In Nicaise and Pignotti [24], the authors examined the problem (P) in the linear situation (i.e. if $g_1(s) = g_2(s) = s$ for all $s \in \mathbb{R}$) and determined suitable relations between $\mu_2$ and $\mu_1$, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the corresponding solution of (1.3) will be instable if $\mu_2 \geq \mu_1$. The main approach used in Nicaise and Pignotti [24] is an observability inequality obtained with a Carleman estimate. The same results were obtained if both the damping and the delay were acting in the boundary domain. We also recall the result by Xu et al. [30], where the authors proved the same result as in Nicaise and Pignotti [24] for the one space dimension by adopting the spectral analysis approach. Very recently, Benaisa and Louhibi [7] extended the result of Nicaise and Pignotti [24] to the non-linear case.

Datko et al. [11] showed that a small delay in a boundary control could turn such well-beake hyperbolic system into a wild one and therefore, delay becomes a source of instability. However, sometimes it can also improve the performance of the systems (see Suh and Bien [29]).

The main purpose of this paper is to prove global solvability and energy decay estimates of the solutions of problem (1.1) when $h$ is of exponential decay rate and $g_1, g_2$ are non-linear. We would like to see the influence of frictional and viscoelastic damping on the rate of decay of solutions in the presence of non-linear degenerate delay term. Of course, the most interesting case occurs when we have delay term and simultaneous and complementary damping mechanisms.

To obtain global solutions of problem (1.1), we use the Galerkin approximation scheme (see Lions [19]) together with the energy estimate method. The technique based on the theory of non-linear semi-groups used in Nicaise and Pignotti [24] does not seem to be applicable in the non-linear case.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by Cavalcanti et al. [9], Daoulati et al. [12], Lasiecka and Doundykov [17] and Lasiecka and Tataru [18], and used by Liu and Zuazua [21], Eller et al. [13] and Alabau-Boussouira [3].

1.2. Statement of results. We use the Sobolev spaces $H^4(\Omega)$, $H^3_0(\Omega)$ and the Hilbert space $L^p(\Omega)$ with their usual scalar products and norms. The prime ' and the subscript $t$ will denote time differentiation and we denote by $(\cdot, \cdot)$ the inner product in $L^2(\Omega)$. The constant $C$ denotes a general positive constant, which may be different in different estimates. Now we introduce, as in the work of in Nicaise and Pignotti [24], the new variable

$$z(x, \rho, t) = u_1(x, t - \tau \rho), \quad x \in \Omega, \ \rho \in (0,1), \ t > 0.$$
Then, we have
\[ \tau z_t(x, \rho, t) + z_{\rho}(x, \rho, t) = 0, \quad \text{in } \Omega \times (0, 1) \times (0, +\infty). \] (1.4)

Therefore, problem (1.1) is equivalent to
\begin{align*}
|u_t|^u_{tt} + \Delta^2 u - \Delta u_{tt} & - \int_0^t h(t - s) \Delta^2 u(s) \, ds \\
+ \mu_1 g_1(u_t(x, t)) + \mu_2 g_2(z(x, 1, t)) = 0 & \quad \text{in } \Omega \times [0, +\infty[,
\end{align*}
\[ \tau z_t(x, \rho, t) + z_{\rho}(x, \rho, t) = 0, \quad \text{in } \Omega \times [0, +\infty[, \quad u(x, t) = 0, \quad \text{on } \partial \Omega \times [0, +\infty[, \]
\[ z(x, 0, t) = u_t(x, t), \quad \text{on } \Omega \times [0, +\infty[, \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \text{in } \Omega, \]
\[ z(x, \rho, 0) = f_0(x, -\rho t), \quad \text{in } \Omega \times [0, 1[. \]

To state and prove our result, we use the following assumptions:
(A1) Assume that \( l \) satisfies
\begin{align*}
0 < l & \leq \frac{2}{n - 2} \quad \text{if } n \geq 3 \\
0 < l & < \infty \quad \text{if } n = 1, 2;
\end{align*}
(A2) \( g_1 : \mathbb{R} \rightarrow \mathbb{R} \) is non decreasing function of class \( C^1 \) and \( H : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is convex, increasing and of class \( C^1(\mathbb{R}_+) \cap C^2([0, +\infty[) \) satisfying
\[ H(0) = 0 \quad \text{and } H \text{ is linear on } [0, \varepsilon] \text{ or} \]
\[ H'(0) = 0 \quad \text{and } H'' > 0 \text{ on } [0, \varepsilon] \text{ such that} \]
\[ |g_1(s)| \leq c_2 |s| \quad \text{if } |s| \geq \varepsilon \]
\[ g_1^2(s) \leq H^{-1}(sg_1(s)) \quad \text{if } |s| \leq \varepsilon, \]
where \( H^{-1} \) denotes the inverse function of \( H \) and \( \varepsilon, c_2 \) are positive constants. \( g_2 : \mathbb{R} \rightarrow \mathbb{R} \) is an odd no decreasing function of class \( C^1(\mathbb{R}) \) such that there exist \( c_3, \alpha_1, \alpha_2 > 0, \)
\[ |g_2'(s)| \leq c_3, \]
\[ \alpha_1 s g_2(s) \leq G(s) \leq \alpha_2 s g_1(s), \]
where \( G(s) = \int_0^s g_2(r) \, dr; \)
(A3) \( \alpha_2 \mu_2 < \alpha_1 \mu_1; \)
(A4) For the relaxation function \( h : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a bounded \( C^1 \) function such that
\[ \int_0^\infty h(s) \, ds = \beta < 1, \]
and we assume that there exist a positive constant \( \zeta \) satisfying
\[ h'(t) \leq -\zeta h(t). \]

We define the energy associated with the solution of system (1.5) by
\[ E(t) = \frac{1}{t + 2} \|u_t\|_{L^2}^2 + \frac{1}{2} \left( 1 - \int_0^t h(s) \, ds \right) \|\Delta u\|_2^2 + \frac{1}{2} \|\nabla u_t\|_2^2 \\
+ \frac{1}{2} (h \circ \Delta u)(t) + \xi \int_\Omega \int_0^1 G(z(x, \rho, t)) \, d\rho \, dx, \]
(1.11)
where $\xi$ is a positive constant such that
\[
\frac{\tau \mu_2 (1 - \alpha_1)}{\alpha_1} < \xi < \frac{\tau \mu_1 - \alpha_2 \mu_2}{\alpha_2},
\]
\[
(h \circ v)(t) = \int_0^t h(t - s)\|v(t) - v(s)\|_2^2 \, ds.
\]

Now we have the existence of a global solution.

**Theorem 1.1.** Let $u_0 \in H^4(\Omega) \cap H^2_0(\Omega)$, $u_1 \in H^2_0(\Omega)$ and $f_0 \in H^2_0(\Omega, H^2(0, 1))$ satisfy the compatibility condition $f(\cdot, 0) = u_1$. Assume that (A1)-(A4) hold. Then (1.1) admits a weak solution $u \in L^\infty([0, \infty); H^4(\Omega) \cap H^2_0(\Omega))$, $u_t \in L^\infty([0, \infty); H^2_0(\Omega))$, $u_{tt} \in L^2([0, \infty); H^1_0(\Omega))$.

Also we have a uniform decay rates for the energy.

**Theorem 1.2.** Assume that (A1)-(A4) hold. Then, there exist positive constants $w_1, w_2, w_3$ and $\varepsilon_0$ such that the solution of (1.1) satisfies
\[
E(t) \leq w_3 H^{-1}_1(w_1 t + w_2) \quad \forall t > 0,
\]
where
\[
H_1(t) = \int_t^1 \frac{1}{H_2(s)} \, ds, \quad (1.12)
\]
\[
H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0, \varepsilon] \\ t H'(\varepsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } ]0, \varepsilon[, \end{cases}
\]
here, $H_1$ is strictly decreasing and convex on $(0, 1]$ with $\lim_{t \to 0} H_1(t) = +\infty$.

### 2. Preliminaries

Let $\lambda_1$ be the first eigenvalue of the spectral Dirichlet problem
\[
\Delta^2 u = \lambda_1 u, \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \eta} = 0 \quad \text{in } \Gamma,
\]
\[
\|\nabla u\|_2 \leq \frac{1}{\sqrt{\lambda_1}} \|\Delta u\|_2. \quad (2.1)
\]

Next we have a Sobolev-Poincaré inequality [1].

**Lemma 2.1.** Let $q$ be a number with $2 \leq q < +\infty(n = 1, 2)$ or $2 \leq q \leq 2n/(n - 2)(n \geq 3)$, then there exists a constant $C_q = C_q(\Omega, q)$ such that
\[
\|u\|_q \leq C_q \|\nabla u\|_2 \quad \text{for } u \in H^1_0(\Omega).
\]

**Lemma 2.2.** For $h, \Psi \in C^1([0, +\infty[; \mathbb{R})$ we have
\[
\int_{\Omega} h \ast \Psi \, dx = -\frac{1}{2} h(t)\|\Psi(t)\|_2^2 + \frac{1}{2} (h' \circ \Psi)(t) - \frac{1}{2} \int_0^t \frac{d}{dt} \left[(h \circ \Psi)(t) - \left(\int_0^t h(s) \, ds\right)\|\Psi\|_2^2\right].
\]
Remark 2.3. Let us denote by $\Phi^*$ the conjugate function of the differentiable convex function $\Phi$, i.e.,
$$
\Phi^*(s) = \sup_{t \in \mathbb{R}^+} (st - \Phi(t)).
$$

Then $\Phi^*$ is the Legendre transform of $\Phi$, which is given by (see Arnold [5] p. 61-62)
$$
\Phi^*(s) = s(\Phi')^{-1}(s) - \Phi([\Phi']^{-1}(s)), \quad \text{if } s \in (0, \Phi'(r)],
$$
and $\Phi^*$ satisfies the generalized Young inequality
$$
AB \leq \Phi^*(A) + \Phi(B), \quad \text{if } A \in (0, \Phi'(r)], \ B \in (0, r]. \quad (2.2)
$$

Lemma 2.4. Let $(u, z)$ be a solution of the problem (1.5). Then, the energy functional defined by (1.11) satisfies
$$
E'(t) \leq \beta_1 \int_{\Omega} u_t g_1(u_t) \, dx - \beta_2 \int_{\Omega} z(x, 1, t)g_2(z(x, 1, t)) \, dx
- \frac{1}{2} h(t)\|\Delta u(t)\|^2 + \frac{1}{2} (h' \circ \Delta u)(t) \leq 0,
$$
where $\beta_1 = \mu_1 - \frac{\xi \omega_2}{\tau} - \mu_2 \alpha_2$ and $\beta_2 = \frac{\xi \alpha_1}{\tau} - \mu_2 (1 - \alpha_1)$.

Proof. By multiplying the first equation in (1.5) by $u_t$, integrating over $\Omega$ and using integration by parts, we obtain
$$
\frac{d}{dt} \left[ \frac{1}{2} \|u_t\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\Delta u\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u_t\|_{L^2(\Omega)}^2 \right] = \mu_1 \int_{\Omega} u_t(x, t)g_1(u_t(x, t)) \, dx
+ \mu_2 \int_{\Omega} u_t(x, t)g_2(z(x, 1, t)) \, dx
\quad (2.4)
$$
$$
= \int_{\Omega} \int_0^t h(t - s)\Delta u(s)\Delta u_t(t) \, ds \, dx.
$$

By applying the Lemma 2.2, the term on the right-hand side of (2.4) can be rewritten as
$$
\int_{\Omega} \int_0^t h(t - s)\Delta u(s)\Delta u_t(t) \, ds \, dx + \frac{1}{2} h(t)\|\Delta u(t)\|_{L^2(\Omega)}^2
\quad = \frac{1}{2} \frac{d}{dt} \left[ \int_0^t h(s) \, ds \|\Delta u(t)\|_{L^2(\Omega)}^2 - (h \circ \Delta u)(t) \right] + \frac{1}{2} (h' \circ \Delta u)(t).
$$

Consequently, (2.4) becomes
$$
\frac{d}{dt} \left[ \frac{1}{2} \|u_t\|_{L^2(\Omega)}^2 + \frac{1}{2} \left( 1 - \int_0^t h(s) \, ds \right) \|\Delta u\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u_t\|_{L^2(\Omega)}^2 + \frac{1}{2} (h \circ \Delta u)(t) \right]
\quad = -\mu_1 \int_{\Omega} u_t(x, t)g_1(u_t(x, t)) \, dx - \mu_2 \int_{\Omega} u_t(x, t)g_2(z(x, 1, t)) \, dx
\quad - \frac{1}{2} h(t)\|\Delta u(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} (h' \circ \Delta u)(t) \quad (2.5)
$$

We multiply the second equation in (1.5) by $\xi g_2(z)$, we integrate the result over $\Omega \times (0, 1)$, to obtain
$$
\xi \int_{\Omega} \int_0^1 z_t(x, \rho, t)g_2(z(x, \rho, t)) \, d\rho \, dx = -\frac{\xi}{\tau} \int_{\Omega} \int_0^1 z_{\rho}(x, \rho, t)g_2(z(x, \rho, t)) \, d\rho \, dx
\quad = -\frac{\xi}{\tau} \int_{\Omega} \int_0^1 \frac{\partial}{\partial \rho} \left( G(z(x, \rho, t)) \right) \, d\rho \, dx
$$
and by recalling (1.8), we obtain

\[ H = -\frac{\xi}{\tau} \int_\Omega (G(z(x,1,t)) - G(z(x,0,t))) \, dx. \]

Hence

\[ \frac{d}{dt} \int_\Omega \int_0^1 G(z(x,\rho,t)) \, d\rho \, dx = -\frac{\xi}{\tau} \int_\Omega G(z(x,1,t)) \, dx + \frac{\xi}{\tau} \int_\Omega G(u_t(x,t)) \, dx. \quad (2.6) \]

By combining (2.5) and (2.6), we obtain

\[ E'(t) = -\frac{1}{2} h(t) \|\Delta u(t)\|_2^2 + \frac{1}{2} (h' \circ \Delta u)(t) - \mu_1 \int_\Omega u_t(x,t)g_1(u_t(x,t)) \, dx \]

\[ - \mu_2 \int_\Omega u_t(x,t)g_2(z(x,1,t)) \, dx - \frac{\xi}{\tau} \int_\Omega G(z(x,1,t)) \, dx + \frac{\xi}{\tau} \int_\Omega G(u_t(x,t)) \, dx, \]

and by recalling (1.8), we obtain

\[ E'(t) \leq -\left( \mu_1 - \frac{\xi \alpha_2}{\tau} \right) \int_\Omega u_t(x,t)g_1(u_t(x,t)) \, dx - \frac{1}{2} h(t) \|\Delta u(t)\|_2^2 + \frac{1}{2} (h' \circ \Delta u)(t) \]

\[ + \mu_2 \int_\Omega (G(u_t(x,t)) + G^*(g_2(z(x,1,t)))) \, dx - \frac{\xi}{\tau} \int_\Omega G(z(x,1,t)) \, dx \]

\[ \leq -\left( \mu_1 - \frac{\xi \alpha_2}{\tau} - \mu_2 \alpha_2 \right) \int_\Omega u_t(x,t)g_1(u_t(x,t)) \, dx \]

\[ - \left( \frac{\xi \alpha_1}{\tau} - \mu_2 (1 - \alpha_1) \right) \int_\Omega z(x,1,t)g_2(z(x,1,t)) \, dx \]

\[ - \frac{1}{2} h(t) \|\Delta u(t)\|_2^2 + \frac{1}{2} (h' \circ \Delta u)(t) \leq 0. \]

This completes the proof. \( \square \)

3. Proofs of main results

3.1. Proof of Theorem 1.1

Throughout this section we assume \( u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), \( u_1 \in H^2_0(\Omega) \) and \( j_0 \in H^2(\Omega, H^2(0,1)) \). We will use the Faedo-Galerkin method to prove the existence of a global solution. Let \( T > 0 \) be fixed and let
The nonlinear terms in (3.1) are locally Lipschitz continuous. Note that \( u_t \), so that it can be extended outside \([0, \infty)\). We can find a positive constant \( C \) for all \( t > C \), \( t \in (3.1) \) makes sense. The standard theory of ODE guarantees that the system (3.1)-(3.5) has a unique solution in \([0, t_k)\), with \( 0 < t_k < T \), by Zorn lemma since the nonlinear terms in (3.1) are locally Lipschitz continuous. Note that \( u^k(t) \) is of class \( \mathcal{C}^2 \).

In the next step, we obtain a priori estimates for the solution of the system (3.1)-(3.5), so that it can be extended outside \([0, t_k)\) to obtain one solution defined for all \( t > 0 \), using a standard compactness argument for the limiting procedure. **First estimate.** Since the sequences \( u^k_0, u^k_1 \) and \( z^k_0 \) converge and from Lemma 2.4, we can find a positive constant \( C_1 \) independent of \( k \) such that

\[
E^k(t) - E^k(0) \leq -\beta_1 \int_0^t \int_\Omega u^k_t g_1(u^k_t) \, dx \, ds - \beta_2 \int_0^t \int_\Omega z^k(x, 1, s) g_2(z^k(x, 1, s)) \, dx \, ds
\]
\[-\frac{1}{2} \int_0^t h(s) \|\Delta u^k(s)\|^2 ds + \frac{1}{2} \int_0^t (h' \circ \Delta u^k)(s) \, ds \leq -\beta_1 \int_0^t \int_\Omega u^k g_1(u^k_t) \, dx \, ds - \beta_2 \int_0^t \int_\Omega z^k(x, 1, s) g_2(z^k(x, 1, s)) \, dx \, ds.\]

As \( h \) is a positive non increasing function, so we obtain
\[E^k(t) + \beta_1 \int_0^t \int_\Omega u^k g_1(u^k_t) \, dx \, ds + \beta_2 \int_0^t \int_\Omega z^k(x, 1, s) g_2(z^k(x, 1, s)) \, dx \, ds \leq E^k(0) \leq C_1,\]

where
\[
E^k(t) = \frac{1}{l + 2} \|u^k_t\|_{L^2}^2 + \frac{1}{2} \left(1 - \int_0^t h(s) \, ds\right) \|\Delta u^k\|_2^2 + \frac{1}{2} \|\nabla u^k\|_2^2 + \frac{1}{2} (h \circ \Delta u^k)(t) + \frac{1}{2} \int_\Omega \int_0^1 G(z^k(x, \rho, t)) \, d\rho \, dx,\]

and \( C_1 \) is a positive constant depending only on \( \|u_0\|_{H^2_0} \) and \( \|u_1\|_{H^2_0} \). Noting (1.9) and (3.6), we obtain the first estimate:
\[
\|u^k_t\|_{L^2}^2 + \|\Delta u^k\|_2^2 + \|\nabla u^k\|_2^2 + (h \circ \Delta u^k)(t) + \int_\Omega \int_0^1 G(z^k(x, \rho, t)) \, d\rho \, dx \leq C_2,\]

where \( C_2 \) is a positive constant depending only on \( \|u_0\|_{H^2_0} \), \( \|u_1\|_{H^2_0} \), \( l \), \( \beta \), \( \xi \), \( \tau \), \( \beta_1 \) and \( \beta_2 \). These estimates imply that the solution \((u^k, z^k)\) exists globally in \([0, +\infty)\).

Estimate (3.7) yields that
\[
\begin{align*}
    u^k & \text{ is bounded in } L^\infty_{loc}(0, \infty, H^2_0(\Omega)), \\
    u^k_t & \text{ is bounded in } L^\infty_{loc}(0, \infty, H^1_0(\Omega)), \\
    G(z^k(x, \rho, t)) & \text{ is bounded in } L^\infty_{loc}(0, \infty, L^1(\Omega \times (0, 1))), \\
    u^k_t(t) g_1(u^k_t(t)) & \text{ is bounded in } L^1(\Omega \times (0, T)), \\
    z^k(x, 1, t) g_2(z^k(x, 1, t)) & \text{ is bounded in } L^1(\Omega \times (0, T)).
\end{align*}
\]
Replacing $\phi^j$ by $-\Delta_x \phi^j$ in (3.4), multiplying by $d t^k$ and summing over $j$ from 1 to $k$, it follows that

$$\tau \int_{\Omega} \nabla_k z^k \nabla_k z^k \, dx + \int_{\Omega} \nabla_k z^k \nabla_k z^k \, dx = 0.$$  

Then, we obtain

$$\frac{\tau}{2} \frac{d}{dt} \|\nabla z^k\|^2_2 + \frac{1}{2} \frac{d}{d \rho} \|\nabla z^k\|^2_2 = 0.$$  

We integrate over $(0, 1)$ to find that

$$\frac{\tau}{2} \frac{d}{dt} \int_0^1 \|\nabla z^k(x, \rho, t)\|^2_2 \, d \rho + \frac{1}{2} \|\nabla z^k(x, 1, t)\|^2_2 - \frac{1}{2} \|\nabla u^k(t)\|^2_2 = 0. \quad (3.15)$$  

Combining (3.13)-(3.15) and using Lemma 2.2, we obtain

$$\frac{1}{2} \frac{d}{dt} \left[ \left(1 - \int_0^t h(s) \, ds\right) \|\nabla \Delta u^k\|^2_2 + \|\Delta u^k\|^2_2 + (h \circ \nabla \Delta u^k) \right]$$  

$$+ \tau \int_0^1 \|\nabla z^k(x, \rho, t)\|^2_2 \, d \rho + 2 \int_{\Omega} |u^k(t)|^2 |\nabla u^k| \, dx + \frac{1}{2} \|\nabla z^k(x, 1, t)\|^2_2$$  

$$= (l + 1) \int_{\Omega} |u^k| |\nabla u^k| \nabla u^k \, dx - \mu_1 \int_{\Omega} |\nabla u^k|^2 g^2(u^k) \, dx$$  

$$- \mu_2 \int_{\Omega} \nabla u^k \nabla z^k(x, 1, t) g^2(z^k(x, 1, t)) \, dx + \frac{1}{2} \|\nabla u^k\|^2_2$$  

$$- \frac{1}{2} h(t) \|\nabla \Delta u^k\|^2_2 + \frac{1}{2} (h' \circ \nabla \Delta u^k). \quad (3.16)$$  

From the first estimate (3.7) and Young’s inequality, we obtain

$$(l + 1) \int_{\Omega} |u^k| |\nabla u^k| \nabla u^k \, dx \leq (l + 1) C_2^{2/(l+2)+1/2} \|\nabla u^k\|_2$$  

$$\leq \eta \|\nabla u^k\|^2_2 + \frac{(l + 1)^2 C_2^{2/(l+2)+1}}{4 \eta}, \quad \eta > 0. \quad (3.17)$$  

By using (1.7), (3.7) and Young’s inequality, we obtain

$$\mu_2 \int_{\Omega} \nabla u^k \nabla z^k(x, 1, t) g^2(z^k(x, 1, t)) \, dx$$  

$$\leq \eta \|\nabla z^k(x, 1, t)\|^2_2 + \frac{1}{4 \eta} \|\nabla u^k\|^2_2 \quad (3.18)$$  

Taking into account (3.17), (3.18) into (3.16) yields

$$\frac{1}{2} \frac{d}{dt} \left[ \left(1 - \int_0^t h(s) \, ds\right) \|\nabla \Delta u^k\|^2_2 + \|\Delta u^k\|^2_2 + (h \circ \nabla \Delta u^k) \right]$$  

$$+ \tau \int_0^1 \|\nabla z^k(x, \rho, t)\|^2_2 \, d \rho + 2 \int_{\Omega} |u^k(t)|^2 |\nabla u^k| \, dx$$  

$$+ \mu_1 \int_{\Omega} |\nabla u^k|^2 g^2(u^k) \, dx + \left( \frac{1}{2} - \eta \right) \|\nabla z^k(x, 1, t)\|^2_2$$  

$$\leq \eta \|\nabla u^k\|^2_2 - \frac{1}{2} h(t) \|\nabla \Delta u^k\|^2_2 + \frac{1}{2} (h' \circ \nabla \Delta u^k) + C_2(\eta). \quad (3.19)$$
Multiplying (3.1) by \( c_{ij}^{jk} \) and summing over \( j \) from 1 to \( k \), it follows that

\[
\int_{\Omega} |u_t^j|^2 |u_t^k|^2 \, dx + \| \nabla u_t^k \|_2^2 \\
= - \int_{\Omega} \Delta^2 u^k u_t^k \, dx + \int_0^t h(t-s) \int_{\Omega} \Delta u^k(s) \Delta u_t^k(t) \, dx \, ds \\
- \mu_1 \int_{\Omega} u_t^k g_1(u_t^k) \, dx - \mu_2 \int_{\Omega} u_t^k g_2(z^k(x,1,t)) \, dx.
\]  \( (3.20) \)

Differentiating (3.4) with respect to \( t \), we obtain

\[
(\tau z_t^k + z_{i\nu}^k) = 0,
\]

Multiplying by \( d_t^j k \) and summing over \( j \) from 1 to \( k \), it follows that

\[
\frac{\tau}{2} \frac{d}{dt} \| z_t^k \|_2^2 + \frac{1}{2} \frac{d}{dt} \| z_t^k \|_2^2 = 0,
\]

Integrating over \((0,1)\) with respect to \( \rho \), we obtain

\[
\frac{\tau}{2} \frac{d}{dt} \int_0^1 \| z_t^k \|_2^2 \, d\rho + \frac{1}{2} \frac{d}{dt} \| z_t^k(x,1,t) \|_2^2 - \frac{1}{2} \| u_t^k(x,t) \|_2^2 = 0. \tag{3.21} \]

Summing (3.20) and (3.21), we obtain

\[
\int_{\Omega} |u_t^j|^2 |u_t^k|^2 \, dx + \| \nabla u_t^k \|_2^2 + \frac{\tau}{2} \frac{d}{dt} \int_0^1 \| z_t^k \|_2^2 \, d\rho + \frac{1}{2} \| z_t^k(x,1,t) \|_2^2 \\
= - \int_{\Omega} \Delta^2 u^k u_t^k \, dx + \int_0^t h(t-s) \int_{\Omega} \Delta u^k(s) \Delta u_t^k(t) \, dx \, ds \tag{3.22} \\
+ \frac{1}{2} \| u_t^k(x,t) \|_2^2 - \mu_1 \int_{\Omega} u_t^k g_1(u_t^k) \, dx - \mu_2 \int_{\Omega} u_t^k g_2(z^k(x,1,t)) \, dx.
\]

By using Young’s inequality, the right hand side of (3.22) can be estimated as follows:

\[
\int_{\Omega} \Delta^2 u^k u_t^k \, dx \leq \eta \| \nabla u_t^k \|_2^2 + \frac{1}{4\eta} \| \nabla \Delta u^k \|_2^2, \quad \eta > 0, \tag{3.23}
\]

and

\[
\int_0^t h(t-s) \int_{\Omega} \Delta u^k(s) \Delta u_t^k(t) \, dx \, ds \\
= - \int_0^t h(t-s) \int_{\Omega} \nabla \Delta u^k(s) \nabla u_t^k(t) \, dx \, ds \\
\leq \eta \| \nabla u_t^k \|_2^2 + \frac{1}{4\eta} \int_{\Omega} \left( \int_0^t (h(t-s) \| \nabla \Delta u^k(s) \| \, ds \right)^2 dx \\
\leq \eta \| \nabla u_t^k \|_2^2 + \frac{1}{4\eta} \int_{\Omega} \left( \int_0^t (h(t-s) \| \nabla \Delta u^k(s) \\
- \nabla \Delta u^k(t) + \| \nabla \Delta u^k(t) \| \, ds \right)^2 dx, \tag{3.24}
\]

Then we use Young’s inequality to obtain, for any \( \eta > 0 \),

\[
\int_{\Omega} \left( \int_0^t (h(t-s) \| \nabla \Delta u^k(s) - \nabla \Delta u^k(t) \| + \| \nabla \Delta u^k(t) \| \, ds \right)^2 dx
\]
Using (1.9), we obtain
\[
\int_\Omega \left( \int_0^t h(t-s)|\nabla \Delta u^k(s) - \nabla \Delta u^k(t)| \, ds \right)^2 \, dx \\
+ \int_\Omega \left( \int_0^t h(t-s)|\nabla \Delta u^k(t) \, ds \right)^2 \, dx \\
+ 2 \int_\Omega \left( \int_0^t h(t-s)|\nabla \Delta u^k(s) - \nabla \Delta u^k(t) \, ds \right) \left( \int_0^t h(t-s)|\nabla \Delta u^k(t) \, ds \right) \, dx \\
\leq (1 + \eta) \int_\Omega \left( \int_0^t h(t-s)|\nabla \Delta u^k(t) \, ds \right)^2 \, dx \\
+ \left( 1 + \frac{1}{\eta} \right) \int_\Omega \left( \int_0^t h(t-s)|\nabla \Delta u^k(s) - \nabla \Delta u^k(t) \, ds \right)^2 \, dx,
\]

Using (1.9), we obtain
\[
\int_\Omega \left( \int_0^t h(t-s)|\nabla \Delta u^k(s) - \nabla \Delta u^k(t) \right) \, ds \, |\nabla \Delta u^k(t) \, dx \right)^2 \, dx \\
\leq \beta^2 (1 + \eta) ||\nabla \Delta u^k(t)||_2^2 + \beta (1 + \frac{1}{\eta})(h \circ \nabla \Delta u^k).
\]

By Young’s inequality, we obtain
\[
\mu_1 \int_\Omega u_i^k g_1(u_i^k) \, dx \leq \eta \int_\Omega |u_i^k|^2 \, dx + \frac{\mu_1^2}{4 \eta} \int_\Omega |g_1(u_i^k)|^2 \, dx \\
\leq \eta C_x^2 \|\nabla u_i^k\|_2^2 + \frac{\mu_1^2}{4 \eta} \int_\Omega |g_1(u_i^k)|^2 \, dx
\]

By Young’s inequality, we obtain
\[
\int_\Omega |u_i^k| |u_i^k| \, dx + \left( 1 - 2 \eta (1 + C_x^2) - \frac{C_x^2}{2} \right) ||\nabla u_i^k||_2^2 \\
+ \frac{\tau}{2} \int_0^1 \|z^k\|_2^2 \, d\rho + \frac{1}{2} ||z^k(x, 1, t)||_2^2 \\
\leq \beta^2 (1 + \eta) ||\nabla \Delta u^k||_2^2 + \beta \frac{1}{\eta} (1 + \frac{1}{\eta})(h \circ \nabla \Delta u^k)
\]

Thus, from (3.19) and (3.28), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left[ \left( 1 - \int_0^t h(s) \, ds \right) ||\nabla \Delta u^k||_2^2 + ||\Delta x u_i^k||_2^2 + (h \circ \nabla \Delta u^k) \right] \\
+ \tau \int_0^1 \|\nabla x z^k(x, \rho, t)\|_2^2 \, d\rho + 2 \int \|u^k_i(t)\| \|\nabla x u^k_i\|_2^2 \, dx + \tau \int_0^1 \|z^k\|_2^2 \, d\rho \\
+ \mu_1 \int_\Omega |\nabla x u^k_i|^2 g_1(u_i^k) \, dx + c^2 \|\nabla x z^k(x, 1, t)\|_2^2 + \int \|u^k_i\| \|u_i^k\|_2^2 \, dx \\
+ \left( 1 - \eta (3 + 2C_x^2) - \frac{C_x^2}{2} \right) ||\nabla u_i^k||_2^2 + \frac{1}{2} ||z^k(x, 1, t)||_2^2.
Using (1.10), Jensen’s inequality and the concavity of $H^{-1}$, we obtain
\[
\int_{\Omega} |g_1(u_k^t)|^2 \, dx \leq \int_{|u_k^t| \geq \epsilon} |g_1(u_k^t)|^2 \, dx + \int_{|u_k^t| \leq \epsilon} |g_1(u_k^t)|^2 \, dx
\leq \int_{|u_k^t| \geq \epsilon} u_k^t g_1(u_k^t) \, dx + \int_{\Omega} H^{-1}(u_k^t g_1(u_k^t)) \, dx
\leq \int_{|u_k^t| \geq \epsilon} u_k^t g_1(u_k^t) \, dx + cH^{-1}(\int_{\Omega} u_k^t g_1(u_k^t) \, dx),
\]
\[
\int_{\Omega} |g_1(u_k^t)|^2 \, dx \leq \int_{|u_k^t| \geq \epsilon} u_k^t g_1(u_k^t) \, dx + c' H^*(1) + c'' \int_{\Omega} u_k^t g_1(u_k^t) \, dx
\leq c' H^*(1) + c' \int_{\Omega} u_k^t g_1(u_k^t) \, dx
\leq c' H^*(1) + c(-E')
\]
and
\[
\int_{\Omega} |g_2(z^k(x, 1, t))|^2 \, dx \leq c' \int_{\Omega} z^k(x, 1, t) g_2(z^k(x, 1, t)) \, dx \leq c(-E')
\]
Using Gronwall’ Lemma, we obtain
\[
\leq \frac{1}{2} h(t) \|\nabla \Delta u^k\|_2^2 + \frac{1}{2} (h' \circ \nabla \Delta u^k) + \frac{\beta^2 (1 + \eta)}{4\eta} \|\nabla \Delta u^k\|_2^2
\]
\[
+ \frac{\eta}{4\eta} (h \circ \nabla \Delta u^k) + \frac{\mu^2}{4\eta} \int_{\Omega} |g_1(u_k^t)|^2 \, dx
\]
\[
+ \frac{\mu^2}{4\eta} \int_{\Omega} |g_2(z^k(x, 1, t))|^2 \, dx + C_2(\eta).
\]

By choosing $\eta$ small enough such that $1 - \eta(3 + 2C^2_s) > 0$, integrating over $(0, t)$ and using (1.10), we obtain
\[
\left(1 - \int_0^t h(s) \, ds\right) \|\nabla \Delta u^k\|_2^2 + \|\Delta_x u^k\|_2^2 + (h \circ \nabla \Delta u^k) + \tau \int_0^t \|\nabla_x z^k(x, \rho, t)\|_2^2 \, d\rho
\]
\[
+ 2 \int_{\Omega} |u_k^t(t)|^2 |\nabla_x u_k^t|^2 \, dx + \tau \int_0^t \|z_k^t\|^2_2 \, d\rho + \mu_1 \int_{\Omega} \int_{\Omega} |\nabla_x u_k^t|^2 g_1(u_k^t) \, dx \, ds
\]
\[
+ c_2 \int_0^t \int_{\Omega} |\nabla_x z^k(x, 1, t)|^2 \, ds + \int_0^t \int_{\Omega} |u_k^t|^2 |u_k^t| \, dx \, ds
\]
\[
+ \left(1 - \eta(3 + 2C^2_s) - \frac{C_2^2}{2}\right) \int_0^t \|\nabla u_{\Omega t}^k\|^2_2 \, ds + \frac{\mu^2}{4\eta} \int_{\Omega} \int_{\Omega} |g_2(z^k(x, 1, t))|^2 \, dx \, ds + C_2(\eta) T
\]
We observe that the estimate (3.7) and (3.30) that there exists a subsequence \( \{u^m\} \) of \( \{u^k\} \) and a function \( u \) such that

\[
u^m \rightharpoonup u \text{ weakly star in } L^\infty(0, T, H^4(\Omega) \cap H_0^2(\Omega)), \tag{3.31}\]

\[
u_t^m \rightharpoonup u_t \text{ weakly star in } L^\infty(0, T, H_0^2(\Omega)), \tag{3.32}\]

\[g_1(u_t^m) \rightharpoonup \chi \text{ weakly star in } L^2(\Omega \times (0, T)), \tag{3.33}\]

\[
u_{tt}^m \rightharpoonup u_{tt} \text{ weakly star in } L^2(0, T, H_0^1(\Omega)), \tag{3.34}\]

\[z^m \rightharpoonup z \text{ weakly star in } L^\infty(0, T, H_0^2(\Omega, L^2(0, 1))), \tag{3.35}\]

\[z_t^m \rightharpoonup z_t \text{ weakly star in } L^\infty(0, T, L^2(\Omega \times (0, 1))), \tag{3.36}\]

\[g_2(z_t^m(x, 1, t)) \rightharpoonup \psi \text{ weakly star in } L^2(\Omega \times (0, T)) \tag{3.37}\]

From the first estimate (3.7) and Lemma 2.1, we deduce

\[\|u^k_t\|_{L^2(0, T, L^2(\Omega))} \leq \left( \frac{C_k}{\sqrt{\lambda}} \right)^{2(l+1)} \int_0^T \|\Delta u^k_t\|_{L^2}^{2(l+1)} \leq \left( \frac{C_k}{\sqrt{\lambda}} \right)^{2(l+1)} C_3^{2(l+1)} T.\]

On the other hand, from Aubin-Lions theorem, (see Lions [19]), we deduce that there exists a subsequence \( \{u^m\} \) of \( \{u^k\} \) such that

\[u_t^m \rightharpoonup u_t \text{ strongly in } L^2(0, T, L^2(\Omega)), \tag{3.38}\]

which implies

\[u_{tt}^m \rightharpoonup u_{tt} \text{ almost everywhere in } A. \tag{3.39}\]

Hence

\[|u_t^m| |u_t^m| \rightharpoonup |u_t| |u_t| \text{ almost everywhere in } A \tag{3.40}\]

where \( A = \Omega \times (0, T) \). Thus, using (3.38), (3.40) and Lions Lemma, we derive

\[|u_t^m| |u_t^m| \rightharpoonup |u_t| |u_t| \text{ weakly in } L^2(0, T, L^2(\Omega)) \tag{3.41}\]

and

\[z^m \rightharpoonup z \text{ strongly in } L^2(0, T, L^2(\Omega)) \]

which implies \( z^m \rightharpoonup z \) almost everywhere in \( A \).

**Lemma 3.1.** For each \( T > 0 \), \( g_1(u_t), g_2(z(x, 1, t)) \in L^1(A) \) and \( \|g_1(u^t)\|_{L^1(A)} \leq K \), \( \|g_2(z(x, 1, t))\|_{L^1(A)} \leq K \), where \( K \) is a constant independent of \( t \).

**Proof.** By (A2) and (3.39), we have

\[g_1(u_t^m(x, t)) \rightharpoonup g_1(u_t(x, t)) \text{ almost everywhere in } A, \]

\[0 \leq u_t^k(x, t) g_1(u_t^m(x, t)) \rightharpoonup u_t(x, t) g_1(u_t(x, t)) \text{ almost everywhere in } A. \]

Hence, by (3.11) and Fatou’s Lemma, we have

\[\int_0^T \int_\Omega u_t(x, t) g_1(u_t(x, t)) \, dx \, dt \leq K_1 \text{ for } T > 0 \tag{3.42}\]

Now, we can estimate \( \int_0^T \int_\Omega |g_1(u_t(x, t))| \, dx \, dt \). By Cauchy-Schwarz inequality and using (3.29), (3.42), we have

\[\int_0^T \int_\Omega |g_1(u_t(x, t))| \, dx \, dt \leq c |A|^{1/2} \left( \int_0^T \int_\Omega u_t(x, t) g_1(u_t(x, t)) \, dx \, dt \right)^{1/2} \]
that

By multiplying (3.1) by $\theta$ Vitali’s convergence theorem we deduce that

Similarly, we have

By applying (3.4) and (3.45) to obtain

Lemma 3.2. $g_1(u^k_t) \to g_1(u_t)$ in $L^1(\Omega \times (0,T))$ and $g_2(z^k) \to g_2(z)$ in $L^1(\Omega \times (0,T))$

Proof. Let $E \subset \Omega \times [0,T]$ and set

where $|E|$ is the measure of $E$. If $M(r) = \inf \{|s| : s \in \mathbb{R} \text{ and } |g(s)| \geq r\}$

By applying (3.11) we deduce that $\sup_k \int_E |g_1(u^k_t)| \, dx \, dt \to 0$ as $|E| \to 0$. From Vitali’s convergence theorem we deduce that

Similarly, we have

This completes the proof.

Hence

By multiplying (3.1) by $\theta(t) \in \mathcal{D}(0,T)$ and by integrating over $(0,T)$, it follows that

and multiplying (3.4) by $\theta(t) \in \mathcal{D}(0,T)$ and integrating over $(0,T) \times (0,1)$, it follows that

The convergence of (3.31, 3.37), (3.41), (3.43) and (3.44) are sufficient to pass to the limit in (3.45) and (3.46) to obtain

$$
- \frac{1}{l+1} \int_0^T (|u^k_t|^l u^k_t, w^j) \theta(t) \, dt + \int_0^T \mu_1 (\tau_{\tau z^k} + z^k, w^j) \theta(t) \, dt = 0.
$$

The convergence of (3.31, 3.37), (3.41), (3.43) and (3.44) are sufficient to pass to the limit in (3.45) and (3.46) to obtain

$$
- \frac{1}{l+1} \int_0^T (|u^k_t|^l u^k_t, w) \theta(t) \, dt + \int_0^T (\Delta_x u, \Delta_x w) \theta(t) \, dt = 0.
$$
Integrating by parts, we have

\[ + \int_0^T (\nabla_x u_{tt}, \nabla_x w) \theta(t) \, dt - \int_0^T \int_0^t h(t-s)(\Delta u(s), \Delta w) \theta(t) \, ds \, dt \]

\[ + \mu_1 \int_0^T (g_1(u_t), w) \theta(t) \, dt + \mu_2 \int_0^T (g_2(z(., 1)), w) \theta(t) \, dt = 0, \]

and

\[ \int_0^T \int_0^1 (\tau z_t + z, \phi) \theta(t) \, dt \, dp = 0. \]

By integrating, we have

\[ \int_0^T \left( |u_t|^l u_{tt} + \Delta^2 u - \Delta u_{tt} - \int_0^t h(t-s) \Delta^2 u(s) \, ds \right. \]

\[ + \mu_1 g_1(u_t) + \mu_2 g_2(z(., 1), u) \theta(t) \, dt = 0, \]

This completes the proof of Theorem 1.1.

3.3. Proof of Theorem 1.2

To prove our main result, we define the functionals

\[ \psi(t) = \int_0^1 \int e^{-2\tau r} G(z(x, \rho, t)) \, d\rho \, dx, \quad (3.47) \]

\[ \phi(t) = \frac{1}{l+1} \int_0^1 |u_t|^l u_t \, dx + \int_0^T \nabla u_t \nabla u \, dx, \quad (3.48) \]

\[ \varphi(t) = \int_\Omega \left( \Delta u_t - \frac{1}{l+1} |u_t|^l u_t \right) \int_0^t h(t-s)(u(t) - u(s)) \, ds \, ds. \quad (3.49) \]

Set

\[ F(t) = ME(t) + \varepsilon_1 \psi(t) + \varepsilon_2 \phi(t) + \varphi(t), \quad (3.50) \]

where \( M, \varepsilon_1 \) and \( \varepsilon_2 \) are suitable positive constants to be determined later.

Lemma 3.3. There exist two positive constants \( \kappa_0 \) and \( \kappa_1 \) depending on \( \varepsilon_1, \varepsilon_2 \) and \( M \) such that for all \( t > 0 \)

\[ \kappa_0 E(t) \leq F(t) \leq \kappa_1 E(t). \quad (3.51) \]

Proof. Using (1.11), we have

\[ |\psi(t)| \leq \frac{1}{\xi} E(t). \quad (3.52) \]

From Young’s inequality and Lemma 2.1, we deduce

\[ |\phi(t)| \]

\[ \leq \frac{1}{l+2} \|u_t\|_{l+2}^2 + \frac{(l+1)^{-1}}{l+2} \|u_t\|_{l+2}^{l+2} + \frac{1}{2} \|\Delta u\|_2^2 + \frac{1}{2} \|\nabla u_t\|_2^2 \]

\[ \leq \frac{1}{l+2} \|u_t\|_{l+2}^2 + \frac{(l+1)^{-1}}{l+2} \left( \frac{C_s}{\sqrt{\lambda_1}} \right)^{l+2} \|\Delta u\|_2^{l+2} + \frac{1}{2\lambda_1} \|\Delta u\|_2^2 + \frac{1}{2} \|\nabla u_t\|_2^2 \quad (3.53) \]

\[ \leq \frac{1}{l+2} \|u_t\|_{l+2}^2 + \frac{(l+1)^{-1}}{l+2} \left( \frac{C_s}{\sqrt{\lambda_1}} \right)^{l+2} \left( \frac{2E(0)}{1-\beta} \right)^{l/2} + \frac{1}{2\lambda_1} \|\Delta u\|_2^2 \]

\[ + \frac{1}{2} \|\nabla u_t\|_2^2. \]

Integrating by parts, we have

\[ \varphi(t) = - \int_\Omega \nabla u_t \int_0^t h(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \]
we use Young’s inequality applied with the conjugate exponents $\frac{l+2}{l+1}$ and $l+2$, the second term in the right hand side can be estimated as

$$
- \int_{\Omega} \frac{1}{t+1} |u_t|^l u_t \int_0^t h(t-s)(u(t)-u(s)) \, ds \, dx,
$$

By combining (3.52), (3.53) and (3.56), we have

$$
| - \int_{\Omega} \frac{1}{t+1} |u_t|^l u_t \int_0^t h(t-s)(u(t)-u(s)) \, ds \, dx |
\leq \frac{1}{l+2} \|u_t\|_{l+2}^2 + \frac{(l+1)^{-1}}{l+2} \int_{\Omega} \left( \int_0^t h(t-s) |u(t)-u(s)| \, ds \right)^{l+2} \, dx
$$

By combining (3.54) and (3.55), we deduce that

$$
|\varphi(t)| \leq \frac{1}{l+2} \|u_t\|_{l+2}^2 + \frac{1}{2} \|\nabla u_t\|_2^2 + \left\{ \frac{(l+1)^{-1}}{l+2} \left( \frac{C_s}{\sqrt{\lambda_1}} \right)^{l+2} \left( \frac{4E(0)}{1-\beta} \right)^{l/2} + \frac{\beta}{2\lambda_1} \right\} (h \circ \Delta u)(t).
$$

By combining (3.52), (3.53) and (3.56), we have

$$
F(t) \leq (M + \frac{\varepsilon_1}{\xi}) E(t) + \frac{\varepsilon_2 + 1}{l+2} \|u_t\|_{l+2}^2
$$

By combining (3.52), (3.53) and (3.56), we have

$$
F(t) \leq (M - \frac{\varepsilon_1}{\xi}) E(t) - \frac{\varepsilon_2 + 1}{l+2} \|u_t\|_{l+2}^2
$$

Similarly,

$$
F(t) \geq (M - \frac{\varepsilon_1}{\xi}) E(t) - \frac{\varepsilon_2 + 1}{l+2} \|u_t\|_{l+2}^2
$$
Lemma 3.5. Let \( \phi \) obtain \( \psi \).

\[
\begin{align*}
\text{Proof.} & \quad \text{By differentiating (3.47) with respect to } t \quad \text{and using (1.4) and (1.8), we obtain} \\
& \quad \psi'(t) = -2\psi(t) - \frac{\alpha_1 e^{-2\tau}}{\tau} \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) dx \\
& \quad \quad + \frac{\alpha_2}{\tau} \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) dx.
\end{align*}
\]

(3.57)

The proof is complete. \( \square \)

Lemma 3.4. Let \((u, z)\) be the solution to (1.3). Then

\[
\psi'(t) \leq -2\psi(t) - \frac{\alpha_1 e^{-2\tau}}{\tau} \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) dx \\
+ \frac{\alpha_2}{\tau} \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) dx.
\]

(3.58)

for \( M \) large enough. \( \square \)
Proof. Differentiating (3.48) with respect to $t$ and using the first equation of (1.5), we obtain

$$\phi'(t) = \frac{1}{l+1} \int_{\Omega} (|u_t|^l u_t)' u \, dx + \frac{1}{l+1} \int_{\Omega} |u_t|^{l+2} \, dx + \int_{\Omega} \nabla u_t \nabla u \, dx$$

$$+ \int_{\Omega} \nabla u_t \nabla u_t \, dx$$

$$= \int_{\Omega} |u_t|^l u_t u \, dx + \frac{1}{l+1} \int_{\Omega} |u_t|^{l+2} - \int_{\Omega} \Delta u u_t u \, dx + \| \nabla u_t \|^2$$

$$= \int_{\Omega} \left( |u_t|^l u_t - \Delta u \right) u \, dx + \frac{1}{l+1} \int_{\Omega} |u_t|^{l+2} + \| \nabla u_t \|^2$$

$$= \frac{1}{l+1} \| u_t \|^{l+2}_2 + \| \nabla u_t \|^2 - \int_{\Omega} \left( \Delta^2 u + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(z(x,1,t)) \right)$$

$$- \int_{\Omega} h(t-s) \Delta^2 u(s) \, ds \, dx$$

$$= \frac{1}{l+1} \| u_t \|^{l+2}_2 + \| \nabla u_t \|^2 - \| \Delta u \|^2_2 + \int_{\Omega} \Delta u(t) \int_{0}^{t} h(t-s) \Delta u(s) \, ds \, dx$$

$$- \mu_1 \int_{\Omega} u g_1(u_t(x,t)) \, dx - \mu_2 \int_{\Omega} u g_2(z(x,1,t)) \, dx$$

By using Young's inequality and Sobolev embedding, we can estimate the fourth term in the right side as follows:

$$\int_{\Omega} \Delta u(t) \int_{0}^{t} h(t-s) \Delta u(s) \, ds \, dx$$

$$\leq \int_{0}^{t} h(s) \, ds \| \Delta u(t) \|^2_2 + \int_{0}^{t} \int_{0}^{t} h(t-s) |\Delta u(t)| \Delta u(s) - \nabla u(t) \, ds \, dx$$

$$\leq \int_{0}^{t} h(s) \, ds \| \Delta u(t) \|^2_2 + \eta \| \Delta u(t) \|^2_2 + \frac{\beta}{4\eta} (h \circ \Delta u)(t)$$

$$\leq (\beta + \eta) \| \Delta u(t) \|^2_2 + \frac{\beta}{4\eta} (h \circ \Delta u)(t)$$

Since

$$\int_{\Omega} u g_1(u_t) \, dx \leq \eta C_2^2 \frac{C_2^2}{\lambda_1} \| \Delta u \|^2_2 + \frac{1}{4\eta} \int_{\Omega} |g_1(u_t)|^2 \, dx, \quad (3.59)$$

$$\int_{\Omega} u g_2(z(x,1,t)) \, dx \leq \eta C_2^2 \frac{C_2^2}{\lambda_1} \| \Delta u \|^2_2 + \frac{1}{4\eta} \int_{\Omega} |g_2(z(x,1,t))|^2 \, dx. \quad (3.60)$$

This completes the proof. \qed

Lemma 3.6. Let $(u, z)$ be a solution of (1.5). Then, for any $\delta > 0$,

$$\phi'(t) \leq \delta (2\beta^2 + 1) \| \Delta u(t) \|^2_2 + \left( \delta + \frac{\delta a_0}{l+1} - \int_{0}^{t} h(s) \, ds \right) \| \nabla u_t \|^2_2$$

$$+ \beta \left( \frac{2\delta}{l+1} + \frac{\mu_1 C_2^2}{4\delta \lambda_1} + \frac{\mu_2 C_2^2}{4\delta \lambda_1} \right) (h \circ \Delta u)(t) + \mu_1 \delta \| g_1(u_t(x,t)) \|^2_2$$

$$- \frac{h(0)}{4\delta \lambda_1} (1 + \frac{C_2^2}{l+1}) (h' \circ \Delta u)(t) + \mu_2 \delta \| g_2(z(x,1,t)) \|^2_2$$
In what follows we will estimate \( I \).

Proof. By using the Liebnnitz formula, and the first equation of \( \text{(1.5)} \), we have

\[
\varphi'(t) = -\int_{\Omega} \left( \int_0^t h(t-s) \Delta u(s) \, ds \right) \left( \int_0^t h(t-s)(\Delta u(t) - \Delta u(s)) \, ds \right) \, dx \\
+ \int_{\Omega} \Delta u(t) \left( \int_0^t h(t-s)(\Delta u(t) - \Delta u(s)) \, ds \right) \, dx \\
+ \mu_1 \int_{\Omega} g_1(u_t(x, t)) \int_0^t h(t-s)(u(t) - u(s)) \, ds \, dx \\
+ \mu_2 \int_{\Omega} g_2(z(x, 1, t)) \int_0^t h(t-s)(u(t) - u(s)) \, ds \, dx \\
- \int_{\Omega} \nabla u_t \int_0^t h'(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx \\
- \frac{1}{l+1} \int_{\Omega} |u_t|^l u_t \int_0^t h'(t-s)(u(t) - u(s)) \, ds \, dx \\
- \int_0^t h(s) \, ds \| \nabla u_t(t) \|^2 - \frac{1}{l+1} \int_0^t h(s) \, ds \| u_t(t) \|^2_{l+2} \\
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 - \int_0^t h(s) \, ds \| \nabla u_t(t) \|^2 \\
- \frac{1}{l+1} \int_0^t h(s) \, ds \| u_t(t) \|^2_{l+2}.
\]

In what follows we will estimate \( I_1, \ldots, I_6 \). So for \( \delta > 0 \), we have

\[
|I_1| \leq \delta \int_{\Omega} \left( \int_0^t h(t-s) |\Delta u(s)| \, ds \right)^2 \, dx \\
+ \frac{1}{4\delta} \int_{\Omega} \left( \int_0^t h(t-s) |\Delta u(t) - \Delta u(s)| \, ds \right)^2 \, dx \\
\leq \delta \int_{\Omega} \left( \int_0^t h(t-s) (|\Delta u(s) - \Delta u(t)| + |\Delta u(t)|) \, ds \right)^2 \, dx \\
+ \frac{1}{4\delta} \left( \int_0^t h(s) \, ds \right) (h \circ \Delta u)(t) \\
\leq 2\delta \left( \int_0^t h(t) \, ds \right)^2 \| \Delta u(t) \|^2_2 + \left( 2\delta + \frac{1}{4\delta} \right) \int_0^t h(s) \, ds \left( h \circ \Delta u \right)(t) \\
\leq 2\delta \beta^2 \| \Delta u(t) \|^2_2 + \beta \left( 2\delta + \frac{1}{4\delta} \right) \left( h \circ \Delta u \right)(t).
\]

Similarly,

\[
|I_2| \leq \delta \| \Delta u(t) \|^2_2 + \frac{\beta}{4\delta} (h \circ \Delta u)(t), \\
|I_3| \leq \delta \mu_1 \| g_1(u_t(x, t)) \|^2_2 + \frac{\mu_1 \beta C^2}{4\delta \lambda_1} (h \circ \Delta u)(t), \\
|I_4| \leq \delta \mu_2 \| g_2(z(x, 1, t)) \|^2_2 + \frac{\mu_2 \beta C^2}{4\delta \lambda_1} (h \circ \Delta u)(t),
\]
\[ |I_5| \leq \delta \int_{\Omega} |\nabla u_t|^2 \, dx + \frac{1}{4\delta} \int_{\Omega} \left( \int_0^t |h'(t-s)||\nabla u(t) - \nabla u(s)| \, ds \right)^2 \, dx \]
\[ \leq \delta \|\nabla u_t\|^2 + \frac{1}{4\delta} \int_{\Omega} \int_0^t -h'(s) \, ds \int_0^t -h'(t-s)||\nabla u(t) - \nabla u(s)|^2 \, ds \, dx \] (3.66)
\[ \leq \delta \|\nabla u_t\|^2 - \frac{h(0)}{4\delta\lambda_1} (h' \circ \Delta u)(t), \]
\[ |I_6| \leq \frac{1}{l+1} \left[ \delta \int_{\Omega} \|u_t\|^2 \, dx + \frac{1}{4\delta} \int_{\Omega} \left( \int_0^t h'(t-s)(u(t) - u(s)) \, ds \right)^2 \, dx \right] \]
\[ \leq \frac{1}{l+1} \left[ \delta \|u_t\|_2^{2(l+1)} - \frac{h(0)C_s^2}{4\delta\lambda_1} (h' \circ \Delta u)(t) \right] \] (3.67)
\[ \leq \frac{\delta a_0}{l+1} \|\nabla u_t\|^2 - \frac{h(0)C_s^2}{4\delta\lambda_1(l+1)} (h' \circ \Delta u)(t), \]
where \( a_0 = C_s^{2(l+1)}(2E(0)) \).

**Lemma 3.7.** Let \((u, z)\) be a solution of (1.5) and assume that (A1)–(A4) hold. Then \(F(t)\) satisfies the following estimate, along the solution and for some positive constants \(m, a_6 > 0\),
\[ F'(t) \leq -mE(t) + a_6 \|g_1(u_t(x, t))\|^2_2. \] (3.68)

**Proof.** From (3.50), (3.57), (3.57) and (3.58), we conclude that for any \(t \geq t_0 > 0\),
\[ F'(t) = ME'(t) + \varepsilon_1 \psi'(t) + \varepsilon_2 \phi'(t) + \varphi'(t) \]
\[ \leq -(M\beta_1 - \varepsilon_1 \frac{\alpha_2}{\tau}) \int_{\Omega} u_t(x, t)g_1(u_t(x, t)) \, dx \]
\[ - \left( \frac{M\beta_2 - \varepsilon_2}{2\delta} \right) \frac{\alpha_1 e^{-2\delta}}{\tau} \int_{\Omega} z(x, 1, t)g_2(z(x, 1, t)) \, dx \]
\[ - 2\varepsilon_1 \psi(t) - \frac{1}{l+1} (h_0 - \varepsilon_2) \|u_t\|^{l+2}_2 - \left( h_0 - \varepsilon_2 - \delta \left( 1 + \frac{a_0}{l+1} \right) \right) \|\nabla u_t\|^2_2 \]
\[ - \left( \frac{Mh_1}{2} + \varepsilon_2 \left( 1 - \beta - \delta - \frac{\delta C_s^2}{\lambda_1} (\mu_1 + \mu_2) \right) \varphi(t) - \delta (2\beta^2 + 1) \right) \|\Delta u\|^2_2 \]
\[ + \left( \frac{M}{2} - \frac{h(0)}{4\delta\lambda_1} \left( 1 + \frac{C_s^2}{l+1} \right) \right) (h' \circ \Delta u)(t) \]
\[ + \left( \frac{\beta \varepsilon_2}{2\delta} + 2\beta \delta + \frac{\beta}{2\delta} + \frac{C_s^2 \beta}{2\delta \lambda_1} (\mu_1 + \mu_2) \right) g_1(u_t(x, t)) \|\Delta u\|_2^2 \]
where \( h_0 = \int_0^{t_0} h(s) \, ds > 0 \) and \( h_1 = \min\{h(t)\} \) for all \( t \geq t_0 \). We take \( \varepsilon_2 < h_0 \) and \( \delta > 0 \) sufficiently small such that
\[ a_1 = \frac{1}{l+1} (h_0 - \varepsilon_2) > 0, \quad a_2 = h_0 - \varepsilon_2 - \delta \left( 1 + \frac{a_0}{l+1} \right) > 0. \]
We choose \( M \) large enough such that
\[ a_3 = \frac{Mh_1}{2} + \varepsilon_2 \left( 1 - \beta - \delta - \frac{\delta C_s^2}{\lambda_1} (\mu_1 + \mu_2) \right) - \delta (2\beta^2 + 1) > 0, \]
A combination of (3.68), (3.69) and (3.72) yields by using (1.6), we obtain and use Jensen’s inequality and the concavity of

\[ E \]

By recalling that

\[ a_4 = \zeta \left( \frac{M}{2} - \frac{h(0)}{4\delta \lambda_1} \left( \frac{1 + C^2}{l + 1} \right) \right) - \left( \frac{\beta \varepsilon_2}{4\eta} + 2\beta \delta + \frac{\beta}{2\delta} + \frac{C^2}{2\delta \lambda_1} \{ \mu_1 + \mu_2 \} \right) > 0, \]

\[ M \beta_1 - \varepsilon_1 \frac{c_2}{\beta} > 0, \quad M \beta_2 - c_3 \mu_2 \{ \delta + \frac{\varepsilon_2}{4\delta} \} - \varepsilon_1 \frac{C}{\tau} > 0. \]

Then

\[ F'(t) \leq -a_1 \| u_t \|^2_{L^2} - a_2 \| \nabla u_t \|^2_{L^2} - a_3 \| \Delta u_t \|^2_{L^2} - a_4 (h \circ \Delta u)(t) \]

\[ - a_5 \int_{\Omega} \int_0^1 G(z(x, \rho, t)) \, d\rho + \frac{\varepsilon_1}{\xi} E(t) + a_6 \| g_1(u_t(x, t)) \|^2_{L^2}, \]

where \( a_5 = 2 \varepsilon_1 \) and \( a_6 = \mu_1 (\delta + \frac{\varepsilon_2}{4\delta}). \)

**Proof of Theorem 1.2.** As in Komornik [16], we consider the following partition of \( \Omega, \)

\[ \Omega_1 = \{ x \in \Omega : |u_t| > \varepsilon \}, \quad \Omega_2 = \{ x \in \Omega : |u_t| \leq \varepsilon \} \]

By using (1.6), we have

\[ \int_{\Omega_1} |g_1(u_t)|^2 \, dx \leq c_2 \int_{\Omega_1} u_t g_1(u_t) \, dx \leq -cE'(t). \quad (3.69) \]

**Case 1.** \( H \) is linear on \([0, \varepsilon]\). In this case, one can easily check that there exists \( c_1 > 0 \), such that \( |g_1(s)| \leq c_1 s \) for all \( s \leq \varepsilon \), and thus,

\[ \int_{\Omega_2} |g_1(u_t)|^2 \, dx \leq c_1 \int_{\Omega_2} u_t g_1(u_t) \, dx \leq -cE'(t), \quad (3.70) \]

\[ (F(t) + cE(t))' \leq -m H_2(E(t)). \quad (3.71) \]

**Case 2.** \( H'(0) \) and \( H'' > 0 \) on \([0, \varepsilon]\) we define

\[ I(t) = \frac{1}{|\Omega_2|} \int_{\Omega_2} u_t g(u_t) \, dx, \]

and use Jensen’s inequality and the concavity of \( H^{-1} \) to obtain

\[ H^{-1}(I(t)) \geq c \int_{\Omega_2} H^{-1}(u_t g(u_t)) \, dx, \]

by using (1.6), we obtain

\[ \int_{\Omega_2} |g_1(u_t)|^2 \, dx \leq c \int_{\Omega_2} H^{-1}(u_t g(u_t)) \, dx \]

\[ \leq c H^{-1}(I(t)) \leq c H^{-1}(-cE'(t)). \quad (3.72) \]

A combination of (3.68), (3.69) and (3.72) yields

\[ (F(t) + cE(t))' \leq -m E(t) + c H^{-1}(-cE'(t)), \quad t \geq t_0. \quad (3.73) \]

By recalling that \( E' \leq 0, \quad H' > 0, \) and \( H'' > 0 \) on \((0, \varepsilon]\) and using (3.73), we obtain

\[ \left( H'(\varepsilon_0 E(t)) \{ F(t) + cE(t) \} + cE(t) \right)' \]

\[ = \varepsilon_0 E'(t) H'(\varepsilon_0 E(t))(F(t) + cE(t)) + H'(\varepsilon_0 E(t))(F(t) + cE(t))' + cE'(t) \]

\[ \leq -m H'(\varepsilon_0 E(t)) E(t) + c H'(\varepsilon_0 E(t)) H^{-1}(-cE'(t)) + cE'(t), \quad (3.74) \]

\[ \leq -m H'(\varepsilon_0 E(t)) E(t) + c H'(\varepsilon_0 E(t)) H^{-1}(-cE'(t)) + cE'(t), \]
by using Remark 2.3 with $H^*$, the convex conjugate of $H$ in the sense of Young, we obtain
\[
\left( H'(\varepsilon_0 E(t)) \{F(t) + cE(t)\} + cE(t) \right) \quad \leq \quad -mH'(\varepsilon_0 E(t))E(t) + cH^*(H'(\varepsilon_0 E(t))) \\
\leq \quad -mH'(\varepsilon_0 E(t))E(t) + c\varepsilon_0 H'(\varepsilon_0 E(t))E(t) \\
\leq \quad -cH'(\varepsilon_0 E(t))E(t) = -cH_2(E(t)).
\] (3.75)

Let
\[
\tilde{F}(t) = \begin{cases} F(t) + cE(t) & \text{if } H \text{ is linear on } [0, \varepsilon], \\ H'(\varepsilon_0 E(t)) \{F(t) + cE(t)\} + cE(t) & \text{if } H'(0) > 0 \text{ and } H'' > 0 \text{ on } [0, \varepsilon], \end{cases}
\] (3.76)

From (3.71) and (3.75), it follows that
\[
\frac{d}{dt} \tilde{F}(t) \leq -cH_2(E(t)), \quad \forall t \geq t_0.
\]

On the other hand, after choosing $M > 0$ larger if needed, we can observe from Lemma 3.3 that $F(t)$ is equivalent to $E(t)$. So, $\tilde{F}(t)$ is also equivalent to $E(t)$, for some positive constants $\tilde{c}_1$ and $\tilde{c}_2$
\[
\tilde{c}_1 E(t) \leq \tilde{F}(t) \leq \tilde{c}_2 E(t).
\] (3.77)

By setting $L(t) = \epsilon \tilde{F}(t)$ for $\epsilon < 1/\tilde{c}_2$, we easily see that, by (3.77), we have $L(t) \sim E(t)$ and
\[
L'(t) \leq \epsilon \tilde{F}(t) \leq -cH_2(E(t)) \\
\leq -cH_2\left( \frac{1}{\tilde{c}_2} \tilde{F}(t) \right) \\
\leq -cH_2\left( \epsilon \tilde{F}(t) \right) \\
\leq -cH_2(L(t)).
\]

Then
\[
\frac{L'(t)}{H_2(L(t))} \leq -c \epsilon
\] (3.78)

By recalling (1.12), we deduce that $H_2(t) = -1/H_1'(t)$, hence
\[
L'(t)H_1'(L(t)) \geq c \epsilon, \quad \forall t \geq t_0.
\]

A simple integration over $(t_0, t)$ yields
\[
H_1(L(t)) \geq H_1(L(t_0)) + c \epsilon (t - t_0).
\]

By choosing $\epsilon > 0$ sufficiently small such that $H_1(L(t_0)) - c \epsilon t_0 > 0$, and exploiting the fact that $H_1^{-1}$ is decreasing, we infer that
\[
L(t) \leq H_1^{-1}(c \epsilon t + H_1(L(t_0)) - c \epsilon t_0).
\] (3.79)

Consequently, the equivalence of $F$, $\tilde{F}$, $L$ and $E$ yields the estimate
\[
E(t) \leq w_3 H_1^{-1}(w_1 t + w_2).
\]

This completes the proof. \qed
References


Nadia Mezouar  
Laboratoire of Mathematics, Djillali Liabes University, P.O. Box 89, Sidi Bel Abbes 22000, Algeria  
E-mail address: nadia_dz12@yahoo.fr

Mama Abdelli  
Laboratoire of Mathematics, Djillali Liabes University, P.O. Box 89, Sidi Bel Abbes 22000, Algeria  
E-mail address: abdelli_mama@yahoo.fr

Amira Rachah  
Laboratoire de Mathématiques pour l'Industrie et la Physique, Institut de Mathématiques de Toulouse, Université Paul Sabatier, F-31062 Toulouse Cedex 9, France  
E-mail address: amira.rachah@math.univ-toulouse.fr