CLASSIFICATION AND EVOLUTION OF BIFURCATION CURVES FOR THE ONE-DIMENSIONAL PERTURBED GELFAND EQUATION WITH MIXED BOUNDARY CONDITIONS II

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Abstract. In this article, we study the classification and evolution of bifurcation curves of positive solutions for the one-dimensional perturbed Gelfand equation with mixed boundary conditions,

\[ u''(x) + \lambda \exp \left( \frac{au}{a + u} \right) = 0, \quad 0 < x < 1, \]
\[ u(0) = 0, \quad u'(1) = -c < 0, \]

where \( 4 \leq a < a_1 \approx 4.107 \). We prove that, for \( 4 \leq a < a_1 \), there exist two nonnegative \( c_0 = c_0(a) < c_1 = c_1(a) \) satisfying \( c_0 > 0 \) for \( 4 \leq a < a^* \approx 4.069 \), and \( c_0 = 0 \) for \( a^* \leq a < a_1 \), such that, on the \( (\lambda, \|u\|_\infty) \)-plane, (i) when \( 0 < c < c_0 \), the bifurcation curve is strictly increasing; (ii) when \( c = c_0 \), the bifurcation curve is monotone increasing; (iii) when \( c_0 < c < c_1 \), the bifurcation curve is \( S \)-shaped; (iv) when \( c \geq c_1 \), the bifurcation curve is \( \subset \)-shaped. This work is a continuation of the work by Liang and Wang [8] where authors studied this problem for \( a \geq a_1 \), and our results partially prove a conjecture on this problem for \( 4 \leq a < a_1 \) in [8].

1. Introduction

In this article, we study the classification and evolution of bifurcation curves of positive solutions for the one-dimensional perturbed Gelfand equation with mixed (or more precisely, Dirichlet-Neumann) boundary conditions given by

\[ u''(x) + \lambda \exp \left( \frac{au}{a + u} \right) = 0, \quad 0 < x < 1, \]
\[ u(0) = 0, \quad u'(1) = -c < 0, \]

where \( \lambda > 0 \) is treated as a bifurcation parameter, \( c > 0 \) is treated as an evolution parameter, and constant \( a \) satisfies \( 4 \leq a < a_1 \approx 4.107 \) where constant \( a_1 \) is defined in [3] (3.23)]. The bifurcation curve of positive solutions of \( (1.1) \) is defined by

\[ \hat{S}_c = \{ (\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of } (1.1) \}. \]
This work is a continuation of our previous work in [8] where we studied (1.1) for $a \geq a_1$. It is worthwhile noting that the classification and evolution of bifurcation curves $\tilde{S}_c$ of (1.1) is closely related to the one resulting from the same differential equation in (1.1) with zero Dirichlet boundary conditions [2, 5, 8], that is,

$$u''(x) + \lambda \exp \left( \frac{au}{a + u} \right) = 0, \quad 0 < x < 1,$$

$$u(0) = 0, \quad u(1) = 0.$$

The bifurcation curve of positive solutions of (1.2) is defined by

$$S = \{ (\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.2)} \}.$$

Before going into further discussions on problems (1.1) and (1.2), we first give some terminologies in this paper for the shapes of bifurcation curves $\tilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane (Following terminology also hold for $S$ if $\tilde{S}_c$ is replaced by $S$.)

![Figure 1](image.png)

**Figure 1.** Three different types of exactly $S$-shaped bifurcation curves $\tilde{S}_c$ with $\lambda > 0$ and $\|u_\lambda\|_\infty > 0$. (i) Type 1. (ii) Type 2. (iii) Type 3.

**$S$-shaped:** The bifurcation curve $\tilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane is said to be $S$-shaped if $\tilde{S}_c$ has at least two turning points, say $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, satisfying $\lambda_* < \lambda^*$ and $\|u_{\lambda_*}\|_\infty < \|u_{\lambda^*}\|_\infty$, and

(i) $\tilde{S}_c$ starts at some point $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ and initially continues to the right,

(ii) at $(\lambda^*, \|u_{\lambda^*}\|_\infty)$, $\tilde{S}_c$ turns to the left,

(iii) at $(\lambda_*, \|u_{\lambda_*}\|_\infty)$, $\tilde{S}_c$ turns to the right,

(iv) $\tilde{S}_c$ tends to infinity as $\lambda \to \infty$. That is, $\lim_{\lambda \to \infty} \|u_\lambda\|_\infty = \infty$.

**Exactly $S$-shaped:** The bifurcation curve $\tilde{S}_c$ on the $(\lambda, \|u\|_\infty)$-plane is said to be exactly $S$-shaped if $\tilde{S}_c$ is $S$-shaped and it has exactly two turning points; see Figure 1.

**Type 1/2/3 $S$-shaped:** Assume that the bifurcation curve $\tilde{S}_c$ is $S$-shaped on the $(\lambda, \|u\|_\infty)$-plane. Let $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ be the starting point of $\tilde{S}_c$, and

$$\bar{\lambda}_{\min} \equiv \min \{ \lambda : (\lambda, \|u_\lambda\|_\infty) \text{ is a turning point of } \tilde{S}_c \}.$$

Then $\tilde{S}_c$ is said to be type 1 (resp., type 2 and type 3) $S$-shaped if $\lambda_0 < \bar{\lambda}_{\min}$ (resp., $\lambda_0 = \bar{\lambda}_{\min}$ and $\lambda_0 > \bar{\lambda}_{\min}$ ); see Figure 1(i) (resp., Figure 1(ii) and 1(iii)).
The bifurcation curve $\tilde{S}_c$ on the $(\lambda, \|u\|_{\infty})$-plane is said to be \textit{C-shaped} if $\tilde{S}_c$ has at least one turning point $(\lambda_*, \|u_{\lambda_*}\|_{\infty})$, and

(i) $\tilde{S}_c$ starts at some point $(\lambda_0, \|u_{\lambda_0}\|_{\infty})$ and initially continues to the left,
(ii) at $(\lambda_*, \|u_{\lambda_*}\|_{\infty})$, $\tilde{S}_c$ turns to the right,
(iii) $\lambda_* < \lambda_0$ and $\|u_{\lambda_0}\|_{\infty} < \|u_{\lambda_*}\|_{\infty}$,
(iv) $\tilde{S}_c$ tends to infinity as $\lambda \to \infty$. That is, $\lim_{\lambda \to \infty} \|u_{\lambda}\|_{\infty} = \infty$.

Exactly \textit{C-shaped}: The bifurcation curve $\tilde{S}_c$ on the $(\lambda, \|u\|_{\infty})$-plane is said to be \textit{exactly C-shaped} if $\tilde{S}_c$ is \textit{C-shaped} and it has exactly one turning point; see Figure 2.

\textbf{Strictly/Monotone increasing:} The bifurcation curve $\tilde{S}_c$ on the $(\lambda, \|u\|_{\infty})$-plane is said to be \textit{strictly (resp., monotone) increasing} if $\lambda_1 < \lambda_2$ (resp., $\lambda_1 \leq \lambda_2$) for any two points $(\lambda_i, \|u_{\lambda_i}\|_{\infty})$, $i = 1, 2$, lying in $\tilde{S}_c$ with $\|u_{\lambda_1}\|_{\infty} < \|u_{\lambda_2}\|_{\infty}$.

For (1.2), it has been a long-standing conjecture [1, 6, 9] that there exists a positive critical bifurcation value $a^* \approx 4.07 > 4$ such that, on the $(\lambda, \|u\|_{\infty})$-plane, the bifurcation curve $S$ is strictly increasing for $0 < a \leq a^*$ and is exactly type 1 $S$-shaped for $a > a^*$. Very recently, Huang and Wang [3] gave a rigorous proof of this conjecture for (1.2). Their main result is stated in the next theorem.

\textbf{Theorem 1.1} ([3 Theorem 4 and Fig. 1]). Consider (1.2) with varying $a > 0$. Then, on the $(\lambda, \|u\|_{\infty})$-plane, the bifurcation curve $S$ of (1.2) is a continuous curve which starts at the origin and it tends to infinity as $\lambda \to \infty$. Moreover, there exists a critical bifurcation value $a^* \approx 4.069$ satisfying $4 < a^* < a_1 \approx 4.107$ such that the following assertions (i)--(iii) hold:

(i) For $a > a^*$, the bifurcation curve $S$ is exactly type 1 $S$-shaped on the $(\lambda, \|u\|_{\infty})$-plane. Moreover, all positive solutions $u_{\lambda}$ are nondegenerate except that $u_{\lambda_*}$ and $u_{\lambda^*}$ are degenerate for some positive $\lambda_* < \lambda^*$.
(ii) For $a = a^*$, the bifurcation curve $S$ is strictly increasing on the $(\lambda, \|u\|_{\infty})$-plane. Moreover, all positive solutions $u_{\lambda}$ are nondegenerate except that $u_{\lambda_0}$ is degenerate for some positive $\lambda_0$.
(iii) For $0 < a < a^*$, the bifurcation curve $S$ is strictly increasing on the $(\lambda, \|u\|_{\infty})$-plane. Moreover, all positive solutions $u_{\lambda}$ are nondegenerate.

For (1.1), Liang and Wang [8] proved the next theorem with any fixed $a > a_1 \approx 4.107$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Exactly C-shaped bifurcation curve $\tilde{S}_c$ with $\lambda_0 > 0$ and $\|u_{\lambda_0}\|_{\infty} > 0$.}
\end{figure}
Theorem 1.2 (8, Theorem 2.4] and see e.g., Figure 3 with $a = 5$). Consider (1.1) with any fixed $a > a_1 \approx 4.107$. Then, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve $\tilde{S}_c$ of (1.1) is a continuous curve which starts at some point $(\lambda_0, \|u_{\lambda_0}\|_\infty)$ with $\lambda_0 > 0$ and $\|u_{\lambda_0}\|_\infty > 0$ and it tends to infinity as $\lambda \to \infty$. Moreover, there exists $c_1 = c_1(a) > 1.057$ such that the following two assertions (i) and (ii) hold:

(i) For $0 < c < c_1$, the bifurcation curve $\tilde{S}_c$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist three positive $c_{1,1} \leq c_{1,2} \leq c_{1,3}$ on $(0, c_1)$, all depending on $a$, such that the S-shaped bifurcation curve $\tilde{S}_c$ belongs to type 1, type 2 and type 3 when $0 < c < c_{1,1}$, $c = c_{1,2}$ and $c_{1,3} < c < c_1$, respectively.

(ii) For $c \geq c_1$, the bifurcation curve $\tilde{S}_c$ is C-shaped on the $(\lambda, \|u\|_\infty)$-plane.

Figure 3. Numerical simulations of bifurcation curves $S$ and $\tilde{S}_c$ for $a = 5$ and varying $c > 0$ on the $(\lambda, \|u\|_\infty)$-plane of the bi-logarithm coordinates. Here $c_{1,2}^- < c_{1,2} \approx 0.488 < c_{1,2}^+ < c_1 \approx 1.365 < c_1^+ < c_2 \approx 7.718 < c_2^+ < c_3 \approx 47.711 < c_3^+$ (adopted from [8, Fig. 4]).

This article is organized as follows: Section 2 contains statements of the main result. Section 3 contains the proof of the main result.

2. Main result

In this section, we give our main result (Theorem 2.1) for problem (1.1) with $4 \leq a < a_1 \approx 4.107$, where classification and evolution of bifurcation curves $\tilde{S}_c$ for (1.1) with varying $c > 0$ are studied. Theorem 2.1 with $4 \leq a < a_1$ extends Theorem 1.2 with $a \geq a_1$, and we obtain a more complicated evolution of bifurcation curves $\tilde{S}_c$ with varying $c > 0$. Note that some basic properties and ordering properties of bifurcation curves $\tilde{S}_c$ for positive $a$ and $c$, on the $(\lambda, \|u\|_\infty)$-plane have been discussed in [8, Theorems 2.1 and 2.2].
Theorem 2.1 (See Figure 4). Consider (1.1) for any fixed $a$ satisfying $4 \leq a < a_1 \approx 4.107$. Then there exist two nonnegative $c_0 = c_0(a) < c_1 = c_1(a)$ satisfying $c_0 > 0$ for $4 \leq a < a^* \approx 4.069$, $c_0 = 0$ for $a^* \leq a < a_1$, and $c_1 > 1.057$ for $4 \leq a < a_1$, such that the following assertions (I)–(IV) hold:

(i) For $0 < c < c_0$, the bifurcation curve $\hat{S}_c$ is strictly increasing on the $(\lambda, \|u_\lambda\|_\infty)$-plane. Moreover, there exists a positive $\lambda_0$ such that (1.1) has no positive solution for $0 < \lambda < \lambda_0$, and exactly one positive solution for $\lambda \geq \lambda_0$.

(ii) For $c = c_0$, the bifurcation curve $\hat{S}_c$ is monotone increasing on the $(\lambda, \|u_\lambda\|_\infty)$-plane. Moreover, there exists a positive $\lambda_0$ such that (1.1) has no positive solution for $0 < \lambda < \lambda_0$, and at least one positive solution for $\lambda \geq \lambda_0$.

(iii) For $c_0 < c < c_1$, the bifurcation curve $\hat{S}_c$ is $S$-shaped on the $(\lambda, \|u_\lambda\|_\infty)$-plane. More precisely, there exist three positive $c_{1,1} \leq c_{1,2} \leq c_{1,3}$ on $(c_0, c_1)$, all depending on $a$, such that the following three assertions hold:

(a) (See Figure 2(i)) If $c_0 < c < c_{1,1}$, then the bifurcation curve $\hat{S}_c$ is type 1 $S$-shaped on the $(\lambda, \|u_\lambda\|_\infty)$-plane. Moreover, there exist three positive $\lambda_0 < \lambda_* < \lambda^*$ which are all strictly increasing functions of $c$ on $(c_0, c_{1,1})$ such that (1.1) has no positive solution for $0 < \lambda < \lambda_0$, at least one positive solution for $\lambda_0 \leq \lambda < \lambda_*$ and $\lambda > \lambda^*$, at least two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and at least three positive solutions for $\lambda_* < \lambda < \lambda^*$.

(b) (See Figure 2(ii)) If $c = c_{1,2}$, then the bifurcation curve $\hat{S}_c$ is type 2 $S$-shaped on the $(\lambda, \|u_\lambda\|_\infty)$-plane. Moreover, there exist three positive $\lambda_0 = \lambda_* < \lambda^*$ such that (1.1) has no positive solution for $0 < \lambda < \lambda_0$, at least one positive solution for $\lambda > \lambda^*$, at least two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and at least three positive solutions for $\lambda_* < \lambda < \lambda^*$.

(c) (See Figure 2(iii)) If $c_{1,3} < c < c_1$, then the bifurcation curve $\hat{S}_c$ is type 3 $S$-shaped on the $(\lambda, \|u_\lambda\|_\infty)$-plane. Moreover, there exist three positive $\lambda_* < \lambda_0 < \lambda^*$ which are all strictly increasing functions of $c$ on $(c_{1,3}, c_1)$ such that (1.1) has no positive solution for $0 < \lambda < \lambda_*$, at least one positive solution for $\lambda = \lambda_*$ and $\lambda > \lambda^*$, at least two positive solutions for $\lambda^* < \lambda < \lambda_0$, at least one positive solution for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and at least three positive solutions for $\lambda_* < \lambda < \lambda^*$.

(iv) (See Figure 2) For $c \geq c_1$, the bifurcation curve $\hat{S}_c$ is $C$-shaped on the $(\lambda, \|u_\lambda\|_\infty)$-plane. Moreover, there exist two positive $\lambda_* < \lambda_0$ such that (1.1) has no positive solution for $0 < \lambda < \lambda_*$, at least one positive solution for $\lambda = \lambda_*$ and $\lambda > \lambda_0$, and at least two positive solutions for $\lambda_* < \lambda \leq \lambda_0$.

Remark 2.2. By Theorem 2.1 we conclude that, on the $(\lambda, \|u_\lambda\|_\infty)$-plane, (i) For $4.069 \approx a^* \leq a < a_1 \approx 4.107$, since $c_0 = c_0(a) = 0$, the bifurcation curve $\hat{S}_c$ evolves from an $S$-shaped curve to a $C$-shaped curve as the evolution parameter varies from $0^+$ to $\infty$, which shows the same evolution for $a \geq a_1$, as claimed in Theorem 1.2.

It then implies, by Theorem 1.1, that such evolution is persistent whenever the bifurcation curve $S$ of (1.2) is exactly type 1 $S$-shaped on the $(\lambda, \|u_\lambda\|_\infty)$-plane; (ii) For $4 \leq a < a^*$, since $c_0 > 0$, the bifurcation curve $\hat{S}_c$ evolves from a strictly increasing curve to a monotone increasing curve, then to an $S$-shaped curve, and
Figure 4. Numerical simulations of bifurcation curves $S$ and $\tilde{S}_c$ for $a = 4$ and varying $c > 0$ on the $(\lambda, \|u\|_\infty)$-plane of the bi-logarithm coordinates. Here $0 < c_0^- < c_0 \approx 0.10 < c_{1,2}^- < c_{1,2} \approx 0.85 < c_{1,2}^+ < c_1 \approx 1.39 < c_{11}^+ < c_{12}^+$ (adopted from [8, Fig. 7]).

finally to a $\subset$-shaped curve when $c$ varying from $0^+$ to $\infty$. It partially verifies a conjecture on problem (1.1) for $4 \leq a < a^*$ proposed in [8, Theorem 2.3] and shows the emergence of more complicated evolution of bifurcation curves $\tilde{S}_c$ with varying $c > 0$.

3. PROOF OF THE MAIN RESULT

To prove our main result (Theorem 2.1) on problem (1.1), we modify time-map technique (the quadrature method) used in [2, 8]. We shall recall some well-developed results in [8]. First, for fixed $a, c > 0$, we define

$$H_c(\rho, q) = 2 \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{c}{\sqrt{F(\rho) - F(q)}}$$

for $0 \leq q < \rho$, where $f(s) = \exp \left( \frac{as}{s + a} \right)$ and $F(s) = \int_0^s f(t)dt$; see [8, (3.6)]. For fixed $a, c > 0$, let $\rho_0 = \rho_0(c)$ be the unique positive number such that $H_c(\rho_0, 0) = 0$, where the existence and uniqueness of $\rho_0$ are proved in [8, Lemma 3.2(ii)]. Then it can be proved that, for fixed $a, c > 0$ and $\rho \geq \rho_0$, $H_c(\rho, q)$ has a unique zero $q(\rho, c)$ on $[0, \rho]$; see [8, Lemma 3.2(iv)]. Moreover, the time map formula for mixed boundary value problem (1.1) is defined as

$$H_c(\rho, q(\rho, c)) \equiv \frac{c^2}{2[F(\rho) - F(q(\rho, c))]}$$

for $\rho \geq \rho_0(c)$, see [8, (3.26)]. Then it can be easily derived, by similar arguments as given in [2, Theorem 3.3] or [8, (3.26) and (3.27)], that positive solutions $u$ of (1.1) correspond
Thus studying the shape of the bifurcation curve $S_c$ of (1.1) for $a,c > 0$ is equivalent to studying the shape of the time map $H_c(\rho,q(\rho,c))$ for $\rho \geq \rho_0$.

To prove Theorem 2.1, we need the following Lemmas 3.1–3.4. First, in Lemma 3.1 we record some results on the time map formula $H_c(\rho,q(\rho,c))$ in [8].

**Lemma 3.1.** Fix $a \geq 4$ and consider $H_c(\rho,q(\rho,c))$ for $c > 0$ and $\rho \geq \rho_0$. Then the following assertions (i)–(ix) hold:

(i) [8] Lemma 3.2(iv)] For $c > 0$, if $0 < \rho < \rho_0(c)$, then $H_c(\rho,q)$ has no zero $q$ on $[0,\rho)$, while if $\rho \geq \rho_0(c)$, then $H_c(\rho,q)$ has a unique zero $q(\rho,c)$ on $[0,\rho)$, that is,

$$H_c(\rho,q(\rho,c)) = 0.$$  \hspace{1cm} (3.4)

Moreover, $q(\rho,c) = 0$ if and only if $\rho = \rho_0(c)$.

(ii) [8] Lemma 3.2(vi)] For $c > 0$ and $\rho \geq \rho_0$,

$$0 < \rho - q(\rho,c) \leq \frac{c^2 e^\varepsilon}{4\rho}. \hspace{1cm} (3.5)$$

(iii) [8] Lemma 3.2(vii)] $\rho_0(c) \in C(0,\infty)$ is a strictly increasing function of $c$ on $(0,\infty)$.

(iv) [8] Lemma 3.2(viii)] For $\rho > 0$, $q(\rho,c) \in C(0,\hat{c}] \cap C^1(0,\hat{c})$ is a strictly decreasing function of $c$ on $[0,\hat{c}]$. Here $\hat{c} = \sqrt{2F'(\rho)G(\rho)}$.

(v) [8] Lemma 3.4(i)] For any two positive numbers $\tilde{c}_1 < \tilde{c}_2$, $H_{\tilde{c}_1}(\rho,q(\rho,\tilde{c}_1)) < H_{\tilde{c}_2}(\rho,q(\rho,\tilde{c}_2))$ for $\rho \geq \rho_0(\tilde{c}_2)$.

(vi) [8] Lemma 3.5(i)] There exists a unique positive $c_1 = c_1(a)$ such that

$$\lim_{\rho \to \rho_0(c)} \frac{d}{d\rho} H_c(\rho,q(\rho,c)) \begin{cases} > 0 & \text{when } c \in (0,c_1), \\ = 0 & \text{when } c = c_1, \\ < 0 & \text{when } c \in (c_1,\infty). \end{cases} \hspace{1cm} (3.6)$$

(vii) [8] Lemma 3.5(ii)] For $c \geq c_1$, there exists $\bar{\rho}(c) > \rho_0(c)$ such that $\frac{d}{d\rho} H_c(\rho,q(\rho,c)) < 0$ for $\rho_0(c) < \rho < \bar{\rho}(c)$.

(viii) [8] Lemma 3.5(iii)] For $0 < c < c_1$ and $\rho_0(c) < \rho < \rho_0(c_1)$, $\frac{d}{d\rho} H_c(\rho,q(\rho,c)) > 0$.

On the other hand, for zero Dirichlet boundary value problem (1.2), its time map formula is defined as

$$G(\rho) = \sqrt{2} \int_0^\rho \frac{ds}{\sqrt{F'(\rho)-F'(s)}} \text{ for } \rho > 0, \hspace{1cm} (3.7)$$

see [11][12][13]. Then positive solutions $u$ of (1.2) correspond to

$$||u||_\infty = \rho \text{ and } G(\rho) = \sqrt{\lambda}. \hspace{1cm} (3.8)$$

Thus studying the shape of the bifurcation curve of (1.2) for $a > 0$ is equivalent to studying the shape of the time map $G(\rho)$ on $[0,\infty)$. It is worthwhile to point out that the first term of $H_c(\rho,q)$ defined in the right hand side of (3.1) is equal to $\sqrt{2G(\rho)}$, which implies that $G(\rho)$ has an influence on $H_c(\rho,q(\rho,c))$ (or say that the shape of the bifurcation curve $S_c$ of (1.1) is correlated with the shape of the bifurcation curve $S$ of (1.2)).
In the next Lemma 3.2, we record some results on the relationship between \( H_c(\rho, q(\rho, c)) \) and \( G(\rho) \) in [3].

**Lemma 3.2.** Fix \( a > 0 \) and consider \( G(\rho) \) for \( \rho > 0 \) and \( H_c(\rho, q(\rho, c)) \) for \( \rho \geq \rho_0 \) and \( c > 0 \). Then the following two assertions hold:

(i) **[3 Lemma 3.3(i)]** For \( c > 0 \) and \( \rho \geq \rho_0 \), \( H_c(\rho, q(\rho, c)) \leq [G(\rho)]^2 \), and the equality holds if and only if \( \rho = \rho_0 \).

(ii) **[3 Lemma 3.6]** If \( G'(\rho) \leq 0 \) for some \( \rho > 0 \), then \( \frac{d}{d\rho} H_c(\rho, q(\rho, c)) < 0 \) for \( 0 < c < \hat{c} \).

In the next lemma we record the sign of derivatives of the time map formula \( G(\rho) \) for \( \rho > 0 \) in [3].

**Lemma 3.3 ([3 Theorem 4]).** Consider \((1.2)\) with varying \( a > 0 \). There exists a critical bifurcation value \( a^* \approx 4.069 \) satisfying \( 4 < a^* < a_1 \approx 4.107 \) such that the following three assertions hold:

(i) For \( 0 < a < a^* \), \( G'(\rho) > 0 \) for all \( \rho > 0 \).

(ii) For \( a = a^* \), there exist a unique positive \( \rho^* \) such that \( G'(\rho^*) = 0 \) and \( G'(\rho) > 0 \) for all \( \rho > 0 \) and \( \rho \neq \rho^* \).

(iii) For \( a > a^* \), there exist two positive \( \hat{\rho}_1 < \hat{\rho}_2 \) such that

\[
G'(\rho) = \begin{cases} 
< 0 & \text{when } \rho \in (\hat{\rho}_1, \hat{\rho}_2), \\
= 0 & \text{when } \rho = \hat{\rho}_1 \text{ or } \hat{\rho}_2, \\
> 0 & \text{when } \rho \in (0, \hat{\rho}_1) \cup (\hat{\rho}_2, \infty). 
\end{cases}
\]  

\( \text{(3.9)} \)

**Lemma 3.4.** Fix \( a \geq 4 \) and consider \( H_c(\rho, q(\rho, c)) \) for \( \rho \geq \rho_0 \) and \( c > 0 \). Then the following three assertions hold:

(i) For any \( c > 0 \), there exists a positive \( \rho_M = \rho_M(a, c) \geq \rho_0 \) such that

\[
\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0 \text{ for } \rho \geq \rho_M.
\]

(ii) For any two positive numbers \( \hat{c}_1 < \hat{c}_2 \) and \( \rho \geq \rho_0(\hat{c}_2) \), if \( \frac{d}{d\rho} H_{\hat{c}_2}(\rho, q(\rho, \hat{c}_2)) \geq 0 \), then \( \frac{d}{d\rho} H_{\hat{c}_1}(\rho, q(\rho, \hat{c}_1)) > 0 \).

(iii) If there exist two positive numbers \( \hat{\rho}_1 < \hat{\rho}_2 \) such that \( G'(\rho) > 0 \) for \( \hat{\rho}_1 \leq \rho \leq \hat{\rho}_2 \), then there exists a positive \( \hat{c} = \hat{c}(a) \) such that \( \frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0 \) for \( \hat{\rho}_1 \leq \rho \leq \hat{\rho}_2 \) and \( 0 < c < \hat{c} \).

**Proof.** Note first that, as computed in [3] (3.3), (3.30), (3.31) and the last equation in the proof of Lemma 3.6,

\[
\frac{d}{d\rho} H_c(\rho, q(\rho, c)) = \frac{c^2 f(q(\rho, c))}{2 [F(\rho) - F(q(\rho, c))]^{1/2} [2 [F(\rho) - F(q(\rho, c))] + c f(q(\rho, c))] } \Psi(\rho, q(\rho, c))
\]

where

\[
\Psi(\rho, q(\rho, c)) = \sqrt{2} G'(\rho) - 2 \int_0^\rho \frac{f'(s) f(\rho)}{f(s)^2 \sqrt{F(\rho) - F(s)}} ds
\]

\[
= \int_0^\rho \frac{\theta(\rho) - \theta(s)}{\rho [F(\rho) - F(s)]^{3/2}} ds - 2 \int_{q(\rho, c)}^\rho \frac{f'(s) f(\rho)}{f(s)^2 \sqrt{F(\rho) - F(s)}} ds
\]
\[ \theta(\rho) - \theta(s) > 0 \quad \text{for } 0 \leq s < \rho, \quad (3.11) \]
\[ \left[ \frac{3}{2}F(\rho) - \rho f(\rho) \right] - \left[ \frac{3}{2}F(s) - sf(s) \right] > 0 \quad \text{for } 0 \leq s < \rho, \quad (3.12) \]
\[ \rho f(\rho) - \frac{f'(s)}{f(s)} \leq \frac{1}{4} \quad \text{for } \rho - 1 < s < \rho. \quad (3.13) \]

The proofs of (3.11)-(3.13) are omitted since they are trivial. Then, for \( \rho > \rho_M \), we have that \( \rho - q(\rho, c) < 1 \) by (3.5), and
\[
\Psi(\rho, q(\rho, c)) = \int_0^\rho \frac{\theta(\rho) - \theta(s)}{\rho [F(\rho) - F(s)]^{3/2}} ds - 2 \int_{q(\rho, c)}^\rho \frac{f'(s)f(\rho)}{f(s)^2 [F(\rho) - F(s)]} ds \\
> \int_{q(\rho, c)}^\rho \frac{2[1 - \rho f(\rho) - f'(s)]F(\rho) - F(s) - [\rho f(\rho) - sf(s)]}{\rho [F(\rho) - F(s)]^{3/2}} ds \quad \text{(by (3.11))} \\
> \int_{q(\rho, c)}^\rho \frac{3F(\rho) - F(s) - [\rho f(\rho) - sf(s)]}{\rho [F(\rho) - F(s)]^{3/2}} ds \quad \text{(by (3.13))} \\
> 0
\]
by (3.12). So Lemma 3.4(i) holds.

(II) We prove Lemma 3.4(ii). Let \( \tilde{c}_1 < \tilde{c}_2 \) be arbitrary two positive numbers and suppose that \( \frac{d}{d\rho} H_{\tilde{c}_2}(\rho, q(\rho, \tilde{c}_2)) \geq 0 \) for some \( \rho \geq \rho_0(\tilde{c}_2) \). Then, since
\[ \frac{\partial}{\partial q} \Psi(\rho, q) = 2 \frac{f'(q)f(\rho)}{[f(q)]^2 [F(\rho) - F(q)]} > 0 \]
and \( q(\rho, \tilde{c}_1) > q(\rho, \tilde{c}_2) \) for all \( \rho \geq \rho_0(\tilde{c}_2) \) by Lemma 3.1(iv), we have
\[ \Psi(\rho, q(\rho, \tilde{c}_1)) > \Psi(\rho, q(\rho, \tilde{c}_2)) \geq 0. \]
Consequently, \( \frac{d}{d\rho} H_{\tilde{c}_2}(\rho, q(\rho, \tilde{c}_1)) > \frac{d}{d\rho} H_{\tilde{c}_2}(\rho, q(\rho, \tilde{c}_2)) \) by (3.10). So Lemma 3.4(ii) holds.

(III) We prove Lemma 3.4(ii). Suppose there exist two positive numbers \( \hat{\rho}_1 < \hat{\rho}_2 \) such that \( G'(\rho) > 0 \) for \( \hat{\rho}_1 \leq \rho \leq \hat{\rho}_2 \). Then there exists \( \epsilon > 0 \) such that \( G'(\rho) \geq \epsilon \) for \( \hat{\rho}_1 \leq \rho \leq \hat{\rho}_2 \). By (3.5), there exists \( \hat{c} > 0 \) such that \( \rho - q(\rho, c) < \frac{\epsilon^2}{16\epsilon} \) for \( \hat{\rho}_1 \leq \rho \leq \hat{\rho}_2 \) and \( 0 < c \leq \hat{c} \). This implies that
\[ \Psi(\rho, q(\rho, c)) \geq \sqrt{2\epsilon} - 2 \int_{q(\rho, c)}^\rho \frac{e^{2a}}{\sqrt{\rho - s}} ds = \sqrt{2\epsilon} - 4e^{2a}\sqrt{\rho - q(\rho, c)} > 0 \]
for \( \hat{\rho}_1 \leq \rho \leq \hat{\rho}_2 \) and \( 0 < c \leq \hat{c} \). So Lemma 3.4(iii) holds. The proof is complete. \( \square \)

We are now in a position to prove Theorem 2.1.

**Proof of Theorem 2.1.** **Case 1.** \( 4 \leq a < a^* \approx 4.069 \). Define set
\[ I = \{ c > 0 : \frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0 \quad \text{on } (\rho_0(c), \infty) \}. \]
We first show that $I$ is nonempty. In fact, let $c_1$ be defined in (3.6)\textsuperscript{[4]} and $\tilde{\rho}_1 = \rho_0(c_1).$ Then, by Lemma 3.1\textsuperscript{(viii)}, we have that, for $0 < c < c_1,$

$$
\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0 \text{ on } (\rho_0(c), \tilde{\rho}_1).
$$

(3.15)

On the other hand, by Lemma 3.4\textsuperscript{(i)–(ii)} and letting $\tilde{\rho}_2 = \rho_M(a, c_1),$ we have that, for $0 < c < c_1,$

$$
\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0 \text{ on } [\tilde{\rho}_2, \infty).
$$

(3.16)

Moreover, by Lemma 3.3\textsuperscript{(i)} and Lemma 3.4\textsuperscript{(iii)}, there exists a positive $\tilde{c}_0 < c_1$ such that, for $0 < c < \tilde{c}_0,$ $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ for $\rho$ slightly larger than $\rho_0(c).$ Hence $I \subset (0, c_1),$ and there exists $\tilde{\rho} > \rho_0(\tilde{c})$ such that $\frac{d}{d\rho} H_c(\tilde{\rho}, q(\tilde{\rho}, c)) < 0.$ Hence, for $0 < c < \tilde{c}_0,$ $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ on $(\rho_0(c), \infty)$ and hence $(0, \tilde{c}_0) \subset I.$ So $I$ is nonempty.

Next, we show that $I$ is a finite connected interval. Note that, by Lemma 3.1\textsuperscript{(vii)}, when $c \geq c_1,$ $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) < 0$ for $\rho$ slightly larger than $\rho_0(c).$ Hence $I \subset (0, c_1).$ Moreover, if there exist $\tilde{c} \in (0, c_1)$ such that $\tilde{c} \notin I,$ then there exists $\tilde{\rho} > \rho_0(\tilde{c})$ such that $\frac{d}{d\rho} H_c(\tilde{\rho}, q(\tilde{\rho}, c)) < 0.$ Then, by (3.15), we have that $\tilde{\rho} > \tilde{\rho}_1.$ It implies, by Lemma 3.4\textsuperscript{(ii)}, that, for $c \in (\tilde{c}, c_1),$ $\tilde{\rho} > \tilde{\rho}_1 = \rho_0(c_1) > \rho_0(c)$ and $\frac{d}{d\rho} H_c(\tilde{\rho}, q(\tilde{\rho}, c)) < 0.$ Consequently, $(\tilde{c}, c_1) \notin I$ and hence $I$ is a finite connected interval.

By the definition of $I,$ above arguments and Lemma 3.1\textsuperscript{(vii)}, we obtain that there exists a positive $c_0 < c_1$ such that

$$
I = (0, c_0).
$$

(3.17)

Moreover, when $c = c_0,$

$$
\frac{d}{d\rho} H_{c_0}(\rho, q(\rho, c_0)) \geq 0 \text{ on } (\rho_0(c_0), \infty),
$$

(3.18)

and there exists $\tilde{\rho} > \rho_0(c_0)$ such that $\frac{d}{d\rho} H_{c_0}(\tilde{\rho}, q(\tilde{\rho}, c_0)) = 0.$ Indeed, such $\tilde{\rho} > \tilde{\rho}_1$ by (3.15). It follows that, by Lemma 3.4\textsuperscript{(ii)}, for $c_0 < c < c_1,$ $\tilde{\rho} > \tilde{\rho}_1 > \rho_0(c)$ and

$$
\frac{d}{d\rho} H_c(\tilde{\rho}, q(\tilde{\rho}, c)) < 0.
$$

(3.19)

By the relationship between bifurcation curves $\tilde{S}_c$ and the time map $H_c$ from (3.2) and (3.3), we have the following conclusions:

**Case (I).** For $0 < c < c_0,$ that is, $c \in I,$ the bifurcation curve $\tilde{S}_c$ is strictly increasing on the $(\lambda, \|u\|_\infty)$-plane since $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ on $(\rho_0(c), \infty).$

**Case (II).** For $c = c_0,$ the bifurcation curve $\tilde{S}_c$ is monotone increasing on the $(\lambda, \|u\|_\infty)$-plane by (3.18).

**Case (III).** For $c_0 < c < c_1,$ the bifurcation curve $\tilde{S}_c$ is S-shaped on the $(\lambda, \|u\|_\infty)$-plane since $\lim_{\rho \to \rho_0(c)^+} \frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ by (3.6), $\frac{d}{d\rho} H_c(\rho, q(\rho, c)) > 0$ on $[\tilde{\rho}_2, \infty)$ by (3.16), and $\frac{d}{d\rho} H_c(\tilde{\rho}, q(\tilde{\rho}, c)) < 0$ by (3.19).

We next show that the S-shaped bifurcation curve $\tilde{S}_c$ could be of either type 1, type 2 or type 3 for some value $c$ on $(c_0, c_1).$
Case (III)(a). The existence of type 1 $S$-shaped bifurcation curves $\tilde{S}_c$. Since 
\[
\frac{d}{dp}H_c(\rho, q(\rho, c)) > 0 \text{ on } (\tilde{\rho}_2, \infty) \text{ by } (3.16),
\] we have that, for $c_0 < c < c_1$,
\[
\min_{\rho \geq \tilde{\rho}_1} H_c(\rho, q(\rho, c)) = \min_{\tilde{\rho}_1 \leq \rho \leq \tilde{\rho}_2} H_c(\rho, q(\rho, c)) > \min_{\tilde{\rho}_1 \leq \rho \leq \tilde{\rho}_2} H_{c_0}(\rho, q(\rho, c_0)) \quad \text{(by Lemma 3.1(v))}
\] (3.20)
\[
= H_{c_0}(\tilde{\rho}_1, q(\tilde{\rho}_1, c_0))
\]
by (3.18). On the other hand, by (3.15) and Lemma 3.1(v), we have that 
\[
H_{c_0}(\rho_0(c), q(\rho_0(c), c_0)) < H_{c_0}(\tilde{\rho}_1, q(\tilde{\rho}_1, c_0))
\]
\[
< H_{c_1}(\tilde{\rho}_1, q(\tilde{\rho}_1, c_1)) = H_{c_1}(\rho_0(c), q(\rho_0(c), c_1))
\]
Consequently, by the intermediate value theorem, there exists $c_{1,1} \in (c_0, c_1)$ such that 
\[
H_{c_{1,1}}(\rho_0(c_{1,1}), q(\rho_0(c_{1,1}), c_{1,1})) = H_{c_0}(\tilde{\rho}_1, q(\tilde{\rho}_1, c_0)).
\] (3.21)
Hence, for $0 < c < c_{1,1}$,
\[
H_c(\rho_0(c), q(\rho_0(c), c)) = G(\rho_0(c)) \quad \text{(by Lemma 3.2(i))}
\]
\[
< G(\rho_0(c_{1,1}) \quad \text{(by Lemma 3.3(i) and Lemma 3.1(iii))}
\]
\[
= H_{c_{1,1}}(\rho_0(c_{1,1}), q(\rho_0(c_{1,1}), c_{1,1})) \quad \text{(by Lemma 3.2(i))}
\]
\[
= H_{c_0}(\tilde{\rho}_1, q(\tilde{\rho}_1, c_0)) \quad \text{(by (3.21))}
\]
\[
< \min_{\rho \geq \tilde{\rho}_1} H_c(\rho, q(\rho, c))
\]
by (3.20). It then follows, by (3.15), that 
\[
H_c(\rho_0(c), q(\rho_0(c), c)) < H_c(\rho, q(\rho, c))
\]
for $\rho > \rho_0(c)$. It implies that, for $0 < c \leq c_{1,1}$, the $S$-shaped bifurcation curve $\tilde{S}_c$ is of type 1 on the $(\lambda, \|u\|_\infty)$-plane.

Case (III)(b). The existence of type 3 $S$-shaped bifurcation curves $\tilde{S}_c$. The proof of this part is the same as that given in [S] Proof of Theorem 2.4, Cases (i)(b)] and hence the proof is omitted.

Case (III)(c). The existence of a type 2 $S$-shaped bifurcation curve $\tilde{S}_c$. The proof of this part is the same as that given in [S] Proof of Theorem 2.4, Case (i)(c)] and hence the proof is omitted.

Case (IV). For $c > c_1$, the bifurcation curve $\tilde{S}_c$ is $S$-shaped on the $(\lambda, \|u\|\infty)$-plane since 
\[
\lim_{\rho \to \rho_0(c)^+} \frac{d}{dp}H_c(\rho, q(\rho, c)) = 0 \quad \text{by (3.6)}
\]
and since 
\[
\frac{d}{dp}H_c(\rho, q(\rho, c)) > 0
\]
for $\rho \geq \rho_M(a, c)$ by Lemma 3.4(i).

Case 2. $a = a^* \approx 4.069$. Let $\rho^*$ be the unique positive number such that 
\[
G'(\rho^*) = 0
\]
as defined in Lemma 3.3(ii). Then, for $c > 0$, 
\[
\frac{d}{dp}H_c(\rho^*, q(\rho^*, c)) < 0 \quad \text{by Lemma 3.2(ii)}
\]
Hence the bifurcation curve $\tilde{S}_c$ must not be monotone increasing on the $(\lambda, \|u\|\infty)$-plane. Or equivalently, $c_0 = 0$ if we similarly define $I = (0, c_0)$ as in (3.14) and (3.17) in Case 1. The remaining parts of the proof in this case followed by similar arguments stated in above Case 1 and hence they are omitted here.

Case 3. $a^* < a < a_1$. Note that, by Lemma 3.3(iii), Equation (3.9) holds for all $a > a^*$. Thus the proof of this part followed by same arguments given as in [S] Proof of Theorem 2.4, part (III)] and the
multiplicity result of positive solutions for (1.1) in each case follows immediately from the definition of shapes of bifurcations curves, see e.g., Figures 1 and 2. The proof is complete.

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References


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