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SPECTRAL DENSITY ON THE QUATERNIONIC HEISENBERG GROUP AND A GREEN KERNEL FOR FRACTIONAL POWERS OF ITS CASIMIR-LAPLACIAN

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Abstract. In this article, we introduce a new integral representation of the resolvent kernel for the Casimir-Laplacian on the quaternionic Heisenberg group which was obtained in [14] and then find its spectral density. Also we obtain the Green kernel for fractional powers of the Casimir-Laplace operator.

1. Introduction

Spectral densities associated with partial differential operators are widely applied in quantum field theory, scattering theory and elsewhere [4]. An explicit expression for spectral density of the Kohn Laplacian for the Heisenberg group $H^3 = \mathbb{C} \times \mathbb{R}$ was obtained by Askour and Mouayn [6].

The first aim of this paper is to look for such formulae for the spectral density for the quaternionic Heisenberg groups. These groups are defined by replacing the complex field $\mathbb{C}$ by the field of quaternions $\mathbb{H}$ in the definition of H^3. More precisely, we make $\mathbb{H} \times \mathbb{R}^3$ into a nilpotent Lie group of step two by suitably defining the group operation.

The second aim is to use the explicit formula for the resolvent kernel to give the Green kernel of the fractional power of the Casimir-Laplace operator $\mathcal{L}$, i.e, $\mathcal{L}^\alpha$ for $\alpha \in]0, 1[$, on the quaternionic Heisenberg group. We prove that its formula is given by a series expansion in terms of the generalized Laguerre polynomials. The techniques are similar to those used in [18] for sub-Laplacians on the quaternionic Heisenberg group, which also works for the power of Casimir-Laplacian.

The plan of the article is as follows. In Section 2, a brief summary of the quaternionic Heisenberg group and some properties of its Casimir-Laplacian are given. We ends this section by establishing a new integral representation of the resolvent kernel, that plays a major role in the following sections. The Section 3 deals with the spectral density for the quaternionic Heisenberg group. In the section 4, we give a formulas for the Green kernel for fractional powers of the Casimir-Laplace operator.

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2. Resolvent kernel on the quaternionic Heisenberg group

2.0.1. *Quaternionic Heisenberg group and associated Casimir Laplacian.* Let $\mathbb{H}$ be the set of all quaternions, $\Im\mathbb{H}$ be the imaginary part of $\mathbb{H}$, then $\Im\mathbb{H} \simeq \mathbb{R}^3$. The standard basis $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ satisfies

$$\mathbf{ij} = -\mathbf{ji} \quad \text{and} \quad \mathbf{ij} = \mathbf{k}.$$

Let $x = x_0 1 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{H}$. The real and imaginary parts are $\Re(x) := x_0$ and $\Im(x) := x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$. The related quaternion, $x_0 1 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$, is known as the quaternion conjugate of x, and is denoted $\overline{x}$. Let $\omega(x, \tilde{x}) = \frac{1}{2}\Im(\tilde{x}\overline{x})$ be a $\mathbb{R}^3$-valued symplectic map on $\mathbb{H}$.

The product space $\mathbb{H} \times \Im\mathbb{H}$ together with the multiplication, noted by "$\cdot_\omega$",

$$(x, z)\cdot_\omega(\tilde{x}, \tilde{z}) = \Big(x + \tilde{x}; z + \tilde{z} + \omega(x, \tilde{x})\Big) \tag{2.1}$$

constitutes a Lie group, called the quaternionic Heisenberg group, and denoted by $\mathcal{H}^7_\omega$ [14]. Its Lie algebra is generated by the left-invariant vector fields

$$\begin{aligned}
\nabla_{e_1,0} &= \frac{\partial}{\partial x_0} + \frac{1}{2}\Big(x_1\frac{\partial}{\partial z_1} + x_2\frac{\partial}{\partial z_2} + x_3\frac{\partial}{\partial z_3}\Big),\\
\nabla_{e_2,0} &= \frac{\partial}{\partial x_1} + \frac{1}{2}\Big(-x_0\frac{\partial}{\partial z_1} - x_3\frac{\partial}{\partial z_2} + x_2\frac{\partial}{\partial z_3}\Big)\\
\nabla_{e_3,0} &= \frac{\partial}{\partial x_2} + \frac{1}{2}\Big(x_3\frac{\partial}{\partial z_1} - x_0\frac{\partial}{\partial z_2} - x_1\frac{\partial}{\partial z_3}\Big),\\
\nabla_{e_4,0} &= \frac{\partial}{\partial x_3} + \frac{1}{2}\Big(-x_2\frac{\partial}{\partial z_1} + x_1\frac{\partial}{\partial z_2} - x_0\frac{\partial}{\partial z_3}\Big)
\end{aligned}$$

and the canonical vector fields

$$\nabla_{0,\varepsilon_1} = \frac{\partial}{\partial z_1}, \quad \nabla_{0,\varepsilon_2} = \frac{\partial}{\partial z_2}, \quad \nabla_{0,\varepsilon_3} = \frac{\partial}{\partial z_3},$$

where (e_1, e_2, e_3, e_4) and $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ are the canonical basis of $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively.

The commutation relations between the generators are

$$\begin{gathered}
[\nabla_{e_i,0}, \nabla_{0,\varepsilon_r}] = [\nabla_{0,\varepsilon_r}, \nabla_{0,\varepsilon_s}] = 0; \quad i = 0, \dots, 3;\ r, s = 1, 2, 3;\\
[\nabla_{e_i,0}, \nabla_{e_j,0}] = 2\nabla_{0,\varepsilon_k},
\end{gathered}$$

where (ijk) is any circular permutation of (123).

We define the Laplacian $\tilde{\mathcal{L}}$ as a Casimir operator on $\mathcal{H}^7_\omega$ by considering

$$\tilde{\mathcal{L}} = \sum_{r=1}^{4} \nabla^2_{e_r,0} + \sum_{s=1}^{3} \nabla^2_{0,\tilde{e}_s}. \tag{2.2}$$

Explicitly, $\tilde{\mathcal{L}}$ takes the form

$$\begin{aligned}
\tilde{\mathcal{L}} = {}& \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \Big(1 + \frac{1}{4}(x_0^2 + x_1^2 + x_2^2 + x_3^2)\Big)\Big(\frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial z_3^2}\Big),\\
&+ \Big(x_1\frac{\partial}{\partial x_0} - x_0\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}\Big)\frac{\partial}{\partial z_1}\\
&+ \Big(x_2\frac{\partial}{\partial x_0} - x_3\frac{\partial}{\partial x_1} - x_0\frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_3}\Big)\frac{\partial}{\partial z_2}\\
&+ \Big(x_3\frac{\partial}{\partial x_0} + x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} - x_0\frac{\partial}{\partial x_3}\Big)\frac{\partial}{\partial z_3}.
\end{aligned}$$

It is noted that $\tilde{\mathcal{L}}$ can be written in terms of the standard Laplacian on $\mathbb{R}^4$,

$$\Delta_{\mathbb{R}^4_x} = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2},$$

and the one on $\mathbb{R}^3$, $\Delta_{\mathbb{R}^3_z} = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial z_3^2}$, as follows

$$\tilde{\mathcal{L}} = \Delta_{\mathbb{R}^4_x} + \langle J(\vec{\partial}_z)x, \vec{\partial}_x\rangle + \frac{1}{4}\|x\|^2\Delta_{\mathbb{R}^3_z} + \Delta_{\mathbb{R}^3_z}, \tag{2.3}$$

which can be also rewritten in terms of the sub-Laplacian $\tilde{\mathcal{L}}_{sub}$ associated to $\mathcal{H}^7_\omega$ as follows

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_{sub} + \Delta_{\mathbb{R}^3_z}, \tag{2.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product and where $\vec{\partial}_x$ and $\vec{\partial}_z$ denote respectively the gradient vectors on $\mathbb{R}^4$ and $\mathbb{R}^3$ given by

$$\vec{\partial}_x = \Big(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big) \quad \text{and} \quad \vec{\partial}_z = \Big(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}\Big),$$

and $J(\vec{\partial}_z)$ is the operator matrix

$$J(\vec{\partial}_z) := \begin{pmatrix} 0 & \frac{\partial}{\partial z_1} & \frac{\partial}{\partial z_2} & \frac{\partial}{\partial z_3} \\ -\frac{\partial}{\partial z_1} & 0 & \frac{\partial}{\partial z_3} & -\frac{\partial}{\partial z_2} \\ -\frac{\partial}{\partial z_2} & -\frac{\partial}{\partial z_3} & 0 & \frac{\partial}{\partial z_1} \\ -\frac{\partial}{\partial z_3} & \frac{\partial}{\partial z_2} & -\frac{\partial}{\partial z_1} & 0 \end{pmatrix}.$$

Note that $J(\vec{\partial}_z)$ is an skew-symmetric matrix whose square is given by

$$J(\vec{\partial}_z)^2 = -\Delta_{\mathbb{R}^3_z} I_4, \tag{2.5}$$

where I_4 is the identity matrix on $\mathbb{R}^4$.

Now we consider the operator $\mathcal{L} := -\tilde{\mathcal{L}}$ which acting on $D(\mathcal{L}) := C_0^\infty(\mathcal{H}^7_\omega)$, the space of complex-valued C^∞- functions with a compact support in $\mathcal{H}^7_\omega$, as its natural regular domain is a densely defined symmetric operator. Precisely, we have

$$\int_{\mathcal{H}^7_\omega} (\mathcal{L}f)\overline{g}\, dm = \int_{\mathcal{H}^7_\omega} f(\overline{\mathcal{L}g})dm; \quad f, g \in D(\mathcal{L}). \tag{2.6}$$

Here $dm(\cdot)$ denotes the Haar measure of $\mathcal{H}^7_\omega$. In fact, $\mathcal{H}^7_\omega$ is a unimodular Lie group on which the Haar measure is just the product of ordinary Lebesgue measures on $\mathbb{R}^4$ and $\mathbb{R}^3$.

Let $(\nabla_i)_{1\le i\le 7}$ are the left-invariant vector fields on $\mathcal{H}^7_\omega$ given above. Then, from (2.2) we have

$$\mathcal{L} = -\sum_{r=1}^{4} \nabla^2_{e_r,0} - \sum_{s=1}^{3} \nabla^2_{0,\varepsilon_s} := -\sum_{i=1}^{7} \nabla_i^2,$$

implying

$$\int_{\mathcal{H}^7_\omega} (\mathcal{L}f)\overline{g}\, dm = -\sum_{i=1}^{7} \int_{\mathcal{H}^7_\omega} (\nabla_i^2 f)\overline{g}\, dm.$$

The formula [10, page. 21] help us obtaining

$$\int_{\mathcal{H}^7_\omega} (\mathcal{L}f)\overline{g}\, dm = -\sum_{i=1}^{7} \int_{\mathcal{H}^7_\omega} f(\overline{\nabla_i^2 g})dm = \int_{\mathcal{H}^7_\omega} f(\overline{\mathcal{L}g})dm,$$

which is exactly the symmetry propriety of $\mathcal{L}$.

By considering the quadratic form defined by
$$Q(f,g)=\langle \mathcal{L}f,g\rangle,\quad f,g\in D(\mathcal{L}), \tag{2.7}$$
and using the formula [10, p. 21], we have
$$\begin{aligned}
Q(f,f)=\langle \mathcal{L}f,f\rangle &= \int_{\mathcal{H}^7_\omega}(\mathcal{L}f)\overline{f}\,dm = -\sum_{i=1}^{7}\int_{\mathcal{H}^7_\omega}(\nabla_i\nabla_i f)\overline{f}\,dm\\
&= \sum_{i=1}^{7}\int_{\mathcal{H}^7_\omega}(\nabla_i f)(\overline{\nabla_i f})dm = \sum_{i=1}^{7}\|\nabla_i f\|^2_{L^2(\mathcal{H}^7_\omega,dm)}\geq 0.
\end{aligned}$$
It follows that $\mathcal{L}$ is a positive operator on $L^2(\mathcal{H}^7_\omega,dm)$.

Following the theory of operators associated with forms, the operator $\mathcal{L}$ admits a unique self-adjoint extension (Friedrichs Extension) which is also a non negative operator (see [5, p. 107] for the general theory). Here this extension will be also denoted by $\mathcal{L}$.

Positivity of $\mathcal{L}$ ensures that its spectrum is a closed unbounded set contained in the positive real-axis [15, p. 178]. Then, by the general spectral theory for unbounded self-adjoint operator, $\mathcal{L}$ admits a spectral decomposition $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$ [5, 8, 9].

The $\{E_\lambda\}_\lambda$ is an increasing family of projectors that satisfy
$$I=\int_0^\infty dE_\lambda$$
where I is the identity operator, and $\mathcal{L}=\int_0^\infty \lambda dE_\lambda$ in the week sense; that is,
$$\langle \mathcal{L}f,g\rangle=\int_0^\infty \lambda d\langle E_\lambda f,g\rangle$$
for $f\in D(\mathcal{L})$ and $g\in L^2(\mathcal{H}^7_\omega,dm)$, where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\mathcal{H}^7_\omega,dm)$.

The Casimir-Laplace operator $\mathcal{L}$ can be also expressed in the divergence form as follows
$$\mathcal{L}=-\operatorname{div}(\mathrm{M}\vec{\nabla}), \tag{2.8}$$
where div is the vector operator divergence and $\vec{\nabla}$ is the gradient vector on $\mathbb{R}^7$, i.e.,
$$\vec{\nabla}=\Big(\frac{\partial}{\partial x_0},\dots,\frac{\partial}{\partial x_3},\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_3}\Big)^t, \tag{2.9}$$
and M is the non-degenerate symmetric matrix given by
$$\mathrm{M}:=\left(\begin{array}{c|c} I_4 & K\\ \hline K^\top & (4+\|x\|^2)/4I_3\end{array}\right), \tag{2.10}$$
where I_3 and I_4 are the identity matrix on $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively and
$$K:=\frac{1}{2}\begin{pmatrix} x_2 & x_3 & x_4\\ -x_1 & -x_4 & x_3\\ x_4 & -x_1 & -x_2\\ -x_3 & x_2 & -x_1\end{pmatrix} \tag{2.11}$$
and $K^\top$ is the transpose.

According to Sylvester's criterion [11], M is a positive definite matrix. Hence, the operator $\mathcal{L}$ is elliptic. The positivity of our operator $\mathcal{L}$ also ensures that the semi group $e^{-t\mathcal{L}}$ is well defined. Its integral kernel was obtained in [14] and it is given by

$$\begin{aligned} k_t(x,z;\tilde{x},\tilde{z}) = \int_{\mathbb{R}^3} & e^{-i\langle z-\tilde{z},\lambda\rangle} e^{-t\|\lambda\|^2} e^{-\frac{i}{2}\langle J(\lambda)\,x,\tilde{x}\rangle} \\ & \times \Big(\frac{\|\lambda\|}{\sinh(t\|\lambda\|)}\Big)^2 e^{-\frac{\|\lambda\|}{4}\coth(t\|\lambda\|)\|x-\tilde{x}\|^2}\,dm(\lambda), \end{aligned}$$

where $x,\tilde{x}\in\mathbb{R}^4$ and $z,\tilde{z}\in\mathbb{R}^3$.

2.0.2. *Resolvent of the operator $\mathcal{L}$.* It is well known that for a positive self-adjoint operator A, the resolvent operator $R(\zeta) := (\zeta - A)^{-1}$ is related to the the semi group heat kernel by means of the Laplace transform [7, p. 56]

$$(\zeta - A)^{-1} = \int_0^\infty e^{-\zeta s}\, e^{-sA} ds; \quad \Re(\zeta) < 0, \tag{2.12}$$

Thanks to this relation, the resolvent kernel for the operator $\mathcal{L}$ was obtained in [14]. Here, we give a more simple representation of the obtained kernel.

Proposition 2.1. *Fix a complex number ζ; $\Re\zeta < 0$. Then, the resolvent $R(\zeta) := (\zeta-\mathcal{L})^{-1}$ of the operator $\mathcal{L}$ is given by*

$$R(\zeta)f(x,z) = \int_{\mathcal{H}^7_\omega} R(\zeta;(x,z),(\tilde{x},\tilde{z}))f(\tilde{x},\tilde{z})\,dm(\tilde{x},\tilde{z}), \tag{2.13}$$

where

$$\begin{aligned} & R(\zeta;(x,z),(\tilde{x},\tilde{z})) \\ & = \tfrac{-4\sqrt{2}\pi}{\|x-\tilde{x}\|\|z-\tilde{z}-\Im(\tilde{x}\overline{x})/2\|} \int_0^\infty \Gamma(\tfrac{-\zeta}{2t}+\tfrac{t}{2}+1) W_{\frac{\zeta}{2t}-\frac{t}{2},\frac{-1}{2}}(t\|x-\tilde{x}\|^2/2) \\ & \quad \times \sin\Big(\|z-\tilde{z}-\Im(\tilde{x}\overline{x})/2\|t\Big) t^{3/2}\,dt, \end{aligned} \tag{2.14}$$

where $\Gamma(\cdot)$ is Euler's Gamma-function and where $W_{\kappa,\mu}(\cdot)$ is the Whittaker function [1, p.505].

Proof. To prove (2.14), we recall first that for $\zeta\in\mathbb{C}$ with $\Re(\zeta)>0$, the resolvent kernel for $\tilde{\mathcal{L}}$ has been obtained in [14] as

$$\begin{aligned} \tilde{R}(\zeta;(x,z),(\tilde{x},\tilde{z})) = \tfrac{\sqrt{2}}{\|x-\tilde{x}\|} \int_{\mathbb{R}^3} & e^{-i\langle z-\tilde{z},\lambda\rangle} e^{-\frac{i}{2}\langle J(\lambda)x,\tilde{x}\rangle} \Gamma\big(\tfrac{\zeta}{2\|\lambda\|}+\tfrac{\|\lambda\|}{2}+1\big) \\ & \times \|\lambda\|^{1/2} W_{\frac{-\zeta}{2\|\lambda\|}-\frac{\|\lambda\|}{2},\frac{-1}{2}}(\|\lambda\|\|x-\tilde{x}\|^2/2)\,dm(\lambda). \end{aligned} \tag{2.15}$$

The Whittaker function can be defined as

$$W_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+\frac{1}{2}} U(\mu-\kappa+\frac{1}{2}, 1+2\mu; z), \tag{2.16}$$

where $U(a,b;z)$ is the Kummer's Function of the second kind [1, p.504].

This implies that the resolvent kernel for $\mathcal{L} := -\tilde{\mathcal{L}}$; for $\Re(\zeta)<0$, is given by

$$\begin{aligned} R(\zeta;(x,z),(\tilde{x},\tilde{z})) = \tfrac{-\sqrt{2}}{\|x-\tilde{x}\|} \int_{\mathbb{R}^3} & e^{-i\langle z-\tilde{z},\lambda\rangle} e^{-\frac{i}{2}\langle J(\lambda)x,\tilde{x}\rangle} \Gamma\big(\tfrac{-\zeta}{2\|\lambda\|}+\tfrac{\|\lambda\|}{2}+1\big) \\ & \times \|\lambda\|^{1/2} W_{\frac{\zeta}{2\|\lambda\|}-\frac{\|\lambda\|}{2},-\frac{1}{2}}(\|\lambda\|\|x-\tilde{x}\|^2/2)\,dm(\lambda). \end{aligned} \tag{2.17}$$

Taking the polar coordinates $\lambda = r\omega$; $\omega \in \mathbb{S}^2$, the above kernel becomes

$$\begin{aligned} R(\zeta;(x,z),(\tilde{x},\tilde{z})) = \frac{-\sqrt{2}}{\|x-\tilde{x}\|} \int_0^{+\infty} & \Gamma(\tfrac{-\zeta}{2r} + \tfrac{r}{2} + 1) I_{x,\tilde{x},z,\tilde{z}}(r) \\ & \times W_{\frac{\zeta}{2r}-\frac{r}{2},-\frac{1}{2}}(r\|x-\tilde{x}\|^2/2)\, r^{5/2}\, dr, \end{aligned} \tag{2.18}$$

where

$$I_{x,\tilde{x},z,\tilde{z}}(r) = \int_{\mathbb{S}^2} e^{-ir\langle z-\tilde{z},\omega\rangle} e^{-\frac{ir}{2}\langle J(\omega)\,x,\tilde{x}\rangle} d\sigma(\omega). \tag{2.19}$$

By a direct calculation, we obtain $\langle J(\omega)\,x,\tilde{x}\rangle = \langle u,\omega\rangle$, where $u := -\Im(\tilde{x}\overline{x}) \in \Im\mathbb{H} \simeq \mathbb{R}^3$. Hence, we can rewrite (2.19) as

$$I_{x,\tilde{x},z,\tilde{z}}(r) = \int_{\mathbb{S}^2} e^{-ir\langle z-\tilde{z}+u/2,\omega\rangle}\, d\sigma(\omega).$$

Using the identity [16, p.347]

$$\int_{\mathbb{S}^2} e^{-i\langle v,\omega\rangle}\, d\sigma(\omega) = 4\pi \frac{\sin(\|v\|)}{\|v\|}, \quad \text{for all } v \in \mathbb{R}^3,$$

it follows that

$$I_{x,\tilde{x},z,\tilde{z}}(r) = 4\pi \frac{\sin(\|z-\tilde{z}+u/2\|r)}{\|z-\tilde{z}+u/2\|r}. \tag{2.20}$$

Substituting (2.20) into the expression of the resolvent kernel in (2.18), we finally obtain

$$\begin{aligned} & R(\zeta;(x,z),(\tilde{x},\tilde{z})) \\ & = \frac{-4\sqrt{2}\pi}{\|x-\tilde{x}\|\|z-\tilde{z}-\Im(\tilde{x}\overline{x})/2\|} \int_0^{\infty} \Gamma(\tfrac{-\zeta}{2t} + \tfrac{t}{2} + 1) W_{\frac{\zeta}{2t}-\frac{t}{2},-\frac{1}{2}}(t\|x-\tilde{x}\|^2/2) \\ & \quad \times \sin\Big(\|z-\tilde{z}-\Im(\tilde{x}\overline{x})/2\|t\Big) t^{3/2}\, dt, \end{aligned}$$

as required. □

3. Spectral density of the Casimir-Laplace operator $\mathcal{L}$

In this section we shall use the technique developed in [6], to derive an explicit expression of the spectral density of the Casimir-Laplace operator $\mathcal{L}$ of the quaternionic Heisenberg group. We denote this spectral density by "e_λ" and it is given by

$$e_\lambda = \frac{dE_\lambda}{d\lambda},$$

where $\{E_\lambda\}$ is the spectral decomposition associated with the self-adjoint operator $\mathcal{L}$ which understood as an operator-valued distribution, i.e., an element of the space $D'(\mathbb{R}, L(D(\mathcal{L}), L^2(\mathcal{H}^7_\omega)))$. Here $L(D(\mathcal{L}), L^2(\mathcal{H}^7_\omega))$ is the space of bounded operators from $D(\mathcal{L})$ to $L^2(\mathcal{H}^7_\omega, dm)$.

Since the operator $\mathcal{L}$ is elliptic, its spectral density e_λ has an associated kernel $e(\lambda;.,.)$, an element of the space $D'(\mathbb{R}, D'(\mathcal{H}^7_\omega \times \mathcal{H}^7_\omega))$ [8, 9]. More precisely, we have the following result.

Theorem 3.1. *The spectral density* $e_\lambda = \frac{dE_\lambda}{d\lambda}$ *of* $\mathcal{L}$ *is the operator valued distribution* $\varphi \to \langle e_\lambda, \varphi\rangle$ *from* $\mathcal{D}(\mathbb{R})$ *to* $L(D(\mathcal{L}), L^2(\mathcal{H}^7_\omega))$ *given by*

$$\langle e_\lambda, \varphi\rangle f(x,z) = \int_{\mathcal{H}^7_\omega} \Big[\int_0^\infty e(\lambda; (x,z), (\tilde{x},\tilde{z}))\varphi(\lambda)d\lambda\Big] f(\tilde{x},\tilde{z})\, dm(\tilde{x},\tilde{z}),$$

where

$$\begin{aligned} e(\lambda; (x,z), (\tilde{x},\tilde{z})) &= \frac{2\sqrt{2}\pi}{\|x-\tilde{x}\|\|z-\tilde{z}+\Im(x\overline{\tilde{x}})/2\|} \\ &\quad \times \sum_{j=0}^{+\infty} [(\lambda+j^2)^{1/2}-j]^{\frac{3}{2}} L_j^{(-1)}([(\lambda+j^2)^{1/2}-j]\|x-\tilde{x}\|^2/2) \\ &\quad \times (\lambda+j^2)^{-\frac{1}{2}} \sin\Big([(\lambda+j^2)^{1/2}-j]\|z-\tilde{z}-\Im(\tilde{x}\overline{x})/2\|\Big) \\ &\quad \times ([(\lambda+j^2)^{1/2}-j]^2-\zeta) e^{-[(\lambda+j^2)^{1/2}-j]\|x-\tilde{x}\|^2/4} \end{aligned}$$

and

$$L_n^{(-1)}(t) = \frac{-t}{n}\sum_{k=0}^{n-1}\binom{n}{n-1-k}\frac{(-t)^k}{k!}. \tag{3.1}$$

Proof. By using the recurrence formulas $\Gamma(\gamma+1) = \gamma\Gamma(\gamma)$, we can write the resolvent kernel of $\mathcal{L}$ given in (2.14), as

$$\begin{aligned} &R(\zeta; (x,z), (\tilde{x},\tilde{z})) \\ &= \tfrac{-2\sqrt{2}\pi}{\tau\rho}\int_0^\infty (\tfrac{-\zeta}{t}+t)\Gamma(\tfrac{-\zeta}{2t}+\tfrac{t}{2}) W_{\frac{\zeta}{2t}-\frac{t}{2},-\frac{1}{2}}(t\rho^2/2)\sin(\tau t)t^{3/2}\, dt, \end{aligned} \tag{3.2}$$

where

$$\rho := \|x-\tilde{x}\| \quad \text{and} \quad \tau := \|z-\tilde{z}-\Im(\tilde{x}\overline{x})/2\|. \tag{3.3}$$

Substituting the summation formula [19, p.28]

$$U(a,b,z) = \frac{1}{\Gamma(a)}\sum_{j=0}^{+\infty}\frac{1}{j+a}L_j^{(b-1)}(z)$$

into the expression of the Whittaker function (2.16), we get

$$\Gamma(\mu-\kappa+\frac{1}{2})W_{\kappa,\mu}(z) = e^{-z/2}\, z^{\mu+\frac{1}{2}}\sum_{j=0}^{+\infty}\frac{1}{j+\mu-\kappa+\frac{1}{2}}L_j^{(2\mu)}(z). \tag{3.4}$$

For the parameters $\kappa = \frac{\zeta}{2t}-\frac{t}{2}, \mu = -\frac{1}{2}$ and $z = t\rho^2/2$, the integral in (3.2) takes the form

$$\begin{aligned} R(\zeta; (x,z), (\tilde{x},\tilde{z})) &= \tfrac{-4\sqrt{2}\pi}{\tau\rho}\sum_{j=0}^{+\infty}\int_0^\infty e^{-t\rho^2/4}L_j^{(-1)}(t\rho^2/2)\frac{(t^2-\zeta)t^{1/2}}{j+\frac{-\zeta}{2t}+\frac{t}{2}}\sin(\tau t)\, dt \\ &= \tfrac{-4\sqrt{2}\pi}{\tau\rho}\sum_{j=0}^{+\infty}\int_0^\infty e^{-t\rho^2/4}L_j^{(-1)}(t\rho^2/2)\frac{(t^2-\zeta)t^{3/2}}{2jt+t^2-\zeta}\sin(\tau t)\, dt, \end{aligned}$$

where the generalized Laguerre polynomials are given in [17, p.102] by

$$L_n^{(-m)}(x) = (-x)^m\frac{(n-m)!}{n!}\sum_{k=0}^{n-m}\binom{n}{n-m-k}\frac{(-x)^k}{k!}; \quad m \geq 1.$$

For each fix $j \geq 1$, by performing the change of variables $\lambda = 2jt + t^2$, we rewrite the above kernel as

$$
\begin{aligned}
&R(\zeta;(x,z),(\tilde{x},\tilde{z}))\\
&= \tfrac{-2\sqrt{2}\pi}{\tau\rho} \sum_{j=0}^{+\infty} \int_0^{\infty} \Big([(\lambda+j^2)^{1/2}-j]^2-\zeta\Big)[(\lambda+j^2)^{1/2}-j]^{3/2} e^{-[(\lambda+j^2)^{1/2}-j]\rho^2/4}\\
&\times (\lambda+j^2)^{-\frac{1}{2}} L_j^{(-1)}([(\lambda+j^2)^{1/2}-j]\rho^2/2) \sin\Big([(\lambda+j^2)^{1/2}-j]\tau\Big)\frac{d\lambda}{\lambda-\zeta}\\
&= \tfrac{2\sqrt{2}\pi}{\tau\rho} \int_0^{\infty} \Big[\sum_{j=0}^{+\infty} \Big([(\lambda+j^2)^{1/2}-j]^2-\zeta\Big)[(\lambda+j^2)^{1/2}-j]^{3/2} e^{-[(\lambda+j^2)^{1/2}-j]\rho^2/4}\\
&\times (\lambda+j^2)^{-\frac{1}{2}} L_j^{(-1)}([(\lambda+j^2)^{1/2}-j]\rho^2/2) \sin\Big([(\lambda+j^2)^{1/2}-j]\tau\Big)\Big]\frac{d\lambda}{\zeta-\lambda}.
\end{aligned}
$$

Setting

$$
\begin{aligned}
&e(\lambda;(x,z),(\tilde{x},\tilde{z}))\\
&= \tfrac{2\sqrt{2}\pi}{\tau\rho} \sum_{j=0}^{+\infty} ([(\lambda+j^2)^{1/2}-j]^2-\zeta)\,[(\lambda+j^2)^{1/2}-j]^{3/2} e^{-[(\lambda+j^2)^{1/2}-j]\rho^2/4}\\
&\times (\lambda+j^2)^{-\frac{1}{2}} L_j^{(-1)}([(\lambda+j^2)^{1/2}-j]\rho^2/2) \sin\Big([(\lambda+j^2)^{1/2}-j]\tau\Big),
\end{aligned}
\tag{3.5}
$$

we can write

$$
R(\zeta;(x,z),(\tilde{x},\tilde{z})) = \int_0^{\infty} \frac{e(\lambda;(x,z),(\tilde{x},\tilde{z}))}{\zeta-\lambda}\, d\lambda. \tag{3.6}
$$

The convergence of the series given in (3.5) can be seen as follows. For j sufficiently large, we have

$$
\begin{aligned}
&\Big|([(\lambda+j^2)^{1/2}-j]^2-\zeta)\,[(\lambda+j^2)^{1/2}-j]^{3/2} e^{-[(\lambda+j^2)^{1/2}-j]\rho^2/4}\\
&\times (\lambda+j^2)^{-1/2} L_j^{(-1)}([(\lambda+j^2)^{1/2}-j]\rho^2/2) \sin\Big([(\lambda+j^2)^{1/2}-j]\tau\Big)\Big|\\
&\leq c(\lambda) j^{-4} L_j^{(-1)}([(\lambda+j^2)^{1/2}-j]\rho^2/2),
\end{aligned}
\tag{3.7}
$$

where $c(\lambda)$ is a positive constant. Using the asymptotic formula [13, p.248],

$$
L_j^{(-1)}([(\lambda+j^2)^{1/2}-j]\rho^2/2) = O(j^{-1}),
$$

the inequality (3.7) becomes

$$
\begin{aligned}
&\Big|([(\lambda+j^2)^{1/2}-j]^2-\zeta)[(\lambda+j^2)^{1/2}-j]^{3/2}\,(\lambda+j^2)^{-1/2} e^{-[(\lambda+j^2)^{1/2}-j]\rho^2/4}\\
&\times L_j^{(-1)}([(\lambda+j^2)^{1/2}-j]\rho^2/2) \sin\Big([(\lambda+j^2)^{1/2}-j]\tau\Big)\Big| \leq c(\lambda) j^{-5}.
\end{aligned}
$$

Hence, the quantity $e(\lambda;(x,z),(\tilde{x},\tilde{z}))$ defined by the series above makes a sense.

Using the polynomial expression of $L_j^{(-1)}([(\lambda+j^2)^{1/2}-j]\|x-\tilde{x}\|^2/2)$ given in (3.1), we can prove that the associated kernel $e(\lambda;(x,z),(\tilde{x},\tilde{z}))$ of the spectral density e_λ is smooth with respect to the spacial variables $((x,z),(\tilde{x},\tilde{z})) \in \mathcal{H}_\omega^7 \times \mathcal{H}_\omega^7$.

Now, in view of (3.6) for $\Re(\zeta) < 0$ and $f, g \in L^2(\mathcal{H}_\omega^7, dm)$, we have

$$
\langle R(\zeta)f, g\rangle
$$

$$= \int_0^\infty \frac{d\lambda}{\zeta-\lambda} \int_{\mathcal{H}^7_\omega} \Big[\int_{\mathcal{H}^7_\omega} e(\lambda;(x,z),(\tilde{x},\tilde{z})) f(\tilde{x},\tilde{z})\, dm(\tilde{x},\tilde{z}) \Big] \overline{g(x,z)}\, dm(x,z).$$

On the other hand, by recalling the formula rewritten in the weak sense [15, p. 93]

$$\langle R(\zeta) f, g\rangle = \int_0^\infty \frac{1}{\zeta-\lambda} d\langle E_\lambda f, g\rangle.$$

and uniqueness of the spectral measure, we get

$$\frac{d\langle E_\lambda, g\rangle}{d\lambda} = \langle K_\lambda f, g\rangle,$$

where K_λ is the operator

$$K_\lambda f(x,z) = \int_{\mathcal{H}^7_\omega} e(\lambda;(x,z),(\tilde{x},\tilde{z})) f(\tilde{x},\tilde{z})\, dm(\tilde{x},\tilde{z}), \quad f \in L^2(\mathcal{H}^7_\omega, dm).$$

Finally, an interpretation argument of the operator K_λ in terms of the operators valued distribution see [8, p. 9] completes the proof. $\square$

Remark 3.2. Using the sandwiched resolvent formula [4, p. 57] we have

$$AR(\lambda - i0)B - AR(\lambda + i0)B = 2i\pi\, Ae_\lambda B,$$

where $R(\cdot)$ is the resolvent operator given in (2.13), for all $A, B \in \mathfrak{L}_2(L^2(\mathcal{H}^7_\omega))$ the set of all Hilbert-Schmidt operators.

4. Green kernel for fractional powers of $\mathcal{L}$

For $0 < \alpha < 1$ one defines the (fractional) power $\mathcal{L}^\alpha$ by the usual functional calculus. It is still an unbounded self-adjoint positive operator. As application of the formula obtained for the resolvent kernel of $\mathcal{L}$, we give the Green kernel of the fractional power operator $\mathcal{L}^\alpha$ for $\alpha \in]0,1[$. More precisely, we have the following result.

Theorem 4.1. *Let $\alpha \in]0,1[$. Then the Green kernel of the fractional power operator $\mathcal{L}^\alpha$ is*

$$\begin{aligned} G_\alpha((x,z),(\tilde{x},\tilde{z})) = \frac{-2\sqrt{2}\pi\|x-\tilde{x}\|}{\|z-\tilde{z}-\Im(\tilde{x}\overline{x})/2\|} & \int_0^\infty e^{-\|x-\tilde{x}\|^2 t/4} W_\alpha(t) \\ & \times \sin\Big(\|z-\tilde{z}-\Im(\tilde{x}\overline{x})/2\|\, t\Big) t^{\frac{7}{2}-\alpha}\, dt, \end{aligned} \tag{4.1}$$

where

$$W_\alpha(t) = \sum_{k=0}^\infty (t+2k+2)^{-\alpha} L_k^{(1)}(\|x-\tilde{x}\|^2 t/2). \tag{4.2}$$

Proof. Since $\mathcal{L}$ is a positive self-adjoint operator, its resolvent [12, p.21 or p.83] satisfies

$$\|R(-s)\| \le \frac{1}{s}. \tag{4.3}$$

This estimate enables us to define the fractional powers $\mathcal{L}^\alpha$, $\alpha \in]0,1[$ according to the formula [12, p.127]

$$\mathcal{L}^\alpha g = \frac{\sin\pi\alpha}{\pi} \int_0^\infty s^{\alpha-1} R(-s)\mathcal{L} g\, ds, \quad g \in D(\mathcal{L}). \tag{4.4}$$

Thanks to Kato's formula [12, p.124], the resolvent operator $R_\alpha(\gamma) = (\gamma - \mathcal{L}^\alpha)^{-1}$, $\alpha\pi < |\arg\gamma| < \pi$, is given by

$$R_\alpha(\gamma) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\zeta^\alpha R(-\zeta)}{\zeta^{2\alpha} - 2\zeta^\alpha\gamma\cos\pi\alpha + \gamma^2} d\zeta. \tag{4.5}$$

The action of $R_\alpha(\gamma)$ on a function $f \in L^2(\mathcal{H}^7_\omega)$ is

$$R_\alpha(\gamma)f(x,z) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\zeta^\alpha R(-\zeta)f(x,z)}{\zeta^{2\alpha} - 2\zeta^\alpha\gamma\cos\pi\alpha + \gamma^2} d\zeta,\ dm(x,z)$$

-almost every where. Then the resolvent kernel of $\mathcal{L}^\alpha$ is

$$R_\alpha(\gamma;(x,z),(\tilde{x},\tilde{z})) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\zeta^\alpha R(-\zeta;(x,z),(\tilde{x},\tilde{z}))}{\zeta^{2\alpha} - 2\zeta^\alpha\gamma\cos\pi\alpha + \gamma^2}\, d\zeta. \tag{4.6}$$

The limit value $\gamma = 0$ in (4.6) gives a Green kernel of $\mathcal{L}^\alpha$:

$$G_\alpha := R_\alpha(0;(x,z),(\tilde{x},\tilde{z})) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \zeta^{-\alpha} R(-\zeta;(x,z),(\tilde{x},\tilde{z}))\, d\zeta. \tag{4.7}$$

Using the expression in (2.14), we obtain that

$$G_\alpha = \frac{-2\sqrt{2}\sin(\pi\alpha)}{\tau\rho}\int_0^\infty N_{\alpha,\rho}(t)\sin(\tau t)\, t^{3/2}\, dt, \tag{4.8}$$

where ρ, τ are defined in (3.3) and

$$N_{\alpha,\rho}(t) = \int_0^\infty \zeta^{-\alpha}\Gamma(\tfrac{\zeta}{2t} + \tfrac{t}{2} + 1) W_{-\frac{\zeta}{2t} - \frac{t}{2}, -\frac{1}{2}}(t\rho^2/2)\, d\zeta.$$

By substituting a by $t\rho^2/2$ and using the identity $W_{\kappa,-\mu}(a) = W_{\kappa,\mu}(a)$ [2, p.265], the last integral becomes

$$N_{\alpha,\rho}(t) = \int_0^\infty \zeta^{-\alpha}\Gamma(\tfrac{\zeta}{2t} + \tfrac{t}{2} + 1) W_{-\frac{\zeta}{2t} - \frac{t}{2}, \frac{1}{2}}(a)\, d\zeta. \tag{4.9}$$

Next, using the integral representation [3, p.147],

$$\Gamma(\nu) W_{\frac{1}{2} - \frac{p}{2} - \nu, -\frac{p}{2}}(z) = z^{1/2 - p/2}\, e^{\frac{z}{2}}\int_0^\infty e^{-ps}(1 - e^{-s})^{\nu - 1} e^{-ze^s}\, ds; \quad \Re z, \Re\nu > 0.$$

In our case $z = a$, $\nu = \frac{\zeta}{2t} + \frac{t}{2} + 1$ and $p = -1$, and therefore (4.9) reads

$$N_{\alpha,\rho}(t) = a\, e^{a/2}\int_0^\infty e^s(1 - e^{-s})^{t/2} e^{-ae^s}\Big\{\int_0^\infty \zeta^{-\alpha}(1 - e^{-s})^{\zeta/2t} d\zeta\Big\}\, ds.$$

By directly computation, we have

$$\int_0^\infty \zeta^{-\alpha}(1 - e^{-s})^{\zeta/2t} d\zeta = (2t)^{1-\alpha}\Gamma(1 - \alpha)\Big[-\log(1 - e^{-s})\Big]^{\alpha - 1}.$$

It follows that

$$N_{\alpha,\rho}(t) = (2t)^{1-\alpha}\Gamma(1 - \alpha) a\, e^{a/2}\, I_\alpha(t), \tag{4.10}$$

where

$$I_\alpha(t) = \int_0^\infty e^s\, (1 - e^{-s})^{t/2} e^{-ae^s}[-\log(1 - e^{-s})]^{\alpha - 1}\, ds. \tag{4.11}$$

Making the change of variable $e^{-s} = 1 - e^{-y}$, equation (4.11) becomes

$$I_\alpha(t) = e^{-a}\int_0^\infty y^{\alpha - 1}(1 - e^{-y})^{-2} e^{-\frac{ae^{-y}}{1 - e^{-y}}}\, e^{-(\frac{t}{2} + 1)y}\, dy. \tag{4.12}$$

By using the identity

$$(1-w)^{-\beta-1}e^{-\frac{zw}{1-w}} = \sum_{k=0}^{\infty} L_k^{(\beta)}(z)\, w^k, \quad \beta, z \in \mathbb{C},\ |w|<1 \tag{4.13}$$

[17, p. 101] for $\beta = 1$, $w = e^{-y}$ and $z = a$, the integral $I_\alpha(t)$ may therefore be written as

$$I_\alpha(t) = e^{-a}\sum_{k=0}^{\infty} L_k^{(1)}(a)\int_0^{\infty} y^{\alpha-1}e^{-(\frac{t}{2}+k+1)y}\, dy. \tag{4.14}$$

The change of variables $\lambda = (\frac{t}{2}+k+1)y$ gives

$$\begin{aligned} I_\alpha(t) &= e^{-a}\sum_{k=0}^{\infty} L_k^{(1)}(a)(\frac{t}{2}+k+1)^{-\alpha}\int_0^{\infty} \lambda^{\alpha-1}e^{-\lambda}\, d\lambda \\ &= e^{-a}\,\Gamma(\alpha)\sum_{k=0}^{\infty} L_k^{(1)}(a)(\frac{t}{2}+k+1)^{-\alpha}. \end{aligned} \tag{4.15}$$

Hence, (4.10) becomes

$$\begin{aligned} N_{\alpha,\rho}(t) &= 2^{1-\alpha}\Gamma(\alpha)\Gamma(1-\alpha)ae^{-a/2}t^{1-\alpha}\sum_{k=0}^{\infty} L_k^{(1)}(a)(\frac{t}{2}+k+1)^{-\alpha} \\ &= 2\Gamma(\alpha)\Gamma(1-\alpha)a\, e^{-a/2}t^{1-\alpha}\sum_{k=0}^{\infty} L_k^{(1)}(a)(t+2k+2)^{-\alpha}. \end{aligned} \tag{4.16}$$

Substituting (4.16) into the expression of G_α in (4.8) and using the Euler's reflection formula [20, p. 239]

$$\Gamma(\gamma)\Gamma(1-\gamma) = \frac{\pi}{\sin(\pi\gamma)}$$

for any complex number γ, the integral G_α in (4.8) becomes

$$G_\alpha = \frac{-2\sqrt{2}\pi\rho}{\tau}\int_0^{\infty} e^{-\rho^2t/4}\Big\{\sum_{k=0}^{\infty}(t+2k+2)^{-\alpha}L_k^{(1)}(\rho^2t/2)\Big\}t^{\frac{7}{2}-\alpha}\sin(\tau t)dt,$$

where

$$W_\alpha(t) = \sum_{k=0}^{\infty}(t+2k+2)^{-\alpha}L_k^{(1)}(\rho^2t/2). \tag{4.17}$$

Using the asymptotic formula [13, p.248]

$$L_k^{(1)}(\rho^2t/2) = O(k), \tag{4.18}$$

we see that the series expansion in (4.17) is well defined. This completes the proof. $\square$

Remark 4.2. When α approaches 1 in (4.1), we recover the expression of the Green function which was obtained in [14]. We hope to return to the case $\alpha > 1$ in a future work.

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