Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 66, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A FRACTIONAL DIFFERENTIAL INCLUSION WITH NON-SMOOTH POTENTIAL

FARIBA FATTAHI, MOHSEN ALIMOHAMMADY

Communicated by Vicentiu Radulescu

Abstract. In this article, we use non-smooth critical point theory and variational methods to study the existence solutions for a fractional boundary-value problem. We provide some intervals for positive parameters in which the problem possess infinitely many solutions.

1. Introduction

In this article, we consider the boundary-value problem

$$\begin{gathered} {}_tD_T^\alpha({}_0D_t^\alpha u(t)) \in \lambda\partial F(u(t)) + \mu\partial G(u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{gathered} \tag{1.1}$$

where ${}_0D_t^\alpha$ and ${}_tD_T^\alpha$ are the left and right Riemann-Liouville fractional derivatives of order α with $0 < \alpha \le 1$, and where $\lambda > 0$ and $\mu \ge 0$ are two parameters. $F, G : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functions, where $F(\omega) = \int_0^\omega f(s)ds$, $G(\omega) = \int_0^\omega g(s)ds$, $\omega \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ are locally essentially bounded functions. $\partial F(u(t))$ denotes the generalized Clarke gradient of the function $F(u(t))$ at $u \in \mathbb{R}$.

We consider the following problem: Find $u \in E_0^\alpha[0,T]$, called a *weak solution* of (1.1), such that for any $v \in E_0^\alpha[0,T]$, we have

$$\int_0^T [{}_0D_t^\alpha u(t) \cdot {}_0D_t^\alpha v(t)]dt = \lambda \int_0^T u_1^*(t)v(t)dt + \mu \int_0^T u_2^*(t)v(t)dt, \tag{1.2}$$

where $u_1^*(t) \in \partial F(u(t))$ and $u_2^*(t) \in \partial G(u(t))$. Fractional differential problems were studied by many authors, see for example [11, 12, 17, 23]. Recently, fractional differential inclusions were considered by many authors: Ahamd et al. [1] studied the existence of solutions for impulsive fractional differential inclusions with anti-periodic boundary conditions. Ntouyas et al. [15] studied the existence of solutions for boundary value problems for nonlinear fractional differential inclusions with mixed type integral boundary conditions. More recently, the study of differential equations by variational method and critical point theory has attracted a lot of attention; see for example [9, 18, 20, 21]. Variational-hemivariational inequalities

2010 *Mathematics Subject Classification.* 35J87, 26A33, 49J40, 49J52.
Key words and phrases. Non-smooth critical point theory; fractional differential inclusion; infinitely many solutions.

Submitted August 21, 2016. Published March 4, 2017.

have been extensively studied in recent years via variational methods; see [2, 3, 4, 5, 19].

Here, we investigate the existence of infinitely many solutions for a fractional differential inclusion under some hypotheses on the behavior of the locally Lipschitz functions F and G in theorem 3.1. We prove the existence of infinitely many solutions for a variational-hemivariational inequality depending on two parameters. Also, we list some consequences of theorem 3.1 and give an example. Finally, we consider the uniform convergence of a sequence of solutions to zero in theorem 3.6.

2. Preliminaries

In this section, first we recall some basic definitions of fractional calculus and locally Lipschitz functions.

Definition 2.1 ([8]). Let f be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order α of f are denoted by ${}_aD_t^{-\alpha}f(t)$ and ${}_tD_b^{-\alpha}f(t)$, respectively, and defined by

$$
\begin{aligned}
{}_aD_t^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \quad t\in[a,b],\ \alpha>0,\\
{}_tD_b^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)ds, \quad t\in[a,b],\ \alpha>0,
\end{aligned}
$$

provided the right-hand sides are pointwise defined on $[a,b]$, where $\Gamma(\alpha)$ is the gamma function.

Definition 2.2 ([8]). Let f be a function defined on $[a,b]$. For $n-1\le\alpha<n$ $(n\in\mathbb{N})$, the left and right Riemann-Liouville fractional derivatives of order α for function f denoted by ${}_aD_t^{\alpha}f(t)$ and ${}_tD_b^{\alpha}f(t)$, respectively, are defined by

$$
\begin{aligned}
{}_aD_t^{\alpha}f(t) &= \frac{d^n}{dt^n}\,{}_aD_t^{\alpha-n}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}f(s)ds, \quad t\in[a,b],\\
{}_tD_b^{\alpha}f(t) &= (-1)^n\frac{d^n}{dt^n}\,{}_tD_b^{\alpha-n}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (s-t)^{n-\alpha-1}f(s)ds,\ t\in[a,b].
\end{aligned}
$$

Proposition 2.3 ([8, 22]). *We have the following property of fractional integration*

$$
\int_a^b [{}_aD_t^{-\alpha}f(t)]g(t)dt = \int_a^b [{}_tD_b^{-\alpha}g(t)]f(t)dt, \quad \alpha>0, \tag{2.1}
$$

provided that $f\in L^p([a,b],\mathbb{R}^N)$, $g\in L^q([a,b],\mathbb{R}^N)$ *and* $p\ge1$, $q\ge1$, $\frac{1}{p}+\frac{1}{q}\le 1+\alpha$ *or* $p\ne1$, $q\ne1$, $\frac{1}{p}+\frac{1}{q}=1+\alpha$.

Definition 2.4 ([8, 16]). For $n\in\mathbb{N}$, $n-1\le\alpha<n$ $(n\in\mathbb{N})$ and a function $f\in AC^n([a,b],\mathbb{R}^N)$, we define

$$
\begin{aligned}
{}_aD_t^{\alpha}f(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds + \Sigma_{j=0}^{n-1}\frac{f^j(a)}{\Gamma(j-\alpha+1)}(t-a)^{j-\alpha},\\
{}_tD_b^{\alpha}f(t) &= \frac{1}{\Gamma(n-\alpha)}\int_t^b \frac{f^{(n)}(s)}{(s-t)^{\alpha+1-n}}ds + \Sigma_{j=0}^{n-1}\frac{(-1)^j f^j(b)}{\Gamma(j-\alpha+1)}(b-t)^{j-\alpha},
\end{aligned}
$$

where $t\in[a,b]$.

Definition 2.5 ([8, 16]). *Let $0 < \alpha \leq 1$. The fractional derivative space $E_0^\alpha[0,T]$ is defined as the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the norm*

$$\|u\|_\alpha = \Big(\int_0^T |{}_0D_t^\alpha u(t)|^2 dt + \int_0^T |u(t)|^2 dt\Big)^{1/2}, \quad \forall u \in E_0^\alpha[0,T].$$

Clearly, the fractional derivative space $E_0^\alpha[0,T]$ is the space of functions $u \in L^2[0,T]$ having an $\alpha-$order fractional derivative ${}_0D_t^\alpha u(t) \in L^2[0,T]$ and $u(0) = u(T) = 0$.

Proposition 2.6 ([6]). *Let $0 < \alpha \leq 1$. The fractional derivative space $E_0^\alpha[0,T]$ is reflexive and separable Banach space.*

Proposition 2.7 ([6]). *Let $0 < \alpha \leq 1$. Then for all $u \in E_0^\alpha[0,T]$,*

$$\|u\|_{L^2} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u(t)\|_{L^2}, \tag{2.2}$$

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2(\alpha-1)+1)^{1/2}} \|{}_0D_t^\alpha u(t)\|_{L^2}. \tag{2.3}$$

According to (2.2), one can consider $E_0^\alpha[0,T]$ with the equivalent norm

$$\|u\|_\alpha = \Big(\int_0^T |{}_0D_t^\alpha u(t)|^2 dt\Big)^{1/2} = \|{}_0D_t^\alpha u\|_{L^2}, \quad \forall u \in E_0^\alpha[0,T].$$

Definition 2.8. A function $u : [0,T] \to \mathbb{R}$ is called a solution for (1.1) if

(1) ${}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t))$ and ${}_0D_t^{\alpha-1} u(t)$ exist for almost all $t \in [0,T]$;
(2) u satisfies in (1.1).

Definition 2.9. A function $u \in E_0^\alpha[0,T]$ is called a weak solution of (1.1) if there exist $u_1^*(x) \in \partial F(u)$, $u_2^*(x) \in \partial G(u)$, such that $u_1^* v,\ u_1^* v \in L^1[0,T]$ and

$$\int_0^T [{}_0D_t^\alpha u(t) \cdot {}_0D_t^\alpha v(t)] dt = \lambda \int_0^T u_1^*(x) v(x) dx + \mu \int_0^T u_2^*(x) v(x) dx, \tag{2.4}$$

for all $v \in E_0^\alpha[0,T]$.

For $\alpha > 1/2$, propositions 2.7 and 2.8, imply that

$$\|u\|_\infty \leq \mathcal{M}\Big(\int_0^T |{}_0D_t^\alpha u(t)|^2 dt\Big)^{1/2} = \mathcal{M}\|u\|_\alpha, \quad u \in E_0^\alpha[0,T],$$

where

$$\mathcal{M} = \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2(\alpha-1)+1)^{1/2}}.$$

Here, we recall some definitions and basic notation of the theory of generalized differentiation for locally Lipschitz functions. We refer the reader to [3, 4, 13, 14, 18] for more details. Let X be a Banach space and $X^\star$ its topological dual. By $\|\cdot\|$ we denote the norm in X and by $\langle\cdot,\cdot\rangle_X$ the duality brackets for the pair $(X, X^\star)$. A function $h : X \to \mathbb{R}$ is said to be locally Lipschitz if for any $x \in X$ there correspond a neighborhood V_x of x and a constant $L_x \geq 0$ such that

$$|h(z) - h(w)| \leq L_x\|z - w\|, \forall z, w \in V_x.$$

For a locally *Lipschitz* function $h : X \to \mathbb{R}$, the generalized directional derivative of h at $u \in X$ in the direction $\gamma \in X$ is defined by

$$h^0(u;\gamma) = \limsup_{w\to u, t\to 0^+} \frac{h(w+t\gamma) - h(w)}{t}.$$

The generalized gradient of h at $u \in X$ is

$$\partial h(u) = \{x^\star \in X^\star : \langle x^\star, \gamma\rangle_X \le h^0(u;\gamma),\ \forall \gamma \in X\},$$

which is non-empty, convex and $w^\star$-compact subset of $X^\star$, where $< \cdot,\cdot\rangle_X$ is the duality pairing between $X^\star$ and X.

Proposition 2.10 ([4]). *Let $h, g : X \to \mathbb{R}$ be locally Lipschitz functionals. Then, for any $u, v \in X$ the following hold:*

(1) *$h^0(u;\cdot)$ is subadditive, positively homogeneous;*
(2) *∂h is convex and weak* compact;*
(3) *$(-h)^0(u;v) = h^0(u;-v)$;*
(4) *the set-valued mapping $h : X \to 2^{X^*}$ is weak* u.s.c.;*
(5) *$h^0(u;v) = \max\{< \xi, v >: \xi \in \partial h(u)\}$;*
(6) *$\partial(\lambda h)(u) = \lambda \partial h(u)$ for every $\lambda \in \mathbb{R}$;*
(7) *$(h+g)^0(u;v) \le h^0(u;v) + g^0(u;v)$;*
(8) *the function $m(u) = \min_{\nu\in\partial h(u)} \nu_{X^*}$ exists and is lower semicontinuous; i.e., $\liminf_{u\to u_0} m(u) \ge m(u_0)$;*
(9) *$h^0(u;v) = \max_{u^*\in\partial h(u)}\langle u^*, v\rangle \le L\|v\|$.*

Definition 2.11 ([5]). An element $u \in X$ is called a critical point for functional h if

$$h^0(u; v-u) \ge 0, \quad \forall v \in X.$$

Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous and coercive, $\Upsilon : X \to \mathbb{R}$ a sequentially weakly upper semicontinuous, λ a positive real parameter. Moreover, assumeing that Φ and Υ are locally Lipschitz functionals, we set $\mathcal{L}_\lambda := \Phi - \lambda\Upsilon$. For every $r > \inf_X \Phi$, we define

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Upsilon(v)\right) - \Upsilon(u)}{r - \Phi(u)},$$
$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

First, we need to the following non-smooth version of a critical point theorem.

Theorem 2.12 ([10]). *Under the assumptions stated for X, Φ and Υ, the following statements hold:*

(a) *For any $r > \inf_X \Phi$ and $\lambda \in (0, \frac{1}{\varphi(r)})$, the restriction of the functional $\mathcal{L}_\lambda = \Phi - \lambda\Upsilon$ to $\Phi^{-1}(-\infty, r)$ admits a global minimum which is a critical point (local minimum) of $\mathcal{L}_\lambda$ in X.*
(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, \frac{1}{\gamma})$, the following alternative holds: either*
 (b1) *$\mathcal{L}_\lambda$ possesses a global minimum, or*
 (b2) *there is a sequence $\{u_n\}$ of critical points (local minima) of $\mathcal{L}_\lambda$ such that*
$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$
(c) *If $\delta < +\infty$, then for each $\lambda \in (0, \frac{1}{\delta})$, the following alternative holds: either*

(c1) *there is a global minimum of* Φ *which is a local minimum of* $\mathcal{L}_\lambda$, *or*
(c2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* $\mathcal{L}_\lambda$ *that converges weakly to a global minimum of* Φ.

3. Main results

Set

$$C(T,\alpha)=\frac{16}{T^2}\Big(\int_0^{T/4} t^{2(1-\alpha)}dt+\int_{T/4}^{3T/4}(t^{1-\alpha}-(t-\frac{T}{4})^{1-\alpha})^2dt \\ +\int_{\frac{3T}{4}}^{T}(t^{1-\alpha}-(t-\frac{T}{4})^{1-\alpha}-(t-\frac{3T}{4})^{1-\alpha})^2dt\Big).$$

and

$$A:=\liminf_{\omega\to+\infty}\frac{\max_{|x|\le\omega}F(x)}{\omega^2},\quad B:=\limsup_{\omega\to+\infty}\frac{F(\omega)}{\omega^2}.$$

Theorem 3.1. *Let* $\frac{1}{2}<\alpha\le 1$. *Assume*

(i) *that* $A<\frac{B}{\mathcal{M}^2C(T,\alpha)}$ *and* $f:\mathbb{R}\to\mathbb{R}$ *is a locally essentially bounded function such that* $F(t)=\int_0^t f(\xi)d\xi$ *for all* $t\in\mathbb{R}$.

then, for each $\lambda\in(\lambda_1,\lambda_2)$, *where*

$$\lambda_1=\frac{C(T,\alpha)}{BT},\quad \lambda_2=\frac{1}{\mathcal{M}^2TA},$$

and for any locally essentially bounded function $g:\mathbb{R}\to\mathbb{R}$ *whose potential* $G(t)=\int_0^t g(\xi)d\xi$ *for all* $t\in\mathbb{R}$ *is a non-negative function satisfying the condition*

$$G_\infty=\limsup_{\omega\to+\infty}\frac{\max_{|x|\le\omega}G(x)}{\omega^2}<+\infty \tag{3.1}$$

and for any $\mu\in[0,\mu_{G,\lambda})$, *where*

$$\mu_{G,\lambda}=\frac{1}{\mathcal{M}^2TG_\infty}(1-\lambda\mathcal{M}^2TA).$$

Then problem (1.1) *has a sequence of weak solutions for every* $\mu\in[0,\mu_{G,\lambda})$.

Proof. Our purpose is to apply theorem 2.12(b). Fix $\bar\lambda\in(\lambda_1,\lambda_2)$ and G satisfying our assumptions. Since $\bar\lambda<\lambda_2$, it implies that

$$\mu_{G,\bar\lambda}=\frac{1}{\mathcal{M}^2TG_\infty}(1-\bar\lambda\mathcal{M}^2TA)>0.$$

Fix $\bar\mu\in[0,\mu_{G,\bar\lambda})$ and define the functionals $\Phi,\Upsilon:X\to\mathbb{R}$ for each $u\in X$ as follows:

$$\Phi(u)=\frac{1}{2}\|u\|_\alpha^2 \tag{3.2}$$

$$\Upsilon(u)=\int_0^T[F(u(t))]dt+\frac{\bar\mu}{\bar\lambda}\int_0^T[G(u(t))]dt. \tag{3.3}$$

Put $\mathcal{L}_{\bar\lambda}(u):=\Phi(u)-\bar\lambda\Upsilon$. The critical points of the functional $\mathcal{L}_{\bar\lambda}$ are the weak solutions of problem (1.1). According to [6], Φ is continuous and convex, so it is weakly sequentially lower semicontinuous, also Φ is continuously Gâteaux differentiable and coercive. By standard argument, Υ is sequentially weakly continuous.

First, we claim that $\bar{\lambda} < 1/\gamma$. Note that $\Phi(0) = \Upsilon(0) = 0$, then for n large enoughlarge,

$$\begin{aligned}\varphi(r) &= \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r[)}\Upsilon(v)\right)-\Upsilon(u)}{r-\Phi(u)}\\ &\le \frac{\sup_{v\in\Phi^{-1}(]-\infty,r[)}\Upsilon(v)}{r}.\end{aligned}$$

Let $\{\omega_n\}$ be a sequence of positive numbers in X such that $\lim_{n\to+\infty}\omega_n = +\infty$ and

$$A = \lim_{n\to+\infty}\frac{\max_{|x|\le\omega_n}F(x)}{\omega_n^2}.$$

Set

$$r_n = \frac{\omega_n^2}{\mathcal{M}^2},\ n\in\mathbb{N}.$$

Hence,

$$\begin{aligned}\varphi(r_n) &\le \frac{\max_{|x|\le\omega_n}T[F(x)+\frac{\bar{\mu}}{\bar{\lambda}}G(x)]}{\frac{\omega_n^2}{\mathcal{M}^2}}\\ &\le \mathcal{M}^2T\frac{\max_{|x|\le\omega_n}[F(x)+\frac{\bar{\mu}}{\bar{\lambda}}G(x)]}{\omega_n^2}\\ &\le \mathcal{M}^2T\Big[\frac{\max_{|x|\le\omega_n}F(x)}{\omega_n^2}+\frac{\bar{\mu}}{\bar{\lambda}}\frac{\max_{|x|\le\omega_n}G(x)}{\omega_n^2}\Big].\end{aligned}$$

Moreover, from assumption (i) and the condition (3.1), it follows that

$$\frac{\max_{|x|\le\omega_n}F(x)}{\omega_n^2}+\frac{\bar{\mu}}{\bar{\lambda}}\frac{\max_{|x|\le\omega_n}G(x)}{\omega_n^2} < +\infty.$$

Then

$$\gamma \le \liminf_{n\to+\infty}\varphi(r_n) \le \mathcal{M}^2T(A+\frac{\bar{\mu}}{\bar{\lambda}}G_\infty) < +\infty.$$

It is clear that, for any $\bar{\mu}\in[0,\mu_{G,\bar{\lambda}})$,

$$\gamma \le \mathcal{M}^2TA + \frac{(1-\bar{\lambda}\mathcal{M}^2TA)}{\bar{\lambda}};$$

therefore,

$$\bar{\lambda} = \frac{1}{\mathcal{M}^2TA+(1-\bar{\lambda}\mathcal{M}^2A)/\bar{\lambda}} < \frac{1}{\gamma}.$$

We claim that the functional $\mathcal{L}_{\bar{\lambda}}$ is unbounded from below. We can consider a sequence $\{\tau_n\}$ of positive numbers such that $\tau_n\to+\infty$. Let $\{\xi_n\}$ be a sequence in X for all $n\in\mathbb{N}$, defined by

$$\xi_n(t) = \begin{cases}\frac{4\Gamma(2-\alpha)\tau_n}{T}t & t\in[0,\frac{T}{4}]\\ \Gamma(2-\alpha)\tau_n & t\in[\frac{T}{4},\frac{3T}{4}]\\ \frac{4\Gamma(2-\alpha)\tau_n}{T}(T-t) & t\in[\frac{3T}{4},T],\end{cases} \tag{3.4}$$

Clearly, $\xi_n(0)=\xi_n(T)=0$ and $\xi_n\in L^2[0,T]$. A direct calculation shows that

$${}_0D_t^\alpha\xi_n(t) = \begin{cases}\frac{4\tau_n}{T}t^{1-\alpha} & t\in[0,\frac{T}{4})\\ \frac{4\tau_n}{T}(t^{1-\alpha}-(t-\frac{T}{4})^{1-\alpha}) & t\in[\frac{T}{4},\frac{3T}{4}]\\ \frac{4\tau_n}{T}(t^{1-\alpha}-(t-\frac{T}{4})^{1-\alpha}-(t-\frac{3T}{4})^{1-\alpha}) & t\in(\frac{3T}{4},T].\end{cases} \tag{3.5}$$

Moreover,

$$\begin{aligned}
&\int_0^T ({}_0D_t^\alpha \xi_n(t))^2 dt \\
&= \int_0^{T/4} + \int_{T/4}^{3T/4} + \int_{\frac{3T}{4}}^T ({}_0D_t^\alpha \xi_n(t))^2 dt \\
&= \frac{16\tau_n^2}{T^2}\Big(\int_0^{T/4} t^{2(1-\alpha)}dt + \int_{T/4}^{3T/4}(t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha})^2 dt \\
&\quad + \int_{\frac{3T}{4}}^T (t^{1-\alpha} - (t-\frac{T}{4})^{1-\alpha} - (t-\frac{3T}{4})^{1-\alpha})^2 dt\Big)
\end{aligned} \tag{3.6}$$

$$= C(T,\alpha)\tau_n^2,$$

for each $n \in \mathbb{N}$. Thus, $\xi_n \in E_0^\alpha[0,T]$.

Since G is non-negative and by the definition of Υ, we have

$$\begin{aligned}
\Upsilon(\xi_n) &= \int_0^T [F(\xi_n(t)) + \frac{\bar{\mu}}{\bar{\lambda}} G(\xi_n(t))]dt \ge \int_0^T F(\xi_n(t))dt \\
&\ge \int_{T/4}^{3T/4} F(\xi_n(t))dt \ge F(\Gamma(2-\alpha)\tau_n)\int_{T/4}^{3T/4} dt.
\end{aligned} \tag{3.7}$$

Let

$$B = \limsup_{\omega\to+\infty} \frac{F(\omega)}{\omega^2}. \tag{3.8}$$

If $B < +\infty$, set $\epsilon \in (0, B - \frac{C(T,\alpha)}{\lambda\Gamma^2(2-\alpha)T})$. Then from 3.8 there exists N_1 such that

$$\int_{T/4}^{3T/4} F(\Gamma(2-\alpha)\tau_n)dt > (B-\epsilon)\Gamma^2(2-\alpha)\tau_n^2\frac{T}{2}, \quad \forall n > N_1.$$

According to (3.6) and (3.7),

$$\begin{aligned}
\mathcal{L}_{\bar{\lambda}}(\xi_n) &\le \frac{1}{2}C(T,\alpha)\tau_n^2 - \bar{\lambda}(B-\epsilon)\Gamma^2(2-\alpha)\tau_n^2\frac{T}{2} \\
&= \tau_n^2(\frac{1}{2}C(T,\alpha) - \bar{\lambda}(B-\epsilon)\Gamma^2(2-\alpha)\frac{T}{2}),
\end{aligned} \tag{3.9}$$

for $n > \mathbb{N}_1$. Choosing a suitable ϵ and $\lim_{n\to+\infty}\tau_n = +\infty$, it results that

$$\lim_{n\to+\infty} \mathcal{L}_{\bar{\lambda}}(\xi_n) = -\infty.$$

If $B = +\infty$, we fix $\nu > \frac{C(T,\alpha)}{\lambda\Gamma^2(2-\alpha)T}$ and from 3.8 there exists N_ν such that

$$\int_{T/4}^{3T/4} F(\Gamma(2-\alpha)\tau_n)dt > \nu\Gamma^2(2-\alpha)\tau_n^2\frac{T}{2}, \quad \forall n > N_\nu.$$

Hence,

$$\mathcal{L}_{\bar{\lambda}}(\xi_n) \le \frac{1}{2}C(T,\alpha)\tau_n^2 - \bar{\lambda}F(\Gamma(2-\alpha)\tau_n\int_{T/4}^{3T/4} dt < \tau_n^2(\frac{1}{2}C(T,\alpha) - \bar{\lambda}\nu\Gamma^2(2-\alpha)\frac{T}{2}),$$

for all $n > N_\nu$. Taking into account the choice of ν, it leads to $\lim_{n\to+\infty}\Phi(u_n) - \bar{\lambda}\Psi(u_n) = -\infty$. Hence, the functional $\mathcal{L}_{\bar{\lambda}}$ is unbounded from below, and it follows that $\mathcal{L}_{\bar{\lambda}}$ has no global minimum. Therefore, applying theorem (2.12) there exists a sequence $\{u_n\} \in X$ of critical points of $\mathcal{L}_{\bar{\lambda}}$ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$. From (3.2) it follows that $\sqrt[2]{\Phi(u_n)} = \|u_n\|_\alpha$ such that $\lim_{n\to+\infty} \|u_n\|_\alpha = +\infty$. □

Lemma 3.2. *Every critical point $u \in E_0^\alpha[0,T]$ of $\mathcal{L}_\lambda$ is a solution of problem* (1.1).

Proof. We suppose that $u \in E_0^\alpha[0,T]$ is a critical point of $\mathcal{L}_\lambda$. There exist $u_1^* \in \partial F(u)$ and $u_2^* \in \partial G(u)$ satisfying

$$\int_0^T ({}_0D_t^\alpha u(t) \cdot {}_0D_t^\alpha v(t))dt - \lambda \int_0^T u_1^*(t)v(t)dt - \mu \int_0^T u_2^*(t)v(t)dt = 0, \tag{3.10}$$

for all $v \in E_0^\alpha[0,T]$. Since $u_1^* v, u_2^* v \in L^1([0,T],\mathbb{R}^N)$, it follows that ${}_tD_T^{-\alpha}u_1^*$, ${}_tD_T^{-\alpha}u_2^* \in L^1([0,T],\mathbb{R}^N)$.

Set $k_1(t) = {}_tD_T^{-\alpha}u_1^*(t)$ and $k_2(t) = {}_tD_T^{-\alpha}u_2^*(t), t \in [0,T]$. From the definition of left and right Riemann-Liouville fractional derivatives

$$\begin{aligned}
&\int_0^T (k_1(t) \cdot {}_0D_t^\alpha v(t))dt + \int_0^T (k_2(t) \cdot {}_0D_t^\alpha v(t))dt \\
&= \int_0^T ({}_tD_T^{-\alpha}u_1^*(t) \cdot {}_0D_t^\alpha v(t))dt + \int_0^T ({}_tD_T^{-\alpha}u_2^*(t) \cdot {}_0D_t^\alpha v(t))dt \\
&= \int_0^T (u_1^*(t) \cdot {}_0D_t^{-\alpha}({}_0D_t^\alpha v(t)))dt + \int_0^T (u_2^*(t) \cdot {}_0D_t^{-\alpha}({}_0D_t^\alpha v(t)))dt \\
&= \int_0^T (u_1^*(t) \cdot v(t))dt + \int_0^T (u_2^*(t) \cdot v(t))dt.
\end{aligned}$$

From (3.10),

$$\begin{aligned}
&\int_0^T ({}_0D_t^\alpha u(t) - \lambda k_1(t) - \mu k_2(t)) \cdot {}_0D_t^\alpha v(t))dt \\
&= \int_0^T \Big({}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t) - \lambda k_1(t) - \mu k_2(t)) \cdot v'(t)\Big)dt = 0,
\end{aligned} \tag{3.11}$$

for all $v \in C_0^\infty([0,T],\mathbb{R}^N)$. Using the argument in [7] we obtain

$$ {}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t) - \lambda k_1(t) - \mu k_2(t)) = C, \quad \forall t \in [0,T].$$

In view of $u_1^*, u_2^* \in L^1([0,T],\mathbb{R}^N)$, we identify the equivalence class ${}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t))$ and its continuous representative

$$ {}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t)) = \lambda \int_t^T u_1^*(t)dt + \mu \int_t^T u_2^*(t)dt + C, \quad \forall t \in [0,T]. \tag{3.12}$$

By properties of the left and right Riemann-Liouville fractional derivatives, we have ${}_tD_T^\alpha({}_0D_t^\alpha u(t)) = -({}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t)))' \in L^1([0,T],\mathbb{R}^N)$. Hence, it follows from (3.5), that

$$ {}_tD_T^\alpha({}_0D_t^\alpha u(t)) = \lambda u_1^*(t) + \mu u_2^*(t), \quad \text{a.e. } t \in [0,T].$$

Moreover, $u \in E_0^\alpha[0,T]$ implies that $u(0) = u(T) = 0$. □

Remark 3.3. Under the following two conditions

$$\liminf_{\omega\to+\infty} \frac{\max_{|x|\le\omega} F(x)}{\omega^2} = 0, \quad \limsup_{\omega\to+\infty} \frac{F(\omega)}{\omega^2} = +\infty,$$

according to Theorem 3.1, for each $\lambda > 0$ and each $\mu \in [0, \frac{1}{\mathcal{M}^2 T G_\infty}[$, problem (1.1) admits infinitely many solutions in $E_0^\alpha[0,T]$. In addition, if $G_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

Example. Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences defined by

$$a_n = ne^n, \quad b_n = (n+1)e^n.$$

We define a sequence of non-negative functions

$$F_n(x) = \begin{cases} (|x|+n)^2 \exp\big(-|\frac{1}{((|x|-ne^e-e^n)^2-(e^{2n}))}|\big) & ne^n < |x| < (n+2)e^n \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

A direct computation shows that

$$\lim_{n\to+\infty} \frac{\max_{|x|\leq a_n} F_n(x)}{a_n^2} = 0, \quad \lim_{n\to+\infty} \frac{F_n(b_n)}{b_n^2} < +\infty.$$

Then Theorem 3.1, implies that for any non-negative function $g : \mathbb{R} \to \mathbb{R}$, whose potential $G(\omega) = \int_0^\omega g(t)dt$ satisfies the condition (3.1), problem (1.1) possesses a sequence of solutions.

An immediate consequence of theorem 3.1 is a special case when $\mu = 0$.

Theorem 3.4. *Assume that the assumptions in theorem 3.1 hold. Then, for each*

$$\lambda \in \Big]\frac{C(T,\alpha)}{T\limsup_{\omega\to+\infty}\frac{F(\omega)}{\omega^2}}, \frac{1}{\mathcal{M}^2 T \liminf_{\omega\to+\infty}\frac{\max_{|x|\leq\omega}F(x)}{\omega^2}}\Big[,$$

the problem

$$_tD_T^\alpha(_0D_t^\alpha u(t)) \in \lambda\partial F(u(t)) \quad \textit{a.e. } t \in [0,T],$$

has an unbounded sequence of solutions in $E_0^\alpha[0,T]$.

Theorem 3.5. *Let* $l_1 : \mathbb{R} \to \mathbb{R}$ *be a locally essentially bounded function and denote* $L_1(x) = \int_0^x l_1(s)ds$ *for all* $s \in \mathbb{R}$. *Suppose that*

- (i1) $\liminf_{\omega\to+\infty}\frac{L_1(\omega)}{\omega^2} < +\infty$,
- (i2) $\limsup_{\omega\to+\infty}\frac{L_1(\omega)}{\omega^2} = +\infty$.

Then, for any locally essentially bounded functions $l_i : \mathbb{R} \to \mathbb{R}$ *for* $2 \leq i \leq n$ *such that*

- (i3) $\max\big\{\sup_{\omega\in\mathbb{R}} L_i(\omega); 2 \leq i \leq n\big\} \leq 0$ *and*
- (i4) $\min\big\{\liminf_{\omega\to+\infty}\frac{L_i(\omega)}{\omega^2}; 2 \leq i \leq n\big\} > -\infty$, *where* $L_i(x) = \int_0^x l_i(s)ds$, $x \in \mathbb{R}$, $2 \leq i \leq n$, *for each*

$$\lambda \in \Big]0, \frac{1}{\mathcal{M}^2 T \liminf_{\omega\to+\infty}\frac{L_1(\omega)}{\omega^2}}\Big[$$

and for any locally essentially bounded function $g : \mathbb{R} \to \mathbb{R}$ *(whose potential* $G(x) = \int_0^x g(s)ds$ *for every* $x \in \mathbb{R}$*) satisfying* (3.1), *and for every* $\mu \in [0, \mu_{G,\lambda}[$, *where*

$$\mu_{G,\lambda} = \frac{1}{\mathcal{M}^2 T G_\infty}(1 - \lambda\mathcal{M}^2 T \liminf_{\omega\to+\infty}\frac{L_1(\omega)}{\omega^2}),$$

then the problem

$$ {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) \in \lambda\partial\Sigma_{i=1}^{n}L_i(u(t)) + \mu\partial G(u(t)) \quad a.e.\ t\in[0,T],\ u(0)=u(T)=0, $$

admits an unbounded sequence of solutions in $E_0^{\alpha}[0,T]$.

Proof. Set $F(x)=\Sigma_{i=1}^{n}L_i(x)$ for all $x\in\mathbb{R}$. In view of (i2) and (i4),

$$ \limsup_{\omega\to+\infty}\frac{F(\omega)}{\omega^2} = \limsup_{\omega\to+\infty}\frac{\Sigma_{i=1}^{n}L_i(\omega)}{\omega^2} = +\infty. $$

Conditions (i1) and (i3) imply that

$$ \liminf_{\omega\to+\infty}\frac{\max_{|x|\le\omega}F(x)}{\omega^2} \le \liminf_{\omega\to+\infty}\frac{L_1(\omega)}{\omega^2} < +\infty. $$

By using theorem 3.1, we complete the proof. □

Theorem 3.6. *Let f be a locally essentially bounded function and suppose that*

$$ \liminf_{\omega\to0^+}\frac{F(\omega)}{\omega^2} < \frac{1}{\mathcal{M}^2C(T,\alpha)}\limsup_{\omega\to0^+}\frac{F(\omega)}{\omega^2}. \tag{3.14} $$

Then for any $\lambda\in\Lambda_1:=]\lambda_3,\lambda_4[$, where

$$ \lambda_3 = \frac{C(T,\alpha)}{T\limsup_{\omega\to0^+}\frac{F(\omega)}{\omega^2}}, \quad \lambda_4 = \frac{1}{\mathcal{M}^2T\liminf_{\omega\to0^+}\frac{F(\omega)}{\omega^2}}, $$

and for any $g:\mathbb{R}\to\mathbb{R}$ such that

$$ G_0 = \limsup_{\omega\to0^+}\frac{\max_{|x|\le\omega}G(x)}{\omega^2} < +\infty \tag{3.15} $$

and

$$ \mu_{G,\lambda}^1 = \frac{1}{\mathcal{M}^2TG_0}\Big(1-\lambda\mathcal{M}^2T\liminf_{\omega\to0^+}\frac{F(\omega)}{\omega^2}\Big) $$

($\mu_{G,\lambda}^1=+\infty$, when $G_0=0$). Problem 1.1 has a sequence of solutions, which converges strongly to 0 in $E_0^{\alpha}[0,T]$.

Proof. Fix $\lambda\in\Lambda_1$ and pick $\mu\in[0,\mu_{G,\lambda}^1[$. Suppose that Φ,Υ are the functionals defined by (3.2) and (3.3). Let l_n be a sequence of positive numbers such that $\lim_{n\to\infty}l_n=0$ and

$$ \lim_{n\to\infty}\frac{F(l_n)}{l_n^2} = \liminf_{\omega\to0^+}\frac{F(\omega)}{\omega^2}. $$

As in theorem 3.1, we set $r_n=\frac{l_n^2}{\mathcal{M}^2}$, $n\in\mathbb{N}$. It follows that $\delta<+\infty$. First we show that

$$ \Phi-\lambda\Upsilon \text{ does not have a local minimum at zero.} \tag{3.16} $$

Let $\{\theta_n\}$ be a sequence of positive numbers such that $\theta_n\to0$ in $]0,\theta[$, $\theta>0$ and $\{\xi_n\}$ be the sequence defined in 3.4. According to the non-negativity of G it leads that 3.7 satisfies. Using 3.14, $\lambda_3<\lambda_4$. Let

$$ B_1 = \limsup_{\omega\to0^+}\frac{F(\omega)}{\omega^2}. $$

If $B_1<+\infty$, then 3.9 holds. By the choice of ϵ,

$$ \lim_{n\to+\infty}(\Phi(\xi_n)-\lambda\Upsilon(\xi_n)) < 0 = \Phi(0)-\lambda\Upsilon(0). $$

Therefore, $\Phi - \lambda\Upsilon$ does not have a local minimum at zero, in view of fact that $\|\xi_n\| \to 0$.

An argument similar to the one in the proof of theorem 3.1 ,for the case $B_1 = 0$, imply 3.16. Since $\min_X \Phi = \Phi(0)$, in view of theorem 2.12(c) the consequence is obtained. $\square$

Next we show an application of theorem 3.1 for obtaining infinitely many solutions. Consider

$$\begin{gathered} {}_tD_T^\alpha({}_0D_t^\alpha u(t)) \in -\lambda\theta(t)\partial F(u) - \mu\vartheta(t)\partial G(u) \quad t \in [0,T] \\ u(0) = u(T) = 0, \end{gathered} \tag{3.17}$$

where λ, μ are real parameters, $\lambda > 0, \mu \geq 0$ and $F, G : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz functions given by $F(\omega) = \int_0^\omega f(t)dt$, $G(\omega) = \int_0^\omega g(t)dt$, $\omega \in \mathbb{R}$ such that $f, g : \mathbb{R} \to \mathbb{R}$ are measurable (not necessarily continuous) functions. Moreover, $\theta, \vartheta \in L^1[0,T]$ and $\theta, \vartheta \geq 0$ will given. Our result is stated as follows.

Theorem 3.7. *Assume the following two conditions:*

(i1')

$$\liminf_{\omega\to+\infty} \frac{\max_{|x|\leq\omega} F(x)}{\omega^2} < \frac{\theta^* \limsup_{\omega\to+\infty} \frac{F(\omega)}{\omega^2}}{\mathcal{M}^2 C(T,\alpha)},$$

(i2')

$$\lambda_1 = \frac{C(T,\alpha)}{T\theta^* \limsup_{\omega\to+\infty} \frac{F(\omega)}{\omega^2}}, \quad \lambda_2 = \frac{1}{T\mathcal{M}^2 \liminf_{\omega\to+\infty} \frac{\max_{|x|\leq\omega} F(x)}{\omega^2}},$$

where $\theta^* = \int_0^T \theta(t)dt$ *and* $\vartheta^* = \int_0^T \vartheta(t)dt$.

Then for any $\mu \in [0, \mu_{G,\lambda})$, *problem* (3.17) *has an unbounded sequence of solutions in* $E_0^\alpha[0,T]$.

Proof. Define the functionals $\Phi, \mathcal{E} : X \to \mathbb{R}$ for each $u \in X$ as follows:

$$\begin{gathered} \Phi(u) = \frac{1}{2}\|u\|_\alpha^2, \\ \Upsilon(u) = \int_0^T \theta(t)F(u(t))dt + \frac{\mu}{\lambda}\int_0^T \vartheta(t)G(u(t))dt, \\ \mathcal{E}(u) = \Upsilon(u) - \chi(u), \quad \mathcal{L}_\lambda(u) := \Phi(u) - \lambda\mathcal{E}(u). \end{gathered}$$

As in theorem 3.1, we show that $\bar{\lambda} < \frac{1}{\gamma}$. Note that

$$\begin{aligned} \varphi(r_n) &\leq \frac{\max_{|x|\leq\omega_n}[\theta^* F(x) + \frac{\bar{\mu}}{\bar{\lambda}}\vartheta^* G(x)]}{\frac{\omega_n^2}{\mathcal{M}^2}} \\ &\leq \mathcal{M}^2 \frac{\max_{|x|\leq\omega_n}[\theta^* F(x) + \frac{\bar{\mu}}{\bar{\lambda}}\vartheta^* G(x)]}{\omega_n^2} \\ &\leq \mathcal{M}^2 \Big[\frac{\max_{|x|\leq\omega_n} \theta^* F(x)}{\omega_n^2} + \frac{\bar{\mu}}{\bar{\lambda}} \frac{\max_{|x|\leq\omega_n} \vartheta^* G(x)}{\omega_n^2}\Big], \end{aligned}$$

Therefore,

$$\gamma \leq \liminf_{n\to+\infty} \varphi(r_n) \leq \mathcal{M}^2(\theta^* A + \frac{\bar{\mu}}{\bar{\lambda}}\vartheta^* G_\infty) < +\infty.$$

It is clear that, for every $\bar{\mu} \in [0, \mu_{G,\bar{\lambda}})$,

$$\gamma \le \mathcal{M}^2\theta^* A + \vartheta^* \frac{(1-\bar{\lambda}\mathcal{M}^2 TA)}{T\bar{\lambda}}.$$

Then

$$\bar{\lambda} = \frac{1}{\mathcal{M}^2\theta^* A + \vartheta^*(1-\bar{\lambda}\mathcal{M}^2 TA)/T\bar{\lambda}} < \frac{1}{\gamma}.$$

We claim that the functional $\mathcal{L}_{\bar{\lambda}}$ is unbounded from below.

Since G is non-negative, from the definition of Υ we have

$$\begin{aligned}
\Upsilon(u) &= \int_0^T \theta(t)F(u(t))dt + \frac{\mu}{\lambda}\int_0^T \vartheta(t)G(u(t))dt \\
&\ge \int_{T/4}^{3T/4} \theta(t)F(\xi_n(t))dt \\
&\ge F(\Gamma(2-\alpha)\tau_n)\int_{T/4}^{3T/4} \theta(t)dt = F(\Gamma(2-\alpha)\tau_n)\theta',
\end{aligned} \tag{3.18}$$

where $\theta' = \int_{T/4}^{3T/4} \theta(t)dt$. Set

$$B = \limsup_{\omega\to+\infty} \frac{F(\omega)}{\omega^2}. \tag{3.19}$$

If $B < +\infty$, let $\epsilon \in (0, B - \frac{C(T,\alpha)}{2\lambda\theta'\Gamma^2(2-\alpha)})$, then from 3.19 there exists N_1 such that

$$\theta' F(\Gamma(2-\alpha)\tau_n) > \theta'(B-\epsilon)\Gamma^2(2-\alpha)\tau_n^2, \quad \forall n > N_1.$$

According to (3.18),

$$\begin{aligned}
\mathcal{L}_{\bar{\lambda}}(\xi_n) &\le \frac{1}{2}C(T,\alpha)\tau_n^2 - \bar{\lambda}(\beta-\epsilon)\theta'\Gamma^2(2-\alpha)\tau_n^2 \\
&= \tau_n^2\Big(\frac{1}{2}C(T,\alpha) - \bar{\lambda}(\beta-\epsilon)\theta'\Gamma^2(2-\alpha)\Big),
\end{aligned} \tag{3.20}$$

for $n > N_1$. Choosing a suitable ϵ and using that $\lim_{n\to+\infty} \tau_n = +\infty$, it results that

$$\lim_{n\to+\infty} \mathcal{L}_{\bar{\lambda}}(\xi_n) = -\infty.$$

If $B = +\infty$, we fix $\nu > \frac{C(T,\alpha)}{2\bar{\lambda}\theta'\Gamma^2(2-\alpha)}$ and from 3.19, there exists N_ν such that

$$\theta' F(\Gamma(2-\alpha)\tau_n) > \theta'\nu\Gamma^2(2-\alpha)\tau_n^2, \quad \forall n > N_\nu.$$

Hence,

$$\mathcal{L}_{\bar{\lambda}}(\xi_n) \le \frac{1}{2}C(T,\alpha)\tau_n^2 - \bar{\lambda}\theta' F(\Gamma(2-\alpha)\tau_n) < \tau_n^2\Big(\frac{1}{2}C(T,\alpha) - \bar{\lambda}\theta'\nu\Gamma^2(2-\alpha)\Big),$$

for all $n > N_\nu$. Taking into account the choice of ν, implies that $\lim_{n\to+\infty} \Phi(u_n) - \bar{\lambda}\Psi(u_n) = -\infty$. From theorem 2.12, there is a sequence $\{u_n\} \in X$ of critical points of $\mathcal{L}_{\bar{\lambda}}$ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$. □

Acknowledgements. The authors are very grateful to the anonymous referee for the valuable suggestions.

References

[1] B. Ahmad, J. J. Nieto; *A study of impulsive fractional differential inclusions with anti-periodic boundary conditions,* Fractional Differential Calculus, **1** (2012), 1–15.

[2] M. Alimohammady, F. Fattahi; *Existence of solutions to hemivaritional inequalities involving the $p(x)$-biharmonic operator,* Electron. J. Diff. Equ., Vol. 2015 (2015), 1–12.

[3] S. Carl, V. K. Le, D. Motreanu; *Nonsmooth variational problems and their inequalities,* Springer Monographs in Mathematics, Springer, New York, 2007.

[4] F. H. Clarke; *Optimization and nonsmooth analysis,* Wiley, 1983.

[5] A. Iannizzotto; *Three critical points for perturbed nonsmooth functionals and applications,* Nonlinear Analysis, Theory, Methods & Applications , **72** (2010), 1319–1338.

[6] F. Jiao, Y. Zhou; *Existence results for fractional boundary value problem via critical point theory,* International Journal of Bifurcation and Chaos, **22** (2012), 1–17.

[7] F. Jiao, Y. Zhou; *Existence of solutions for a class of fractional boundary value problems via critical point theory,* Comput. Math. Appl. **62** (2011), 1181–1199.

[8] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations,* Elsevier, Amsterdam, 2006.

[9] L. Li, V. Rădulescu, D. Repovs; *Nonlocal Kirchhoff superlinear equations with indefinite nonlinearity and lack of compactness,* Int. J. Nonlinear Sci. Numer. Simul., **17** (2016), 325-332.

[10] S. A. Marano, D. Motreanu; *Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving the p-Laplacian,* J. Differential Equations, **182 (1)** (2002), 108–120.

[11] G. Molica Bisci, V. Rădulescu; *Ground state solutions of scalar field fractional Schrödinger equations,* Calculus of Variations and Partial Differential Equations, **54** (2015), 2985–3008.

[12] G. Molica Bisci, D. Repovs; *On doubly nonlocal fractional elliptic equations,* Rend. Lincei Mat. Appl., **26** (2015), 161-176.

[13] D. Motreanu, P. D. Panagiotopoulos; *Minimax theorems and qualitative properties of the solutions of hemivariational inequalities,* Kluwer Academic Publishers, Dordrecht, 1999.

[14] D. Motreanu, V. Rădulescu; *Variational and non-variational methods in nonlinear analysis and boundary value problems,* Kluwer Academic Publishers, Boston-Dordrecht-London, 2003.

[15] S. K. Ntouyas, S. Etemad, J, Tariboon; *Existence of solutions for fractional differential inclusions with integral boundary conditions,* Boundary Value Problems, (2015), 1–14. DOI:10.1186/s13661-015-0356-y.

[16] I. Podlubny, *Fractional Differential Equations,* Academic Press, San Diego, 1999.

[17] P. Pucci, M. Xiang, B. Zhang; *Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations,* Adv. Nonlinear Anal., **5** (2016), no. 1, 27–55.

[18] V. Rădulescu; *Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations,* Contemporary Mathematics and Its Applications, Vol. 6. Hindawi Publ Corp., New York, 2008.

[19] V. Rădulescu, B. Alleche; *Equilibrium problems techniques in the qualitative analysis of quasi-hemivariational inequalities,* Optimization **64** (2015), 1855–1868.

[20] V. Rădulescu, D. Repovš; *Partial differential equations with variable exponents:* Variational Methods and Qualitative Analysis, CRC Press, Taylor & Francis Group, Boca Raton FL, 2015.

[21] V. Rădulescu; *Nonlinear elliptic equations with variable exponent: old and new,* Nonlinear Analysis: Theory, Methods and Applications, **121**(2015), 336–369.

[22] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional integral and derivatives: theory and applications,* Gordon and Breach, Longhorne, PA, 1993.

[23] X. Zhang, B. Zhang, M. Xiang, *Ground states for fractional Schrödinger equations involving a critical nonlinearity,* Adv. Nonlinear Anal., **5**(2016), no. 3, 293-314.

Fariba Fattahi
Department of Mathematics, University of Mazandaran, Babolsar, Iran
E-mail address: `F.Fattahi@stu.umz.ac.ir`

Mohsen Alimohammady
Department of Mathematics, University of Mazandaran, Babolsar, Iran
E-mail address: `Amohsen@umz.ac.ir`