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PROPERTIES OF SCALES OF KATO CLASSES, BESSEL POTENTIALS, MORREY SPACES, AND A WEAK HARNACK INEQUALITY FOR NON-NEGATIVE SOLUTIONS OF ELLIPTIC EQUATIONS

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Abstract. In this article, we study some basic properties of the scale of Kato classes related with the Bessel kernel, Lorentz spaces, and Morrey spaces. Also we characterize the weak Harnack inequality for non-negative solutions of elliptic equations in terms of the Bessel kernel and the Kato classes of order α.

1. Introduction

In this article we prove a weak Harnack inequality for non-negative solutions of elliptic differential equations of divergence form with potentials from the Kato class of order α. Namely, given a bounded domain Ω in $\mathbb{R}^n$, we consider the Shrödinger operator

$$Lu + Vu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}u(x)\Big) + V(x)u(x), \quad x \in \Omega,$$

where the matrix $A(x) = \big(a_{ij}(x)\big)$ is symmetric, bounded, measurable and positive uniformly in x, i.e.,

$$\lambda|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \Lambda|\xi|^2, \quad x \in \Omega,\ \xi \in \mathbb{R}^n$$

for some $0 < \lambda \le \Lambda$. Given $V \in L^1_{\rm loc}(\Omega)$, a function $u \in H^1(\Omega)$ is a weak solution of $Lu + Vu = 0$ if and only if

$$\int_\Omega \langle A\nabla u, \nabla\varphi\rangle dx + \int_\Omega V u dx = 0, \quad \text{for all } \varphi \in H^1_0(\Omega).$$

In this study, we use a class of potential more general than the one considered by Mohammed [5]. The study there is based heavily on the use and properties of the approximation of the Green function and the Green function of the corresponding operator. We substitute the approximate Green function by an approximate kernel of Bessel potentials denoted by G^r_α, and the Green function by the Kernel of the Bessel potentials. Also, we relate the Kato class of order α with the Bessel and Riesz potentials.

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The Kato class K_n on the n-dimensional space $\mathbb{R}^n$ was introduced and studied by Aizenman and Simon [1, 7]. The definition of K_n is based on a condition considered by Kato [4]. Similar function classes were defined by Schechter [6] and Stummed [10]. We refer the reader to [2, 3, 6, 7] for more information concerning to Kato class and its applications. We set

$$\phi(V,r) = \sup_{x\in\mathbb{R}^n} \int_{B(x,r)} \frac{|V(y)|}{|x-y|^{n-2}} dy,$$

where $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. The Kato class K_n consists of locally integrable functions V on $\mathbb{R}^n$ such that

$$\lim_{r\to 0} \phi(V,r) = 0.$$

Davies and Hinz [3] introduced the scale $K_{n,\alpha}$ of the Kato class of order α. For $\alpha > 0$ we set

$$\eta(V)(r) = \sup_{x\in\mathbb{R}^n} \int_{B(x,r)} \frac{|V(y)|}{|x-y|^{n-\alpha}} dy.$$

The Kato class of order α consists of locally integrable functions V on $\mathbb{R}^n$ such that

$$\lim_{r\to 0} \eta(V)(r) = 0.$$

2. Kato class of order α

In this section, we gather definitions and notation that will be used later. By $L_{\mathrm{loc},u}(\mathbb{R}^n)$ we denote the space of functions V such that

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,1)} |V(y)| dy < \infty.$$

Definition 2.1. The distribution function D_V of a measurable function V is given by

$$D_V(\lambda) = m(\{x \in \mathbb{R}^n : |V(x)| > \lambda\})$$

where m denotes the Lebesgue measure on $\mathbb{R}^n$.

Definition 2.2. Let V be a measurable function in $\mathbb{R}^n$. The decreasing rearrangement of V is the function V^* defined on $[0,\infty)$ by

$$V^*(t) = \inf\{\lambda : D_V(\lambda) \le t\} \quad (t \ge 0).$$

Definition 2.3 (Lorentz spaces)**.** Let V be a measurable function, we say that V belongs to $L(n/\alpha, 1)$ ($\alpha > 0$) if

$$\int_0^\infty t^{\frac{\alpha}{n}-1} V^*(t) dt < \infty$$

and V belongs to $L(\frac{n}{n-\alpha}, \infty)$ if

$$\sup_{t>0} t^{1-\frac{\alpha}{n}} V^*(t) < \infty.$$

Definition 2.4 (Morrey spaces)**.** Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, for $q \ge 0$, we say that V belongs to $L^{1,n/q}(\mathbb{R}^n)$ if

$$\sup_{x\in\mathbb{R}^n} \frac{1}{r^{n/q}} \int_{B(x,r)} |V(y)| dy = \|V\|_{L^{1,n/q}(\mathbb{R}^n)} < \infty.$$

The following definition is a slight variant of the scale $K_{n,\alpha}$ Kato class.

Definition 2.5. Let $V \in L^1_{\rm loc}(\mathbb{R}^n)$, we say that V belongs to $\widetilde{K}_{n,\alpha}$ if

$$\eta(V)(r) = \sup_{x\in\mathbb{R}^n} \int_{B(x,r)} \frac{|V(y)|}{|x-y|^{n-\alpha}} dy < \infty.$$

Next, we study some properties of the class $\widetilde{K}_{n,\alpha}$.

Lemma 2.6. *$\widetilde{K}_{n,\alpha} \subset L^1_{{\rm loc},u}(\mathbb{R}^n)$.*

Proof. Let $V \in \widetilde{K}_{n,\alpha}$ and fix $r_0 > 0$. Then there exits a positive constant $C > 0$ such that $\eta(V)(r) \le C$. It follows that

$$\sup_{x\in\mathbb{R}^n} \frac{1}{r_0^{n-\alpha}} \int_{B(x,r_0)} |V(y)| dy \le \sup_{x\in\mathbb{R}^n} \int_{B(x,r_0)} \frac{|V(y)|}{|x-y|^{n-\alpha}} dy, \quad (\alpha > 0).$$

Therefore,

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,r_0)} |V(y)| dy < AC$$

where $A = 1/r_0^{n-\alpha}$. Finally, let $B(x,1) \subset \cup_{k=1}^n B(x_k, r_0)$, then

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,1)} |V(y)| dy \le \sum_{k=1}^n \sup_{x\in\mathbb{R}^n} \int_{B(x_k,r_0)} |V(y)| dy.$$

Thus,

$$\sup_{x\in\mathbb{R}^n} \int_{B(x,1)} |V(y)| dy < \infty.$$

Therefore, $\widetilde{K}_{n,\alpha} \subset L^1_{{\rm loc},u}(\mathbb{R}^n)$. □

Lemma 2.7. *$L(n/\alpha, 1) \subset \widetilde{K}_{n,\alpha}$, $(\alpha > 0)$.*

Proof. Let $V \in L(n/\alpha, 1)$ $(\alpha > 0)$, then

$$\int_0^\infty t^{\frac{\alpha}{n}-1} V^*(t) dt < \infty.$$

Since $|V|\chi_{B(x,\varepsilon)} \le |V|$, we have $(V\chi_{B(x,\varepsilon)}) \le V^*(t)$. Then

$$\int_0^\infty t^{\frac{\alpha}{n}-1} (V\chi_{B(x,\varepsilon)})^*(t) dt \le \int_0^\infty t^{\frac{\alpha}{n}-1} V^*(t) dt.$$

Thus, $V\chi_{B(x,\varepsilon)} \in L(n/\alpha, 1)$.

On the other hand, letting $g(x) = |x|^{\alpha-n}$, we have

$$\begin{aligned} m(\{x : g(x) > \lambda\}) &= m(\{x : |x|^{\alpha-n} > \lambda\}) \\ &= m\Big(\{x : |x| < (\frac{1}{\lambda})^{\frac{1}{n-\alpha}}\}\Big) \\ &= C_n (\frac{1}{\lambda})^{\frac{n}{n-\alpha}}, \end{aligned}$$

where $C_n = m(B(0,1))$. Next, we set $t = C_n(\frac{1}{\lambda})^{\frac{n}{n-\alpha}}$, then $\lambda = C_n^{\frac{n-\alpha}{n}} t^{\frac{\alpha}{n}-1}$. Thus $g^*(t) = C_n t^{\frac{\alpha}{n}-1}$, from this we obtain

$$\|g\|_{(\frac{n}{n-\alpha},\infty)} = \|\frac{1}{|\cdot|^{n-\alpha}}\|_{(\frac{n}{n-\alpha},\infty)} = \sup_{t>0} C_n^{\frac{n-\alpha}{n}} t^{1-\frac{\alpha}{n}} t^{\frac{\alpha}{n}-1} = C_n^{\frac{n-\alpha}{n}}.$$

Finally,

$$\int_{B(x,\varepsilon)} \frac{|V(y)|}{|x-y|^{n-\alpha}} dt \le \|V\chi_{B(x,\varepsilon)}\|_{(\frac{n}{\alpha},1)} \Big\| \frac{1}{|\cdot|^{n-\alpha}} \Big\|_{(\frac{n}{n-\alpha},\infty)} \le C_n^{\frac{n-\alpha}{n}} \|V\chi_{B(x,\varepsilon)}\|_{(\frac{n}{\alpha},1)}.$$

Thus, $V \in \widetilde{K}_{n,\alpha}$. □

Example 2.8. Regarding the functions that belong to $\widetilde{K}_{n,\alpha}$, we claim that

$$V(x) = \frac{1}{|x|^\alpha (\log|x|)^{2\alpha}} \in \widetilde{K}_{n,\alpha}.$$

Proof of the Claim. It will be sufficient to show that $V \in L(n/\alpha, 1)$, to do this let us consider

$$m\Big(\Big\{x : \frac{1}{|x|^\alpha(\log|x|)^{2\alpha}} > \lambda\Big\}\Big) = m\Big(\Big\{x : |x|^\alpha(\log|x|)^{2\alpha} < \frac{1}{\lambda}\Big\}\Big).$$

Putting $\varphi(|x|) = |x|^\alpha(\log|x|)^{2\alpha}$, we have

$$\begin{aligned} m\Big(\big\{x : |x|^\alpha(\log|x|)^{2\alpha} < \frac{1}{\lambda}\big\}\Big) &= m\Big(\big\{x : \varphi(|x|) < \frac{1}{\lambda}\big\}\Big) \\ &= m\Big(\big\{x : |x| < \varphi^{-1}(\frac{1}{\lambda})\big\}\Big) \\ &= C_n\Big(\varphi^{-1}(\frac{1}{\lambda})\Big)^n. \end{aligned}$$

Let $t = C_n(\varphi^{-1}(\frac{1}{\lambda}))^n$, thus $C_n^{1/n}\varphi^{-1}(\frac{1}{\lambda}) = t^{1/n}$, then $\varphi^{-1}(\frac{1}{\lambda}) = C(n)t^{1/n}$, where $C(n) = C_n^{-1/n}$ so $\frac{1}{\lambda} = \varphi(C(n)t^{1/n})$, hence

$$\lambda = \frac{1}{\varphi\big(C(n)t^{1/n}\big)} = \frac{C(n)}{|t|^{\frac{\alpha}{n}}(\log|t|)^{2\alpha}}.$$

Therefore

$$V^*(t) = \frac{C(n)}{|t|^{\frac{\alpha}{n}}(\log|t|)^{2\alpha}} = \frac{C(n)}{t^{\frac{\alpha}{n}}(\log t)^{2\alpha}}\,.$$

Note taht

$$\int_0^\infty t^{\frac{\alpha}{n}-1} V^*(t)dt = C(n)\int_0^\infty t^{\frac{\alpha}{n}-1}\frac{dt}{t^{\frac{\alpha}{n}}(\log t)^{2\alpha}} = C(n)\int_0^\infty \frac{dt}{t(\log t)^{2\alpha}} < \infty,$$

then $V \in L(n/\alpha, 1)$, hence $V \in \widetilde{K}_{n,\alpha}$. □

Lemma 2.9. *If $V \in L^{1,n/q}(\mathbb{R}^n)$ and $p > n/\alpha$, $1 \le p \le \infty$, then*

$$\int_{B(x,\delta)} \frac{|V(y)|}{|x-y|^{n-\alpha}} dy \le \delta^{\alpha - n/p}\|V\|_{L^{1,n/q}(\mathbb{R}^n)}.$$

Moreover $L^{1,n/q}(\mathbb{R}^n) \subset \widetilde{K}_{n,\alpha}$.

Proof. Let $V \in L^{1,n/q}(\mathbb{R}^n)$. Note that

$$\int_{B(x,\delta)} \frac{|V(y)|}{|x-y|^{n-\alpha}} dy = \int_{\mathbb{R}^n} \frac{|V(y)\chi_{B(x,\delta)}(y)|}{|x-y|^{n-\alpha}} dy = \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}}, \tag{2.1}$$

where $d\mu(y) = |V(y)|\chi_{B(x,\delta)}(y)dy$, from this we have

$$\mu(B(x,r)) \int_{B(x,r)} |V(y)|\chi_{B(x,\delta)}(y)dy.$$

Going back to (2.1), we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}} =&(n-\alpha)\int_0^\infty r^{\alpha-n-1}\mu(B(x,r))dr\\
=&(n-\alpha)\int_0^\infty r^{\alpha-n-1}\int_{B(x,r)\cap B(x,\delta)}|V(y)|dydr\\
=&(n-\alpha)\int_0^\delta r^{\alpha-n-1}\int_{B(x,r)}|V(y)|dydr\\
&+(n-\alpha)\int_\delta^\infty r^{\alpha-n-1}\int_{B(x,\delta)}|V(y)|dydr\\
\le&(n-\alpha)\int_0^\delta r^{\alpha-n+\frac{n}{q}-1}\Big(\frac{1}{r^{n/q}}\int_{B(x,r)}|V(y)|dy\Big)dr\\
&+(n-\alpha)\delta^{n/q}\int_\delta^\infty r^{\alpha-n-1}\Big(\frac{1}{\delta^{n/q}}\int_{B(x,r)}|V(y)|dy\Big)dr\\
\le&(n-\alpha)\Big[\int_0^\delta r^{\alpha-\frac{n}{p}-1}dr+\delta^{n/q}\int_\delta^\infty r^{\alpha-n-1}dr\Big]\|V\|_{L^{1,n/q}(\mathbb{R}^n)}\\
=&(n-\alpha)\Big[[\frac{r^{\alpha-n/p}}{\alpha-\frac{n}{p}}\Big|_0^\delta+\delta^{n/q}\frac{r^{\alpha-n}}{\alpha-n}\Big|_0^\infty\Big]\|V\|_{L^{1,n/q}(\mathbb{R}^n)}\\
=&(n-\alpha)\Big[\frac{p(n-\alpha)+(\alpha p-n)}{(\alpha-n/p)(n-\alpha)}\Big]\delta^{\alpha-\frac{n}{p}}\|V\|_{L^{1,n/q}(\mathbb{R}^n)}\\
=&\Big[\frac{np-n}{p\alpha-n}\Big]\delta^{\alpha-n/p}\|V\|_{L^{1,n/q}(\mathbb{R}^n)}.
\end{aligned}
$$

Therefore $L^{1,n/q}(\mathbb{R}^n)\subset\widetilde{K}_{n,\alpha}$. □

3. Space of functions of bounded mean oscillation (BMO)

In the same sense that the Hardy space $H^1(\mathbb{R}^n)$ is a substitute for $L^1(\mathbb{R}^n)$, it will turn out that the space $BMO(\mathbb{R}^n)$ (the space of "bounded mean oscillation") is the corresponding natural substitute for the space $L^\infty(\mathbb{R}^n)$ of bounded functions on $\mathbb{R}^n$.

A locally integrable function f belongs to BMO if

$$\frac{1}{m(B_r)}\int_{B_r}|f(x)-f_{B_r}|dm\le A \tag{3.1}$$

holds for all balls $B_r=B(x,r)$, here

$$f_{B_r}=\frac{1}{m(B_r)}\int_{B_r}fdm=\fint_{B_r}fdm$$

denotes the mean value of f over the ball and m stand for the Lebesgue measure on $\mathbb{R}^n$. The inequality (3.1) asserts that over any ball B, the average oscillation of f is bounded. The smallest bound A for which (3.1) is satisfied is then taken to be the norm of f in this space, and is denoted by $\|f\|_{BMO}$. Let us begin by making some remarks about functions that are in BMO.

The following result is due to Jhon-Niremberg. If $f\in BMO$ then there exist positive constants C_1 and C_2 so that, for every $r>0$ and every ball B_r

$$m(\{x\in B_r:|f(x)-f_{B_r}|>\lambda\})\le Ce^{-C_2\lambda/\|f\|_{BMO}}m(B_r).$$

One consequence of the above result is the following corollary.

Corollary 3.1. *If $f \in BMO$, then there exist positive constants C_1 and C_2 such that*
$$\int_{B_r} e^{C|f(x)-f_{B_r}|} dm \leq \Big(\frac{C_1 C}{C_2 - C} + 1\Big) m(B_r)$$
for every ball B_r and $0 < C < C_2$.

Proof. Let us define $\varphi(x) = e^x - 1$. Notice that $\varphi(0) = 0$, and hence
$$\begin{aligned}\int_{B_r} (e^{C|f(x)-f_{B_r}|} - 1) &= C \int_0^\infty e^{C\lambda} m(\{x \in B_r : |f(x) - f_{B_r}| > \lambda\}) d\lambda \\ &\leq C C_1 \Big[\int_0^\infty e^{-(C_2 - C)\lambda} d\lambda\Big] m(B_r).\end{aligned}$$
From the above inequality we have
$$\int_{B_r} e^{C|f(x)-f_{B_r}|} dm \leq \Big(\frac{CC_1}{C_2 - C} + 1\Big) m(B_r).$$
□

4. p bounded mean oscillation

A locally integrable function f belongs to BMO_p if for $1 \leq p < \infty$
$$\|f\|_{BMO_p} = \sup_{B_r} \Big(\frac{1}{m(B_r)} \int_{B_r} |f(x) - f_{B_r}|^p dm\Big)^{1/p} < \infty.$$

Theorem 4.1. *If $f \in BMO_p$ then there exists a positive constant C depending on p such that*
$$\|f\|_{BMO} \leq C_p \|f\|_{BMO_p}.$$

Proof. Let $f \in BMO_p$ by virtue of the Hölder inequality we have
$$\int_{B_r} |f(x) - f_{B_r}| dm \leq [m(B_r)]^{1-1/p} \Big(\int_{B_r} |f(x) - f_{B_r}|^p dm\Big)^{1/p}.$$
Hence
$$\frac{1}{m(B_r)} \int_{B_r} |f(x) - f_{B_r}| dm \leq \sup_{B_r} \Big(\frac{1}{m(B_r)} \int_{B_r} |f(x) - f_{b_r}|^p dm\Big)^{1/p}$$
for any ball B_r. □

5. Bessel kernel

The connection between the Bessel and Riesz potential was observed by Stein [8, 9]. We will develop the basic properties of the Bessel kernel.

Here $F : S' \to S'$ denotes the Fourier transform on S' where S' represent the set of all tempered distributions. S' is thus the dual of the Schwartz space S. For $f \in L^1(\mathbb{R}^n)$ we have
$$F(f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx.$$
The Riesz kernel, I_α, $0 < \alpha < n$, is defined by
$$I_\alpha(x) = \frac{|x|^{\alpha - n}}{\gamma(\alpha)}, \tag{5.1}$$

where

$$\gamma(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma(\frac{n}{2} - \alpha/2)}$$

Γ denotes the gamma function.

We begin by deriving the kernel of the Bessel potential. First let us consider

$$t^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty e^{-ts} \delta^a \frac{d\delta}{\delta}. \tag{5.2}$$

After a suitable change of variables is not difficult to obtain (5.2). Using (5.2) with $\alpha/2 > 0$ we have

$$(4\pi)^{\alpha/2}(1 + 4\pi^2|x|^2)^{-\alpha/2} = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-\frac{\delta}{4\pi}(1+4\pi^2|x|^2)} \delta^{\alpha/2} \frac{d\delta}{\delta}. \tag{5.3}$$

Now we want to compute

$$F\{(1 + 4\pi^2|x|^2)^{-\alpha/2}\}(\xi) = \int_{\mathbb{R}^n} (1 + 4\pi^2|x|^2)^{-\alpha/2} e^{-2\pi i x\cdot\xi} dx.$$

By (5.3) we obtain

$$\begin{aligned} &\frac{1}{(4\pi)^{\alpha/2}} \int_{\mathbb{R}^n} \Big(\frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-\frac{\delta}{4\pi}(1+4\pi^2|x|^2)} \delta^{\alpha/2} \frac{d\delta}{\delta} \Big) e^{-2\pi i x\cdot\xi} dx \\ &= \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\frac{\pi|\xi|^2}{\delta}} e^{-\frac{\delta}{4\pi}} \delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta}; \end{aligned}$$

therefore

$$F\{(1 + 4\pi^2|x|^2)^{-\alpha/2}\}(\xi) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\frac{\pi|\xi|^2}{\delta}} e^{-\frac{\delta}{4\pi}} \delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta}.$$

5.1. **Bessel kernel.** We define the Bessel kernel

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\frac{\pi|x|^2}{\delta}} e^{-\frac{\delta}{4\pi}} \delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta}. \tag{5.4}$$

Lemma 5.1. (a) *For each $\alpha > 0$, $G_\alpha(x) \in L^1(\mathbb{R}^n)$.*
(b) *$F(G_\alpha(x)) = (1 + 4\pi^2|x|^2)^{-\alpha/2}$.*

Proof. (a) By (5.4) we obtain

$$\int_{\mathbb{R}^n} G_\alpha(x) dx = \int_{\mathbb{R}^n} \Big(\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\frac{\pi|x|^2}{\delta}} e^{-\frac{\delta}{4\pi}} \delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta} \Big) dx.$$

Since $\int_{\mathbb{R}^n} e^{-\frac{\pi|x|^2}{\delta}} dx = \delta^{n/2}$ and using Fubini, we set

$$\int_{\mathbb{R}^n} G_\alpha(x) dx = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\frac{\delta}{4\pi}} \delta^{\frac{\alpha-n}{2}} \Big(\int_{\mathbb{R}^n} e^{-\pi|x|^2/\delta} dx \Big) \frac{d\delta}{\delta}.$$

After a suitable change of variable we have

$$\int_{\mathbb{R}^n} G_\alpha(x) dx = 1,$$

and so $G_\alpha(x) \in L^1(\mathbb{R}^n)$.

(b) In the sense of distributions we have whenever $\varphi \in S$,

$$\int_{\mathbb{R}^n} f(x) F(\varphi(x)) dx = \int_{\mathbb{R}^n} F(f(x)) \varphi(x) dx. \tag{5.5}$$

Let us consider the function

$$f(x)=e^{-\frac{\delta}{4\pi}}e^{-\pi|x|^2};\quad \text{then } F(f(x))=e^{-\frac{\delta}{4\pi}}e^{-\frac{\pi|x|^2}{\delta}}\delta^{-n/2}.$$

By (5.5) we have

$$\int_{\mathbb{R}^n} e^{-\frac{\delta}{4\pi}(1+4\pi^2|x|^2)}\hat{\varphi}(x)dx=\int_{\mathbb{R}^n} e^{-\frac{\delta}{4\pi}}e^{-\frac{\pi|x|^2}{\delta}}\delta^{-n/2}\varphi(x)dx,$$

where $\hat{\varphi}(x)=F(\varphi(x))$, then

$$\begin{aligned}&\int_0^\infty\Big(\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_{\mathbb{R}^n} e^{-\frac{\delta}{4\pi}(1+4\pi^2|x|^2)}\hat{\varphi}(x)dx\Big)\delta^{\alpha/2}\frac{d\delta}{\delta}\\ &=\int_0^\infty\Big(\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_{\mathbb{R}^n} e^{-\frac{\delta}{4\pi}}e^{-\frac{\pi|x|^2}{\delta}}\delta^{-n/2}\varphi(x)dx\Big)\delta^{\alpha/2}\frac{d\delta}{\delta}\,.\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}&\int_{\mathbb{R}^n}\Big(\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty e^{-\frac{\delta}{4\pi}(1+4\pi^2|x|^2)}\delta^{\alpha/2}\frac{d\delta}{\delta}\Big)\hat{\varphi}(x)dx\\ &=\int_{\mathbb{R}^n}\Big(\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty e^{-\frac{\delta}{4\pi}}e^{-\frac{\pi|x|^2}{\delta}}\delta^{\frac{\alpha-n}{2}}\frac{d\delta}{\delta}\Big)\varphi(x)dx.\end{aligned}$$

That is,

$$\int_{\mathbb{R}^n}(1+4\pi^2|x|^2)^{-\alpha/2}\hat{\varphi}(x)dx=\int_{\mathbb{R}^n}G_\alpha(x)\varphi(x)dx;$$

therefore $F(G_\alpha(x))=(1+4\pi^2|x|^2)^{-\alpha/2}$. □

Remark 5.2. From Lemma 5.1(b) we have $G_\alpha * G_\beta=G_{\alpha+\beta}$.

Lemma 5.3. $F\{\int_0^\infty e^{-\pi\delta|x|^2}\delta^a\frac{d\delta}{\delta}\}=\int_0^\infty e^{-\pi\frac{|x|^2}{\delta}}\delta^{-n/2}\delta^a\frac{d\delta}{\delta}$.

Proof. By definition

$$\begin{aligned}F\Big\{\int_0^\infty e^{-\pi\delta|x|^2}\delta^a\frac{d\delta}{\delta}\Big\}&=\int_{\mathbb{R}^n}(\int_0^\infty e^{-\pi\delta|x|^2}\delta^a\frac{d\delta}{\delta})e^{-2\pi ix\cdot\xi}dx\\ &=\int_0^\infty(\int_{\mathbb{R}^n} e^{-\pi\delta|x|^2}e^{-2\pi ix\cdot\xi}dx)\delta^a\frac{d\delta}{\delta}\\ &=\int_0^\infty e^{-\pi\delta|x|^2/\delta}\delta^{-n/2}\delta^a\frac{d\delta}{\delta}.\end{aligned}$$

Therefore

$$F\Big\{\int_0^\infty e^{-\pi\delta|x|^2}\delta^a\frac{d\delta}{\delta}\Big\}=\int_0^\infty e^{-\pi\delta|x|^2/\delta}\delta^{-n/2}\delta^a\frac{d\delta}{\delta}.$$

□

Proposition 5.4. $\frac{|x|^{\alpha-n}}{\gamma(\alpha)}=\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty e^{-\pi|x|^2/\delta}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta}$.

Proof. We have

$$\begin{aligned}&\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty e^{-\pi|x|^2/\delta}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta}\\ &=\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty e^{-\pi/u}(|x|^2u)^{\frac{\alpha-n}{2}-1}|x|^2du\\ &=\frac{|x|^{\alpha-n}}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty e^{-\pi/u}u^{\frac{\alpha-n}{2}-1}du\end{aligned}$$

$$
\begin{aligned}
&= \frac{|x|^{\alpha-n}}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_\infty^0 e^{-w}\big(\frac{\pi}{w}\big)^{\frac{\alpha-n}{2}-1}\big(-\frac{\pi}{w^2}\big)dw \\
&= \frac{\pi^{(\alpha-n)/2}|x|^{\alpha-n}}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-w} w^{\frac{n-\alpha}{2}-1}dw \\
&= \frac{\Gamma(\frac{n}{2}-\frac{\alpha}{2})}{2^\alpha\pi^{n/2}\Gamma(\alpha/2)}|x|^{\alpha-n} \\
&= \frac{|x|^{\alpha-n}}{\gamma(\alpha)};
\end{aligned}
$$

therefore

$$
\frac{|x|^{\alpha-n}}{\gamma(\alpha)} = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta}.
$$

□

Remark 5.5. By (5.1) and Proposition 5.4 we can define $I_\alpha(x)$ as follows

$$
I_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}}\Gamma(\alpha/2) \int_0^\infty e^{-\pi|x|^2/\delta}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta}. \tag{5.6}
$$

Comparing the formulas (5.4) and (5.6) it follows immediately that $G_\alpha(x)$ is positive, and

$$
0 < G_\alpha(x) < I_\alpha(x) \quad \text{for } 0<\alpha<n.
$$

Proposition 5.6. *$G_\alpha(x) = \frac{|x|^{\alpha-n}}{\gamma(\alpha)} + \mathcal{O}(|x|^{\alpha-n})$, as $|x|\to\infty$.*

Proof. For $\varepsilon>0$ we have

$$
\begin{aligned}
\int_\varepsilon^\infty e^{-\pi|x|^2/\delta}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta} &= \int_{\frac{\varepsilon}{x|^2}}^\infty e^{-\pi/u}(|x|^2u)^{\frac{\alpha-n}{2}-1}|x|^2du \\
&= |x|^{\alpha-n}\int_{\frac{\varepsilon}{x|^2}}^\infty e^{-\pi/u}u^{\frac{\alpha-n}{2}-1}du \\
&= |x|^{\alpha-n}\int_{|x|^2\frac{\pi}{\varepsilon}}^0 e^{-w}\big(\frac{\pi}{w}\big)^{\frac{\alpha-n}{2}-1}\big(-\frac{\pi}{w^2}\big)dw \\
&= |x|^{\alpha-n}\pi^{(\alpha-n)/2}\int_0^{|x|^2\frac{\pi}{\varepsilon}} e^{-w}w^{\alpha-n+1-2}dw.
\end{aligned}
$$

Let us define $\varphi(x,\varepsilon) = \int_0^{|x|^2\frac{\pi}{\varepsilon}} e^{-w}w^{\alpha-n+1-2}dw$, note that $\varphi(x,\varepsilon)\to 0$ as $x\to 0$. So we can write

$$
\int_\varepsilon^\infty e^{-\pi|x|^2/\delta}\delta^{(\alpha-n)/2}\frac{d\delta}{\delta} = C|x|^{\alpha-n}\varphi(x,\varepsilon),
$$

where $C=\pi^{(\alpha-n)/2}$.

Now we have to prove that for every $\tau>0$ there exists $\lambda>0$ such that if $|x|<\lambda$ then

$$
\big|G_\alpha(x) - \frac{|x|^{\alpha-n}}{\gamma(\alpha)}\big| \le \tau|x|^{\alpha-n}\,.
$$

To do that let us consider

$$
G_\alpha(x) - \frac{|x|^{\alpha-n}}{\gamma(\alpha)} = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta}[e^{\delta/4\pi}-1]\delta^{\frac{\alpha-n}{2}}\frac{d\delta}{\delta},
$$

since $\frac{e^{\delta/4\pi}-1}{e^{\delta/4\pi}} \to 0$ as $\delta \to 0$, we have $e^{-\delta/4\pi} = 1 + \mathcal{O}(e^{\delta/4\pi})$ as $\delta \to 0$.

Taking $\tau > 0$ there exists $\varepsilon > 0$ such that

$$\begin{aligned}
&\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta}[e^{-\delta/4\pi} - 1]\delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta} \\
&\leq \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta} \frac{\tau}{2} e^{-\delta/4\pi} \delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta} \\
&\leq \frac{\gamma(\alpha)\tau|x|^{\alpha-n}}{2\gamma(\alpha)} = \frac{\tau}{2}|x|^{\alpha-n};
\end{aligned}$$

therefore

$$\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta}[e^{-\delta/4\pi} - 1]\delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta} \leq \frac{\tau}{2}|x|^{\alpha-n}. \tag{5.7}$$

Since $\varepsilon > 0$ has been chosen we take $|x| < \lambda$ such that $\varphi(x,\varepsilon) \leq \frac{\tau}{2c}$. Then we obtain

$$\begin{aligned}
&\frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_\varepsilon^\infty e^{-\pi|x|^2/\delta}[e^{-\delta/4\pi} - 1]\delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta} \\
&\leq \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_\varepsilon^\infty e^{-\pi|x|^2/\delta} \frac{\tau}{2} e^{-\delta/4\pi} \delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta} \\
&\leq \frac{\tau}{2(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_\varepsilon^\infty e^{-\pi|x|^2/\delta} \delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta} \\
&\leq \frac{\tau}{2(4\pi)^{\alpha/2}\Gamma(\alpha/2)} C|x|^{(\alpha-n)}\varphi(x,\varepsilon) \\
&\leq \frac{\tau}{2}|x|^{(\alpha-n)}.
\end{aligned}$$

Finally

$$\begin{aligned}
\big|G_\alpha(x) - \frac{|x|^{(\alpha-n)}}{\gamma(\alpha)}\big| &\leq \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \int_0^\infty e^{-\pi|x|^2/\delta}[e^{-\delta/4\pi} - 1]\delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta} \\
&= \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)} \Big[\int_0^\varepsilon e^{-\pi|x|^2/\delta}[e^{-\delta/4\pi} - 1]\delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta} \\
&\quad + \int_\varepsilon^\infty e^{-\pi|x|^2/\delta}[e^{-\delta/4\pi} - 1]\delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta} \Big]
\end{aligned}$$

from (5.6) and (5.7) we obtain

$$\big|G_\alpha(x) - \frac{|x|^{(\alpha-n)}}{\gamma(\alpha)}\big| \leq \tau|x|^{\alpha-n};$$

therefore

$$G_\alpha(x) - \frac{|x|^{(\alpha-n)}}{\gamma(\alpha)} = \mathcal{O}(|x|^{\alpha-n}) \quad \text{as } |x| \to 0 \text{ for } 0 < \alpha < n.$$

□

On the other hand by differentiating formula (5.4) we obtain

$$\begin{aligned}
\big|\frac{\partial G_\alpha(x)}{\partial x_j}\big| &= \Big| C \int_0^\infty \frac{\partial}{\partial x_j} \Big(e^{-\frac{\pi|x|^2}{\delta}} e^{-\frac{\delta}{4\pi}} \delta^{\frac{\alpha-n}{2}} \frac{d\delta}{\delta} \Big) \Big| \\
&\leq C|x_j| \int_0^\infty e^{-\frac{\pi|x|^2}{\delta}} \delta^{\frac{\alpha-n-2}{2}} \frac{d\delta}{\delta}
\end{aligned}$$

by Proposition 5.4, the above expression is less than or equal to $C|x_j||x|^{\alpha-n-2}$. Thus

$$\left|\frac{\partial G_\alpha(x)}{\partial x_j}\right| \leq C|x|^{\alpha-n-1}.$$

Proposition 5.7. *$G_\alpha(x) = \mathcal{O}(e^{-\frac{|x|}{2}})$ as $|x| \to \infty$, which shows that the kernel G_α is rapidly decreasing as $|x| \to \infty$.*

Proof. Let

$$f(\delta) = e^{-\frac{\pi|x|^2}{\delta}-\frac{\delta}{4\pi}}.$$

After a not too difficult calculation we obtain

$$f(2\pi|x|) = e^{-|x|},$$

which is a maximum value. Also if $|x| \geq 1$ then clearly

$$e^{-\pi|x|^2/\delta}e^{-\delta/4\pi} \leq e^{-\pi/\delta}e^{-\delta/4\pi},$$
$$e^{-\pi|x|^2/\delta}e^{-\delta/4\pi} \leq e^{-|x|} \quad \text{for } \delta \neq 2\pi|x|.$$

Now let us consider

$$\min(e^{-|x|}, e^{-\pi/\delta-\delta/4\pi}) = \begin{cases} e^{-|x|} & \text{if } |x| \geq \frac{\pi}{\delta}+\frac{\delta}{4\pi}, \\ e^{-\pi/\delta-\frac{\delta}{4\pi}} & \text{if } |x| \leq \frac{\pi}{\delta}+\frac{\delta}{4\pi}. \end{cases}$$

Note $\frac{\pi}{\delta}+\frac{\delta}{4\pi} \leq 1$ since $a+b \geq 2\sqrt{ab}$; therefore we have

$$-|x| \leq -\frac{|x|}{2} - \frac{\pi}{2\delta} - \frac{\delta}{8\pi}\,.$$

Finally when $|x| \geq 1$,

$$\min(e^{-|x|}, e^{-\pi/\delta-\delta/4\pi}) \leq e^{-\frac{x}{2}} - \frac{\pi}{2\delta}e^{-\frac{\delta}{\delta\pi}}\,.$$

From this we obtain

$$e^{-\pi|x|^2/\delta}e^{-\delta/4\pi} \leq e^{-\frac{|x|}{2}}e^{-\frac{\pi}{2\delta}}e^{-\frac{\delta}{8\pi}}.$$

Therefore,

$$G_\alpha(x) \leq \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty e^{-\frac{|x|}{2}}e^{-\frac{\pi}{2\delta}}e^{-\frac{\delta}{8\pi}}\delta^{\frac{\alpha-n}{2}}\frac{d\delta}{\delta}$$

so $|G_\alpha(x)| \leq Me^{-\frac{|x|}{2}}$, where

$$M = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty e^{-\frac{\pi}{2\delta}}e^{-\frac{\delta}{8\pi}}\delta^{\frac{\alpha-n}{2}}\frac{d\delta}{\delta}.$$

□

Remark 5.8. From Proposition 5.6, if $0 < \alpha < n$ then there exist $C_\alpha > 0$ and $\tilde{C}_\alpha > 0$ such that

$$\tilde{C}_\alpha|x|^{\alpha-n} \leq G_\alpha(x) \leq C_\alpha|x|^{\alpha-n}$$

for all x with $0 < |x| < 1$.

Also from Proposition 5.7 we observe that, for every $\alpha > 0$ there exist $M_\alpha > 0$ such that

$$G_\alpha(x) \leq M_\alpha e^{C|x|}$$

for all $x \in \mathbb{R}^n$ with $|x| > 1$.

From these two observations we can write

$$G_\alpha(x) \le C_\alpha\Big(\frac{\chi_{B(0,1)}(x)}{|x|^{n-\alpha}} + e^{-C|x|}\Big) \quad \text{for all } x \in \mathbb{R}^n.$$

Next we use the Bessel kernel to build a explicit weak solution for the Schrödinger operator. Let φ be a function belonging to $C_0^\infty(\mathbb{R}^n)$ and such that

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ 0 & \text{if } |x| \ge 2 \end{cases}$$

with $0 \le \varphi(x) \le 1$ for every $x \in \mathbb{R}^n$, we set $\varphi_r(x) = \varphi(\frac{x}{r})$ and define

$$G_\alpha^r(x) = G_\alpha(x)\varphi_r(x)$$

for $|x| \le r$. Observe that

$$G_\alpha^r(x) \to G_\alpha(x) \quad \text{as } r \to 0$$

and that $G_\alpha^r \in H_0(\Omega) \cap L^\infty(\Omega)$ with

$$\int_\Omega \langle A\nabla G_\alpha^r, \nabla\varphi\rangle = ⨍_{B_r} V G_\alpha^r \varphi dm \tag{5.8}$$

for any $\varphi \in H_0^1(\Omega)$ and $V \in L^1_{\rm loc}(\Omega)$. G_α^r will be called an approximate Bessel kernel. (5.8) tell us that G_α^r is a weak solution of $LG_\alpha^r + VG_\alpha^r = 0$.

Also, for a real function f we write $f^+ = \max\{f, 0\}$ for $x \in \Omega$, and $r > 0$ with $B_r = B(x, r)$.

6. Main result

In this section we give a characterization of the weak Harnack inequality for nonnegative solutions of elliptic equations in terms of the Bessel kernel and Kato class of order α. We start this section with the following result.

Lemma 6.1. *Let u be a non-negative weak solution of $Lu+Vu = 0$. If $\phi(V)(r_0) < \infty$ for some $r_0 < 0$, then there exists a constant $C > 0$ such that*

$$⨍_{B_r} \Big|\log u - ⨍_{B_r} \log u\Big|^2 dm \le C$$

for $B_{2r} \subseteq \Omega$ with $0 < r \le r_0$.

Proof. Let $\varphi \in C_0^\infty(B_{2r})$ with $\varphi \equiv 1$ on B_r. Then

$$\int A\nabla u\nabla\Big(\frac{\varphi^2}{u}\Big)dm = -\int A\varphi^2\frac{\nabla u\cdot\nabla u}{u^2}dm + 2\int a\frac{\varphi}{u}\nabla u\cdot\nabla\varphi dm.$$

Thus,

$$\int A\varphi^2\frac{\nabla u\cdot\nabla u}{u^2}dm = -\int A\nabla u\nabla\Big(\frac{\varphi^2}{u}\Big)dm + 2\int A\frac{\varphi}{u}\nabla u\cdot\nabla\varphi dm$$

$$\begin{aligned}\lambda\int\varphi^2|\nabla\log u|^2dm =&\lambda\int\varphi^2\left|\frac{\nabla u}{u}\right|^2dm\\ \le&\int A\varphi^2\frac{\nabla u\cdot\nabla u}{u^2}dm\\ \le&\int A\varphi^2\frac{\nabla u\cdot\nabla u}{u^2}dm\\ =&\int Vu\frac{\varphi^2}{u}dm+2\int A\frac{\varphi}{u}\nabla u\nabla\varphi dm\\ =&\int V\varphi^2dm+2\int A\frac{\varphi}{u}\nabla u\nabla\varphi dm.\end{aligned}$$

Since $V\in\widetilde{K}_{n,\alpha}$ there exists $C>0$ such that $\eta(V)(2r)\le C$. Now, it follows that

$$\frac{1}{(2r)^{n-\alpha}}\int_{B(x,2r)}|V(y)|dy\le\int_{B(x,2r)}\frac{|V(y)|}{|x-y|^{n-\alpha}}dy.$$

Thus,

$$\int_{B(x,2r)}|V(y)|dy\le\eta(V)(2r)(2r)^{n-\alpha}\le Cr^{n-\alpha}.$$

This immediately gives us

$$\int_{B(x,2r)}|\nabla\log u|^2dm\le Cr^{n-2}.$$

From this and the Poincaré inequality we obtain

$$⨍\Big|\log u-⨍_{B_r}\log u\Big|^2dm\le C⨍|\nabla\log u|^2dm\le Cr^{n-\alpha}.$$

□

The above lemma and Theorem 4.1 tell us that $\log u\in BMO$. Then by Corollary 3.1 there exits a positive constant C such that for $\beta>0$,

$$\frac{1}{m(B_r)}\int_{B_r}e^{|f(x)-f_{B_r}|}dm\le C,$$

where $f=\log u$. Using this we conclude that

$$\begin{aligned}&\frac{1}{m(B_r)}\Big(\int_{B_r}e^{\beta f}dm\Big)\frac{1}{m(B_r)}\Big(\int_{B_r}e^{-\beta f}dm\Big)\\ &=\frac{1}{(m(B_r))^2}\Big(\int_{B_r}e^{\beta(f-f_{B_r})}dm\Big)\Big(\int_{B_r}e^{-\beta(f-f_{B_r})}dm\Big)\\ &\le\Big(\frac{1}{m(B_r)}\int_{B_r}e^{\beta|f-f_{B_r}|}dm\Big)^2\le C,\end{aligned}$$

which implies

$$\Big(\int_{B_r}e^{\beta f}dm\Big)\Big(\int_{B_r}e^{-\beta f}dm\Big)\le C[m(B_r)]^2,$$

hence

$$\Big(\int_{B_r}|u|^\beta dm\Big)\Big(\int_{B_r}|u|^{-\beta}dm\Big)\le C[m(B_r)]^2. \tag{6.1}$$

Proposition 6.2. *Suppose that* (6.1) *holds, then there exits a positive constant* C *such that*
$$\int_{B_{2r}} |u|^\beta dm \le C \int_{B_r} |u|^\beta dm,$$
where $B_{2r} \subseteq \Omega$. *The above inequality is known as doubling condition.*

Proof. If (6.1) holds, then we have
$$\Big(\int_{B_r} |u|^\beta dm \Big))^{1/2} \Big(\int_{B_r} |u|^{-\beta} dm \Big)^{1/2} \le C^{1/2} m(B_r);$$
from this inequality we obtain
$$\Big(\int_{B_r} |u|^{-\beta} dm \Big)^{1/2} \le C^{1/2} m(B_r) \Big(\int_{B_r} |u|^\beta dm \Big)^{-1/2}. \tag{6.2}$$
On the other hand, by Schwartz's inequality and (6.2) we have
$$\begin{aligned}
m(B_r) &\le \int_{B_r} |u|^{\beta/2} |u|^{-\beta/2} dm \\
&\le \Big(\int_{B_r} |u|^\beta dm \Big)^{1/2} \Big(\int_{B_r} |u|^{-\beta} dm \Big)^{1/2} \\
&\le \Big(\int_{B_r} |u|^\beta dm \Big)^{1/2} \Big(\int_{B_{2r}} |u|^{-\beta} dm \Big)^{1/2} \\
&\le C^{1/2} m(B_r) \Big(\int_{B_r} |u|^\beta dm \Big)^{1/2} \Big(\int_{B_{2r}} |u|^\beta dm \Big)^{-1/2}.
\end{aligned}$$
Thus
$$m(B_r) \le C^{1/2} m(B_r) \Big(\frac{\int_{B_r} |u|^\beta dm}{\int_{B_{2r}} |u|^\beta dm} \Big).$$
Finally
$$\int_{B_{2r}} |u|^\beta dm \le C \int_{B_r} |u|^\beta dm.$$
□

We need the following mean-value inequality (see, [2]).

Theorem 6.3. *Let* u *be a weak solution of* $Lu + Vu = 0$ *in* Ω. *Given* $0 < p < \infty$, *there are positive constants* δ *and* C *such that*
$$\sup_{B_r} |u| \le C \Big(⨍_{B_r} |u|^p dm \Big)^{1/p}$$
whenever $\phi(V)(r) \le \delta$.

Let $J : \mathbb{R} \to \mathbb{R}$ be a smooth function. In our next result, we consider a weak solution of $Lu + VJ(u)$ in Ω such that $0 \le J(u) \le u$ in Ω. The proof of the following theorem follows along the same lines as the corresponding proof on [5].

Theorem 6.4 (Weak Harnack Inequality)**.** *Let* u *be a non-negative weak solution of* $Lu + Vu = 0$, *and let* $B_r = B(x, r)$ *such that* $4B_r \subseteq \Omega$. *Then there are positive constants* δ_0 *and* C *such that*
$$\Big(⨍_{B_r} u^\beta dm \Big)^{1/\beta} \le C \inf_{B_r} fu,$$
where β *is the constant in* (6.1), *whenever* $\eta(V)(r) \le \delta_0$.

Proof. For $t > 0$, we write $\Omega_t^r = \{x \in \Omega : G_\alpha^r(x) > t\}$ and $\Omega_t = \{x \in \Omega : G_\alpha(x) > t\}$, and also define the function

$$H(r,t) = \Big(\frac{G_\alpha^r}{t} - 1\Big) - \log^+\Big(\frac{G_\alpha^r}{t} - 1\Big).$$

On the one hand, we have $(\log^2 x)/2 \le x - 1 - \log x$ for $x \le 1$. On the other hand, we have

$$\frac{1}{2}\Big[\log^+(\frac{G_\alpha^r}{t})\Big]^2 \le H(r,t) \le \frac{G_\alpha^r}{t}$$

and that $H(r,t)$ is supported on Ω_t^r for all $t > 0$. Now, we claim that given $\beta > 0$ there is a positive constant $C = C(\beta, \lambda, L)$ such that for any $t > 0$

$$\int_{\Omega_t^r} \Big|\nabla\Big(u^{\beta/2}\log^+(\frac{G_\alpha^r}{t})\Big)\Big|^2 dm \le \frac{C}{t}\Big[\int_{\Omega_t^r} |V| G_\alpha^r u^\beta dm + \fint_{B_r} u^\beta dm\Big]. \tag{6.3}$$

We first prove the claim for a solution of $Lu + VJ(u) = 0$ such that $0 \le J(u) \le u$ and $\inf_\Omega u > 0$. In the definition (5.8) we take

$$\varphi = \Big(\frac{1}{t} - \frac{1}{G_\alpha^r}\Big)^+ u^\beta$$

as a test function (taking into account that $\inf_\Omega u > 0$). Then, we find that

$$\begin{aligned} &\int_{\Omega_t^r} \langle A\nabla G_\alpha^r, \nabla G_\alpha^r\rangle \frac{u^\alpha}{(G_\alpha^r)^2} dm + \alpha\int_\Omega \langle A\nabla G_\alpha^r, \nabla u\rangle \Big(\frac{1}{t} - \frac{1}{G_\alpha^r}\Big)^+ u^\beta dm \\ &= \fint_{B_r} (\frac{1}{t} - \frac{1}{G_\alpha^r})^+ u^\beta dm. \end{aligned} \tag{6.4}$$

Using

$$\nabla(H(r,t)u^{\beta-1}) + (1-\beta)u^{\beta-2}H(r,t)\nabla u = u^{\beta-1}(\frac{1}{t} - \frac{1}{G_\alpha^r})^+ \nabla G_\alpha^r$$

in (6.4) follows by application of (6.1) that

$$\begin{aligned} &\int_{\Omega_t^r} \langle A\nabla G_\alpha^r, \nabla G_\alpha^r\rangle \frac{u^\beta}{(G_\alpha^r)^2} dm + \alpha\int_\Omega \langle A\nabla(u^{\beta/2}), \nabla(u^{\beta/2})\rangle \big[\log^+\big(\frac{G_\alpha^r}{t}\big)\big]^2 dm \\ &\le \fint_{B_r} \frac{u^\beta}{t} dm - \beta\int_\Omega \langle A\nabla u, \nabla(H(r,t)u^{\beta-1}) dm \end{aligned}$$

from this we have

$$\int_{\Omega_t^r} \Big|\nabla u^{\beta/2}\log^+(\frac{G_\alpha^r}{t})\Big|^2 dm \le C(\beta,\lambda)\Big[\fint_{B_r} \frac{u^\beta}{t} dm - \beta\int_\Omega VJ(u)H(r,t)u^{\beta-1} dm\Big].$$

Recalling that $0 \le J(u) \le u$ and using (6.1) we have (6.3). Now, let u be a non-negative weak solution of $Lu + Vu = 0$ in Ω. Then for any $\varepsilon > 0$, and $J(u) = u - \varepsilon$, we can see that $w = u + \varepsilon$ is a weak solution of $Lw + VJ(w) = 0$ in Ω with $0 \le J(w) \le w$ such that $\inf_\Omega w > 0$. Therefore using (6.4) for w, letting $\varepsilon \to 0$ and using the fact that u is locally bounded, we should apply the Fatou's Lemma and the Lebesgue dominated convergence theorem to have the full statement of the claim.

Let $R_j = (\frac{C_j}{t})^{1/(n-\alpha)}$ for $j = 1, 2$ and C_1 and C_2 the constants in Remark 5.8. Then the following inclusions are direct consequences of the inequalities in Remark 5.8

$$B_{R_2} \subseteq \Omega_t^r, \quad \Omega_t^t \subseteq B_{R_1}.$$

Since u, G_α^r belong to $L^1_{\rm loc}(\Omega)$, we shall apply the Sobolev inequality to (6.4) to obtain the following chain of inequalities:

$$\begin{aligned}\frac{C}{R_1^2}\int_{\Omega_{2t}^r}\big|\log^+(\frac{G_\alpha^r}{t})\big|^2u^\beta dm &\le \frac{C}{R_1^2}\int_{\Omega_t^r}\Big|\log^+(\frac{G_\alpha^r}{t})\Big|^2u^\beta dm\\ &\le \int_{\Omega_t^r}\Big|\nabla u^{\beta/2}\log^+(\frac{G_\alpha^r}{t})\Big|^2dm\\ &\le \frac{C}{t}\int_{\Omega_t^r}|V|G_\alpha^r u^\beta dm+\frac{1}{m(B_r)}\int_{B_r}u^\beta dm.\end{aligned}$$

Next, using Remark 5.8 and (6.4) in the last inequality, we obtain

$$\frac{1}{R_1^2}\int_{\Omega_{2t}^r}u^\beta dm\le\frac{C}{t}\sup_{B_{R_1}}u^\beta\int_{B_{R_1}}|V||x-y|^{\alpha-n}dy+\frac{C}{m(B_r)}\int_{B_r}u^\beta dm.$$

Since $G_\alpha^r(x)\to G(x)$ as $r\to0$, we observe that $\chi_{\Omega_t}\le\liminf_r\chi_{\Omega_t}r$ from this and the Fatou's Lemma we deduce

$$\frac{1}{R_1^2}\int_{\Omega_{2t}^r}u^\beta dm\le\frac{C}{t}\big[\eta(V)(R_1)\sup_{B_{R_1}}u^\beta+u^\beta(x)\big]$$

by (6.4) we obtain

$$\frac{1}{R_1^2}\int_{B_{R_2}}u^\beta dm\le\frac{C}{t}\big[\eta(V)(R_1)\sup_{B_{R_1}}u^\beta+u^\beta(x)\big]$$

and thus, let $r>0$ such that $B_{4r}\subseteq\Omega$. We choose t such that $t=\max\{C_1,C_2\}r^{\alpha-n}$ and observe that (6.4) holds if $C_1\le C_2$ then $R_2=r$ and $R_1\le r$. If $C_2<C_1$, then $R_2=(\frac{C_2}{C_1})^{\frac{1}{n-\alpha}}r$ and $R_1=r$. In either case, we use the doubling property of u^α and Theorem 6.3 to conclude that

$$\fint_{B_r}\frac{u^\beta}{t}dm\le\eta(V)\fint_{B_r}\frac{u^\beta}{t}dm+Cu^\beta,$$

by choosing r sufficiently small, we conclude that

$$\fint_{B_r}u^\beta\le Cu^\beta(x),$$

which gives the desired result. □

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