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MULTIPLICITY OF SOLUTIONS FOR NONPERIODIC PERTURBED FRACTIONAL HAMILTONIAN SYSTEMS

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Abstract. In this article, we prove the existence and multiplicity of nontrivial solutions for the nonperiodic perturbed fractional Hamiltonian systems

$$-{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t x(t)) - \lambda L(t)\cdot x(t) + \nabla W(t,x(t)) = f(t),$$
$$x \in H^{\alpha}(\mathbb{R},\mathbb{R}^N),$$

where $\alpha \in (1/2,1]$, $\lambda > 0$ is a parameter, $t \in \mathbb{R}, x \in \mathbb{R}^N$, ${}_{-\infty}D^{\alpha}_t$ and ${}_tD^{\alpha}_{\infty}$ are left and right Liouville-Weyl fractional derivatives of order α on the whole axis $\mathbb{R}$ respectively, the matrix $L(t)$ is not necessary positive definite for all $t \in \mathbb{R}$ nor coercive, $W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ and $f \in L^2(\mathbb{R},\mathbb{R}^N)\backslash\{0\}$ small enough. Replacing the Ambrosetti-Rabinowitz Condition by general superquadratic assumptions, we establish the existence and multiplicity results for the above system. Some examples are also given to illustrate our results.

1. Introduction

Hamiltonian systems form a significant field of nonlinear functional analysis, since they arise in phenomena studied in several fields of applied science such as physics, astronomy, chemistry, biology, engineering and other fields of science. Since Newton wrote the differential equation describing the motion of the planet and derived the Kepler ellipse as its solution, the complex dynamical behavior of the Hamiltonian system has attracted a wide range of mathematicians and physicists. The variational methods to investigate Hamiltonian system were first used by Poincaré, who used the minimal action principle of the Jacobi form to study the closed orbits of a conservative system with two degrees of freedom. Ambrosetti and Rabinowitz in [1] proved "Mountain Pass Theorem", "Saddle Point Theorem", "Linking Theorem" and a series of very important minimax form of critical point theorem. The study of Hamiltonian systems makes a significant breakthrough, due to critical point theory. Critical point theorem was first used by Rabinowitz [20] to obtain the existence of periodic solutions for first order Hamiltonian systems, while the first multiplicity result is due to Ambrosetti and Zelati [2]. Since then, there is a large number of literatures on the use of critical point theory and variational methods to prove the existence of homoclinic or heteroclinic orbits of Hamiltonian systems see for example [6, 9, 17] and the references therein.

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On the other hand, fractional calculus has received increased popularity and importance in the past decades to describe long-memory processes. For more details, we refer the reader to the monographs [7, 13, 15] and the reference therein. Recently, the critical point theory has become an effective tool in studying the existence of solutions to fractional differential equations by constructing fractional variational structures.

Recently, in Jiao and Zhou [11] showed that critical point theory is an effective approach to tackle the existence of solutions for the fractional boundary-value problem

$$\begin{gathered} {}_tD_T^\alpha({}_0D_t^\alpha x(t)) = \nabla W(t, x(t)), \quad \text{a.e. } t \in [0, T], \\ x(0) = x(T), \end{gathered}$$

where $\alpha \in (1/2, 1)$, $x \in \mathbb{R}^N$, $W \in C^1([0, T] \times \mathbb{R}^N, \mathbb{R}), \nabla W(t, x)$ is the gradient of W at x, and obtained the existence of at least one nontrivial solution. Inspired by this paper, Torres [22] studied the fractional Hamiltonian system

$$\begin{gathered} -{}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha x(t)) - L(t) \cdot x(t) + \nabla W(t, x(t)) = 0, \\ x \in H^\alpha(\mathbb{R}, \mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $\alpha \in (\frac{1}{2}, 1)$, $t \in \mathbb{R}, x \in \mathbb{R}^N$, ${}_{-\infty}D_t^\alpha$ and ${}_tD_\infty^\alpha$ are left and right Liouville-Weyl fractional derivatives of order α on the whole axis $\mathbb{R}$ respectively, $L(t) \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is symmetric and positive definite matrix for all $t \in \mathbb{R}$ and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. The author showed that (1.1) possesses at least one nontrivial solution via Mountain Pass Theorem, by assuming that L and W satisfy the following hypotheses:

- (A1) $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$;
- (A2) the smallest eigenvalue of $L(t) \to +\infty$ as $t \to \infty$;
- (A3) $|\nabla W(t, x)| = o(|x|)$ as $|x| \to 0$ uniformly in $t \in \mathbb{R}$;
- (A4) there is $\overline{W} \in C(\mathbb{R}^N, \mathbb{R})$ such that
$$|W(t, x)| + |\nabla W(t, x)| \le |\overline{W}(x)|, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N.$$
- (A5) there exists a constant $\mu > 2$ such that
$$0 < \mu W(t, x) \le \nabla W(t, x) \cdot x, \quad \forall t \in \mathbb{R},\ x \in \mathbb{R}^N \backslash \{0\}.$$

When $\alpha = 1$, (1.1) reduces to the standard second-order Hamiltonian systems

$$\ddot{x}(t) - L(t)x(t) + \nabla W(t, x(t)) = 0. \tag{1.2}$$

When $L(t)$ is a symmetric matrix valued function for all $t \in \mathbb{R}$ and $W(t, x)$ satisfies the so-called global Ambrosetti-Rabinowitz Condition (A5), the existence and multiplicity of homoclinic solutions for Hamiltonian systems (1.2) have been extensively investigated in many recent papers see for example [2, 6, 9, 21] and the references therein. If $L(t)$ and $W(t, x)$ are neither periodic in t, the problem of existence of homoclinic orbits for (1.2) is quite different from the ones just described, because of lack of compactness of Sobolev embedding. In [21] and without periodicity assumptions on both L and W, Rabinowitz and Tanaka first studied system (1.2) and prove the existence of one nontrivial homoclinic orbit of (1.2) under assumptions (A1)–(A5).

Remark 1.1. Although the technical coercively assumption (A2) plays a key role to guarantee the compactness of the Sobolev embedding, it is somewhat restrictive and eliminates many functions.

Motivated by the above works , in this article, when $f \in L^2(\mathbb{R},\mathbb{R}^N)\backslash\{0\}$, $L(t) \in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix but not necessary positive definite for all $t \in \mathbb{R}$ not coercive, $W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ and replacing the Ambrosetti-Rabinowitz condition by general superquadratic assumptions, we establish the existence and multiplicity results for the nonperiodic perturbed fractional Hamiltonian system

$$\begin{gathered} -{}_tD^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t x(t)) - \lambda L(t)\cdot x(t) + \nabla W(t,x(t)) = f(t), \\ x \in H^{\alpha}(\mathbb{R},\mathbb{R}^N), \end{gathered} \tag{1.3}$$

where $\alpha \in (1/2,1], \lambda > 0$ is a parameter. Precisely, we suppose that

(A6) $\min_{x\in\mathbb{R}^N,|x|=1} L(t)x\cdot x \geq 0$ and there is $b>0$ such that $\text{meas}(\{L \ngeq b\}) < 1/c_\alpha^2$, where meas$(\cdot)$ is the Lebesgue measure, $\{L \ngeq b\} = \{t\in\mathbb{R} : L(t) \ngeq b\}$ and c_α defined the Sobolev constant (see section 2);

(A7) $W(t,0)=0$ and for any $0<\alpha_1<\alpha_2$,
$$C^{\alpha_1}_{\alpha_2} := \inf\Big\{\frac{\widetilde{W}(t,x)}{|x|^2}; t\in\mathbb{R}, \alpha_1<|x|<\alpha_2\Big\} > 0,$$
where $\widetilde{W}(t,x) := \frac{1}{2}\nabla W(t,x)\cdot x - W(t,x)$;

(A8) there exist $c_1>0$, $R_1>1$ and $\beta\in(1,2)$ such that
$$\nabla W(t,x)\cdot x \leq c_1\widetilde{W}(t,x)|x|^{2-\beta}, \quad \forall t\in\mathbb{R}, |x|\geq R_1;$$

(A9) there exist a constants $T_0>0$ and $x_0\in\mathbb{R}^N\backslash\{0\}$ such that
$$\int_{-T_0}^{T_0} \lambda L(t)x_0\cdot x_0 - W(t,x_0)\, dt < 0.$$

Our main results reads as follows.

Theorem 1.2. *Assume that $f\in L^2(\mathbb{R},\mathbb{R}^N)\backslash\{0\}$ and* (A3), (A4), (A6)–(A9) *hold. Then, there exist constants $f_0,\lambda_0>0$ such that, for any $\lambda>\lambda_0$ system* (1.3) *possesses at least two nontrivial solutions whenever $\|f\|_{L^2}<f_0$.*

Corollary 1.3. *Assume that $f\in L^2(\mathbb{R},\mathbb{R}^N)\backslash\{0\}$,* (A3), (W2), (A6)–(A8) *are satisfied and*

(A9')
$$\lim_{|x|\to+\infty}\frac{W(t,x)}{|x|^2} = +\infty, \quad \textit{uniformly for a.e. } t\in\mathbb{R}.$$

Then, there exist constants $f_0,\lambda_0>0$ such that, for any $\lambda>\lambda_0$ system (1.3) *possesses at least two nontrivial solutions whenever $\|f\|_{L^2}<f_0$.*

Remark 1.4. Assumption (A5) implies (A8), (A9) and (A9'). In fact assuming (A5) is satisfied, it is clear that (A9) and (A9') hold. Choose $R_1\geq 1$ so large that
$$\frac{1}{\mu} < \frac{1}{2} - \frac{1}{|x|^{2-\beta}} \quad \text{whenever } |x|\geq R_1.$$
Then, for such $|x|$, we have
$$W(t,x) \leq \Big(\frac{1}{2}-\frac{1}{|x|^{2-\beta}}\Big)\nabla W(t,x)\cdot x,$$
and it follows that
$$\nabla W(t,x)\cdot x \leq |x|^{2-\beta}\Big(\frac{1}{2}\nabla W(t,x)\cdot x - W(t,x)\Big) = |x|^{2-\beta}\widetilde{W}(t,x).$$

Here and in the following $x \cdot y$ denotes the inner product of $x, y \in \mathbb{R}^N$ and $|\cdot|$ denotes the associated norm. Throughout this article, we denote by c, c_i the various positive constants which may vary from line to line and are not essential to the problem.

2. Preliminaries

2.1. **Liouville-Weyl Fractional Calculus.**

Definition 2.1. The left and right Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined by

$$
\begin{aligned}
{}_{-\infty}I_x^{\alpha} u(x) &:= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-\xi)^{\alpha-1} u(\xi) d\xi, \\
{}_{x}I_{\infty}^{\alpha} u(x) &:= \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi-x)^{\alpha-1} u(\xi) d\xi\,,
\end{aligned}
$$

respectively, where $x \in \mathbb{R}$.

Definition 2.2. The left and right Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined by

$$
{}_{-\infty}D_x^{\alpha} u(x) := \frac{d}{dx} {}_{-\infty}I_x^{1-\alpha} u(x), \tag{2.1}
$$

$$
{}_{x}D_{\infty}^{\alpha} u(x) := -\frac{d}{dx} {}_{x}I_{\infty}^{1-\alpha} u(x)\,, \tag{2.2}
$$

respectively, where $x \in \mathbb{R}$.

Remark 2.3. Definitions (2.1) and (2.2) may be written in the alternative forms:

$$
\begin{aligned}
{}_{-\infty}D_x^{\alpha} u(x) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{u(x)-u(x-\xi)}{\xi^{\alpha+1}} d\xi, \\
{}_{x}D_{\infty}^{\alpha} u(x) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{u(x)-u(x+\xi)}{\xi^{\alpha+1}} d\xi\,.
\end{aligned}
$$

Recall that the Fourier transform $\widehat{u}(w)$ of $u(x)$ is defined by

$$
\widehat{u}(w) = \int_{-\infty}^{\infty} e^{-ix.w} u(x) dx.
$$

We establish the Fourier transform properties of the fractional integral and fractional operators as follows:

$$
\begin{aligned}
\widehat{{}_{-\infty}I_x^{\alpha} u(x)}(w) &:= (iw)^{-\alpha} \widehat{u}(w), \\
\widehat{{}_{x}I_{\infty}^{\alpha} u(x)}(w) &:= (-iw)^{-\alpha} \widehat{u}(w), \\
\widehat{{}_{-\infty}D_x^{\alpha} u(x)}(w) &:= (iw)^{\alpha} \widehat{u}(w), \\
\widehat{{}_{x}D_{\infty}^{\alpha} u(x)}(w) &:= (-iw)^{\alpha} \widehat{u}(w).
\end{aligned}
$$

2.2. **Fractional derivative spaces.** Let us recall for any $\alpha > 0$, the semi-norm

$$|u|_{I^{\alpha}_{-\infty}} := \|_{-\infty}D^{\alpha}_{x}u\|_{L^2},$$

and the norm

$$\|u\|_{I^{\alpha}_{-\infty}} := \Big(\|u\|^2_{L^2} + |u|^2_{I^{\alpha}_{-\infty}}\Big)^{1/2}.$$

Let the space $I^{\alpha}_{-\infty}(\mathbb{R})$ denote the completion of $C^{\infty}_0(\mathbb{R})$ with respect to the norm $\|\cdot\|_{I^{\alpha}_{-\infty}}$, i.e.,

$$I^{\alpha}_{-\infty}(\mathbb{R}) = \overline{C^{\infty}_0(\mathbb{R})}^{\|\cdot\|_{I^{\alpha}_{-\infty}}}.$$

Next, we define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$ in terms of the Fourier transform. For $0 < \alpha < 1$, define the semi-norm

$$|u|_{\alpha} = \||w|^{\alpha}\widehat{u}\|_{L^2},$$

and the norm

$$\|u\|_{\alpha} = (\|u\|^2_{L^2} + |u|^2_{\alpha})^{1/2},$$

and let

$$H^{\alpha}(\mathbb{R}) := \overline{C^{\infty}_0(\mathbb{R})}^{\|\cdot\|_{\alpha}}.$$

We note that a function $u \in L^2(\mathbb{R})$ belongs to $I^{\alpha}_{-\infty}(\mathbb{R})$ if and only if

$$|w|^{\alpha}\widehat{u} \in L^2(\mathbb{R}).$$

In particular, $|u|_{I^{\alpha}_{-\infty}} = \||w|^{\alpha}\widehat{u}\|_{L^2(\mathbb{R})}$. Therefore $H^{\alpha}(\mathbb{R})$ and $I^{\alpha}_{-\infty}(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm (see [22]).

Analogous to $I^{\alpha}_{-\infty}(\mathbb{R})$, we introduce $I^{\alpha}_{\infty}(\mathbb{R})$. Let us define the semi-norm

$$|u|_{I^{\alpha}_{\infty}} := \|_{x}D^{\alpha}_{\infty}\|_{L^2(\mathbb{R})},$$

and norm

$$\|u\|_{I^{\alpha}_{\infty}} := (\|u\|^2_{L^2} + |u|^2_{I^{\alpha}_{\infty}})^{1/2},$$

and let

$$I^{\alpha}_{-\infty}(\mathbb{R}) = \overline{C^{\infty}_0(\mathbb{R})}^{\|\cdot\|_{I^{\alpha}_{-\infty}}}.$$

Moreover $I^{\alpha}_{\infty}(\mathbb{R})$ and $I^{\alpha}_{-\infty}(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm.

Lemma 2.4 ([22]). *If $\alpha > 1/2$, then $H^{\alpha}(\mathbb{R}) \subset C(\mathbb{R})$ and there is a constant $C = C_{\alpha}$ such that*

$$\|u\|_{L^{\infty}} = \sup_{x\in\mathbb{R}} |u(x)| \le C\|u\|_{\alpha} \tag{2.3}$$

where $C(\mathbb{R})$ denote the space of continuous functions on $\mathbb{R}$.

Remark 2.5. If $u \in H^{\alpha}(\mathbb{R})$, then $u \in L^q(\mathbb{R})$ for all $q \in [2,\infty]$, since

$$\int_{\mathbb{R}} |u(x)|^q dx \le \|u\|^{q-2}_{L^{\infty}}\|u\|^2_{L^2}.$$

In what follows, we introduce the fractional space in which we will construct the variational framework of (1.3). Let

$$X^{\alpha} = \big\{x \in H^{\alpha}(\mathbb{R},\mathbb{R}^n) : \int_{\mathbb{R}} |_{-\infty}D^{\alpha}_{t}x(t)|^2 + L(t)x(t)\cdot x(t)dt < \infty\big\}.$$

The space X^{α} is a reflexive and separable Hilbert space with the inner product

$$(x,y)_{X^{\alpha}} = \int_{\mathbb{R}} ({}_{-\infty}D^{\alpha}_{t}x(t).{}_{-\infty}D^{\alpha}_{t}y(t)) + L(t)x(t)\cdot y(t)dt,$$

and the corresponding norm is

$$\|x\|_{X^\alpha} = \sqrt{(x,x)_{X^\alpha}}.$$

For $\lambda > 0$, we also need the following inner product

$$(x,y)_\lambda = \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha x(t) \cdot {}_{-\infty}D_t^\alpha y(t) + \lambda L(t)x(t) \cdot y(t))dt,$$

and the corresponding norm

$$\|x\|_\lambda = \sqrt{(x,x)_\lambda}.$$

Set $X_\lambda^\alpha = (X^\alpha, \|\cdot\|_\lambda)$. Observing $\|x\|_\lambda \geq \|x\|_{X^\alpha}$ for all $\lambda \geq 1$.

Lemma 2.6. *If L satisfies* (A6) *then, X^α is continuously embedded in $H^\alpha(\mathbb{R},\mathbb{R}^n)$.*

Proof. By (A6) and (2.3) we have

$$\begin{aligned}
&\int_{\mathbb{R}} |x(t)|^2 dt \\
&= \int_{\{L<b\}} |x(t)|^2 dt + \int_{\{L\geq b\}} |x(t)|^2 dt \\
&\leq \|x\|_{L^\infty}^2 \operatorname{meas}(\{L<b\}) + \frac{1}{b}\int_{\{L\geq b\}} L(t)x(t)\cdot x(t)dt \\
&\leq c_\alpha^2 \operatorname{meas}(\{L<b\})\Big(\int_{\mathbb{R}} (|{}_{-\infty}D_t^\alpha x(t)|^2 + |x(t)|^2)dt\Big) + \frac{1}{b}\int_{\{L\geq b\}} L(t)x(t)\cdot x(t)dt.
\end{aligned}$$

Therefore,

$$\|x\|_{L^2}^2 \leq \frac{\max\{c_\alpha^2 \operatorname{meas}(\{L<b\}), \frac{1}{b}\}}{1 - c_\alpha^2 \operatorname{meas}(\{L<b\})} \|x\|_{X^\alpha}^2 \tag{2.4}$$

and

$$\begin{aligned}
\|x\|_\alpha^2 &= \int_{\mathbb{R}} (|{}_{-\infty}D_t^\alpha x(t)|^2 + |x(t)|^2)dt \\
&\leq \Big(1 + \frac{\max\{c_\alpha^2 \operatorname{meas}(\{L<b\}), \frac{1}{b}\}}{1 - c_\alpha^2 \operatorname{meas}(\{L<b\})}\Big) \|x\|_{X^\alpha}^2,
\end{aligned} \tag{2.5}$$

which yields that the embedding $X^\alpha \hookrightarrow H^\alpha(\mathbb{R},\mathbb{R}^N)$ is continuous. ◻

Remark 2.7. Using the same conditions and techniques in (2.4) and (2.5), for all $\lambda \geq \frac{1}{bc_\alpha^2 \operatorname{meas}(\{L<b\})}$, we also obtain

$$\|x\|_{L^2}^2 \leq \frac{c_\alpha^2 \operatorname{meas}(\{L<b\})}{1 - c_\alpha^2 \operatorname{meas}(\{L<b\})} \|x\|_\lambda^2, \tag{2.6}$$

$$\|x\|_\alpha^2 \leq \Big(1 + \frac{c_\alpha^2 \operatorname{meas}(\{L<b\})}{1 - c_\alpha^2 \operatorname{meas}(\{L<b\})}\Big) \|x\|_\lambda^2. \tag{2.7}$$

Furthermore, using (2.3), (2.5) and (2.6), for every $p \in (2,\infty)$ and

$$\lambda \geq \frac{1}{bc_\alpha^2 \operatorname{meas}(\{L<b\})},$$

we have

$$
\begin{aligned}
&\int_{\mathbb{R}} |x(t)|^p dt \\
&\le \|x\|_{L^\infty}^{p-2} \int_{\mathbb{R}} |x(t)|^2 dt \\
&\le c_\alpha^{p-2} \Big(\int_{\mathbb{R}} (|{}_{-\infty}D_t^\alpha x(t)|^2 + |x(t)|^2) dt \Big)^{\frac{p-2}{2}} \frac{c_\alpha^2 \operatorname{meas}(\{L<b\})}{1-c_\alpha^2 \operatorname{meas}(\{L<b\})} \|x\|_\lambda^2 \\
&\le c_\alpha^{p-2} \Big(1 + \frac{c_\alpha^2 \operatorname{meas}(\{L<b\})}{1-c_\alpha^2 \operatorname{meas}(\{L<b\})} \Big)^{\frac{p-2}{2}} \|x\|_\lambda^{p-2} \frac{c_\alpha^2 \operatorname{meas}(\{L<b\})}{1-c_\alpha^2 \operatorname{meas}(\{L<b\})} \|x\|_\lambda^2 \qquad (2.8) \\
&= \Big(1 + \frac{c_\alpha^2 \operatorname{meas}(\{L<b\})}{1-c_\alpha^2 \operatorname{meas}(\{L<b\})} \Big)^{\frac{p-2}{2}} \frac{c_\alpha^p \operatorname{meas}(\{L<b\})}{1-c_\alpha^2 \operatorname{meas}(\{L<b\})} \|x\|_\lambda^p \\
&= \operatorname{meas}(\{L<b\}) \Big(\frac{c_\alpha^2}{1-c_\alpha^2 \operatorname{meas}(\{L<b\})} \Big)^{p/2} \|x\|_\lambda^p \\
&:= \delta_p^p \|x\|_\lambda^p .
\end{aligned}
$$

3. Proof of Theorem 1.2 and Corollary 1.3

For this purpose, we establish the corresponding variational framework to obtain solutions of (1.3). To this end, define the functional $I_\lambda : X_\lambda^\alpha \to \mathbb{R}$ by

$$
\begin{aligned}
I_\lambda(x) &= \int_{\mathbb{R}} \Big[\frac{1}{2} |{}_{-\infty}D_t^\alpha x(t)|^2 + \frac{\lambda}{2} L(t)x(t)\cdot x(t) - W(t,x(t)) + f(t)\cdot x(t) \Big] dt \\
&= \frac{1}{2}\|x\|_\lambda^2 - \int_{\mathbb{R}} W(t,x(t)) dt + \int_{\mathbb{R}} f(t)\cdot x(t) dt.
\end{aligned}
$$

Under assumptions (A3), (A4), (A6)–(A8), we see that I_λ is a continuously Fréchet-differentiable functional defined on X_λ^α; i.e., $I_\lambda \in C^1(X_\lambda^\alpha, \mathbb{R})$. Moreover, we have

$$
\begin{aligned}
&I_\lambda'(x)y \\
&= \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha x(t) . {}_{-\infty}D_t^\alpha y(t)) + \lambda L(t)x(t)\cdot y(t) - \nabla W(t,x(t))\cdot y(t) + f(t)\cdot y(t)] dt,
\end{aligned} \tag{3.1}
$$

for all $x, y \in X_\lambda^\alpha$, which yields

$$
I_\lambda'(x)x = \|x\|_\lambda^2 - \int_{\mathbb{R}} \nabla W(t,x(t))\cdot x(t) dt + \int_{\mathbb{R}} f(t)\cdot x(t) dt. \tag{3.2}
$$

We know that to find a solutions of (1.3), it suffices to obtain the critical points of I_λ; see [22]. For this purpose the lemma below is useful.

Recall that $\phi \in C^1(E,\mathbb{R})$ satisfy the Palais-Smale condition (PS) if any sequence $(x_n) \subset E$, for which $(\phi(x_n))$ is bounded and $\phi'(x_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence in E.

Lemma 3.1 ([20]). *Let E be a real Banach space and $\phi \in C^1(E,\mathbb{R})$ satisfying the Palais-Smale condition. If ϕ satisfies the following conditions:*

(i) $\phi(0) = 0$,
(ii) *there exist constants $\rho, \gamma > 0$ such that $\phi_{/\partial B_\rho(0)} \ge \gamma$,*
(iii) *there exist $e \in E \backslash \overline{B}_\rho(0)$ such that $\phi(e) \le 0$.*

Then ϕ possesses a critical value $c \geq \gamma$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} \phi(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0, g(1) = e\}.$$

To find the critical points of I_λ, we shall show that I_λ satisfies the (PS) condition.

Because of the lack of the compactness of the Sobolev embedding, we require the following convergence result.

Lemma 3.2. *Assume that $x_n \rightharpoonup x$ in X^α_λ,* (A3), (A4), (A7) *are satisfied and $f \in L^2$. Then*

$$I_\lambda(x_n - x) = I_\lambda(x_n) - I_\lambda(x) + o(1) \quad \text{as } n \to +\infty, \tag{3.3}$$

$$I'_\lambda(x_n - x) = I'_\lambda(x_n) - I'_\lambda(x) + o(1) \quad \text{as } n \to +\infty. \tag{3.4}$$

In particular, if (x_n) is a (PS) sequence of I_λ such that $I_\lambda(x_n) \to c$ for some $c \in \mathbb{R}$ then

$$I_\lambda(x_n - x) \to c - I_\lambda(x) \quad \text{as } n \to +\infty, \tag{3.5}$$

$$I'_\lambda(x_n - x) \to 0 \quad \text{as } n \to +\infty, \tag{3.6}$$

after passing to a subsequence.

Proof. As $x_n \rightharpoonup x$ in X^α_λ, we have $(x_n, x)_\lambda \to (x, x)_\lambda$ as $n \to \infty$. Then

$$\begin{aligned}\|x_n\|^2_\lambda &= (x_n - x, x_n - x)_\lambda + (x, x_n)_\lambda + (x_n - x, x)_\lambda \\ &= \|x_n - x\|^2_\lambda + \|x\|^2_\lambda + o(1).\end{aligned}$$

Obviously,

$$(x_n, z)_\lambda = (x_n - x, z)_\lambda + (x, z)_\lambda, \quad \forall z \in X^\alpha_\lambda.$$

Hence, to show (3.3) and (3.4) it suffices to prove that

$$\int_{\mathbb{R}} (W(t, x_n) - W(t, x_n - x) - W(t, x))dt = o(1), \tag{3.7}$$

$$\sup_{\varphi\in X^\alpha_\lambda, \|\varphi\|_\lambda=1} \int_{\mathbb{R}} (\nabla W(t, x_n) - \nabla W(t, x_n - x) - \nabla W(t, x)) \cdot \varphi dt = o(1). \tag{3.8}$$

Here, we only prove (3.8) the proof of (3.7) is similar. Setting $y_n := x_n - x$, then $y_n \rightharpoonup 0$ in X^α_λ and $y_n(t) \to 0$ a.e. $t \in \mathbb{R}$. From (A3), for every $\varepsilon > 0$, there exist $\sigma = \sigma(\varepsilon) \in (0, 1)$ such that

$$|\nabla W(t, u)| \leq \varepsilon|u|, \quad \forall t \in \mathbb{R}, |u| \leq \sigma. \tag{3.9}$$

By $(A4)$ and (3.9), we have

$$|\nabla W(t, u)| \leq \varepsilon|u| + c_\varepsilon|u|^2, \quad \forall t \in \mathbb{R}, |u| \leq N_1, \tag{3.10}$$

where

$$N_1 := \sup_n\{\|y_n\|_{L^\infty},\ \|y_n + x\|_{L^\infty},\ \|x\|_{L^\infty} + 1\}, \quad c_\varepsilon = \max_{|u|\in[\sigma,N_1]} \overline{W}(u)\sigma^{-2}.$$

By (3.10) and the Young Inequality, for each $\varphi \in X^\alpha_\lambda$ with $\|\varphi\|_\lambda = 1$, we have

$$\begin{aligned}&|(\nabla W(t, y_n + x) - \nabla W(t, y_n)).\varphi| \\ &\leq \varepsilon(|y_n + x| + |y_n|)|\varphi| + c_\varepsilon(|y_n + x|^2 + |y_n|^2)|\varphi|,\end{aligned}$$

$$\begin{aligned}&\le c(\varepsilon|y_n||\varphi|+\varepsilon|x||\varphi|+c_\varepsilon|y_n|^2|\varphi|+c_\varepsilon|x|^2|\varphi|),\\&\le c(\varepsilon|y_n|^2+\varepsilon|x|^2+\varepsilon|\varphi|^2+\varepsilon|y_n|^3+c'_\varepsilon|\varphi|^3+c''_\varepsilon|x|^3),\end{aligned}$$

and

$$\begin{aligned}&|(\nabla W(t,y_n+x)-\nabla W(t,y_n)-\nabla W(t,x))\cdot\varphi|\\&\le c(\varepsilon|y_n|^2+\varepsilon|x|^2+\varepsilon|\varphi|^2+\varepsilon|y_n|^3+c'_\varepsilon|\varphi|^3+c''_\varepsilon|x|^3).\end{aligned}\tag{3.11}$$

If we take

$$\psi_n(t):=\max\{|(\nabla W(t,y_n+x)-\nabla W(t,y_n)-\nabla W(t,x))\varphi|-c\varepsilon(|y_n|^2+|y_n|^3),0\},$$

we obtain

$$0\le\psi_n(t)\le c(\varepsilon|x|^2+\varepsilon|\varphi|^2+c'_\varepsilon|\varphi|^3+c''_\varepsilon|x|^3)\in L^1(\mathbb{R},\mathbb{R}^N).$$

The Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}}\psi_n(t)\,dt\to 0\,\text{as}\,n\to\infty.\tag{3.12}$$

It follows from the definition of $\psi_n(t)$ that

$$|(\nabla W(t,y_n+x)-\nabla W(t,y_n)-\nabla W(t,x)).\varphi|\le\psi_n(t)+\varepsilon c(|y_n|^2+|y_n|^3),$$

and then

$$\begin{aligned}&|\int_{\mathbb{R}}(\nabla W(t,y_n+x)-\nabla W(t,y_n)-\nabla W(t,x))\cdot\varphi\,dt|\\&\le\|\psi_n(t)\|_{L^1}+\varepsilon c(\|y_n\|_{L^2}^2+\|y_n\|_{L^3}^3),\end{aligned}$$

for all n. Because φ is arbitrary in X^α_λ, we obtain

$$\begin{aligned}&\sup_{\varphi\in X^\alpha_\lambda,\,\|\varphi\|_\lambda=1}\Big|\int_{\mathbb{R}}(\nabla W(t,y_n+x)-\nabla W(t,y_n)-\nabla W(t,x))\cdot\varphi\ dt\Big|\\&\le\|\psi_n(t)\|_{L^1}+\varepsilon c(\|y_n\|_{L2}^2+\|y_n\|_{L^3}^3),\end{aligned}$$

which, jointly with (2.8) and (3.12) shows that

$$\sup_{\varphi\in X^\alpha_\lambda,\,\|\varphi\|_\lambda=1}\Big|\int_{\mathbb{R}}(\nabla W(t,y_n+x)-\nabla W(t,y_n)-\nabla W(t,x))\cdot\varphi\,dt\Big|\le\varepsilon c,$$

for n sufficiently large. Therefore, (3.8) holds.

If moreover $I_\lambda(x_n)\to c$ and $I'_\lambda(x_n)\to 0$ as $n\to\infty$, equations (3.3) and (3.4) respectively, imply that

$$I_\lambda(x_n-x)\to c-I_\lambda(x)+o(1),$$

and

$$I'_\lambda(x_n-x)=-I'_\lambda(x)\,\text{as}\,n\to+\infty.$$

We show that $I'_\lambda(x)=0$. For every $\zeta\in C_0^\infty(\mathbb{R},\mathbb{R}^N)$, we have

$$I'_\lambda(x)\zeta=\lim_{n\to\infty}I'_\lambda(x_n)\zeta=0.$$

Consequently, $I'_\lambda(x)=0$ and (3.6) holds. □

Lemma 3.3. *Suppose that $f\in L^2$ and* (A3), (A4), (A6), (A8) *are satisfied. Then, there exists $\lambda_0>0$ such that any bounded (PS) sequence of I_λ has a convergent subsequence when $\lambda>\lambda_0$.*

Proof. Let (x_n) be a bounded sequence such that $(I_\lambda(x_n))$ is bounded and $I'_\lambda(x_n) \to 0$ as $n \to \infty$. Then, after passing to a subsequence, we have $x_n \rightharpoonup x$ in X^α_λ and $y_n \to 0$ in $L^2(\{L(t) < b\})$ where $y_n := x_n - x$. Moreover,

$$\|y_n\|^2_{L^2} \le \frac{1}{\lambda b}\int_{\{L\ge b\}} \lambda L(t) y_n \cdot y_n dt + \int_{\{L<b\}} |y_n|^2 dt \le \frac{1}{\lambda b}\|y_n\|^2_\lambda + o(1). \tag{3.13}$$

Setting $N_2 := \sup_n \|y_n\|_{L^\infty}$. By (A4), we obtain

$$|\widetilde{W}(t,y_n)| = \big|\frac{1}{2}\nabla W(t,y_n)\cdot y_n - W(t,y_n)\big| \le \max_{|u|\in[0,N_2]} \overline{W}(u)(N_2+1), \quad \forall n,$$

which, jointly with (3.13) and (A8) yields

$$\begin{aligned} \int_{|y_n|\ge R_1} \nabla W(t,y_n)\cdot y_n dt &\le c_1 \int_{|y_n|\ge R_1} \widetilde{W}(t,y_n)|y_n|^{2-\beta} dt \\ &\le c c_1 R_1^{-\beta} \int_{|y_n|\ge R_1} |y_n|^2 dt \\ &\le \frac{cc_1}{\lambda b}\|y_n\|^2_\lambda + o(1). \end{aligned} \tag{3.14}$$

Furthermore, using (A4), (3.9) and (3.13), we have

$$\begin{aligned} &\int_{|y_n|<R_1} \nabla W(t,y_n) y_n dt \\ &\le \int_{|y_n|\le\sigma} \varepsilon |y_n|^2 dt + \int_{\sigma<|y_n|<R_1} |\nabla W(t,y_n)||y_n| dt \\ &\le \varepsilon \int_{|y_n|\le\delta} |y_n|^2 dt + \max_{|u|\in[\sigma,R_1]} \overline{W}(u)\sigma^{-1} \int_{\mathbb{R}} |y_n|^2 dt \\ &\le \frac{c}{\lambda b}\|y_n\|^2_\lambda + o(1). \end{aligned} \tag{3.15}$$

Because $f \in L^2$, one has, for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\Big(\int_{|t|\ge T_\varepsilon} |f(t)|^2 dt\Big)^{1/2} < \varepsilon.$$

Using (2.8) and the Hölder inequality, we have

$$\big|\int_{|t|\ge T_\varepsilon} f(t) y_n dt\big| \le \Big(\int_{|t|\ge T_\varepsilon} |f(t)|^2 dt\Big)^{1/2}\Big(\int_{\mathbb{R}} |y_n|^2 dt\Big)^{1/2} \le c\varepsilon \quad \forall n. \tag{3.16}$$

Obviously

$$\int_{|t|<T_\varepsilon} f(t)\cdot y_n dt \le \Big(\int_{\mathbb{R}} |f(t)|^2 dt\Big)^{1/2}\Big(\int_{|t|<T_\varepsilon} |y_n|^2 dt\Big)^{1/2} \to 0, \tag{3.17}$$

as $n \to \infty$. By (3.16) and (3.17), we have

$$\int_{\mathbb{R}} f(t)\cdot y_n(t) dt \to 0, \tag{3.18}$$

as $n \to \infty$. Consequently, a combination of (3.4), (3.14), (3.15) and (3.18) implies

$$\begin{aligned} o(1) = I'_\lambda(y_n) y_n &= \|y_n\|^2_\lambda - \int_{\mathbb{R}} \nabla W(t,y_n)\cdot y_n dt + \int_{\mathbb{R}} f(t)\cdot y_n dt \\ &\ge (1 - \frac{cc_1}{\lambda b} - \frac{c}{\lambda b})\|y_n\|^2_\lambda + o(1). \end{aligned}$$

Choosing $\lambda_0 > 0$ large enough such the term $(1 - \frac{cc_1}{\lambda b} - \frac{c}{\lambda b})$ is positive. When $\lambda > \lambda_0$, we obtain $y_n \to 0$ and then $x_n \to x$ in X_λ^α. □

Lemma 3.4. *If $f \in L^2$ and (A3), (A4), (A6)–(A8) are satisfied, then I_λ satisfies the (PS) condition whenever $\lambda > \lambda_0$.*

Proof. Let (x_n) be a (PS) sequence of I_λ. By Lemma 3.3, it suffices to prove that (x_n) is bounded. Indeed, assume that $\|x_n\|_\lambda \to \infty$ as $n \to \infty$ and setting $y_n := \frac{x_n}{\|x_n\|_\lambda}$. Then $\|y_n\|_\lambda = 1$ and $\|y_n\|_{L^p} \le \delta_p$ for $p \in [2, +\infty]$. Moreover, we have

$$o(1) = \frac{I'_\lambda(x_n)x_n}{\|x_n\|_\lambda^2} = 1 - \int_{\mathbb{R}} \frac{\nabla W(t, x_n) \cdot x_n}{\|x_n\|_\lambda^2} dt + o(1),$$

as $n \to \infty$. We obtain

$$\int_{\mathbb{R}} \frac{\nabla W(t, x_n) \cdot y_n}{|x_n|} |y_n| dt = \int_{\mathbb{R}} \frac{\nabla W(t, x_n) \cdot x_n}{\|x_n\|_\lambda^2} dt \to 1, \tag{3.19}$$

as $n \to \infty$. Let $0 \le \alpha_1 < \alpha_2$ and $\omega_n^{\alpha_1,\alpha_2} := \{t \in \mathbb{R}; \alpha_1 \le |x_n(t)| < \alpha_2\}$. By (2.8) and because (x_n) is a (PS) sequence of I_λ, then there exists $N_0 > 0$ such that for $n \ge N_0$ we have

$$\begin{aligned} c + \|x_n\|_\lambda &\ge I_\lambda(x_n) - \frac{1}{2} I'_\lambda(x_n)x_n \\ &\ge \int_{\mathbb{R}} \widetilde{W}(t, x_n) dt + \frac{1}{2} \int_{\mathbb{R}} f(t) \cdot x_n dt \\ &\ge \int_{\mathbb{R}} \widetilde{W}(t, x_n) dt - \frac{\delta_2}{2} \|f\|_{L^2} \|x_n\|_\lambda. \end{aligned}$$

This implies, for $n \ge N_0$, that

$$\begin{aligned} &c(1 + \|x_n\|_\lambda) \\ &\ge \int_{\mathbb{R}} \widetilde{W}(t, x_n) dt \\ &= \int_{\omega_n^{0,\alpha_1}} \widetilde{W}(t, x_n) dt + \int_{\omega_n^{\alpha_1,\alpha_2}} \widetilde{W}(t, x_n) dt + \int_{\omega_n^{\alpha_2,\infty}} \widetilde{W}(t, x_n) dt. \end{aligned} \tag{3.20}$$

By (A3), for any $\varepsilon > 0 (\varepsilon < \frac{1}{3})$ there exists $\kappa_\varepsilon > 0$ such that

$$|\nabla W(t, u)| \le (\frac{\varepsilon}{\delta_2^2}) |u|, \quad \forall |u| \le \kappa_\varepsilon, t \in \mathbb{R}.$$

Thus,

$$\int_{\omega^{0,\kappa_\varepsilon}} \frac{|\nabla W(t, x_n)|}{|x_n|} |y_n|^2 dt \le \int_{\omega^{0,\kappa_\varepsilon}} \frac{\varepsilon}{\delta_2^2} |y_n|^2 dt \le \frac{\varepsilon}{\delta_2^2} \|y_n\|_{L^2}^2 \le \varepsilon, \quad \forall n. \tag{3.21}$$

Because $\beta > 1$ and by (A8), (2.8) and (3.20) we can choose $\theta_\varepsilon \geq R_1$ large enough such that

$$\begin{aligned} \int_{\omega^{\theta_\varepsilon,+\infty}} \frac{\nabla W(t,x_n)x_n}{\|x_n\|_\lambda^2} dt &\leq \int_{\omega^{\theta_\varepsilon,+\infty}} c_1 \frac{|y_n|\widetilde{W}(t,x_n)}{|x_n|^{\beta-1}\|x_n\|_\lambda} dt \\ &\leq c_1\|y_n\|_{L^\infty} \int_{\omega^{\theta_\varepsilon,+\infty}} \frac{\widetilde{W}(t,x_n)}{\theta_\varepsilon^{\beta-1}\|x_n\|_\lambda} \\ &\leq \frac{cc_1\|y_n\|_{L^\infty}(1+\|x_n\|_\lambda)}{\theta_\varepsilon^{\beta-1}\|x_n\|_\lambda} \\ &\leq \frac{c\delta_\infty}{\theta_\varepsilon^{\beta-1}} < \varepsilon, \quad \forall n \geq N_0. \end{aligned} \tag{3.22}$$

By (A7), we have $\widetilde{W}(t,x_n(t)) \geq C_{\kappa_\varepsilon}^{\theta_\varepsilon}|x_n|^2$ for $t \in \omega_n^{\kappa_\varepsilon,\theta_\varepsilon}$. Noting $C_{\kappa_\varepsilon}^{\theta_\varepsilon} > 0$ it follows from (3.20) that

$$\begin{aligned} \int_{\omega^{\kappa_\varepsilon,\theta_\varepsilon}} |y_n|^2 dt &= \frac{1}{\|x_n\|_\lambda^2} \int_{\omega^{\kappa_\varepsilon,\theta_\varepsilon}} |x_n|^2 dt \\ &\leq \frac{1}{C_{\kappa_\varepsilon}^{\theta_\varepsilon}\|x_n\|_\lambda^2} \int_{\omega^{\kappa_\varepsilon,\theta_\varepsilon}} \widetilde{W}(t,x_n) dt \\ &\leq \frac{c(1+\|x_n\|_\lambda)}{C_{\kappa_\varepsilon}^{\theta_\varepsilon}\|x_n\|_\lambda^2} \to 0, \end{aligned} \tag{3.23}$$

as $n \to +\infty$, which yields that

$$\int_{\omega^{\kappa_\varepsilon,\theta_\varepsilon}} \frac{|\nabla W(t,x_n)|}{|x_n|}|y_n|^2 dt \leq \tau_\varepsilon \int_{\omega^{\kappa_\varepsilon,\theta_\varepsilon}} |y_n|^2 dt \to 0, \tag{3.24}$$

as $n \to \infty$, where $\tau_\varepsilon = \max_{|u|\in[\kappa_\varepsilon,\theta_\varepsilon]} \overline{W}(u)\cdot\kappa_\varepsilon$. Hence, by (3.21), (3.22) and (3.24), we have

$$\int_{\mathbb{R}} \frac{\nabla W(t,x_n)\cdot y_n}{|x_n|}|y_n| \leq \int_{\mathbb{R}} \frac{|\nabla W(t,x_n)|}{|x_n|}|y_n|^2 \leq 3\varepsilon < 1,$$

for n large enough, a contradiction with (3.19) and then (x_n) is bounded in X_λ^α. □

Lemma 3.5. *If* (A3) *holds and* $f \in L^2$*, then there exist* $\rho, \gamma, f_0 > 0$ *such that* $I_\lambda(x)_{/\|x\|_\lambda=\rho} \geq \gamma$ *when* $\|f\|_{L^2} < f_0$*.*

Proof. By (A3), for $\varepsilon := \frac{1}{4\delta_2^2}$ there exists $\sigma_1 = \sigma_1(\varepsilon)$ such that

$$|W(t,x)| \leq \varepsilon|x|^2, \forall t \in \mathbb{R}, |x| \leq \sigma_1. \tag{3.25}$$

Thus, for $\|x\|_\lambda \leq \rho := \sigma_1/\delta_\infty$, by (3.25), we obtain

$$I_\lambda(x) \geq \frac{1}{2}\|x\|_\lambda^2 - \varepsilon\int_{\mathbb{R}} |x|^2 dt - \|f\|_{L^2}\|x\|_{L^2} \geq \|x\|_\lambda\Big(\frac{1}{4}\|x\|_\lambda - \|f\|_{L^2}\delta_2\Big).$$

Let $\gamma := \rho(\frac{1}{4\delta_2}\rho - \|f\|_{L^2}\delta_2)$. Then, if $\|f\|_{L^2} < f_0 := \frac{1}{4\delta_2^2}\rho$, we have $I_\lambda(x)_{/\|x\|_\lambda=\rho} \geq \gamma$. □

Lemma 3.6. *If* $\|f\|_{L^2} < f_0$ *and* (A3), (A7) *are satisfied, then there exists* $x_1 \in X_\lambda^\alpha\backslash\{0\}$ *such that* $I'_\lambda(x_1) = 0$*.*

Proof. Since $f \in L^2\backslash\{0\}$, we can choose $\xi \in X_\lambda^\alpha$ such that $\int_{\mathbb{R}} f(t)\cdot\xi(t)dt < 0$. By (A3) and (A7) we have $W \geq 0$ and

$$I_\lambda(s\xi) \leq \frac{s^2}{2}\|\xi\|_\lambda^2 + s\int_{\mathbb{R}} f(t)\cdot\xi(t)dt < 0,$$

for s small enough. Thus $C_1 := \inf\{I_\lambda(x), x \in \overline{B}_\rho(0)\} < 0$, where ρ is the constants given by Lemma 3.5. From Ekeland's variational principle there exists a sequence $(x_n) \subset \overline{B}_\rho$ such that $C_1 \leq I_\lambda(x_n) < C_1 + \frac{1}{n}$. Then, by a standard procedure, we can show that $(x_n) \subset X_\lambda^\alpha$ is bounded (PS) sequence. Consequently, Lemma 3.3 implies that, there exist $x_1 \in X_\lambda^\alpha$ such that $x_n \to x_1 \in X_\lambda^\alpha, I'_\lambda(x_1) = 0$ and $I_\lambda(x_1) = C_1 < 0$ when $\lambda > \lambda_0$. □

3.1. **Proof of Theorem 1.2.** Let $h(s) = s^{-2}W(t, sx_0)$ for $t \in \mathbb{R}, s > 0$. Then, by (A7),

$$h'(s) = s^{-3}[-2W(t, sx_0) + \nabla W(t, sx_0).sx_0] > 0, \quad \text{for } t \in \mathbb{R},\ s > 0.$$

Integrating the above from 1 to η, we obtain

$$W(t, \eta x_0) \geq \eta^2 W(t, x_0), \quad \text{for } t \in \mathbb{R},\ \eta > 1. \tag{3.26}$$

From (3.26), we have for $s > 1$,

$$\begin{aligned} I_\lambda(sx_0) &= \int_{\mathbb{R}} (\lambda s^2 L(t)x_0\cdot x_0 - W(t, sx_0))dt + s\int_{\mathbb{R}} f(t)\cdot x_0 dt \\ &\leq s^2\big(\int_{\mathbb{R}} \lambda L(t)x_0\cdot x_0 - W(t, x_0)dt\big) + s\int_{\mathbb{R}} f(t)\cdot x_0 dt. \end{aligned}$$

Let

$$e(t) = \begin{cases} sx_0, & \text{if } t \in [-T_0, T_0] \\ 0, & \text{if } t \in \mathbb{R}\backslash[-T_0, T_0], \end{cases}$$

By (A9) there exists $s_0 \geq 1$ such that $\|e\|_\lambda > \rho$ and $I_\lambda(e) < 0$. Since $I_\lambda(0) = 0$ and all the assumptions of Lemma 3.1 are satisfied, so I_λ possesses a critical point $x_2 \in X_\lambda^\alpha$ with $I'_\lambda(x_2) = 0$ and $I_\lambda(x_2) = C_2 > 0$ whenever $\lambda > \lambda_0$.

3.2. **Proof of Corollary 1.3.** If (A9') holds, let $e \in C_0^\infty(\mathbb{R})\backslash\{0\}$. Then, by Fatou's Lemma and by $W \geq 0$ we have

$$I_\lambda(se) \leq s^2\Big[\frac{1}{2}\|e\|_\lambda^2 - \int_{e\neq 0} \frac{W(t, se)}{(se)^2} e^2\, dt\Big] + s\int_{\mathbb{R}} f(t).e(t)dt \to -\infty$$

as $s \to +\infty$, which implies that $I_\lambda(se) < 0$ for $s > 0$ large. Combining this with Lemmas 3.4 and 3.5, all the assumptions of Lemma 3.1 are satisfied, so I_λ possesses a critical point $x_3 \in X_\lambda^\alpha$ with $I'_\lambda(x_3) = 0$ and $I_\lambda(x_3) > 0$ whenever $\lambda > \lambda_0$.

4. An example

Let $L(t) = h(t)I_N$ where

$$h(t) = \begin{cases} 0, & \text{if } |t| < 1, \\ 2n^2|t-n|, & \text{if } |t| \geq 1, |t-n| \leq \frac{1}{2n^2}\ (n \in \mathbb{Z}, |n| \geq 1), \\ 1, & \text{elsewhere}, \end{cases}$$

$$W(t, x) = k(t)|x|^2 \ln(1 + |x|^2),$$

where $k : \mathbb{R} \to \mathbb{R}^+$ is a continuous bounded function with $\inf k(t) > 0$. A straightforward computation shows that L and W satisfies Theorem 1.2 and Corollary 1.3 but they do not satisfy the corresponding results on the above papers, in particular W do not satisfy the Ambrosetti- Rabinowitz Condition (A5).

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