MULTIPLICITY OF SOLUTIONS FOR NONPERIODIC PERTURBED FRACTIONAL HAMILTONIAN SYSTEMS

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Abstract. In this article, we prove the existence and multiplicity of nontrivial solutions for the nonperiodic perturbed fractional Hamiltonian systems

\[-tD_\infty^{\alpha}(-\infty D_t^{\alpha} x(t)) - \lambda L(t) \cdot x(t) + \nabla W(t, x(t)) = f(t),\]

\[x \in H^{\alpha}(\mathbb{R}, \mathbb{R}^N),\]

where \(\alpha \in (1/2, 1]\), \(\lambda > 0\) is a parameter, \(t \in \mathbb{R}, x \in \mathbb{R}^N\), \(-\infty D_t^{\alpha}\) and \(t D_\infty^{\alpha}\) are left and right Liouville-Weyl fractional derivatives of order \(\alpha\) on the whole axis \(\mathbb{R}\) respectively, the matrix \(L(t)\) is not necessary positive definite for all \(t \in \mathbb{R}\) nor coercive, \(W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})\) and \(f \in L^2(\mathbb{R}, \mathbb{R}^N) \setminus \{0\}\) small enough. Replacing the Ambrosetti-Rabinowitz Condition by general superquadratic assumptions, we establish the existence and multiplicity results for the above system. Some examples are also given to illustrate our results.

1. Introduction

Hamiltonian systems form a significant field of nonlinear functional analysis, since they arise in phenomena studied in several fields of applied science such as physics, astronomy, chemistry, biology, engineering and other fields of science. Since Newton wrote the differential equation describing the motion of the planet and derived the Kepler ellipse as its solution, the complex dynamical behavior of the Hamiltonian system has attracted a wide range of mathematicians and physicists. The variational methods to investigate Hamiltonian system were first used by Poincaré, who used the minimal action principle of the Jacobi form to study the closed orbits of a conservative system with two degrees of freedom. Ambrosetti and Rabinowitz in [1] proved “Mountain Pass Theorem”, “Saddle Point Theorem”, “Linking Theorem” and a series of very important minimax form of critical point theorem. The study of Hamiltonian systems makes a significant breakthrough, due to critical point theory. Critical point theorem was first used by Rabinowitz [20] to obtain the existence of periodic solutions for first order Hamiltonian systems, while the first multiplicity result is due to Ambrosetti and Zelati [2]. Since then, there is a large number of literatures on the use of critical point theory and variational methods to prove the existence of homoclinic or heteroclinic orbits of Hamiltonian systems see for example [9, 17] and the references therein.

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On the other hand, fractional calculus has received increased popularity and importance in the past decades to describe long-memory processes. For more details, we refer the reader to the monographs [7, 13, 15] and the reference therein. Recently, the critical point theory has become an effective tool in studying the existence of solutions to fractional differential equations by constructing fractional variational structures. We refer the reader to the monographs [7, 13, 15] and the reference therein. If \( L(t) \) and \( W(t) \) are neither periodic in \( t \), the problem of existence of at least one nontrivial solution via Mountain Pass Theorem, by assuming that \( L(t) \) and \( W(t) \) satisfy the following hypotheses:

(A1) \( L \in C(\mathbb{R}, \mathbb{R}^{N^2}) \) is a positive definite symmetric matrix for all \( t \in \mathbb{R} \);

(A2) the smallest eigenvalue of \( L(t) \to +\infty \) as \( t \to \infty \);

(A3) \( |\nabla W(t,x)| = o(|x|) \) as \( |x| \to 0 \) uniformly in \( t \in \mathbb{R} \);

(A4) there is \( \bar{W} \in C(\mathbb{R}^N, \mathbb{R}) \) such that

\[
|W(t,x)| + |\nabla W(t,x)| \leq |\bar{W}(x)|, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.
\]

(A5) there exists a constant \( \mu > 2 \) such that

\[
0 < \mu W(t,x) \leq \nabla W(t,x) \cdot x, \quad \forall t \in \mathbb{R}, \quad x \in \mathbb{R}^N \setminus \{0\}.
\]

When \( \alpha = 1 \), \((1.1)\) reduces to the standard second-order Hamiltonian systems

\[
\ddot{x}(t) - L(t)x(t) + \nabla W(t,x(t)) = 0.
\]

When \( L(t) \) is a symmetric matrix valued function for all \( t \in \mathbb{R} \) and \( W(t,x) \) satisfies the so-called global Ambrosetti-Rabinowitz Condition (A5), the existence and multiplicity of homoclinic solutions for Hamiltonian systems \((1.2)\) have been extensively investigated in many recent papers see for example [2] [6] [9] [21] and the references therein. If \( L(t) \) and \( W(t,x) \) are neither periodic in \( t \), the problem of existence of homoclinic orbits for \((1.2)\) is quite different from the ones just described, because of lack of compactness of Sobolev embedding. In [21] and without periodicity assumptions on both \( L \) and \( W \), Rabinowitz and Tanaka first studied system \((1.2)\) and prove the existence of one nontrivial homoclinic orbit of \((1.2)\) under assumptions (A1)–(A5).

**Remark 1.1.** Although the technical coercively assumption (A2) plays a key role to guarantee the compactness of the Sobolev embedding, it is somewhat restrictive and eliminates many functions.
Motivated by the above works, in this article, when \( f \in L^2(\mathbb{R}, \mathbb{R}^N) \setminus \{0\} \), \( L(t) \in C(\mathbb{R}, \mathbb{R}^{N^2}) \) is a symmetric matrix but not necessary positive definite for all \( t \in \mathbb{R} \) not coercive, \( W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}) \) and replacing the Ambrosetti-Rabinowitz condition by general superquadratic assumptions, we establish the existence and multiplicity results for the nonperiodic perturbed fractional Hamiltonian system

\[
\begin{align*}
-\lambda D_{\alpha}^\alpha(-\infty D_{\alpha}^\alpha x(t)) - \lambda L(t) \cdot x(t) + \nabla W(t, x(t)) &= f(t), \\
x &\in H^\alpha(\mathbb{R}, \mathbb{R}^N),
\end{align*}
\]

(1.3)

where \( \alpha \in (1/2, 1), \lambda > 0 \) is a parameter. Precisely, we suppose that

(A6) \( \min_{x \in \mathbb{R}^N, |x|=1} L(t)x \cdot x \geq 0 \) and there is \( b > 0 \) such that \( \text{meas}(\{L \geq b\}) < 1/c_2^\alpha \), where \( \text{meas}(\cdot) \) is the Lebesgue measure, \( \{L \geq b\} = \{t \in \mathbb{R}: L(t) \geq b\} \) and \( c_\alpha \) defined the Sobolev constant (see section 2);

(A7) \( W(t, 0) = 0 \) and for any \( 0 < \alpha_1 < \alpha_2 \),

\[
C_{\alpha_2}^{\alpha_1} := \inf \left\{ \frac{\bar{W}(t, x)}{|x|^2}; t \in \mathbb{R}, \alpha_1 < |x| < \alpha_2 \right\} > 0,
\]

where \( \bar{W}(t, x) := \frac{1}{2} \nabla W(t, x) \cdot x - W(t, x) \);

(A8) there exist \( c_1 > 0, R_1 > 1 \) and \( \beta \in (1, 2) \) such that

\[
\nabla W(t, x) \cdot x \leq c_1 \bar{W}(t, x)|x|^{2-\beta}, \quad \forall t \in \mathbb{R}, |x| \geq R_1;
\]

(A9) there exist a constants \( T_0 > 0 \) and \( x_0 \in \mathbb{R}^N \setminus \{0\} \) such that

\[
\int_{-T_0}^{T_0} \lambda L(t)x_0 \cdot x_0 - W(t, x_0) \, dt < 0.
\]

Our main results reads as follows.

**Theorem 1.2.** Assume that \( f \in L^2(\mathbb{R}, \mathbb{R}^N) \setminus \{0\} \) and (A3), (A4), (A6)–(A9) hold. Then, there exist constants \( f_0, \lambda_0 > 0 \) such that, for any \( \lambda > \lambda_0 \) system (1.3) possesses at least two nontrivial solutions whenever \( \|f\|_{L^2} < f_0 \).

**Corollary 1.3.** Assume that \( f \in L^2(\mathbb{R}, \mathbb{R}^N) \setminus \{0\} \), (A3), (W2), (A6)–(A8) are satisfied and

(A9′)

\[
\lim_{|x| \to +\infty} \frac{W(t, x)}{|x|^2} = +\infty, \text{ uniformly for a.e. } t \in \mathbb{R}.
\]

Then, there exist constants \( f_0, \lambda_0 > 0 \) such that, for any \( \lambda > \lambda_0 \) system (1.3) possesses at least two nontrivial solutions whenever \( \|f\|_{L^2} < f_0 \).

**Remark 1.4.** Assumption (A5) implies (A8), (A9) and (A9′). In fact assuming (A5) is satisfied, it is clear that (A9) and (A9′) hold. Choose \( R_1 \geq 1 \) so large that

\[
\frac{1}{\mu} < \frac{1}{2} - \frac{1}{|x|^{2-\beta}} \quad \text{whenever } |x| \geq R_1.
\]

Then, for such \( |x| \), we have

\[
W(t, x) \leq \left( \frac{1}{2} - \frac{1}{|x|^{2-\beta}} \right) \nabla W(t, x) \cdot x,
\]

and it follows that

\[
\nabla W(t, x) \cdot x \leq |x|^{2-\beta} \left( \frac{1}{2} \nabla W(t, x) \cdot x - W(t, x) \right) = |x|^{2-\beta} \bar{W}(t, x).
\]
Here and in the following \( x \cdot y \) denotes the inner product of \( x, y \in \mathbb{R}^N \) and \( | \cdot | \) denotes the associated norm. Throughout this article, we denote by \( c, c_i \) the various positive constants which may vary from line to line and are not essential to the problem.

2. Preliminaries

2.1. Liouville-Weyl Fractional Calculus.

**Definition 2.1.** The left and right Liouville-Weyl fractional integrals of order \( 0 < \alpha < 1 \) on the whole axis \( \mathbb{R} \) are defined by

\[
-\infty I_x^\alpha u(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x - \xi)^{\alpha-1} u(\xi) d\xi,
\]

\[
x I_\infty^\alpha u(x) := \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi - x)^{\alpha-1} u(\xi) d\xi,
\]

respectively, where \( x \in \mathbb{R} \).

**Definition 2.2.** The left and right Liouville-Weyl fractional derivatives of order \( 0 < \alpha < 1 \) on the whole axis \( \mathbb{R} \) are defined by

\[
-\infty D_x^\alpha u(x) := \frac{d}{dx} -\infty I_x^{1-\alpha} u(x),
\]

\[
x D_\infty^\alpha u(x) := -\frac{d}{dx} x I_\infty^{1-\alpha} u(x),
\]

respectively, where \( x \in \mathbb{R} \).

**Remark 2.3.** Definitions (2.1) and (2.2) may be written in the alternative forms:

\[
-\infty D_x^\alpha u(x) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_{0}^{\infty} \frac{u(x) - u(x - \xi)}{\xi^{\alpha+1}} d\xi,
\]

\[
x D_\infty^\alpha u(x) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_{0}^{\infty} \frac{u(x) - u(x + \xi)}{\xi^{\alpha+1}} d\xi.
\]

Recall that the Fourier transform \( \hat{u}(w) \) of \( u(x) \) is defined by

\[
\hat{u}(w) = \int_{-\infty}^{\infty} e^{-ixw} u(x) dx.
\]

We establish the Fourier transform properties of the fractional integral and fractional operators as follows:

\[
-\infty I_x^\alpha u(x)(w) := (iw)^{-\alpha} \hat{u}(w),
\]

\[
x I_\infty^\alpha u(x)(w) := (-iw)^{-\alpha} \hat{u}(w),
\]

\[
-\infty D_x^\alpha u(x)(w) := (iw)^{\alpha} \hat{u}(w),
\]

\[
x D_\infty^\alpha u(x)(w) := (-iw)^{\alpha} \hat{u}(w).
\]
2.2. **Fractional derivative spaces.** Let us recall for any $\alpha > 0$, the semi-norm
\[
|u|_{I_{-\infty}^\alpha} := \| -\infty D^\alpha u\|_{L^2},
\]
and the norm
\[
\|u\|_{I_{-\infty}^\alpha} := \left( \| u\|_{L^2}^2 + |u|_{I_{-\infty}^\alpha}^2 \right)^{1/2}.
\]
Let the space $I_{-\infty}^\alpha(\mathbb{R})$ denote the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm $\| \cdot \|_{I_{-\infty}^\alpha}$, i.e.,
\[
I_{-\infty}^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\| \cdot \|_{I_{-\infty}^\alpha}}.
\]
Next, we define the fractional Sobolev space $H^\alpha(\mathbb{R})$ in terms of the Fourier transform. For $0 < \alpha < 1$, define the semi-norm
\[
|u|_{\alpha} = \| |w|^\alpha \hat{u}\|_{L^2},
\]
and the norm
\[
\|u\|_{\alpha} = (\| u\|_{L^2}^2 + |u|_{\alpha}^2)^{1/2},
\]
and let
\[
H^\alpha(\mathbb{R}) := \overline{C_0^\infty(\mathbb{R})}^{\| \cdot \|_{\alpha}}.
\]
We note that a function $u \in L^2(\mathbb{R})$ belongs to $I_{-\infty}^\alpha(\mathbb{R})$ if and only if $|w|^\alpha \hat{u} \in L^2(\mathbb{R})$.

In particular, $|u|_{I_{-\infty}^\alpha} = \| |w|^\alpha \hat{u}\|_{L^2(\mathbb{R})}$. Therefore $H^\alpha(\mathbb{R})$ and $I_{-\infty}^\alpha(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm (see [22]).

Analogous to $I_{-\infty}^\alpha(\mathbb{R})$, we introduce $I_{-\infty}^\alpha(\mathbb{R})$. Let us define the semi-norm
\[
|u|_{I_{-\infty}^\alpha} := \| x D^\alpha u\|_{L^2(\mathbb{R})},
\]
and norm
\[
\|u\|_{I_{-\infty}^\alpha} := (\| u\|_{L^2}^2 + |u|_{I_{-\infty}^\alpha}^2)^{1/2},
\]
and let
\[
I_{-\infty}^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\| \cdot \|_{I_{-\infty}^\alpha}}.
\]
Moreover $I_{-\infty}^\alpha(\mathbb{R})$ and $I_{-\infty}^\alpha(\mathbb{R})$ are equivalent, with equivalent semi-norm and norm.

**Lemma 2.4** ([22]). If $\alpha > 1/2$, then $H^\alpha(\mathbb{R}) \subset C(\mathbb{R})$ and there is a constant $C = C_\alpha$ such that
\[
\|u\|_{L^\infty} = \sup_{x \in \mathbb{R}} |u(x)| \leq C \|u\|_{\alpha} \tag{2.3}
\]
where $C(\mathbb{R})$ denote the space of continuous functions on $\mathbb{R}$.

**Remark 2.5.** If $u \in H^\alpha(\mathbb{R})$, then $u \in L^q(\mathbb{R})$ for all $q \in [2, \infty]$, since
\[
\int_{\mathbb{R}} |u(x)|^q dx \leq \|u\|_{L^\infty}^{q-2} \|u\|_{L^2}^2.
\]

In what follows, we introduce the fractional space in which we will construct the variational framework of (1.3). Let
\[
X^\alpha = \left\{ x \in H^\alpha(\mathbb{R}, \mathbb{R}^n) : \int_{\mathbb{R}} \left| -\infty D^\alpha_t x(t) \right|^2 + L(t)x(t) \cdot x(t) dt < \infty \right\}.
\]
The space $X^\alpha$ is a reflexive and separable Hilbert space with the inner product
\[
(x, y)_{X^\alpha} = \int_{\mathbb{R}} \left( -\infty D^\alpha_t x(t), -\infty D^\alpha_t y(t) \right) + L(t)x(t) \cdot y(t) dt,
\]
and the corresponding norm is
\[ \|x\|_{\mathcal{X}_{\alpha}} = \sqrt{(x,x)_{\mathcal{X}_{\alpha}}} .\]

For \( \lambda > 0 \), we also need the following inner product
\[ (x,y)_\lambda = \int_{\mathbb{R}} (-\infty D^\alpha x(t) \cdot -\infty D^\alpha y(t) + \lambda L(t)x(t) \cdot y(t))dt, \]
and the corresponding norm
\[ \|x\|_\lambda = \sqrt{(x,x)_\lambda} . \]

Set \( \mathcal{X}_{\alpha,\lambda} = (\mathcal{X}_{\alpha}, \| \cdot \|_\lambda) \). Observing \( \| x \|_\lambda \geq \| x \|_{\mathcal{X}_{\alpha}} \) for all \( \lambda \geq 1 \).

Lemma 2.6. If \( L \) satisfies (A6) then, \( \mathcal{X}_{\alpha} \) is continuously embedded in \( H^\alpha(\mathbb{R}, \mathbb{R}^n) \).

Proof. By (A6) and (2.3) we have
\[
\int_{\mathbb{R}} |x(t)|^2 dt \\
= \int_{\{L < b\}} |x(t)|^2 dt + \int_{\{L \geq b\}} |x(t)|^2 dt \\
\leq \|x\|_{L^2}^2 \text{meas}\{L < b\} + \frac{1}{b} \int_{\{L \geq b\}} L(t)x(t) \cdot x(t)dt \\
\leq c^2_{\alpha} \text{meas}\{L < b\}\left( \int_{\mathbb{R}} (|\infty D^\alpha x(t)|^2 + |x(t)|^2) dt \right) + \frac{1}{b} \int_{\{L \geq b\}} L(t)x(t) \cdot x(t)dt.
\]

Therefore,
\[
\|x\|_{L^2}^2 \leq \frac{\max\{c^2_{\alpha} \text{meas}\{L < b\}, \frac{1}{b}\}}{1 - c^2_{\alpha} \text{meas}\{L < b\}}} \|x\|_{\mathcal{X}_{\alpha}}^2 \quad (2.4)
\]
and
\[
\|x\|_{L^2}^2 = \int_{\mathbb{R}} (|\infty D^\alpha x(t)|^2 + |x(t)|^2) dt \\
\leq \left( 1 + \frac{\max\{c^2_{\alpha} \text{meas}\{L < b\}, \frac{1}{b}\}}{1 - c^2_{\alpha} \text{meas}\{L < b\}} \right) \|x\|_{\mathcal{X}_{\alpha}}^2 , \quad (2.5)
\]
which yields that the embedding \( \mathcal{X}_{\alpha} \hookrightarrow H^\alpha(\mathbb{R}, \mathbb{R}^n) \) is continuous. \( \square \)

Remark 2.7. Using the same conditions and techniques in (2.4) and (2.5), for all \( \lambda \geq \frac{1}{bc^2_{\alpha} \text{meas}\{L < b\}} \), we also obtain
\[
\|x\|_{L^2}^2 \leq \frac{c^2_{\alpha} \text{meas}\{L < b\}}{1 - c^2_{\alpha} \text{meas}\{L < b\}}} \|x\|_{\lambda}^2 , \quad (2.6)
\]
\[
\|x\|_{L^2}^2 \leq \left( 1 + \frac{c^2_{\alpha} \text{meas}\{L < b\}}{1 - c^2_{\alpha} \text{meas}\{L < b\}} \right) \|x\|_{\lambda}^2 , \quad (2.7)
\]
Furthermore, using (2.3), (2.5) and (2.6), for every \( p \in (2, \infty) \) and
\[
\lambda \geq \frac{1}{bc^2_{\alpha} \text{meas}\{L < b\}} ,
\]
we have
\[
\int_{\mathbb{R}} |x(t)|^p dt \\
\leq \|x\|_{L^\infty}^{p-2} \int_{\mathbb{R}} |x(t)|^2 dt \\
\leq c_\alpha^{-p} \left( \int_{\mathbb{R}} (|(-\infty) D_{\alpha}^p x(t)|^2 + |x(t)|^2) dt \right)^{\frac{p-2}{2}} c_\alpha^2 \frac{\text{meas}(\{L < b\})}{1 - c_\alpha^2 \text{meas}(\{L < b\})} \|x\|_\lambda^2
\]
\[
\leq c_\alpha^{-p} \left( 1 + \frac{c_\alpha^2}{1 - c_\alpha^2} \frac{\text{meas}(\{L < b\})}{\text{meas}(\{L < b\})} \right) \|x\|_\lambda^{p-2} c_\alpha^2 \frac{\text{meas}(\{L < b\})}{1 - c_\alpha^2 \text{meas}(\{L < b\})} \|x\|_\lambda^2
\]
\[
= \left( 1 + \frac{c_\alpha^2}{1 - c_\alpha^2} \right) \frac{2}{2} c_\alpha^2 \frac{\text{meas}(\{L < b\})}{1 - c_\alpha^2 \text{meas}(\{L < b\})} \|x\|_\lambda^{p-2} c_\alpha^2 \text{meas}(\{L < b\}) \|x\|_\lambda^2
\]
\[
= \text{meas}(\{L < b\}) \left( \frac{2}{2} c_\alpha^2 \frac{\text{meas}(\{L < b\})}{1 - c_\alpha^2 \text{meas}(\{L < b\})} \right)^{p/2} \|x\|_\lambda^p
\]
\[
:= \delta_\mu^p \|x\|_\lambda^p.
\]

3. Proof of Theorem 1.2 and Corollary 1.3

For this purpose, we establish the corresponding variational framework to obtain solutions of \(1.3\). To this end, define the functional \(I_\lambda : X_\lambda^\alpha \rightarrow \mathbb{R}\) by
\[
I_\lambda(x) = \int_{\mathbb{R}} \left[ \frac{1}{2} \left| (-\infty) D_{\alpha}^p x(t) \right|^2 + \frac{\lambda}{2} L(t) x(t) \cdot x(t) - W(t, x(t)) + f(t) \cdot x(t) \right] dt
\]
\[
= \frac{1}{2} \|x\|_\lambda^2 - \int_{\mathbb{R}} W(t, x(t)) dt + \int_{\mathbb{R}} f(t) \cdot x(t) dt.
\]
Under assumptions (A3), (A4), (A6)-(A8), we see that \(I_\lambda\) is a continuously Fréchet-differentiable functional defined on \(X_\lambda^\alpha\); i.e., \(I_\lambda \in C^1(X_\lambda^\alpha, \mathbb{R})\). Moreover, we have
\[
I_\lambda'(x)y = \int_{\mathbb{R}} \left[ (-\infty) D_{\alpha}^p x(t), (-\infty) D_{\alpha}^p y(t) \right] + \lambda L(t) x(t) \cdot y(t) - \nabla W(t, x(t)) \cdot y(t) + f(t) \cdot y(t) | dt,
\]
for all \(x, y \in X_\lambda^\alpha\), which yields
\[
I_\lambda'(x)x = \|x\|_\lambda^2 - \int_{\mathbb{R}} \nabla W(t, x(t)) \cdot x(t) dt + \int_{\mathbb{R}} f(t) \cdot x(t) dt.
\]
We know that to find a solutions of \(1.3\), it suffices to obtain the critical points of \(I_\lambda\); see [22]. For this purpose the lemma below is useful.

Recall that \(\phi \in C^1(E, \mathbb{R})\) satisfy the Palais-Smale condition (PS) if any sequence \((x_n) \subset E\), for which \((\phi(x_n))\) is bounded and \(\phi'(x_n) \rightarrow 0\) as \(n \rightarrow \infty\), possesses a convergent subsequence in \(E\).

**Lemma 3.1** ([20]). Let \(E\) be a real Banach space and \(\phi \in C^1(E, \mathbb{R})\) satisfying the Palais-Smale condition. If \(\phi\) satisfies the following conditions:

(i) \(\phi(0) = 0\),
(ii) there exist constants \(\rho, \gamma > 0\) such that \(\phi/\rho B_{\rho}(0) \geq \gamma\),
(iii) there exist \(e \in E \setminus \overline{B}_{\rho}(0)\) such that \(\phi(e) \leq 0\).
Then \( \phi \) possesses a critical value \( c \geq \gamma \) given by
\[
c = \inf_{g \in \Gamma} \max_{s \in [0,1]} \phi(g(s)),
\]
where
\[
\Gamma = \{ g \in C([0,1], E) : g(0) = 0, g(1) = e \}.
\]

To find the critical points of \( I_{\lambda} \), we shall show that \( I_{\lambda} \) satisfies the (PS) condition. Because of the lack of the compactness of the Sobolev embedding, we require the following convergence result.

**Lemma 3.2.** Assume that \( x_{n} \to x \) in \( X_{\lambda}^{\alpha} \), (A3), (A4), (A7) are satisfied and \( f \in L^{2} \). Then
\[
\begin{align*}
I_{\lambda}(x_{n} - x) &= I_{\lambda}(x_{n}) - I_{\lambda}(x) + o(1) \quad \text{as } n \to +\infty, \quad (3.3) \\
I_{\lambda}'(x_{n} - x) &= I_{\lambda}'(x_{n}) - I_{\lambda}'(x) + o(1) \quad \text{as } n \to +\infty. \quad (3.4)
\end{align*}
\]
In particular, if \( (x_{n}) \) is a (PS) sequence of \( I_{\lambda} \) such that \( I_{\lambda}(x_{n}) \to c \) for some \( c \in \mathbb{R} \) then
\[
\begin{align*}
I_{\lambda}(x_{n} - x) &\to c - I_{\lambda}(x) \quad \text{as } n \to +\infty, \quad (3.5) \\
I_{\lambda}'(x_{n} - x) &\to 0 \quad \text{as } n \to +\infty, \quad (3.6)
\end{align*}
\]
after passing to a subsequence.

**Proof.** As \( x_{n} \to x \) in \( X_{\lambda}^{\alpha} \), we have \( (x_{n}, x)_{\lambda} \to (x, x)_{\lambda} \) as \( n \to \infty \). Then
\[
\begin{align*}
\| x_{n} \|_{\lambda}^{2} &= (x_{n} - x, x_{n} - x)_{\lambda} + (x, x)_{\lambda} \| x_{n} - x \|_{\lambda} \\
&= \| x_{n} - x \|_{\lambda}^{2} + \| x \|_{\lambda}^{2} + o(1).
\end{align*}
\]
Obviously,
\[
(x_{n}, z)_{\lambda} = (x_{n} - x, z)_{\lambda} + (x, z)_{\lambda}, \quad \forall z \in X_{\lambda}^{\alpha}.
\]
Hence, to show (3.3) and (3.4) it suffices to prove that
\[
\sup_{\varphi \in X_{\lambda}^{\alpha}, \| \varphi \|_{\lambda} = 1} \int_{\mathbb{R}} (W(t, x_{n}) - W(t, x_{n} - x) - W(t, x)) \, dt = o(1), \quad (3.7)
\]
\[
\sup_{\varphi \in X_{\lambda}^{\alpha}, \| \varphi \|_{\lambda} = 1} \int_{\mathbb{R}} (\nabla W(t, x_{n}) - \nabla W(t, x_{n} - x) - \nabla W(t, x)) \cdot \varphi \, dt = o(1). \quad (3.8)
\]
Here, we only prove (3.8) the proof of (3.7) is similar. Setting \( y_{n} := x_{n} - x \), then \( y_{n} \to 0 \) in \( X_{\lambda}^{\alpha} \) and \( y_{n}(t) \to 0 \) a.e. \( t \in \mathbb{R} \). From (A3), for every \( \varepsilon > 0 \), there exist \( \sigma = \sigma(\varepsilon) \in (0,1) \) such that
\[
|\nabla W(t, u)| \leq \varepsilon |u|, \quad \forall t \in \mathbb{R}, |u| \leq \sigma. \quad (3.9)
\]
By (A4) and (3.9), we have
\[
|\nabla W(t, u)| \leq \varepsilon |u| + c_{\varepsilon} |u|^{2}, \quad \forall t \in \mathbb{R}, |u| \leq N_{1}, \quad (3.10)
\]
where
\[
N_{1} := \sup_{n} \{ \| y_{n} \|_{L^{\infty}}, \| y_{n} + x \|_{L^{\infty}}, \| x \|_{L^{\infty}} + 1 \}, \quad c_{\varepsilon} = \max_{|u| \in [\sigma, N_{1}]} \frac{\| W(u) \|}{\sigma}.\]

By (3.10) and the Young Inequality, for each \( \varphi \in X_{\lambda}^{\alpha} \) with \( \| \varphi \|_{\lambda} = 1 \), we have
\[
|\nabla W(t, y_{n} + x) - \nabla W(t, y_{n})| \varphi| \\
\leq \varepsilon (|y_{n} + x| + |y_{n}|) |\varphi| + c_{\varepsilon} (|y_{n} + x|^{2} + |y_{n}|^{2}) |\varphi|,
\]
The Dominated Convergence Theorem implies that we obtain
\[
\psi_n(t) = \max\{|(∇W(t, y_n + x) - ∇W(t, y_n) - ∇W(t, x))·ϕ| ≤ \psi_n(t) + εc(|y_n|^2 + |y_n|^3), \ \text{and} \nabla \|ight.
\]
(3.11)

If we take
\[
ψ_n(t) := \max\{|(∇W(t, y_n + x) - ∇W(t, y_n) - ∇W(t, x))·ϕ| - εc(|y_n|^2 + |y_n|^3), 0\},
\]
we obtain
\[
0 ≤ ψ_n(t) ≤ c(ε|x|^2 + ε|ϕ|^2 + c_ε'|ϕ|^3 + c_ε''|x|^3) ∈ L^1(\mathbb{R}, \mathbb{R}^N).
\]
The Dominated Convergence Theorem implies that
\[
\int_\mathbb{R} ψ_n(t) \, dt → 0 \text{ as } n → ∞.
\] (3.12)

It follows from the definition of ψ_n(t) that
\[
|\nabla W(t, y_n + x) - \nabla W(t, y_n) - \nabla W(t, x))·ϕ| ≤ ψ_n(t) + εc(|y_n|^2 + |y_n|^3),
\]
and then
\[
\int_\mathbb{R} |\nabla W(t, y_n + x) - \nabla W(t, y_n) - \nabla W(t, x))·ϕ| \, dt ≤ \|ψ_n(t)\|_{L^1} + εc(∥y_n∥^2_{L^2} + ∥y_n∥^3_{L^3}),
\]
for all n. Because φ is arbitrary in X_λ^∞, we obtain
\[
\sup_{ϕ ∈ X_λ^∞, ||ϕ||_λ = 1} |\int_\mathbb{R} (\nabla W(t, y_n + x) - \nabla W(t, y_n) - \nabla W(t, x))·ϕ| \, dt \leq \|ψ_n(t)\|_{L^1} + εc(∥y_n∥^2_{L^2} + ∥y_n∥^3_{L^3}),
\]
which, jointly with (2.8) and (3.12) shows that
\[
\sup_{ϕ ∈ X_λ^∞, ||ϕ||_λ = 1} |\int_\mathbb{R} (\nabla W(t, y_n + x) - \nabla W(t, y_n) - \nabla W(t, x))·ϕ| \, dt ≤ εc,
\]
for n sufficiently large. Therefore, (3.8) holds.

If moreover I_λ(x_n) → c and I_λ'(x_n) → 0 as n → ∞, equations (3.3) and (3.4) respectively, imply that
\[
I_λ(x_n) → c - I_λ(x) + o(1),
\]
and
\[
I_λ'(x_n) = -I_λ'(x) \text{ as } n → +∞.
\]
We show that I_λ'(x) = 0. For every ζ ∈ C_0^∞(\mathbb{R}, \mathbb{R}^N), we have
\[
I_λ'(x)ζ = \lim_{n→∞} I_λ'(x_n)ζ = 0.
\]
Consequently, I_λ'(x) = 0 and (3.6) holds.

\[\square\]

**Lemma 3.3.** Suppose that f ∈ L^2 and (A3), (A4), (A6), (A8) are satisfied. Then, there exists λ_0 > 0 such that any bounded (PS) sequence of I_λ has a convergent subsequence when λ > λ_0.
Proof. Let \((x_n)\) be a bounded sequence such that \((I_\lambda(x_n))\) is bounded and \(I'_\lambda(x_n) \to 0\) as \(n \to \infty\). Then, after passing to a subsequence, we have \(x_n \to x\) in \(X^\alpha\) and \(y_n \to 0\) in \(L^2(\{L(t) < b\})\) where \(y_n := x_n - x\). Moreover,

\[
\|y_n\|^2_2 \leq \frac{1}{\lambda b} \int_{\{L \geq b\}} \lambda L(t)y_n \cdot y_n dt + \int_{\{L < b\}} |y_n|^2 dt \leq \frac{1}{\lambda b} \|y_n\|^2_\lambda + o(1). \tag{3.13}
\]

Setting \(N_2 := \sup_n \|y_n\|_{L^\infty}\). By (A4), we obtain

\[
|\tilde{W}(t, y_n)| = \left\| \nabla W(t, y_n) \cdot y_n - W(t, y_n) \right\| \leq \max_{|u| \in [0, N_2]} W(u)(N_2 + 1), \quad \forall n,
\]

which, jointly with (3.13) and (A8) yields

\[
\int_{|y_n| \geq R_1} \nabla W(t, y_n) \cdot y_n dt \leq c_1 \int_{|y_n| \geq R_1} \tilde{W}(t, y_n)|y_n|^{2-\beta} dt
\]

\[
\leq \frac{c_1}{\lambda b} \int_{|y_n| \geq R_1} |y_n|^2 dt \leq \frac{cc_1}{\lambda b} \|y_n\|^2_\lambda + o(1). \tag{3.14}
\]

Furthermore, using (A4), (3.9) and (3.13), we have

\[
\int_{|y_n| < R_1} \nabla W(t, y_n) y_n dt
\]

\[
\leq \int_{|y_n| \leq \delta} \varepsilon |y_n|^2 dt + \int_{\sigma < |y_n| < R_1} |\nabla W(t, y_n)||y_n| dt
\]

\[
\leq \varepsilon \int_{|y_n| \leq \delta} |y_n|^2 dt + \max_{|u| \in [\sigma, R_1]} W(u)\sigma^{-1} \int_{\mathbb{R}} |y_n|^2 dt
\]

\[
\leq \frac{c}{\lambda b} \|y_n\|^2_\lambda + o(1). \tag{3.15}
\]

Because \(f \in L^2\), one has, for any \(\varepsilon > 0\), there exists \(T_\varepsilon > 0\) such that

\[
\left( \int_{|t| \geq T_\varepsilon} |f(t)|^2 dt \right)^{1/2} < \varepsilon.
\]

Using (2.8) and the H"older inequality, we have

\[
|\int_{|t| \geq T_\varepsilon} f(t) y_n dt| \leq \left( \int_{|t| \geq T_\varepsilon} |f(t)|^2 dt \right)^{1/2} \left( \int_{\mathbb{R}} |y_n|^2 dt \right)^{1/2} \leq c \varepsilon \quad \forall n. \tag{3.16}
\]

Obviously

\[
\int_{|t| < T_\varepsilon} f(t) \cdot y_n dt \leq \left( \int_{\mathbb{R}} |f(t)|^2 dt \right)^{1/2} \left( \int_{|t| < T_\varepsilon} |y_n|^2 dt \right)^{1/2} \to 0, \tag{3.17}
\]

as \(n \to \infty\). By (3.16) and (3.17), we have

\[
\int_{\mathbb{R}} f(t) \cdot y_n(t) dt \to 0, \tag{3.18}
\]

as \(n \to \infty\). Consequently, a combination of (3.4), (3.14), (3.15) and (3.18) implies

\[
o(1) = I'_\lambda(y_n) y_n = \|y_n\|^2_\lambda - \int_{\mathbb{R}} \nabla W(t, y_n) \cdot y_n dt + \int_{\mathbb{R}} f(t) \cdot y_n dt
\]

\[
\geq \left( 1 - \frac{cc_1}{\lambda b} - \frac{c}{\lambda b} \right) \|y_n\|^2_\lambda + o(1).
\]
Choosing $\lambda_0 > 0$ large enough such that the term $(1 - \frac{\omega_1}{\lambda} - \frac{\omega_2}{\lambda})$ is positive. When $\lambda > \lambda_0$, we obtain $y_n \to 0$ and then $x_n \to x$ in $X_\lambda^\alpha$. \hfill \Box

**Lemma 3.4.** If $f \in L^2$ and (A3), (A4), (A6)–(A8) are satisfied, then $I_\lambda$ satisfies the $(PS)$ condition whenever $\lambda > \lambda_0$.

**Proof.** Let $(x_n)$ be a $(PS)$ sequence of $I_\lambda$. By Lemma 3.3 it suffices to prove that $(x_n)$ is bounded. Indeed, assume that $\|x_n\|_\lambda \to \infty$ as $n \to \infty$ and setting $y_n := \frac{x_n}{\|x_n\|_\lambda}$. Then $\|y_n\|_\lambda = 1$ and $\|y_n\|_{L^p} \leq \delta_p$ for $p \in [2, +\infty]$. Moreover, we have

$$o(1) = \frac{I_\lambda'(x_n)x_n}{\|x_n\|_\lambda^2} = 1 - \int_\mathbb{R} \frac{\nabla W(t, x_n) \cdot x_n}{\|x_n\|_\lambda^2} dt + o(1),$$

as $n \to \infty$. We obtain

$$\int_\mathbb{R} \frac{\nabla W(t, x_n) \cdot y_n}{\|x_n\|} |y_n| dt = \int_\mathbb{R} \frac{\nabla W(t, x_n) \cdot x_n}{\|x_n\|_\lambda^2} dt \to 1,$$

(3.19)
as $n \to \infty$. Let $0 \leq \alpha_1 < \alpha_2$ and $\omega_n^{\alpha_1, \alpha_2} := \{ t \in \mathbb{R}; \alpha_1 \leq |x_n(t)| < \alpha_2 \}$. By (2.8) and because $(x_n)$ is a $(PS)$ sequence of $I_\lambda$, then there exists $N_0 > 0$ such that for $n \geq N_0$ we have

$$c + \|x_n\|_\lambda \geq I_\lambda(x_n) - \frac{1}{2} I_\lambda'(x_n)x_n$$

$$\geq \int_\mathbb{R} \widetilde{W}(t, x_n) dt + \frac{1}{2} \int_\mathbb{R} f(t) \cdot x_n dt$$

$$\geq \int_\mathbb{R} \widetilde{W}(t, x_n) dt - \frac{\delta_2}{2} \|f\|_{L^2} \|x_n\|_\lambda.$$ 

This implies, for $n \geq N_0$, that

$$c(1 + \|x_n\|_\lambda)$$

$$\geq \int_\mathbb{R} \widetilde{W}(t, x_n) dt$$

$$= \int_{\omega_n^{\alpha_1}} \widetilde{W}(t, x_n) dt + \int_{\omega_n^{\alpha_1, \alpha_2}} \widetilde{W}(t, x_n) dt + \int_{\omega_n^{\alpha_2}} \widetilde{W}(t, x_n) dt. \tag{3.20}$$

By (A3), for any $\varepsilon > 0 (\varepsilon < \frac{1}{3})$ there exists $\kappa_\varepsilon > 0$ such that

$$|\nabla W(t, u)| \leq \left(\frac{\varepsilon}{\delta_2^2}\right)|u|, \quad \forall |u| \leq \kappa_\varepsilon, t \in \mathbb{R}.$$

Thus,

$$\int_{\omega_n^{\alpha_1, \alpha_2}} \frac{|\nabla W(t, u)|}{|x_n|} |y_n|^2 dt \leq \int_{\omega_n^{\alpha_1, \alpha_2}} \frac{\varepsilon}{\delta_2^2} |y_n|^2 dt \leq \frac{\varepsilon}{\delta_2^2} \|y_n\|_{L^2}^2 \leq \varepsilon, \quad \forall n. \tag{3.21}$$
Because $\beta > 1$ and by (A8), (2.8) and (3.20) we can choose $\theta_\varepsilon \geq R_1$ large enough such that

$$\int_{\omega_{\theta_\varepsilon}} \frac{\nabla W(t, x_n)}{\|x_n\|_\Lambda^2} dt \leq \int_{\omega_{\theta_\varepsilon}} c_1 \frac{y_n \widetilde{W}(t, x_n)}{\|x_n\|^\beta_1} dt$$

$$\leq c_1 \|y_n\|_L^\infty \int_{\omega_{\theta_\varepsilon}} \frac{\widetilde{W}(t, x_n)}{\theta_\varepsilon^\beta - 1} dt$$

$$\leq \frac{c_1 \|y_n\|_L^\infty (1 + \|x_n\|)}{\theta_\varepsilon^\beta - 1} \|x_n\|_\Lambda$$

$$\leq \frac{c_1 \|y_n\|_L^\infty (1 + \|x_n\|)}{\theta_\varepsilon^\beta - 1} < \varepsilon, \quad \forall n \geq N_0.$$

By (A7), we have $\widetilde{W}(t, x_n(t)) \geq C^\theta_{\kappa_e}|x_n|^2$ for $t \in \omega_{\kappa_e, \theta_\varepsilon}$. Noting $C^\theta_{\kappa_e} > 0$ it follows from (3.20) that

$$\int_{\omega_{\kappa_e, \theta_\varepsilon}} y_n^2 dt = \int_{\omega_{\kappa_e, \theta_\varepsilon}} |x_n|^2 dt$$

$$\leq \frac{1}{C^\theta_{\kappa_e} \|x_n\|_\Lambda^2} \int_{\omega_{\kappa_e, \theta_\varepsilon}} \widetilde{W}(t, x_n) dt$$

$$\leq \frac{c(1 + \|x_n\|)}{C^\theta_{\kappa_e} \|x_n\|_\Lambda^2} \to 0,$$

as $n \to +\infty$, which yields that

$$\int_{\omega_{\kappa_e, \theta_\varepsilon}} |\nabla W(t, x_n)| |y_n|^2 dt \leq \tau_\varepsilon \int_{\omega_{\kappa_e, \theta_\varepsilon}} |y_n|^2 dt \to 0,$$  \hspace{1cm} (3.24)

as $n \to \infty$, where $\tau_\varepsilon = \max_{u \in [\kappa_e, \theta_\varepsilon]} W(u) \cdot \kappa_e$. Hence, by (3.21), (3.22) and (3.24), we have

$$\int_{\mathbb{R}} |\nabla W(t, x_n) \cdot y_n| \leq \int_{\mathbb{R}} |\nabla W(t, x_n)| |y_n|^2 \leq 3 \varepsilon < 1,$$

for $n$ large enough, a contradiction with (3.19) and then $(x_n)$ is bounded in $X_\Lambda^\alpha$. \hfill $\Box$

**Lemma 3.5.** If (A3) holds and $f \in L^2$, then there exist $\rho, \gamma, f_0 > 0$ such that $I_\lambda(x)/\|x\|_\Lambda = \rho \geq \gamma$ when $\|f\|_{L^2} < f_0$.

**Proof.** By (A3), for $\varepsilon := \frac{1}{4\varepsilon^2}$ there exists $\sigma_1 = \sigma_1(\varepsilon)$ such that

$$|W(t, x)| \leq \varepsilon |x|^2, \quad \forall t \in \mathbb{R}, \ |x| \leq \sigma_1.$$  \hspace{1cm} (3.25)

Thus, for $\|x\|_\Lambda \leq \rho := \sigma_1/\delta_\infty$, by (3.25), we obtain

$$I_\lambda(x) \geq \frac{1}{2} \|x\|_\Lambda^2 - \varepsilon \int_{\mathbb{R}} |x|^2 dt - \|f\|_{L^2} \|x\|_{L^2} \geq \|x\|_\Lambda \left(\frac{1}{2} \|x\|_\Lambda - \|f\|_{L^2} \delta_2\right).$$

Let $\gamma := \rho \left(\frac{1}{2\varepsilon^2} - \|f\|_{L^2} \delta_2\right).$ Then, if $\|f\|_{L^2} < f_0 := \frac{1}{4\varepsilon^2} \rho$, we have $I_\lambda(x)/\|x\|_\Lambda = \rho \geq \gamma.$ \hfill $\Box$

**Lemma 3.6.** If $\|f\|_{L^2} < f_0$ and (A3), (A7) are satisfied, then there exists $x_1 \in X_\Lambda^\alpha \setminus \{0\}$ such that $I_\lambda(x_1) = 0.$
Proof. Since \( f \in L^2\setminus\{0\} \), we can choose \( \xi \in X^\alpha_\lambda \) such that \( \int_\mathbb{R} f(t) \cdot \xi(t)dt < 0 \). By (A3) and (A7) we have \( W \geq 0 \) and
\[
I_\lambda(s\xi) \leq \frac{s^2}{2} \|\xi\|^2 + s \int_\mathbb{R} f(t) \cdot \xi(t)dt < 0,
\]
for \( s \) small enough. Thus \( C_1 : = \inf\{I_\lambda(x), x \in \overline{B}_\rho(0)\} < 0 \), where \( \rho \) is the constants given by Lemma 3.5. From Ekeland’s variational principle there exists a sequence \( (x_n) \subset B_\rho \) such that \( C_1 \leq I_\lambda(x_n) < C_1 + \frac{1}{n} \). Then, by a standard procedure, we can show that \( (x_n) \subset X^\alpha_\lambda \) is bounded \((PS)\) sequence. Consequently, Lemma 3.3 implies that, there exist \( x_1 \in X^\alpha_\lambda \) such that \( x_n \rightarrow x_1 \in X^\alpha_\lambda, I'_\lambda(x_1) = 0 \) and \( I_\lambda(x_1) = C_1 < 0 \) when \( \lambda > \lambda_0 \). \( \square \)

3.1. Proof of Theorem 1.2. Let \( h(s) = s^{-2}W(t, sx_0) \) for \( t \in \mathbb{R}, s > 0 \). Then, by (A7),
\[
h'(s) = s^{-3}[−2W(t, sx_0) + \nabla W(t, sx_0).sx_0] > 0, \quad \text{for } t \in \mathbb{R}, s > 0.
\]
Integrating the above from 1 to \( \eta \), we obtain
\[
W(t, \eta x_0) \geq \eta^2 W(t, x_0), \quad \text{for } t \in \mathbb{R}, \eta > 1. \tag{3.26}
\]
From (3.26), we have for \( s > 1 \),
\[
I_\lambda(sx_0) = \int_\mathbb{R} (\lambda s^2 L(t)x_0 \cdot x_0 - W(t, sx_0))dt + s \int_\mathbb{R} f(t) \cdot x_0dt \\
\leq s^2 \left( \int_\mathbb{R} \lambda L(t)x_0 \cdot x_0 - W(t, x_0)dt + s \int_\mathbb{R} f(t) \cdot x_0dt. \right.
\]
Let
\[
e(t) = \begin{cases} 
xs_0, & \text{if } t \in [-T_0, T_0] \\
0, & \text{if } t \in \mathbb{R}\setminus[-T_0, T_0],
\end{cases}
\]
By (A9) there exists \( s_0 \geq 1 \) such that \( \|e\|_\lambda > \rho \) and \( I_\lambda(e) < 0 \). Since \( I_\lambda(0) = 0 \) and all the assumptions of Lemma 3.1 are satisfied, so \( I_\lambda \) possesses a critical point \( x_2 \in X^\alpha_\lambda \) with \( I'_\lambda(x_2) = 0 \) and \( I_\lambda(x_2) = C_2 > 0 \) whenever \( \lambda > \lambda_0 \).

3.2. Proof of Corollary 1.3. If (A9') holds, let \( e \in C^\infty_0(\mathbb{R}\setminus\{0\}) \). Then, by Fatou’s Lemma and by \( W \geq 0 \) we have
\[
I_\lambda(se) \leq s^2 \frac{1}{2} \|e\|^2 \int_{t \neq 0} \frac{W(t, se)}{(se)^2}e^2 dt + s \int_\mathbb{R} f(t).e(t)dt \rightarrow -\infty
\]
as \( s \rightarrow +\infty \), which implies that \( I_\lambda(se) < 0 \) for \( s > 0 \) large. Combining this with Lemmas 3.4 and 3.5 all the assumptions of Lemma 3.1 are satisfied, so \( I_\lambda \) possesses a critical point \( x_3 \in X^\alpha_\lambda \) with \( I'_\lambda(x_3) = 0 \) and \( I_\lambda(x_3) > 0 \) whenever \( \lambda > \lambda_0 \).

4. An example
Let \( L(t) = h(t)I_N \) where
\[
h(t) = \begin{cases} 
0, & \text{if } |t| < 1, \\
2n^2|t - n|, & \text{if } |t| \geq 1, |t - n| \leq \frac{1}{2n^2} \ (n \in \mathbb{Z}, |n| \geq 1), \\
1, & \text{elsewhere},
\end{cases}
\]
\[
W(t, x) = k(t)|x|^2 \ln(1 + |x|^2),
\]
where \( k : \mathbb{R} \to \mathbb{R}^+ \) is a continuous bounded function with \( \inf k(t) > 0 \). A straightforward computation shows that \( L \) and \( W \) satisfies Theorem 1.2 and Corollary 1.3 but they do not satisfy the corresponding results on the above papers, in particular \( W \) do not satisfy the Ambrosetti- Rabinowitz Condition (A5).

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References


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