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DISSIPATIVE STURM-LIOUVILLE OPERATORS WITH A SPECTRAL PARAMETER IN THE BOUNDARY CONDITION ON BOUNDED TIME SCALES

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Abstract. In this article we consider a second-order Sturm-Liouville operator with a spectral parameter in the boundary condition on bounded time scales. We construct a selfadjoint dilation of the dissipative Sturm-Liouville operators. Using by methods of Pavlov [40, 41, 42], we prove the completeness of the system of eigenvectors and associated vectors of the dissipative Sturm-Liouville operators on bounded time scales.

1. Introduction

In the recent years, the study of dynamic equations on time scales have found a noticeable interest and attracted many researches; see for example [1, 2, 10, 45]. The first fundamental results in this area were obtained by Hilger [29]. He introduced the idea of time scales as a way to unify continuous and discrete analysis and it allows a simultaneous treatment of differential and difference equations, extending those theories to so called dynamic equations. The study of time scales has led to several important applications, e.g., in the study of neural networks, heat transfer, and insect population models, phytoremediation of metals, wound healing and epidemic models [2, 32, 45]. For some basic definitions, we refer the reader to consult the reference [8, 20, 21, 22, 28, 34].

The study of problems involving parameter dependent systems is of great interest to a lot of numerous problems in physics and engineering. A boundary-value problem with a spectral parameter in the boundary condition appears commonly in mathematical models of mechanics. There are many studies about parameter dependent problems [3, 7, 16, 23, 27, 30, 38, 39, 44, 45].

The spectral analysis of non-selfadjoint (dissipative) operators is based on ideas of the functional model and dilation theory rather than on traditional resolvent analysis and Riesz integrals. Using a functional model of a non-selfadjoint operator as a principal tool, spectral properties of such operators are investigated. The functional model of non-selfadjoint dissipative operators plays an important role within both the abstract operator theory and its more specialized applications in other

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disciplines. The construction of functional models for dissipative operators, natural analogues of spectral decompositions for selfadjoint operators is based on Sz. Nagy-Foias dilation theory [36] and Lax-Phillips scattering theory [35]. Pavlov's approach [40, 41, 42] to the model construction of dissipative extensions of symmetric operators was followed by Allahverdiev in his works [3, 4, 5, 6, 7] and some others [23, 38, 39, 46, 47]. The theory of the dissipative Schrödinger operator on a finite interval was applied to the problems arising in the semiconductor physics [9, 10, 11]. In [12, 13, 14, 15], Pavlov's functional model was extended to (general) dissipative operators which are finite dimensional extensions of a symmetric operator, and the corresponding dissipative and Lax-Phillips scattering problems were investigated in some detail. We extend the results [3, 4, 5, 6, 7, 38, 39, 46, 47] to the more general eigenvalues problem (2.2)–(2.4) on time scales. While proving our results, we use the machinery and method of [3, 4, 5, 6, 7].

The organization of this document is as follows: In Section 2, some time scale essentials are included for the convenience of the reader. In Section 3, we construct a selfadjoint dilation of dissipative Sturm-Liouville operator on bounded time scales. We present its incoming and outcoming spectral representations which makes it possible to determine the scattering matrix of the dilation according to the Lax and Phillips scheme [35]. A functional model of this operator is constructed by methods of Pavlov [40, 41, 42] and define its characteristic functions. Finally, we proved a theorem on completeness of the system of eigenvectors and associated vectors of dissipative operators.

2. Preliminaries

Let us denote a time scale by $\mathbb{T}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, $t \in \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, $t \in \mathbb{T}$ (see [20, 21]). We have operators $\mu_\sigma : \mathbb{T} \to [0, \infty)$ and $\mu_\rho : \mathbb{T} \to (-\infty, 0]$ defined by $\mu_\sigma(t) = \sigma(t) - t$ and $\mu_\rho(t) = \rho(t) - t$, respectively. A point $t \in \mathbb{T}$ is left scattered if $\mu_\rho(t) \neq 0$ and left dense if $\mu_\rho(t) = 0$, and a point $t \in \mathbb{T}$ is right scattered if $\mu_\sigma(t) \neq 0$ and right dense if $\mu_\sigma(t) = 0$ (see [20, 21]). We introduce the sets $\mathbb{T}^k$, $\mathbb{T}_k$, $\mathbb{T}^*$ which are derived form the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

In [20, 21], $f^\Delta(t)$ the delta (or Hilger) derivative of f at t (or Δ-differentiable at some point $t \in \mathbb{T}$) is defined as follows: assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. $f^\Delta(t)$ is a number (provided it exists) with the property that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that $|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$, $(s \in U)$. Analogously one may define the notion of ∇-differentiability of some function using the backward jump ρ. One can show $f^\Delta(t) = f^\nabla(\sigma(t))$ and $f^\nabla(t) = f^\Delta(\rho(t))$ for continuously differentiable functions [28].

Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$, such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$, then F is a Δ-antiderivative of f. In this case the integral is given by the formula $\int_a^b f(t)\Delta t = F(b) - F(a)$ for $a, b \in \mathbb{T}$. Similarly, one may define the notion of ∇-antiderivative of some function.

Let $L^2_\Delta(\mathbb{T}^*)$ be the space of all functions defined on $\mathbb{T}^*$ such that

$$\|f\| := \Big(\int_a^b |f(t)|^2 \Delta t \Big)^{1/2} < \infty.$$

The space $L^2_\Delta(\mathbb{T}^*)$ is a Hilbert space with the inner product $(f,g) := \int_a^b f(t)\overline{g(t)}\Delta t$, $f,g \in L^2_\Delta(\mathbb{T}^*)$ (see [43]).

Let $a \leq b$ be fixed points in $\mathbb{T}$ and $a \in \mathbb{T}_k, b \in \mathbb{T}^k$. We will consider the Sturm-Liouville equation

$$l(y) := -[p(t)y^\Delta(t)]^\nabla + q(t)y(t), \quad t \in [a,b], \tag{2.1}$$

where $q : \mathbb{T} \to \mathbb{C}$ is continuous function, $p : \mathbb{T} \to \mathbb{C}$ is ∇-differentiable on $\mathbb{T}^k$, $p(t) \neq 0$ for all $t \in \mathbb{T}$, and $p^\nabla : \mathbb{T}_k \to \mathbb{C}$ is continuous. The Wronskian of y, z is defined as $W(y,z)(t) := p(t)[y(t)z^\Delta(t) - y^\Delta(t)z(t)], \quad t \in \mathbb{T}^*$ (see [28]).

Let L_0 denote the closure of the minimal operator generated by (2.1) and by D_0 its domain. Moreover, we denote by D the set of all functions $y(t)$ from $L^2_\Delta(\mathbb{T}^*)$ such that $l(y) \in L^2_\Delta(\mathbb{T}^*)$; D is the domain of the maximal operator L. Furthermore $L = L_0^*$ (see [37]). Suppose that the operator L_0 has defect index $(2,2)$ (see [31]).

For every $y, z \in D$, Lagrange's identity [28] is defined as $(Ly,z) - (y,Lz) = [y,z](b) - [y,z](a)$, where $[y,z] := p(t)[y(t)\overline{z^\Delta(t)} - y^\Delta(t)\overline{z(t)}]$.

Consider the boundary-value problem defined by

$$l(y) = \lambda y, \quad y \in D, \tag{2.2}$$

subject to the boundary conditions

$$y(b) - hp(b)y^\Delta(b) = 0, \quad \operatorname{Im} h > 0, \tag{2.3}$$

$$\alpha_1 y(a) - \alpha_2 p(a)y^\Delta(a) = \lambda(\alpha_1' y(a) - \alpha_2' p(a)y^\Delta(a)) \tag{2.4}$$

where λ is spectral parameter and $\alpha_1, \alpha_2, \alpha_1', \alpha_2' \in \mathbb{R}$ and α is defined by

$$\alpha := \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} = \alpha_1'\alpha_2 - \alpha_1\alpha_2' > 0.$$

For $\lambda = 1$, this problem was investigated by Tuna [47].

On the other hand, when we restrict the time variable to $t \geq 0$, the usual case of nonnegative continuous time, there is some related literature. For instance, Favini et al. [24, 25, 26] obtained some corresponding results in n dimensions. See also [17, 18, 19].

For the sake of simplicity, we define $R_a(y) := \alpha_1 y(a) - \alpha_2 p(a)y^\Delta(a)$, $R_a'(y) := \alpha_1' y(a) - \alpha_2' p(a)y^\Delta(a)$, $N_1^b(y) := y(b)$, $N_2^b(y) := p(b)y^\Delta(b)$, $N_1^a(y) := y(a)$, $N_2^a(y) := p(a)y^\Delta(a)$, $R_b(y) := N_2^b(y) - hN_1^b(y)$.

Lemma 2.1. *For arbitrary $y, z \in D$ suppose that $R_a(\overline{z}) = \overline{R_a(z)}$, $R_a'(\overline{z}) = \overline{R_a'(z)}$. Then*

$$[y,\overline{z}](a) = \frac{1}{\alpha}[R_a(y)\overline{R_a'(z)} - R_a'(y)\overline{R_a(z)}]\,. \tag{2.5}$$

Proof. Note that

$$\begin{aligned}
&\frac{1}{\alpha}[R_a(y)\overline{R_a'(z)} - R_a'(y)\overline{R_a(z)}] \\
&= \frac{1}{\alpha}[(\alpha_1 y(a) - \alpha_2 p(a)y^\Delta(a))\overline{(\alpha_1' z(a) - \alpha_2' p(a)z^\Delta(a))}] \\
&\quad - \frac{1}{\alpha}[(\alpha_1' y(a) - \alpha_2' p(a)y^\Delta(a))\overline{(\alpha_1 z(a) - \alpha_2 p(a)z^\Delta(a))}] \\
&= \frac{1}{\alpha}[(\alpha_1'\alpha_2 - \alpha_1\alpha_2')(y(a)p(a)\overline{z^\Delta(a)} - p(a)y^\Delta(a)\overline{z(a)})] \\
&= [y,z](a).
\end{aligned}$$

□

Let θ_1, θ_2 denote the solutions of (2.1) satisfying the conditions $N_1^a(\theta_2) = \alpha_2 - \alpha_2'\lambda$, $N_2^a(\theta_2) = \alpha_1 - \alpha_1'\lambda$, $N_1^b(\theta_1) = h$, $N_2^b(\theta_1) = 1$. By equation (2.1), we have

$$\begin{aligned}\Delta(\lambda) &= [\theta_1, \theta_2](t) = -[\theta_2, \theta_1](t) = -[\theta_2, \theta_1](a)\\ &= -\frac{1}{\alpha}[R_a(\theta_1)\overline{R_a'(\theta_2)} - R_a'(\theta_1)\overline{R_a(\theta_2)}] = R_a(\theta_2) - \lambda R_a'(\theta_2),\end{aligned}$$

and $\Delta(\lambda) = [\theta_1, \theta_2](t) = -[\theta_2, \theta_1](t) = -[\theta_2, \theta_1](b) = -(N_2^a(\theta_1) - hN_1^a(\theta_1))$. We also let

$$G(t, \xi, \lambda) = \frac{1}{\Delta(\lambda)}\begin{cases}\theta_2(\xi, \lambda)\theta_1(t, \lambda), & t < \xi\\ \theta_1(t, \lambda)\theta_2(\xi, \lambda), & \xi < t\,.\end{cases}$$

It can be shown that $G(t, \xi, \lambda)$ satisfies equation (2.1) and boundary conditions (2.3)-(2.4). $G(t, \xi, \lambda)$ is a Green function of the boundary-value problem (2.2)–(2.4). Thus, we obtain the $G(t, \xi, \lambda)$ is a Hilbert-Schmidt kernel and the solution of the boundary value problem can be expressed by $y(t, \lambda) = \int_a^b G(t, \xi, \lambda)y(\xi, \lambda)d\xi = R_\lambda y$. Thus R_λ is a Hilbert Schmidt operator on space $L^2_\Delta(\mathbb{T}^*)$. The spectrum of the the boundary value problem coincide with the roots of the equation $\Delta(\lambda) = 0$. Since Δ is analytic and not identical to zero, it means that the function Δ has at most a countable number of isolated zeros with finite multiplicity and possible limit points at infinity.

Suppose that $f^{(1)} \in L^2_\Delta(\mathbb{T}^*)$ and $f^{(2)} \in \mathbb{C}$. We denote linear space $H = L^2_\Delta(\mathbb{T}^*) \oplus \mathbb{C}$ with two component of elements of $\widehat{f} = \begin{pmatrix} f^{(1)}(t)\\ f^{(2)}\end{pmatrix}$. If $\alpha > 0$ and $\widehat{f} = \begin{pmatrix} f^{(1)}(t)\\ f^{(2)}\end{pmatrix}$, $\widehat{g} = \begin{pmatrix} g^{(1)}(t)\\ g^{(2)}\end{pmatrix} \in H$, then the formula

$$(\widehat{f}, \widehat{g}) = \int_a^b f^{(1)}(t)\overline{g}^{(1)}(t)\Delta t + \frac{1}{\alpha}f^{(2)}\overline{g}^{(2)}$$

defines an inner product in Hilbert space H. In terms of this inner product, linear space H is a Hilbert space. Let us define operator of $A_h : H \to H$ with equalities suitable for boundary-value problem

$$D(A_h) = \Big\{\widehat{f} = \begin{pmatrix} f^{(1)}\\ f^{(2)}\end{pmatrix} \in H : f^{(1)} \in D, R_b(f^{(1)}) = 0,\ f^{(2)} = R_a'(f^{(1)})\Big\}$$

and $A_h\widehat{f} = \widetilde{l}(\widehat{f}) := \begin{pmatrix} l(f^{(1)})\\ R_a(f^{(1)})\end{pmatrix}$. Recall that a linear operator A_h with domain $D(A_h)$ in Hilbert space H is called dissipative if $\operatorname{Im}(A_h f, f) \geq 0$ for all $f \in D(A_h)$ and maximal dissipative if it does not have a proper extension.

3. Main Results

Theorem 3.1. *The operator A_h is maximal dissipative in the space H.*

Proof. Let $\widehat{y} \in D(A_h)$. From (2.5), we have

$$\begin{aligned}(A_h\widehat{y}, \widehat{y}) - (\widehat{y}, A_h\widehat{y}) = [y_1, y_1](b) - [y_1, y_1](a)\\ +\frac{1}{\alpha}[R_a(y_1)\overline{R_a'(y_1)} - R_a'(y_1)\overline{R_a(y_1)}] = [y_1, y_1](b) = 2\operatorname{Im} h(p(b)y^\Delta(b))^2.\end{aligned}$$

Since $\operatorname{Im}(A_h\widehat{y},\widehat{y}) = \operatorname{Im} h(p(b)y^{\Delta}(b))^2 \geq 0, A_h$ is a dissipative operator in H. Let us prove that A_h is maximal dissipative operator in the space H. To do this, it is enough to control that

$$(A_h - \lambda I)D(A_h) = H, \quad \operatorname{Im}\lambda < 0 \tag{3.1}$$

To prove (3.1), let $F \in H$, $\operatorname{Im}\lambda < 0$ and set

$$\Gamma = \begin{pmatrix} (\widetilde{G}_t, \overline{F}) \\ R'_a[(\widetilde{G}_t, \overline{F})] \end{pmatrix},$$

where

$$\widetilde{G}_{t,\lambda} = \begin{pmatrix} G(t,\xi,\lambda) \\ R'_a[G(t,\xi,\lambda)] \end{pmatrix}, \quad G(t,\xi,\lambda) = \begin{cases} \frac{1}{\Delta(\lambda)}\theta_2(\xi,\lambda)\theta_1(t,\lambda), & t < \xi \\ \frac{1}{\Delta(\lambda)}\theta_1(t,\lambda)\theta_2(\xi,\lambda), & \xi < t\,. \end{cases}$$

The function $t \mapsto (G(t,\xi,\lambda), \overline{F_1})$ satisfies the equation $l(y) - \lambda y = F_1$ $(a < t < b)$ and the boundary conditions (2.3)—(2.4). Moreover, for all $F \in H$ and for $\operatorname{Im}\lambda < 0$, we obtain $\Gamma \in D(A_h)$. For each $F \in H$ and for $\operatorname{Im}\lambda < 0$, we have $(A_h - \lambda I)\Gamma = F$. Consequently, the result is $(A_h - \lambda I)D(A_h) = H$ in the case of $\operatorname{Im}\lambda < 0$. The proof of is complete. □

Definition 3.2. If the system of vectors $y_0, y_1, y_2, \dots, y_n$ corresponding to the eigenvalue λ_0 satisfy

$$\begin{gathered} l(y_0) = \lambda_0 y_0, \quad R_a(y_0) - \lambda R'_a(y_0) = 0, \quad R_b(y_0) = 0, \\ l(y_s) - \lambda_0 y_s - y_{s-1} = 0, \quad R_a(y_s) - \lambda R'_a(y_s) - R'_a(y_{s-1}) = 0, \\ R_b(y_s) = 0, \quad s = 1, 2, \dots, n, \end{gathered} \tag{3.2}$$

then the system $y_0, y_1, y_2, \dots, y_n$ corresponding to the eigenvalue λ_0 is called a chain of eigenvectors and associated vectors of boundary-value problem (2.2)-(2.4) [3, 4, 5, 6, 7, 23, 38, 39, 46, 47].

Since the operator A_h is dissipative in H and from Definition 3.2, we have that the eigenvalue of boundary value problem (2.2)-(2.4) coincides with the eigenvalue of dissipative operator A_h. Additionally each chain of eigenvectors and associated vectors $y_0, y_1, y_2, \dots, y_n$ corresponding to the eigenvalue λ_0 corresponds to the chain eigenvectors and associated vectors $\widehat{y}_0, \widehat{y}_1, \widehat{y}_2, \dots, \widehat{y}_n$ corresponding to the same eigenvalue λ_0 of dissipative A_h operator. In this case, the equality $\widehat{y}_k = \binom{y_k}{R'_a(y_k)}$, $k = 0, 1, 2, \dots, n$, holds.

Now, we first construct the self adjoint dilation of the operator A_h. To do this, let us add the "incoming" and "outgoing" subspaces $D_- = L^2(-\infty, 0]$ and $D_+ = L^2[0, \infty)$ to $H = L^2_{\Delta}(\mathbb{T}^*) \oplus \mathbb{C}$. The orthogonal sum $\mathcal{H} = D_- \oplus H \oplus D_+$ is called main Hilbert space of the dilation. In the space $\mathcal{H}$, we consider the operator $\mathcal{L}_h$ on the set $D(\mathcal{L}_h)$, its elements consisting of vectors $w = \langle \varphi_-, y, \varphi_+ \rangle$, generated by the expression

$$\mathcal{L}_h\langle \varphi_-, \widehat{y}, \varphi_+ \rangle = \langle i\frac{d\varphi_-}{d\xi}, \widetilde{l}(\widehat{y}), i\frac{d\varphi_+}{d\xi} \rangle \tag{3.3}$$

satisfying the conditions: $\varphi_- \in W_2^1(-\infty, 0]$, $\varphi_+ \in W_2^1[0, \infty)$, $\widehat{y} \in H$, $\widehat{y} = \binom{y_1(x)}{y_2}$, $y_1 \in D$, $y_2 = R'_a(y_1)$, $y(b) - hp(b)y^{\Delta}(b) = \beta\varphi_-(0)$, $y(b) - \overline{h}p(b)y^{\Delta}(b) = \beta\varphi_+(0)$ where $W_2^1(\cdot,\cdot)$ are Sobolev spaces and $\beta^2 := 2\operatorname{Im} h$, $\beta > 0$ (see [3, 4, 5, 6, 7, 38, 39, 46, 47]).

Theorem 3.3. *The operator L_h is selfadjoint in H and it is a selfadjoint dilation of the operator A_h.*

Proof. We first prove that $\mathcal{L}_h$ is symmetric in $\mathcal{H}$. Namely $(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} = 0$. Let $f, g \in D(\mathcal{L}_h)$, $f = \langle \varphi_-, \widehat{y}, \varphi_+ \rangle$ and $g = \langle \psi_-, \widehat{z}, \psi_+ \rangle$. Then we have

$$\begin{aligned}
&(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} \\
&= (\mathcal{L}_h \langle \varphi_-, \widehat{y}, \varphi_+ \rangle, \langle \psi_-, \widehat{z}, \psi_+ \rangle) - (\langle \varphi_-, \widehat{y}, \varphi_+ \rangle, \mathcal{L}_h \langle \psi_-, \widehat{z}, \psi_+ \rangle) \\
&= [y_1, z_1](b) - [y_1, z_1](a) + \frac{1}{\alpha}[R_a(y_1)\overline{R_a'(z_1)} - R_a'(y_1)\overline{R_a(z_1)}] \\
&\quad + i\psi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0)
\end{aligned}$$

Therefore,

$$(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} = [y_1, z_1](b) + i\psi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0). \tag{3.4}$$

On the other hand,

$$\begin{aligned}
i\psi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0) &= \frac{i}{\beta^2}(y(b) - hp(b)y^{\Delta}(b))\overline{(z(b) - hp(b)z^{\Delta}(b))} \\
&\quad - \frac{i}{\beta^2}(y(b) - hp(b)y^{\Delta}(b))\overline{(z(b) - \overline{h}p(b)z^{\Delta}(b))} \\
&= \frac{i}{\beta^2}(h - \overline{h})p(b)[y(b)\overline{z^{\Delta}(b)} - z(b)\overline{y^{\Delta}(b)}].
\end{aligned}$$

By (3.4), we obtain

$$i\psi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0) = -[y_1, z_1](b), \tag{3.5}$$

and we have $(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} = 0$. Thus, we prove that $\mathcal{L}_h$ is a symmetric operator. To prove that $\mathcal{L}_h$ is selfadjoint, we need to show that $\mathcal{L}_h^* \subseteq \mathcal{L}_h$. Now, we consider the bilinear form $(\mathcal{L}_h f, g)_{\mathcal{H}}$ on elements $g = \langle \psi_-, \widehat{z}, \psi_+ \rangle \in D(\mathcal{L}_h^*)$, where $f = \langle \varphi_-, \widehat{y}, \varphi_+ \rangle \in D(\mathcal{L}_h)$, $\varphi_\mp \in W_2^1(\mathbb{R}_\mp)$, $\varphi_\mp(0) = 0$. Integrating by parts, we obtain $\mathcal{L}_h^* g = \langle i\frac{d\psi_-}{d\xi}, \widehat{z}^*, i\frac{d\psi_+}{d\xi} \rangle$, where $\psi_\mp \in W_2^1(\mathbb{R}_\mp)$, $\widehat{z}^* \in H$. Similarly, if $f = \langle 0, \widehat{y}, 0 \rangle \in D(\mathcal{L}_h)$, then integrating by parts in $(\mathcal{L}_h f, g)_{\mathcal{H}}$, we obtain

$$\mathcal{L}_h^* g = \mathcal{L}_h^* \langle \psi_-, \widehat{z}, \psi_+ \rangle = \langle i\frac{d\psi_-}{d\xi}, \widetilde{l}(\widehat{z}), i\frac{d\psi_+}{d\xi} \rangle, \quad z_1 \in D,\ z_2 = R_a'(z_1). \tag{3.6}$$

Consequently, we have $(\mathcal{L}_h f, g)_{\mathcal{H}} = (f, \mathcal{L}_h g)_{\mathcal{H}}$, for each $f \in D(\mathcal{L}_h)$ by (3.6), where the operator $\mathcal{L}_h$ is defined by (3.3). Therefore, the sum of the integrated terms in the bilinear form $(\mathcal{L}_h f, g)_{\mathcal{H}}$ must be equal to zero:

$$\begin{aligned}
&[y_1, z_1](b) - [y_1, z_1](a) + \frac{1}{\alpha}[R_a(y_1)\overline{R_a'(z_1)} \\
&- R_a'(y_1)\overline{R_a(z_1)}] + i\varphi_-(0)\overline{\psi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0) = 0.
\end{aligned}$$

Then by (3.5), we obtain

$$[y_1, z_1](b) + i\varphi_-(0)_-\overline{\psi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0) = 0. \tag{3.7}$$

From the boundary conditions for $\mathcal{L}_h$, we have

$$y(b) = \beta\varphi_-(0) - \frac{h}{i\beta}(\varphi_-(0) - \varphi_+(0)),\ p(b)y^{\Delta}(b) = \frac{i}{\beta}(\varphi_-(0) - \varphi_+(0)).$$

Afterwards, by (3.7) we obtain

$$\begin{aligned} &\beta\varphi_-(0) - \frac{h}{i\beta}(\varphi_-(0) - \varphi_+(0))\overline{z(b)} - \frac{i}{\beta}(\varphi_-(0) - \varphi_+(0))\overline{p(b)z^\Delta(b)} \\ &= i\varphi_+(0)\overline{\psi}_+(0) - i\varphi_-(0)\overline{\psi}_-(0). \end{aligned} \tag{3.8}$$

We obtain $\frac{i\beta^2-h}{\beta}z(b) + \frac{1}{\beta}\overline{p(b)z^\Delta(b)} = \varphi_-(0)$ comparing the coefficients of $\varphi_-(0)$ in (3.8) or

$$z(b) - hp(b)z^\Delta(b) = \beta\psi_-(0). \tag{3.9}$$

Similarly, we obtain

$$z(b) - \overline{h}p(b)z^\Delta(b) = \beta\psi_+(0) \tag{3.10}$$

by comparing the coefficients of $\varphi_+(0)$ in (3.8). Consequently, conditions (3.9) and (3.10) imply $D(\mathcal{L}_h^*) \subseteq D(\mathcal{L}_h)$, hence $\mathcal{L}_h = \mathcal{L}_h^*$.

The selfadjoint operator $\mathcal{L}_h$ generates on $\mathcal{H}$ a unitary group $U_t = \exp(i\mathcal{L}_h t)$ ($t \in \mathbb{R} = (-\infty,\infty)$). Let us denote by $P : \mathcal{H} \to H$ and $P_1 : H \to \mathcal{H}$ the mappings defined by $P : \langle\varphi_-, \widehat{y}, \varphi_+\rangle \to \widehat{y}$ and $P_1 : \widehat{y} \to \langle 0, \widehat{y}, 0\rangle$. Let $Z_t := PU_tP_1$ $t \geq 0$, by using U_t. The family $\{Z_t\}$ ($t \geq 0$) of operators is a strongly continuous semigroup of completely non-unitary contraction on H. Let us denote by B_h the generator of this semigroup: $B_h\widehat{y} = \lim_{t\to+0}(it)^{-1}(Z_t\widehat{y} - \widehat{y})$. The domain of B_h consists of all the vectors for which the limit exists. The operator B_h is dissipative. The operator $\mathcal{L}_h$ is called the selfadjoint dilation of B_h (see [3, 4, 5, 6, 7, 38, 39, 46, 47]). We show that $B_h = A_h$, hence $\mathcal{L}_h$ is selfadjoint dilation of B_h. To show this, it is sufficient to verify the equality

$$P(\mathcal{L}_h - \lambda I)^{-1}P_1\widehat{y} = (A_h - \lambda I)^{-1}\widehat{y}, \widehat{y} \in H, \quad \operatorname{Im} h < 0. \tag{3.11}$$

To do this, we set $(\mathcal{L}_h - \lambda I)^{-1}P_1\widehat{y} = g = \langle\psi_-, \widehat{z}, \psi_+\rangle$. Then we have $(\mathcal{L}_h - \lambda I)g = P_1\widehat{y}$, and hence $\widetilde{l}(\widehat{z}) - \lambda\widehat{z} = \widehat{y}, \psi_-(\xi) = \psi_-(0)e^{-i\lambda\xi}$ and $\psi_+(\xi) = \psi_+(0)e^{-i\lambda\xi}$. Since $g \in D(\mathcal{L}_h)$, then $\psi_- \in L^2(-\infty,0)$, it follows that $\psi_-(0) = 0$, and consequently $\widehat{z}$ satisfies the boundary condition $z(b) - hp(b)z^\Delta(b) = 0$. Therefore $\widehat{z} \in D(A_h)$, and since point λ with $\operatorname{Im}\lambda < 0$ cannot be an eigenvalue of dissipative operator, it follows that $\psi_+(0)$ is obtained from the formula $\psi_+(0) = \beta^{-1}(z(b) - \overline{h}p(b)z^\Delta(b))$. Thus, we have

$$(\mathcal{L}_h - \lambda I)^{-1}P_1\widehat{y} = \langle 0, (A_h - \lambda I)^{-1}\widehat{y}, \beta^{-1}(z(b) - \overline{h}p(b)z^\Delta(b))\rangle$$

for $\widehat{y}$ and $\operatorname{Im}\lambda < 0$. By applying the mapping P, we obtain (3.11) and

$$\begin{aligned} (A_h - \lambda I)^{-1} &= P(\mathcal{L}_h - \lambda I)^{-1}P_1 = -iP\int_0^\infty U_t e^{-i\lambda t}dtP_1 \\ &= -i\int_0^\infty Z_t e^{-i\lambda t}dt = (B_h - \lambda I)^{-1}, \operatorname{Im}\lambda < 0, \end{aligned}$$

so this clearly shows that $A_h = B_h$. □

The unitary group $\{U_t\}$ has an important property which makes it possible to apply it to the Lax-Phillips [35]. It can be described as a characteristic function of maximal dissipative operator. The Lax-Phillips scheme has orthogonal incoming and outcoming subspaces $D_- = \langle L^2(-\infty,0), 0, 0\rangle$ and $D_+ = \langle 0, 0, L^2(0,\infty)\rangle$ satisfying the following properties

(1) $U_tD_- \subset D_-$, $t \leq 0$ and $U_tD_+ \subset D_+$, $t \geq 0$;
(2) $\cap_{t\leq 0}U_tD_- = \cap_{t\geq 0}U_tD_+ = \{0\}$;

(3) $\overline{\cup_{t\geq 0}U_tD_-}=\overline{\cup_{t\leq 0}U_tD_+}=\mathcal{H}$;
(4) $D_-\perp D_+$.

Definition 3.4. The linear operator A with domain $D(A)$ acting on the Hilbert space H is called completely nonselfadjoint (or simple) if there is no invariant subspace $M\subseteq D(A)$ ($M\neq\{0\}$) of the operator A on which the restriction A to M is selfadjoint.

Lemma 3.5. *The operator A_h is completely nonselfadjoint (simple).*

Proof. Let $H'\subset H$ be a nontrivial subspace in which A_h induces a selfadjoint operator A_h' with domain $D(A_h')=H'\cap D(A_h)$. If $\widehat{f}\in D(A_h')$, then $\widehat{f}\in D(A_h'^{*})$ and

$$\begin{aligned}\frac{d}{dt}\|e^{iA_h't}\widehat{f}\|_H^2&=\frac{d}{dt}(e^{iA_h't}\widehat{f},e^{iA_h't}\widehat{f})_H\\&=i(A_h'e^{iA_h't}\widehat{f},e^{iA_h't}\widehat{f})_H-i(e^{iA_h't}\widehat{f},A_h'e^{iA_h't}\widehat{f})_H\,.\end{aligned}$$

Taking $\widehat{g}=e^{iA_h't}\widehat{f}$, we have

$$\begin{aligned}0&=i(A_h'\widehat{g},\widehat{g})_H-i(\widehat{g},A_h'\widehat{g})_H\\&=i[g_1,g_1](b)-i[g_1,g_1](a)+\frac{i}{\alpha}[R_a(g_1)\overline{R_a'(g_1)}-R_a'(y_1)\overline{R_a(g_1)}]\\&=-2\operatorname{Im}h(D_{q^{-1}}y_1(a))^2=-\beta^2(p(b)y^{\Delta}(b))^2.\end{aligned}$$

Since $\widehat{f}\in D(A_h')$, A_h' holds condition above. Moreover, eigenvectors of the operator A_h' should also hold this condition. Therefore, for the eigenvectors $\widehat{y}(\lambda)$ of the operator A_h acting in H' and the eigenvectors of the operator A_h', we have $p(b)y^{\Delta}(b)=0$. From the boundary conditions, we obtain $y(b)=0$ and $\widehat{y}(t,\lambda)=0$. Consequently, by the theorem on expansion in the eigenvectors of the selfadjoint operator A_h', we obtain $H'=\{0\}$. Hence the operator A_h is simple. The proof is complete. $\square$

Let us define

$$H_-=\overline{\cup_{t\geq 0}U_tD_-}\,,\quad H_+=\overline{\cup_{t\leq 0}U_tD_+}$$

where $D_-=\langle L^2(-\infty,0),0,0\rangle$ and $D_+=\langle 0,0,L^2(0,\infty)\rangle$. By using Lemma 3.5, one can obtain $H_-+H_+=\mathcal{H}$.

Assume that $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of $l(y)=\lambda y$, satisfying the conditions

$$\begin{gathered}\varphi(a,\lambda)=\frac{\alpha_2'}{\alpha},\quad p(a)\varphi^{\Delta}(a,\lambda)=\frac{\alpha_1'}{\alpha},\\\psi(a,\lambda)=\alpha_2-\alpha_2'\lambda,\quad p(a)\psi^{\Delta}(a,\lambda)=\alpha_1-\alpha_1'\lambda,\end{gathered}$$

Let us adopt the notation

$$\widehat{\psi}(x,\lambda):=\begin{pmatrix}\psi(x,\lambda)\\\alpha\end{pmatrix},\quad n_b(\lambda)=\frac{p(b)\varphi^{\Delta}(b,\lambda)}{\varphi(b,\lambda)},\quad m_b(\lambda)=\frac{\psi(b,\lambda)}{p(b)\psi^{\Delta}(b,\lambda)}.$$

The functions $m_b(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis. Further, it is possible to show that the function $m_b(\lambda)$ possesses the following properties: $\operatorname{Im} m_b(\lambda)\geq 0$ for all $\operatorname{Im}\lambda>0$, and $\overline{m_b(\lambda)}=m_b(\overline{\lambda})$ for all $\lambda\in\mathbb{C}$, except the real poles $m_b(\lambda)$. We set

$$S_h(\lambda):=\frac{m_b(\lambda)-h}{m_b(\lambda)-\overline{h}}.\tag{3.12}$$

$$U_\lambda^-(t,\xi,\zeta) = \langle e^{-i\lambda\xi}, \alpha n_b(\lambda)\{(m_b(\lambda)-h)p(b)\varphi^\Delta(b,\lambda)\}^{-1}\widehat{\psi}(t,\lambda), \overline{S_h}(\lambda)e^{-i\lambda\zeta}\rangle.$$

We note that the vectors $U_\lambda^-(t,\xi,\zeta)$ for real λ do not belong to the space $\mathcal{H}$. However, $U_\lambda^-(t,\xi,\zeta)$ satisfies the equation $\mathcal{L}_h U = \lambda U$ and the corresponding boundary conditions for the operator $\mathcal{L}_H$. By means of vector $U_\lambda^-(t,\xi,\zeta)$, we define the transformation $F_- : f \to \widetilde{f_-}(\lambda)$ by

$$(F_- f)(\lambda) := \widetilde{f_-}(\lambda) := \frac{1}{\sqrt{2\pi}}(f, U_{\overline{\lambda}}^-)_{\mathcal{H}}$$

on the vectors $f = \langle \varphi_-, \widehat{y}, \varphi_+\rangle$ in which $\varphi_-(\xi)$, $\varphi_+(\zeta)$, $y(x)$ are smooth, compactly supported functions

Lemma 3.6. The transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$. For all vectors $f, g \in H_-$ the Parseval equality and the inversion formulae hold:

$$(f,g)_{\mathcal{H}} = (\widetilde{f_-}, \widetilde{g}_-)_{L^2} = \inf_{-\infty}^{\infty} \widetilde{f_-}(\lambda)\overline{\widetilde{g_-}(\lambda)}d\lambda, \quad f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f_-}(\lambda) U_{\overline{\lambda}}^- d\lambda,$$

where $\widetilde{f_-}(\lambda) = (F_- f)(\lambda)$ and $\widetilde{g_-}(\lambda) = (F_- g)(\lambda)$.

Proof. For $f, g \in D_-$, $f = \langle \varphi_-, 0, 0\rangle$, $g = \langle \psi_+, 0, 0\rangle$, with Paley-Wiener theorem, we have

$$\widetilde{f_-}(\lambda) = \frac{1}{\sqrt{2\pi}}(f, U_{\overline{\lambda}}^-)_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \varphi_-(\xi)e^{-i\lambda\xi}d\xi \in H_-^2,$$

and by using the usual Parseval equality for Fourier integrals, we have

$$(f,g)_{\mathcal{H}} = \int_{-\infty}^{\infty} \varphi_-(\xi)\overline{\psi_-(\xi)}d\xi = \int_{-\infty}^{\infty} \widetilde{f_-}(\lambda)\overline{\widetilde{g_-}(\lambda)}d\lambda = (F_- f, F_- g)_{L^2},$$

Here, $H_\pm^2$ denotes the Hardy classes in $L^2(\mathbb{R})$ consisting of the functions analytically extendible to the upper and lower half-planes, respectively. We now extend to the Parseval equality to the whole of H_-. We consider in H_- the dense set of H_-' of the vectors obtained as follows from the smooth, compactly supported functions in $D_- : f \in H_-'$ if $f = U_T\ f_0$, $f_0 = \langle \varphi_-, 0, 0\rangle$, $\varphi_- \in C_0^\infty(-\infty, 0)$, where $T = T_f$ is a nonnegative number depending on f. If $f, g \in H_-'$, then for $T > T_f$ and $T > T_g$ we have $U_{-T}f, U_{-T}g \in D_-$, moreover, the first components of these vectors belong to $C_0^\infty(-\infty, 0)$. Therefore, since the operators U_t $(t \in \mathbb{R})$ are unitary, by the equality $F_- U_t f = (U_t f, U_{\overline{\lambda}}^-)_{\mathcal{H}} = e^{i\lambda t}(f, U_\lambda^-)_{\mathcal{H}} = e^{i\lambda t}F_- f$, we have

$$\begin{aligned}(f,g)_{\mathcal{H}} &= (U_{-T}f, U_{-T}g)_{\mathcal{H}} = (F_- U_{-T}f, F_- U_{-T}g)_{L^2},\\ &(e^{i\lambda T}F_- f, e^{i\lambda T}F_- g)_{L^2} = (\widetilde{f}, \widetilde{g})_{L^2}.\end{aligned} \tag{3.13}$$

By taking the closure (3.13), we obtain the Parseval equality for the space H_-. The inversion formula is obtained from the Parseval equality if all integrals in it are considered as limits in the of integrals over finite intervals. Finally $F_- H_- = \overline{\cup_{t\ge 0} F_- U_t D_-} = \overline{\cup_{t\ge 0} e^{i\lambda t} H_-^2} = L^2(\mathbb{R})$, that is F_- maps H_- onto the whole of $L^2(\mathbb{R})$. The proof is complete. $\square$

We set

$$U_\lambda^+(t,\xi,\zeta) = \langle S_h(\lambda)e^{-i\lambda\xi}, \alpha n_b(\lambda)(m_b(\lambda)-h)p(b)\varphi^\Delta(b,\lambda)\widehat{\psi}(t,\lambda), e^{-i\lambda\zeta}\rangle.$$

We note that the vectors $U_\lambda^+(t,\xi,\zeta)$ for real λ do not belong to the space $\mathcal{H}$. However, $U_\lambda^+(t,\xi,\zeta)$ satisfies the equation $\mathcal{L}_h U = \lambda U$ and the corresponding boundary conditions for the operator $\mathcal{L}_H$. With the help of vector $U_\lambda^+(t,\xi,\zeta)$, we define the transformation $F_+ : f \to \widetilde{f}_+(\lambda)$ by $(F_+f)(\lambda) := \widetilde{f}_+(\lambda) := \frac{1}{\sqrt{2\pi}}(f, U_\lambda^+)_{\mathcal{H}}$ on the vectors $f = \langle \varphi_-, \widehat{y}, \varphi_+ \rangle$ in which $\varphi_-(\xi), \varphi_+(\zeta)$ and $y(x)$ are smooth, compactly supported functions.

Lemma 3.7. *The transformation F_+ isometrically maps H_+ onto $L^2(\mathbb{R})$. For all vectors $f, g \in H_+$ the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathcal{H}} = (\widetilde{f}_+, \widetilde{g}_+)_{L^2} = \int_{-\infty}^{\infty} \widetilde{f}_+(\lambda)\overline{\widetilde{g_+}(\lambda)}d\lambda, \quad f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}_+(\lambda)U_\lambda^+ d\lambda,$$

where $\widetilde{f}_+(\lambda) = (F_+f)(\lambda)$ and $\widetilde{g_+}(\lambda) = (F_+g)(\lambda)$.

The proof of the above lemma is analogous to the Lemma 3.6, and it is omitted. It is obvious that $|S_h(\lambda)| = 1$ for $\lambda \in \mathbb{R}$. Therefore, it explicitly follows from the formulae for the vectors U_λ^- and U_λ^+ that

$$U_\lambda^+ = \overline{S_h}(\lambda)U_\lambda^-. \tag{3.14}$$

It follows from Lemmas 3.6 and 3.7 that $H_- = H_+$. Together with Lemma 3.5, it can be concluded that $H_- = H_+ = \mathcal{H}$.

Thus, the transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$ with the subspace D_- mapped onto H_-^2 and the operators U_t are transformed into the operators of multiplication by $e^{i\lambda t}$. This means that F_- is the incoming spectral representation for the group $\{U_t\}$. Similarly, F_+ is the outgoing spectral representation for the group $\{U_t\}$. It follows from (3.14) that the passage from the F_- representation of an element $f \in \mathcal{H}$ to its F_+ representation is accomplished as $\widetilde{f}_+(\lambda) = \overline{S_h}(\lambda)\widetilde{f}_-(\lambda)$. Consequently, according to [35], we have proved the following Theorem.

Theorem 3.8. *The function $\overline{S_h}(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the selfadjoint operator $\mathcal{L}_h$).*

Let $S(\lambda)$ be an arbitrary non-constant inner function (see [36]) on the upper half-plane (the analytic function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|S(\lambda)| \le 1$ for all $\lambda \in \mathbb{C}_+$ and $|S(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$). Define $K = H_+^2 \ominus SH_+^2$. Then $K \neq \{0\}$ is a subspace of the Hilbert space H_+^2. We consider the semigroup of operators Z_t $(t \ge 0)$ acting in K according to the formula $Z_t\varphi = P[e^{i\lambda t}\varphi]$, $\varphi = \varphi(\lambda) \in K$, where P is the orthogonal projection from H_+^2 onto K. The generator of the semigroup $\{Z_t\}$ is denoted by $T\varphi = \lim_{t\to+0}(it)^{-1}(Z_t\varphi - \varphi)$, which T is a maximal dissipative operator acting in K and with the domain $D(T)$ consisting of all functions $\varphi \in K$, such that the limit exists. The operator T is called *a model dissipative operator* (we remark that this model dissipative operator, which is associated with the names of Lax-Phillips [35], is a special case of a more general model dissipative operator constructed by Nagy and Foiaş [36]). The basic assertion is that $S(\lambda)$ is the *characteristic function* of the operator T.

Let $K = \langle 0, H, 0 \rangle$, so that $\mathcal{H} = D_- \oplus K \oplus D_+$. It follows from the explicit form of the unitary transformation F_- under the mapping F_- that

$$\begin{gathered} \mathcal{H} \to L^2(\mathbb{R}), \quad f \to \widetilde{f_-}(\lambda) = (F_-f)(\lambda), \quad D_- \to H_-^2, \quad D_+ \to S_hH_+^2, \\ K \to H_+^2 \ominus S_hH_+^2, \quad U_t \to (F_-U_tF_-^{-1}\widetilde{f_-})(\lambda) = e^{i\lambda t}\widetilde{f_-}(\lambda). \end{gathered} \tag{3.15}$$

The formulas (3.15) show that operator A_h is a unitarily equivalent to the model dissipative operator with the characteristic function $S_h(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operator coincide (see [36]), we have thus proved following theorem.

Theorem 3.9. *The characteristic function of the maximal dissipative operator A_h coincides with the function $S_h(\lambda)$ defined in* (3.12)*.*

Using the characteristic function, we investigate the spectral properties of the maximal dissipative operator A_h. We know that the characteristic function of a maximal dissipative operator carries information about the spectral properties of this operator. To prove completeness of the system of eigenvectors and associated vectors of the operator A_h in the space H, we must show that there exists no singular factor $s(\lambda)$ of the characteristic function $S_h(\lambda)$ in the factorization $S_h(\lambda) = s(\lambda)B(\lambda)$ ($B(\lambda)$ is a Blaschke product) (see [3, 4, 5, 6, 7, 33, 36]). The characteristic function $S_h(\lambda)$ of the maximal dissipative operator A_h has the form

$$S_h(\lambda) := \frac{m_b(\lambda) - h}{m_b(\lambda) - \overline{h}},$$

where $\operatorname{Im} h > 0$. From (3.12), it is clear that $S_h(\lambda)$ is an inner function in the upper half-plane, and it is meromorphic in the whole complex λ-plane.

Theorem 3.10. *For all the values of h with $\operatorname{Im} h > 0$, except possibly for a single value $h = h_0$, the characteristic function $S_h(\lambda)$ of the maximal dissipative operator A_h is a Blaschke product. The spectrum of $A_h$is purely discrete and belongs to the open upper half-plane. The operator A_h has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of all eigenvectors and associated vectors of the operator A_h is complete in the space H (see* [3, 4, 5, 6, 7, 23, 38, 39, 46, 47]*).*

Proof. Since $S_h(\lambda)$ is an inner function, it can be factored in the form

$$S_h(\lambda) = e^{i\lambda c} B_h(\lambda), \quad c = c(h) \geq 0, \tag{3.16}$$

where $B_h(\lambda)$ is a Blaschke product. It follows from (3.16) that

$$|S_h(\lambda)| = |e^{i\lambda c}|\, |B_h(\lambda)| \leq e^{-c(h)\operatorname{Im}\lambda}, \quad \operatorname{Im}\lambda \geq 0. \tag{3.17}$$

Further, for $m_b(\lambda)$ in terms of $S_h(\lambda)$, we find from (3.12) that

$$m_b(\lambda) = \frac{h - \overline{h} S_h(\lambda)}{S_h(\lambda) - 1}. \tag{3.18}$$

If $c(h) > 0$ for a given value h ($\operatorname{Im} h > 0$), then (3.17) implies that $\lim_{t\to+\infty} S_h(it) = 0$, and then (3.18) gives us that $\lim_{t\to+\infty} m_b(it) = -h$. Since $m_b(\lambda)$ does not depend on h, this implies that $c(h)$ can be nonzero at not more than a single point $h = h_0$ (and further $h_0 = -\lim_{t\to+\infty} m_b(it)$). The proof is complete. $\square$

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