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EXTREMAL NORM FOR POTENTIALS OF STURM-LIOUVILLE EIGENVALUE PROBLEMS WITH SEPARATED BOUNDARY CONDITIONS

HONGJIE GUO, JIANGANG QI

ABSTRACT. For the n-th eigenvalue of a Sturm-Liouville eigenvalue problem with separated boundary conditions, we express the infimum of the $L^1[0,1]$ norm of potentials, in terms of a parameter λ and the boundary conditions. Also we indicate where the infimum can be attained. As an application, we obtain the extremum of the n-th eigenvalue of a problem for potentials on a sphere in $L^1[0,1]$.

1. INTRODUCTION

Consider the Sturm-Liouville eigenvalue problem

$$-y'' + qy = \lambda y, \quad y = y(x), \quad x \in [0,1] \tag{1.1}$$

associated with the separated boundary conditions

$$\cos\alpha\ y(0) - \sin\alpha\ y'(0) = 0, \quad \cos\beta\ y(1) - \sin\beta\ y'(1) = 0, \tag{1.2}$$

where the potential $q \in L^1[0,1]$ and $\alpha, \beta \in [0,\pi)$. We denote by $\lambda_n(q)$ the n-th eigenvalue of the problem (1.1) with (1.2). Then we have

$$\lambda_1(q) < \lambda_2(q) < \dots < \lambda_n(q) < \dots, \quad \lambda_n(q) \to \infty,\ n \to \infty$$

from the classical spectral theory of Sturm-Liouville problems. For presenting article, we write (1.2) in the equivalent form

$$y(0) - k_1 y'(0) = 0, \quad y(1) + k_2 y'(1) = 0, \tag{1.3}$$

where $k_1 = \infty$ or $k_2 = \infty$ are allowed. Particularly, $k_1 = k_2 = 0$ and $k_1 = k_2 = \infty$ yields the Dirichlet condition and the Neumann condition, respectively. For $\lambda \in \mathbb{R}$, let

$$\Omega_n(\lambda) = \big\{q : q \in L^1[0,1], \lambda_n(q) = \lambda\big\},\ n \geq 1 \tag{1.4}$$

denote the set of all potentials in $L^1[0,1]$ with the same n-th eigenvalue λ and let

$$E_n(\lambda) = \inf\big\{\|q\| : q \in \Omega_n(\lambda)\big\}, \tag{1.5}$$

where $\|\cdot\|$ stands for the L^1-norm. Clearly, $E_n(\lambda)$ is the infimum of L^1-norms of elements in $\Omega_n(\lambda)$. In this paper, we give explicit expression of $E_n(\lambda)$ in terms of λ and the parameters k_1, k_2 for $n \geq 1$. Furthermore, we specify the cases where the

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infimum can or cannot be attained in $L^1[0,1]$ and show an immediate application to the extremum problem of eigenvalues on an L^1-sphere.

We denote the spectrum of (1.1)-(1.3) by $\sigma(q)$. For such problem we can study the infimum L^1-norm of the elements in $\Omega(\lambda_1,\dots,\lambda_n)$, where

$$\Omega(\lambda_1,\dots,\lambda_n)=\{q:q\in L^1[0,1],\lambda_j\in\sigma(q),1\le j\le n\}$$

for given n distinct real numbers $\lambda_j, j=1,\dots,n$. Of course, it is a kind of inverse spectral problem: recovering some information of the potential from finite spectral data. The set $\Omega(\lambda_1,\dots,\lambda_n)$ is an infinite and unbounded subset of $L^1[0,1]$ [12, Corollary 1, p.77]. This kind of inverse problem is clearly different from the classical one, which studies how to determine the potential uniquely using suitable spectral data. In 1946, Borg wrote the first important paper [1] on this subject. From then on, a number of publications have been devoted to this topic [4, 5, 6, 7, 12].

On the other hand, the problem investigated in this article may be viewed as the dual to the extremal problems of eigenvalues with potentials on an L^1-sphere. Extremal problems of eigenvalues with potentials in a given class of functions originated from the famous Lagrange problem: find the strongest column which is the body of revolution of a plane curve around some line located in the same plane, see [2]. In [3, 9, 15], the extremal values of the first eigenvalue for Dirichlet condition or separated boundary conditions (1.3) with $k_1=k_2>0$ have been studied. The analogous problem was also considered for equation (1.1) associated with the separated boundary conditions (1.3) with $k_1\ge k_2>0$ in [8]. Such problem for potentials in a weighted integrable function space is studied in [14].

For problem (1.1) associated with Dirichlet or Numann boundary conditions, the extremal values of $\lambda_n(q)$ on an L^1-sphere were obtained in [16, 17] by applying the variational method and (strong) continuity of $\lambda_n(q)$ in q with respect to the weak topology in $L^1[0,1]$. Recently, Qi and Chen [13] gave the formula of $E_n(\lambda)$ for the problem of (1.1) with Dirichlet condition and they also solved the corresponding extremal problems of eigenvalues as an application.

The results of this paper are generalizations of the results in [13]. The main ingredient in the proof of our results is a generalized Lyapunov inequality. Although the idea is similar to that one in [13], two differences should be pointed out. One is that we give a theoretical and simple proof for the generalized Lyapunov inequality for the problem (1.1) with the separated boundary conditions(1.3) by Mercer's theorem instead of the method used in [13], which is hard applied to the present cases. The other is that, for $\lambda\ge\lambda_n(0)$, the technique used in [13] to calculate $E_n(\lambda)$ in a set of positive and even potentials is not applicable to the cases studied in this paper.

Let $\Omega_n(\lambda)$ be defined as in (1.4) and the parameters $k_1,k_2\ge 0$. The main results of the article read as follows.

Theorem 1.1. *Let $\lambda_1(0)$ be the first eigenvalue of problem (1.1)-(1.3) with $q=0$, and $E_1(\lambda)$ be given by (1.5) with $n=1$.*

(i) *If $0\le k_1\le k_2$, then*

$$E_1(\lambda):=E_1(\lambda,k_1,k_2):=\begin{cases}H_1(\lambda,k_1,k_2), & \text{for } \lambda<\lambda_1(0),\\ K_1(\lambda,k_1,k_2), & \text{for } \lambda\ge\lambda_1(0).\end{cases}\tag{1.6}$$

(ii) *If $k_1>k_2\ge 0$, then*

$$E_1(\lambda)=E_1(\lambda,k_2,k_1),\quad \text{for } \lambda\in\mathbb{R}.\tag{1.7}$$

Moreover, $E_1(\lambda)$ is attainable in $\Omega_1(\lambda)$ for $\lambda \in [\lambda_1(0), \infty)$, but not for $\lambda \in (-\infty, \lambda_1(0))$.

Theorem 1.2. *Let $\lambda_n(0)$ be the n-th eigenvalue of problem (1.1)-(1.3) with $q = 0$ and $E_n(\lambda)$ be given by (1.5) with $n \geq 2$.*

(i) *If $0 \leq k_1 \leq k_2$, then*

$$E_n(\lambda) := E_n(\lambda, k_1, k_2) := \begin{cases} H_n(\lambda, k_1, k_2), & \text{for } \lambda < \lambda_n(0), \\ K_n(\lambda, k_1, k_2), & \text{for } \lambda \geq \lambda_n(0). \end{cases} \tag{1.8}$$

(ii) *If $k_1 > k_2 \geq 0$, then*

$$E_n(\lambda) = E_n(\lambda, k_2, k_1), \quad \text{for } \lambda \in \mathbb{R}. \tag{1.9}$$

Moreover, $E_n(\lambda)$ is attainable in $\Omega_n(\lambda)$ for $\lambda \in [\lambda_n(0), \infty)$, but not for $\lambda \in (-\infty, \lambda_n(0))$.

Here the functions $H_1(\lambda, k_1, k_2)$, $K_1(\lambda, k_1, k_2)$, $H_n(\lambda, k_1, k_2)$ and $E_n(\lambda, k_1, k_2)$ are defined as in (3.2), (3.15), (3.18) and (3.19), respectively.

Furthermore, we observe that $H_n(\lambda, k_1, k_2)$ are decreasing and $E_n(\lambda, k_1, k_2)$ are increasing in terms of λ for $n \geq 1$. So from theorems 1.1 and 1.2, we directly have both the infimum and supremum of $\lambda_n(q)$ on $S_r := \{q \in L^1[0, 1] : \|q\| = r\}$ for all $r > 0$.

Corollary 1.3. *Let $q \in L^1[0, 1]$. Denote the n-th eigenvalue of the problem (1.1)-(1.3) by $\lambda_n(q)$. Then we have*

$$\inf\{\lambda_n(q) : q \in S_r\} = H_n^{-1}(r), \tag{1.10}$$

$$\sup\{\lambda_n(q) : q \in S_r\} = K_n^{-1}(r), \quad r > 0,\ n \geq 1, \tag{1.11}$$

where H_n^{-1} and K_n^{-1} are the inverse functions of H_n and K_n, respectively.

As it was pointed out in Theorems 1.1 and 1.2, the infimum of $\lambda_n(q)$ on S_r cannot be attained, while the supermum can be attained. As special cases: $k_1 = k_2 = 0$ and $k_1 = k_2 = \infty$, we immediately obtain the extremal norms of the potentials for Drichlet and Neumann problems, respectively.

Corollary 1.4. *If $k_1 = k_2 = 0$, then*

$$E_n(\lambda) = \begin{cases} 2n\sqrt{\lambda} \cot \frac{\sqrt{\lambda}}{2n}, & \lambda < \pi^2, \\ \sqrt{\lambda}\big(\sqrt{\lambda} - n\pi\big), & \lambda \geq \pi^2. \end{cases} \tag{1.12}$$

If $k_1 = k_2 = \infty$, then

$$\begin{gathered} E_1(\lambda) = \begin{cases} \sqrt{-\lambda} \tanh \sqrt{-\lambda}, & \lambda < 0, \\ \lambda, & \lambda \geq 0, \end{cases} \\ E_n(\lambda) = \begin{cases} 2(n-1)\sqrt{\lambda} \cot \frac{\sqrt{\lambda}}{2(n-1)}, & \lambda < (n-1)^2\pi^2,\ n \geq 2 \\ \sqrt{\lambda}\big(\sqrt{\lambda} - (n-1)\pi\big), & \lambda \geq (n-1)^2\pi^2,\ n \geq 2. \end{cases} \end{gathered} \tag{1.13}$$

Formula (1.12) is one of the results in [13]. Applying formula (1.13) to Corollary 1.3 yields the extremal values of the n-th eigenvalue for Neumann problem, which are the same as the results in [16, 17].

This article is organized as follows. Section 2 is devoted to the Mercer's theorem for Sturm-Liouville eigenvalue problems and the generalized Lyapunov-type inequality. The proofs of theorems 1.1 and 1.2 are given in Section 3.

2. Lyapunov-type inequalities

In this section we derive a generalized Lyapunov inequality for problem (1.1)-(1.3). The classical Lyapunov inequality has been generalized for problem (1.1) with Dirichlet condition in [13]. But the method used in [13] is difficult to apply to the cases studied here. We will use Mercer's theorem [10] to prove our result, which can be restated as follows.

Theorem 2.1 ([10])**.** *Let $K(x,t)$ be a continuous symmetric function in the finite closed square $[a,b]\times[a,b]$ and let T be a linear integral operator on $L^2[a,b]$ with the kernel $K(x,t)$. If T has only positive (or negative) eigenvalues, or if it has only a finite number of eigenvalues of one sign, the expansion*

$$K(x,t)=\sum_{n=1}^{\infty}\mu_n\varphi_n(x)\overline{\varphi_n(t)}$$

is valid and converges absolutely and uniformly, where μ_n are the eigenvalues of T and φ_n are the corresponding orthonormal eigenfunctions for $n\ge 1$.

For self contained, we give the Mercer's theorem for Sturm-Liouville eigenvalue problems. Consider the problem

$$-\left(p(x)y'(x)\right)'+q(x)y(x)=\mu w(x)y(x),\quad x\in[a,b] \tag{2.1}$$

associated with the separated boundary conditions

$$y(a)-k_1y'(a)=0,\quad y(b)+k_2y'(b)=0 \tag{2.2}$$

In what follows, we always assume that

$$p(x)>0,\quad w(x)\ge 0 \text{ a.e. on } [a,b];\quad 1/p,q,w\in L^1[a,b];\ k_1,k_2\in(-\infty,\infty] \tag{2.3}$$

and μ is the spectral parameter.

Lemma 2.2. *Assume that each eigenvalue μ_n of eigenvalue problem* (2.1) *with* (2.2) *is non zero for $n\ge 1$. Then*

$$\int_a^b G(x,x)w(x)\mathrm{d}x=\sum_{n=1}^{\infty}\frac{1}{\mu_n}, \tag{2.4}$$

where $G(x,t)$ is the Green's function at $\mu=0$.

Proof. The idea is to use Theorem 2.1. From the assumption that 0 is not an eigenvalue of problem (2.1) with (2.2), from the theory of boundary value problems, the above problem is equivalent the integral equation

$$y(x)=\mu\int_a^b G(x,t)y(t)w(t)\mathrm{d}t,$$

where $G(x,t)$ is the Green's function at $\mu=0$. Defined the operator T by

$$Ty(x)=\int_a^b G(x,t)y(t)w(t)\mathrm{d}t,\quad \text{for } y\in L^2_w[a,b].$$

On account of the properties of Green's function, we have that T is an operator with the continuous symmetric kernel $G(x,t)$. Then it is a compact and self-adjoint operator in the Hilbert space $L^2_w[a,b]$. By the spectral theory of compact and self-adjoint operators in the Hilbert space, we obtain that all the eigenvalues of T are $1/\mu_n$, $n=1,2,\dots$. Moreover, these eigenvalues have only a finite number of negative values. Denoting by φ_n the orthonormal eigenfunctions corresponding to $1/\mu_n$ for $n \geq 1$, and employing Theorem 2.1 yields that the expansion

$$G(x,x) = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \varphi_n^2(x) \tag{2.5}$$

is valid and converges absolutely and uniformly. Multiplying both sides of (2.5) by $w(x)$, integrating over $[a,b]$, and applying the normality of $\varphi_n(x)$ gives

$$\int_a^b G(x,x)w(x)\mathrm{d}x = \sum_{n=1}^{\infty} \frac{1}{\mu_n},$$

which is the desired formula. □

We now turn to the main result of this section. That is, we give a generalized Lyapunov inequality for the boundary value problem

$$-(p(x)y'(x))' + q(x)y(x) = w(x)y(x), \quad x \in [a,b] \tag{2.6}$$

with separated boundary conditions (2.2).

Theorem 2.3. *Assume that problem* (2.1) *with* (2.2) *has only positive eigenvalues. If problem* (2.6) *with* (2.2) *has non-trivial solutions, then*

$$\int_a^b w(x)\mathrm{d}x > \frac{1}{\max_{x\in[a,b]} G(x,x)} =: M, \tag{2.7}$$

where $G(x,t)$ is the Green's function at $\mu = 0$. Moreover, the inequality (2.7) *is sharp in the sense of distributions.*

To prove that inequality (2.7) is sharp, we need some knowledge of Dirac delta distribution. The Dirac delta distribution δ at point $c \in (a,b)$ is defined by

$$\int_a^b \delta(x-c)f(x)\mathrm{d}x = f(c), \quad \forall f \in C[a,b].$$

A sequence (f_n) from $L^1[a,b]$ is said to converge weakly to $\delta(x-c)$ if

$$\int_a^b f_n(x)\varphi(x)\mathrm{d}x \to \int_a^b \delta(x-c)\varphi(x)\mathrm{d}x, \quad \forall \varphi \in C[a,b].$$

For this we write $f_n \xrightarrow{w} \delta(x-c)$. In particular, if $f_n \xrightarrow{w} \delta(x-c)$, then

$$\int_a^b f_n(x)\mathrm{d}x \to \int_a^b \delta(x-c)\mathrm{d}x = 1.$$

It is well known that there exists a sequence $(f_n) \in L^1[a,b]$ such that $f_n \xrightarrow{w} \delta(x-c)$. For example, we can take

$$f_n(x) = \begin{cases} n/2, & x \in [c-\frac{1}{n}, c+\frac{1}{n}], \\ 0, & \text{otherwise}. \end{cases}$$

Proof of the theorem 2.3. We first prove that the inequality (2.7) holds. Let μ_n be the eigenvalues of the problem (2.1) with (2.2) for $n \geq 1$. It follows from the assumption that $\mu_n > 0$. In addition, 1 is one of the eigenvalues of (2.1) with (2.2) because the problem (2.6) with (2.2) has non-trivial solutions. Thus $0 < \mu_1 \le 1$. This together with Lemma 2.2 gives that

$$\int_a^b G(x,x)w(x)\mathrm{d}x > 1.$$

Using(2.5), $\mu_n > 0$ and the continuity of $G(x,x)$ on $[a,b]$, we have

$$\int_a^b w(x)\mathrm{d}x > \frac{1}{\max_{x\in[a,b]} G(x,x)} =: M.$$

Note that inequality (2.7) is strict, i.e., the integral of w is not minimized in $L^1[a,b]$. However, the lower bound M is best, namely, it is the limit of the integral of a sequence of weight functions in $L^1[a,b]$, which satisfies the condition of Theorem 2.3. To see this, let $u(x)$ and $v(x)$ be linearly independent solutions of $-(py')'+qy = 0$ satisfying (2.2) at a and b, respectively. Then we have

$$G(x,x) = -\frac{1}{W}u(x)v(x), \ \ for\ x \in [a,b],$$

where W is the modified Wronskian determinant of u and v.

Let $G(c,c) = \max\{G(x,x) : x \in [a,b]\} =: 1/M$. Take $w_c(x) = \delta(x-c)M$ and consider the boundary value problem

$$-(py')' + qy = w_c y, \quad y(0) - k_1y'(0) = y(1) + k_2y'(1) = 0. \tag{2.8}$$

The solution of (2.8) is understood as a function $y \in AC[a,b]$ with $py' \in AC([a,c)\cup (c,b])$ satisfying

$$\begin{gathered} -(p(x)y'(x))' + q(x)y(x) = 0, \quad x \in [a,b],\ x \ne c, \\ y(0) - k_1y'(0) = y(1) + k_2y'(1) = 0, \\ (py')(c-0) - (py')(c+0) = y(c)M. \end{gathered} \tag{2.9}$$

It is easy to verify that (2.8) has a non-trivial solution

$$\varphi(x) = \begin{cases} v(c)u(x), & x \in [a,c], \\ u(c)v(x), & x \in [c,b]. \end{cases}$$

Denote by $\mu(w_c)$ the eigenvalue of problem (2.1) with (2.2) for $q = w_c$. Then we have $\mu(w_c) = 1$. Let (w_n) be a sequence such that $w_n \xrightarrow{w} w_c$ and let $\mu(w_n)$ be one of the eigenvalues of (2.1) with (2.2) for $q = w_n$. Using an argument similar to the one used in [11], we can show that $\mu(w_n) \to \mu(w_c) = 1$. Take $\widetilde{w}_n = \mu(w_n)w_n$, then $\widetilde{w}_n$ satisfies the condition of Theorem 2.3 and $\int_a^b \widetilde{w}_n(x)\mathrm{d}x \to \int_a^b w_c(x)\mathrm{d}x = M$. This completes the proof. □

3. Proofs of Theorems 1.1 and 1.2

It suffices to prove the two theorems for the case $k_1 \le k_2$, because the case $k_1 \ge k_2$ can be transformed into the former one by the transformation $\widehat{y}(x) = y(1-x)$. To this end, we prepare some lemmas. First, let us calculate M in (2.7) with $q = -\lambda$ for $\lambda < \lambda_1(0)$, that is, we consider the problem

$$-y''(x) - \lambda y(x) = w(x)y(x), \quad y(0) - k_1y'(0) = 0,\ y(1) + k_2y'(1) = 0, \tag{3.1}$$

where $x \in [0,1]$ and $k_2 \ge k_1 \ge 0$.

Lemma 3.1. *Let $\lambda_1(0)$ be the first eigenvalue of problem* (1.1)-(1.3) *with $q = 0$. If* (3.1) *has a non-trivial solution for $\lambda \in (-\infty, \lambda_1(0))$, then*

$$
\begin{aligned}
&\int_0^1 w(x)\mathrm{d}x \\
&> \begin{cases} 2\sqrt{\lambda}\cot\frac{\sqrt{\lambda}}{2}(1+\theta_1+\theta_2), & \frac{-1}{k_2^2} < \lambda < \lambda_1(0),\ 0 \le \theta_2-\theta_1 < 1, \\ \sqrt{\lambda}\cot\sqrt{\lambda}(1+\theta_1)+\frac{1}{k_2}, & \lambda \le \frac{-1}{k_2^2} \text{ or } \theta_2-\theta_1 \ge 1 \end{cases} \\
&=: H_1(\lambda, k_1, k_2, \theta_1, \theta_2) =: H_1(\lambda, k_1, k_2),
\end{aligned}
\tag{3.2}
$$

where $\theta_j = \theta_j(\lambda, k_j) = \frac{1}{\sqrt{\lambda}}\arctan(k_j\sqrt{\lambda})$ is the principal values of the complex function, for $j = 1, 2$.

Proof. First, we show that the inequality

$$
\int_0^1 w(x)\mathrm{d}x > \frac{1}{\max_{x\in[0,1]} G(x,x)} =: M \tag{3.3}
$$

holds, where $G(x,t)$ is the Green's function of the problem

$$
-y'' - \lambda y = \mu w y, \quad y(0) - k_1 y'(0) = y(1) + k_2 y'(1) = 0 \tag{3.4}
$$

at $\mu = 0$. We need only show that $w(x)$ satisfies the assumption of Theorem 2.3. According to Rayleigh-Ritz principle, the signs of the eigenvalues for the problem (3.4) are the same as those signs for

$$
-y'' - \lambda y = \mu y, \quad y(0) - k_1 y'(0) = y(1) + k_2 y'(1) = 0. \tag{3.5}
$$

However, it is easy to see that the eigenvalues of (3.5) are positive for $\lambda \in (-\infty, \lambda_1(0))$. Thus inequality (3.3) holds by Theorem 2.3.

Now we give the expression of M in terms of λ and parameters k_1, k_2. Note that $\lambda_1(0)$ is nonnegative in case $k_1, k_2 \ge 0$ by a straightforward computation. For $\lambda \in (-\infty, \lambda_1(0))$, we divide the problem into three cases: $0 < \lambda \le \lambda_1(0)$, $\lambda < 0$ and $\lambda = 0$. Set $\rho = \sqrt{\lambda}$.

If $0 < \lambda \le \lambda_1(0)$. Let

$$
\begin{gathered}
u(x) = \sin\rho x + k_1\rho\cos\rho x = \sin\rho(x+\theta_1), \\
v(x) = \sin\rho(1-x) + k_2\rho\cos\rho(1-x) = \sin\rho(1-x+\theta_2),
\end{gathered}
\tag{3.6}
$$

where $\theta_j = (1/\rho)\arctan(k_j\rho)$ for $j = 1, 2$. Then $u(x)$ and $v(x)$ are linearly independent solutions of $-y'' - \lambda y = 0$ which satisfy the boundary condition at 0 and 1, respectively. A direct calculation yields

$$
\begin{aligned}
G(x,x) &= -\frac{1}{W}u(x)v(x) \\
&= \frac{\cos\rho(1+\theta_2-\theta_1-2x) - \cos\rho(1+\theta_1+\theta_2)}{2\rho\sin\rho(1+\theta_1+\theta_2)}, \quad x \in [0,1],
\end{aligned}
\tag{3.7}
$$

and

$$
\begin{aligned}
\int_0^1 w(x)\mathrm{d}x &> \frac{1}{\max_{x\in[0,1]} G(x,x)} \\
&= \begin{cases} 2\rho\cot\frac{\rho}{2}(1+\theta_1+\theta_2), & 0 \le \theta_2-\theta_1 \le 1, \\ \rho\cot\rho(1+\theta_1)+\frac{1}{k_2}, & \theta_2-\theta_1 > 1. \end{cases}
\end{aligned}
\tag{3.8}
$$

If $\lambda < 0$, the proof is analogous to the above case, except substituting $i\rho := i\sqrt{-\lambda}$ for ρ in (3.6). In such a case, $u(x)$ and $v(x)$ have the following forms

$$u(x) = \begin{cases} i\sinh\rho(x+\theta_1), & k_1\rho < 1, \\ ie^{\rho x}, & k_1\rho = 1, \\ i\cosh\rho(x+\theta_1), & k_1\rho > 1, \end{cases} \quad v(x) = \begin{cases} i\sinh\rho(1-x+\theta_2), & k_2\rho < 1, \\ ie^{\rho(1-x)}, & k_2\rho = 1, \\ i\cosh\rho(1-x+\theta_2), & k_1\rho < 1. \end{cases}$$

where $\theta_j := (1/\rho)\operatorname{artanh}(k_j\rho), \infty$, or $(1/\rho)\operatorname{arccoth}(k_j\rho)$ according to $k_j\rho < 1$, $k_j\rho = 1$, or $k_j\rho > 1$ for $j = 1, 2$, respectively. Then some tedious manipulation gives rise to

$$\int_0^1 w(x)\mathrm{d}x > \begin{cases} 2\rho\coth\frac{\rho}{2}(1+\theta_1+\theta_2), & k_1\rho \le k_2\rho < 1,\ 0 \le \theta_2-\theta_1 \le 1, \\ \rho\coth\rho(1+\theta_1) + \frac{1}{k_2}, & k_1\rho \le 1 \le k_2\rho \text{ or } \theta_2-\theta_1 > 1, \\ \rho\tanh\rho(1+\theta_1) + \frac{1}{k_2}, & k_2\rho \ge k_1\rho > 1. \end{cases} \tag{3.9}$$

If $\lambda = 0$, we choose

$$u(x) = x + k_1, \quad v(x) = 1 - x + k_2;$$

then a simple computation yields

$$\int_0^1 w(x)\mathrm{d}x > \begin{cases} \frac{4}{1+k_1+k_2}, & k_2 - k_1 \le 1, \\ \frac{1}{1+k_1} + \frac{1}{k_2}, & k_2 - k_1 > 1. \end{cases} \tag{3.10}$$

This result is exactly the limits of the other two cases as $\lambda \to 0$. Therefore, we obtain the expression of M for all three cases.

To simplify the above formulas, we use complex functions to express (3.8), (3.9) and (3.10). Let θ_j be the principle values of $\theta_j(\rho) = (1/\rho)\arctan(k_j\rho)$ for $\rho = \sqrt{\lambda}$ and $j = 1, 2$. Then we have

$$\begin{aligned} \int_0^1 w(x)\mathrm{d}x &> \begin{cases} 2\sqrt{\lambda}\cot\frac{\sqrt{\lambda}}{2}(1+\theta_1+\theta_2), & \frac{-1}{k_2^2} < \lambda < \lambda_1(0),\ 0 \le \theta_2-\theta_1 < 1, \\ \sqrt{\lambda}\cot\sqrt{\lambda}(1+\theta_1) + \frac{1}{k_2}, & \lambda \le \frac{-1}{k_2^2} \text{ or } \theta_2-\theta_1 \ge 1 \end{cases} \\ &= H_1(\lambda, k_1, k_2). \end{aligned}$$

The proof is complete. □

Next we set $\Omega_n(\lambda) = \{q : q \in L^1[0,1],\ \lambda_n(q) = \lambda\}$ in (1.4), and $E_n(\lambda) = \inf\{\|q\| : q \in \Omega_n(\lambda)\}$ in (1.5). Let $\lambda_n(0)$ be the n-th eigenvalue of (1.1)-(1.3) with $q = 0$. To set up a formula for $E_n(\lambda)$, we calculate it separately in two smaller sets

$$\begin{aligned} \Omega_n^- &= \{q : q \in \Omega_n(\lambda),\ q(x) \le 0 \text{ a.e. on } [0,1]\}, \quad \text{for } \lambda < \lambda_n(0), \\ \Omega_n^+ &= \{q : q \in \Omega_n(\lambda), q(x) \ge 0 \text{ a.e. on } [0,1]\}, \quad \text{for } \lambda \ge \lambda_n(0). \end{aligned}$$

Lemma 3.2. *For $n \ge 1$, we have*

$$E_n(\lambda) = \inf\{\|q\| : q \in \Omega_n^-\}, \quad \textit{for } \lambda < \lambda_n(0), \tag{3.11}$$
$$E_n(\lambda) = \inf\{\|q\| : q \in \Omega_n^+\}, \quad \textit{for } \lambda \ge \lambda_n(0). \tag{3.12}$$

The proof of the above lemma can be obtained as in [13, Lemma 3.2]; we omit it here. With the help of the Lemma 3.2, we need only to calculate $E_n(\lambda)$ in Ω_n^- for $\lambda < \lambda_n(0)$ and that one in Ω_n^+ for $\lambda \ge \lambda_n(0)$ in the following proofs.

Proof of Theorem 1.1. Our first goal is to give the formula of $E_1(\lambda)$ for $\lambda < \lambda_1(0)$, that is, we will evaluate $E_1(\lambda)$ in Ω_1^-. If $q \in \Omega_1^-$ then $q(x) \le 0$ and $\lambda_1(q) = \lambda < \lambda_1(0)$. We rewrite problem (1.1)-(1.3) as

$$-y'' - \lambda_1(q)y = wy, \quad y(0) - k_1y'(0) = y(1) + k_2y'(1) = 0, \tag{3.13}$$

where $\lambda_1(q) = \lambda$ and $w(x) = -q(x) \ge 0$. Then problem (3.13) has non-trivial solutions, it follows from Lemma 3.1 that

$$\begin{aligned} \|q\| &= \int_0^1 w(x)\mathrm{d}x \\ &> \begin{cases} 2\sqrt{\lambda}\cot\frac{\sqrt{\lambda}}{2}(1+\theta_1+\theta_2), & \frac{-1}{k_2^2} < \lambda < \lambda_1(0),\ 0 \le \theta_2 - \theta_1 < 1, \\ \sqrt{\lambda}\cot\sqrt{\lambda}(1+\theta_1) + \frac{1}{k_2}, & \lambda \le \frac{-1}{k_2^2} \text{ or } \theta_2 - \theta_1 \ge 1 \end{cases} \\ &= H_1(\lambda, k_1, k_2, \theta_1, \theta_2) = H_1(\lambda, k_1, k_2). \end{aligned} \tag{3.14}$$

Since the above inequality is sharp by Theorem 2.3, we have $E_1(\lambda) = H_1(\lambda, k_1, k_2)$ for $\lambda < \lambda_1(0)$.

Next, we give the formula of $E_1(\lambda)$ for $\lambda \ge \lambda_1(0)$. Let $\rho = \sqrt{\lambda}$ and $\theta_j = (1/\rho)\arctan(k_j\rho)$ for $j = 1, 2$. It follows the proof of Lemma 3.1 that $\lambda_1(0)$ is non-negative. This together with $\lambda \ge \lambda_1(0)$ yields $0 < \pi/2\rho - \theta_j < 1$ for $j = 1, 2$. Set

$$q_0(x) = \begin{cases} \lambda, & x \in [\frac{\pi}{2\rho} - \theta_1, 1 - \frac{\pi}{2\rho} + \theta_2], \\ 0, & \text{otherwise}. \end{cases}$$

Then, it is easy to check that $q_0 \in \Omega_1^+$. We claim that

$$\begin{aligned} \|q\| \ge \|q_0\| &= \rho\big[\rho(1+\theta_1+\theta_2) - \pi\big] \\ &= \sqrt{\lambda}\Big[\sqrt{\lambda}\big(1+\theta_1+\theta_2\big) - \pi\Big] =: K_1(\lambda, k_1, k_2), \quad \forall q \in \Omega_1^+. \end{aligned} \tag{3.15}$$

Suppose on the contrary that there exists $q \in \Omega_1^+$ such that $\|q_0\| > \|q\|$. Let $u(x)$ be the corresponding eigenfunction of

$$-y'' + q_0y = \lambda y, \quad y(0) - k_1y'(0) = y(1) + k_2y'(1) = 0.$$

In view of the definition of q_0 we can choose $u(x)$ as

$$u(x) = \begin{cases} \sin\rho(x+\theta_1), & x \in [0, \frac{\pi}{2\rho} - \theta_1), \\ 1, & x \in [\frac{\pi}{2\rho} - \theta_1, 1 - \frac{\pi}{2\rho} + \theta_2], \\ \sin\rho(1-x+\theta_2), & x \in (1 - \frac{\pi}{2\rho} + \theta_2, 1] \end{cases}$$

for which $\max\{u^2(x) : x \in [0,1]\} = 1$ and

$$\int_0^1 q_0u^2 = \int_{\frac{\pi}{2\rho}-\theta_1}^{1-\frac{\pi}{2\rho}+\theta_2} q_0 = \|q_0\| > \|q\| = \int_0^1 q \cdot 1 \ge \int_0^1 qu^2. \tag{3.16}$$

Combing the inequality (3.16) and the Rayleigh-Ritz principle, we obtain

$$\begin{aligned} \lambda = \lambda_1(q_0) &= \frac{1}{\int_0^1 |u|^2}\Big(\int_0^1 (u'^2 + q_0u^2) + k_1(u'(0))^2 + k_2(u'(1))^2\Big) \\ &> \frac{1}{\int_0^1 |u|^2}\Big(\int_0^1 (u'^2 + qu^2) + k_1(u'(0))^2 + k_2(u'(1))^2\Big) \\ &\ge \lambda_1(q) = \lambda, \end{aligned}$$

which is a contradiction. This proves $E_1(\lambda) = K_1(\lambda, k_1, k_2)$ for $\lambda \ge \lambda_1(0)$. the proof is complete. □

Proof of Theorem 1.2. Let $\lambda_n(q) = \lambda$ and $v_n(x)$ be the corresponding eigenfunction. Then $v_n(x)$ has exactly $n-1$ zeros t_j in $(0,1)$, $0 < t_1 < t_2 \cdots < t_{n-1} < 1$. Denote $t_0 = 0$ and $t_n = 1$, then $\lambda_n(q)$ is the first eigenvalue of the following problems:

$$\begin{gathered}
-y'' + qy = \mu y, \quad y(t_0) - k_1 y'(t_0) = y(t_1) = 0; \\
-y'' + qy = \mu y, \quad y(t_{j-1}) = y(t_j) = 0, \quad 2 \le j \le n-1; \\
-y'' + qy = \mu y, \quad y(t_{n-1}) = y(t_n) + k_2 y'(t_n) = 0.
\end{gathered}$$

Let $\delta_j = t_j - t_{j-1}$ and $\lambda_1^{\delta_j}(0)$ be the first eigenvalues for the above problems with $q = 0$ for $j = 1, 2, \dots, n$. For $q \in \Omega_n^-$, it follows from $q \le 0$ that

$$\lambda \le \lambda_1^{\delta_j}(0), \quad j = 1, 2, \dots, n.$$

Denote by θ_j the principle values of $\theta_j(\lambda) = \frac{1}{\sqrt{\lambda}} \arctan(k_j \sqrt{\lambda})$ for $\lambda \in \mathbb{R}$ and $j = 1, 2$. Then, by Theorem 1.1 and a variable transformation, we have

$$\begin{gathered}
\int_{t_0}^{t_1} |q(x)| \mathrm{d}x \ge H_1(\delta_1^2 \lambda, 0, k_1/\delta_1, 0, \theta_1/\delta_1)/\delta_1; \\
\int_{t_{j-1}}^{t_j} |q(x)| \mathrm{d}x \ge H_1(\delta_j^2 \lambda, 0, 0, 0, 0)/\delta_j, \quad 2 \le j \le n-1; \\
\int_{t_{n-1}}^{t_n} |q(x)| \mathrm{d}x \ge H_1(\delta_n^2 \lambda, 0, k_2/\delta_n, 0, \theta_2/\delta_n)/\delta_n.
\end{gathered} \tag{3.17}$$

Summing up the above inequalities from $j = 1$ to $j = n$. From (3.8) and (3.9), we see that the functions on the right-hand sides of the inequalities in (3.17) are either cot functions or coth functions. Hence, due to the convexity of these functions, we have

$$\begin{aligned}
&\int_0^1 |q(x)| \mathrm{d}x \\
&> \begin{cases}
2n\sqrt{\lambda} \cot \frac{\sqrt{\lambda}}{2n}(1 + \theta_1 + \theta_2), & \frac{-1}{k_2^2} < \lambda < \lambda_n(0),\ \theta_1 < \frac{1}{2(n-1)}, \theta_2 < \frac{1+\theta_1}{2n-1}, \\
(2n-1)\sqrt{\lambda} \cot \frac{\sqrt{\lambda}}{2n-1}(1 + \theta_1) + \frac{1}{k_2}, & \frac{-1}{k_1^2} < \lambda \le \frac{-1}{k_2^2},\ \theta_1 < \frac{1}{2(n-1)} \text{ or } \theta_2 \ge \frac{1+\theta_1}{2n-1}, \\
2(n-1)\sqrt{\lambda} \cot \frac{\sqrt{\lambda}}{2(n-1)} + \frac{1}{k_1} + \frac{1}{k_2}, & \lambda \le \frac{-1}{k_1^2} \text{ or } \theta_1 \ge \frac{1}{2(n-1)}
\end{cases} \\
&=: H_n(\lambda, k_1, k_2).
\end{aligned} \tag{3.18}$$

Then, we can employ the same argument as used in the case $n = 1$ to prove that $H_n(\lambda, k_1, k_2)$ is the infimum of $\|q\|$ in Ω_n^-.

Similarly, for $q \in \Omega_n^+$, we obtain

$$\|q\| \ge \sqrt{\lambda}\big[\sqrt{\lambda}\big(1 + \theta_1 + \theta_2\big) - n\pi\big] =: K_n(\lambda, k_1, k_2). \tag{3.19}$$

Finally, $K_n(\lambda)$ is the infimum since it is attained at $\hat{q} \in \Omega_n(\lambda)$ defined by

$$\widehat{q}(x) = \begin{cases}
0, & x \in [0, \frac{\pi}{2\rho} - \theta_1) \cup (\frac{1}{n} - \frac{\pi}{2\rho} + \theta_2, \frac{1}{n}], \\
\lambda, & x \in [\frac{\pi}{2\rho} - \theta_1, \frac{1}{n} - \frac{\pi}{2\rho} + \theta_2]
\end{cases}$$

on $[0,1/n]$, and $\widehat{q}(x+\frac{1}{n})=\widehat{q}(x)$ for $x\in[0,\frac{n-1}{n}]$. Therefore, $E_n(\lambda)=K_n(\lambda)$ for $\lambda\geq\lambda_n(0)$. The proof is complete. $\square$

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Hongjie Guo
Department of Mathematics, Shandong University at Weihai, Weihai 264209, China
E-mail address: ghj790629@163.com

Jiangang Qi (corresponding author)
Department of Mathematics, Shandong University at Weihai, Weihai 264209, China
E-mail address: qijiangang@sdu.edu.cn