EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR A NONLINEAR PSEUDOPARABOLIC EQUATION

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Communicated by Dung Le

Abstract. This article concerns the initial-boundary value problem for nonlinear pseudo-parabolic equation
\[ u_t - u_{xxt} - (1 + \mu(u_x))u_{xx} + (1 + \sigma(u_x))u = f(x, t), \quad 0 < x < 1, \quad 0 < t < T, \]
\[ u(0, t) = u(1, t) = 0, \]
\[ u(x, 0) = \tilde{u}_0(x), \]
where \( f, \tilde{u}_0, \mu, \sigma \) are given functions. Using the Faedo-Galerkin method and the compactness method, we prove that there exists a unique weak solution \( u \) such that \( u \in L^\infty(0, T; H^1_0 \cap H^2), u' \in L^2(0, T; H^1_0) \) and \( \|u\|_{L^\infty(Q_T)} \leq \max\{\|\tilde{u}_0\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(Q_T)}\} \). Also we prove that the problem has a unique global solution with \( H^1 \)-norm decaying exponentially as \( t \to +\infty \). Finally, we establish the existence and uniqueness of a weak solution of the problem associated with a periodic condition.

1. Introduction

Consider the following initial-boundary value problem for the pseudo-parabolic equation arising in third-grade fluid flows
\[ u_t - (1 + \mu(u_x))u_{xx} - \alpha u_{xxt} + (\gamma + \beta \sigma(u_x))u = f(x, t), \quad 0 < x < 1, \quad 0 < t < T, \quad (1.1) \]
with the boundary conditions
\[ u(0, t) = u(1, t) = 0, \quad (1.2) \]
and with the initial condition
\[ u(x, 0) = \tilde{u}_0(x), \quad (1.3) \]
or the \( T \)-periodic condition
\[ u(x, 0) = u(x, T), \quad (1.4) \]
where \( \alpha > 0, \beta > 0, \gamma > 0 \) are given constants and \( f, \tilde{u}_0, \mu, \sigma \) are given functions satisfying conditions specified later.
The pseudo-parabolic equation

\[ u_t - u_{xxt} = F(x, t, u_t, u_x), \quad 0 < x < 1, \quad t > 0 \]  

(1.5)

with the initial condition \( u(x, 0) = \tilde{u}_0(x) \) and with the different boundary conditions, has been extensively studied by many authors, see for example [2], [3], [6], [10], [14] among others and the references given therein. In these works, many results about existence, regularity, asymptotic behavior, and decay of solutions were obtained.

Equations of type (1.5) with a one time derivative appearing in the highest order term are called pseudo-parabolic or Sobolev equations, and arise in many areas of mathematics and physics. We refer to the monographs of Al’shin [1], and of Carroll [7] for references and results on pseudoparabolic or Sobolev type equations. Mathematical study of pseudo-parabolic equations goes back to works of Showalter in the seventies, since then, numerous of interesting results about linear and nonlinear pseudo-parabolic equations have been obtained. We also refer to [12] for asymptotic behavior and to [13] for nonlinear problems.

An important special case of the model is the Benjamin-Bona-Mahony-Burgers (BBMB) equation

\[ u_t + u_x + uu_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \]  

(1.6)

it was studied by Amick et al in [2], where \( \nu > 0, \alpha = 1, x \in \mathbb{R}, t \geq 0. \) The authors proved that solution of (1.6) with initial data in \( L^1 \cap H^2 \) decays to zero in \( L^2 \) norm as \( t \to +\infty. \) With \( \nu > 0, \alpha > 0, x \in [0,1], t \geq 0, \) the model has the form (1.6) was also investigated earlier by Bona and Dougalis in [6], where uniqueness, global existence and continuous dependence of solutions on initial and boundary data were established and the solutions were shown to depend continuously on \( \nu \geq 0 \) and on \( \alpha > 0. \)

The Benjamin-Bona-Mahony (BBM) equation is introduced in [5], in 1972, as a model for describing long - wave behavior. Since then, the periodic boundary value problems, the initial value problems and the initial boundary value problems, for various generalized BBM equations have been studied. On the other hand, many people have studied the large time behaviors of solutions to the initial value problems for various generalized BBM equations, such as BBMB equations with a Burgers-type dissipative term appended, see [14]. Medeiros and Miranda [10] studied another special case, namely

\[ u_t + f(u)_x - u_{xxt} = g(x, t), \]  

(1.7)

where \( u = u(x,t), 0 < x < 1, \) and \( t \geq 0 \) is the time. They proved existence, uniqueness of solutions for \( f \) in \( C^1 \) and regularity in the case \( f(s) = s^2/2. \) Arnold et al. [3] considered the following equation from the point of view of periodic solutions

\[ - (au_{xt})_x + cu_t = -(\alpha u_x)_x + \beta u_x + \gamma, \quad x \in \mathbb{R}, \quad t \in [0, T]. \]  

(1.8)

Here, the authors proved the existence, uniqueness and regularity of solutions under the hypothesis that \( \alpha, \beta \) and \( \gamma \) are \( C^1 \)-functions of \( x, t \) and \( u, \) and that they are bounded together with their first derivatives.

It is well known that equation (1.1) arises within frameworks of mathematical models in engineering and physical sciences on third-grade fluid flows, see [4, 8, 11]
and references therein. For example, the following equation of motion for the unsteady flow of third-grade fluid over the rigid plate with porous medium is investigated

\[
\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 \frac{\partial^3 u}{\partial y^2 \partial t} + 6\beta_3 \frac{\partial^2 u}{\partial y^2} \frac{\partial \phi}{k} \left[ \mu + \frac{\partial u}{\partial t} + 2\beta_3 \left( \frac{\partial u}{\partial y} \right)^2 \right] u, \quad (1.9)
\]

for \( y > 0, t > 0 \), where \( u \) is the velocity component, \( \rho \) is the density, \( \mu \) the coefficient of viscosity, \( \alpha_1 \) and \( \beta_3 \) are the material constants, see [4].

Motivated by the above mentioned works, because of mathematical context, we study of the existence, uniqueness and exponential decay of solutions for Dirichlet problem (1.1)-(1.3) or (1.4). This article is organized as follows. In section 2, under appropriate conditions of \( \alpha, \beta, \gamma, f, u_0, \mu, \sigma \) we prove the existence of a unique solution on \((0, T)\) for every \( T > 0 \) and the boundedness of the solution. In section 3, we study exponential decay of solutions. In section 4, we prove the existence and uniqueness of a \( T \)-periodic weak solution.

2. Preliminaries

Without loss of generality, we consider Problem (1.1)–(1.3) with \( \alpha = \beta = \gamma = 1 \).

We put \( \Omega = (0, 1) \) and denote the usual function spaces used in this paper by the notations \( L^p = L^p(\Omega), H^m = H^m(\Omega) \). Let \( \langle \cdot, \cdot \rangle \) be either the scalar product in \( L^2 \) or the dual pairing of a continuous linear functional and an element of a function space. The notation \( \| \cdot \| \) stands for the norm in \( L^2 \) and we denote by \( \| \cdot \|_X \) the norm in the Banach space \( X \). We call \( X' \) the dual space of \( X \).

We denote by \( L^p(0, T; X), 1 \leq p \leq \infty \) for the Banach space of real functions \( u : (0, T) \to X \) measurable, such that

\[
\| u \|_{L^p(0, T; X)} = \left( \int_0^T \| u(t) \|_X^p \, dt \right)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty, \\
\| u \|_{L^\infty(0, T; X)} = \operatorname{ess sup}_{0 < t < T} \| u(t) \|_X \quad \text{for } p = \infty.
\]

On \( H^1 \), we shall use the norm

\[
\| v \|_{H^1} = (\| v \|^2 + \| v_x \|^2)^{1/2}.
\]

The following lemma is well known.

**Lemma 2.1.** The imbedding \( H^1 \hookrightarrow C^0(\Omega) \) is compact and

\[
\| v \|_{C^0(\Omega)} \leq \sqrt{2} \| v \|_{H^1} \quad \text{for all } v \in H^1.
\]

**Remark 2.2.** On \( H^1_0 \), \( \| v \|_{H^1} \) and \( \| v_x \| \) are equivalent norms. Furthermore,

\[
\| v \|_{C^0(\Omega)} \leq \| v_x \| \quad \text{for all } v \in H^1_0.
\]

3. Existence and uniqueness theorem

Without losing of generality, we consider problem (1.1)–(1.3) with \( \alpha = \beta = \gamma = 1 \).

\[
u_t - \nu_{xxx} - \frac{\partial}{\partial x} \left( u_x + \bar{u}(u_x) \right) + (1 + \sigma(u_x))u = f(x, t), \quad 0 < x < 1, \ 0 < t < T, \\
u(0, t) = u(1, t) = 0, \\
u(x, 0) = \tilde{u}_0(x),
\]

(3.1)
where $\bar{\mu}(y) = \int_0^y \mu(z)dz$, $y \in \mathbb{R}$.

The weak formulation of (3.1) can be given in the following manner: Find $u(t)$ defined in the open set $(0, T)$ such that $u(t)$ satisfies the variational problem

$$\langle u_t(t), w \rangle + \langle u_{xt}(t), w_x \rangle + \langle u_x(t) + \bar{\mu}(u_x(t)), w_x \rangle + \langle (1 + \sigma(u_x(t)))u(t), w \rangle = \langle f(t), w \rangle,$$

for all $w \in H^1_0$ and the initial condition

$$u(0) = \tilde{u}_0.$$  \hfill (3.3)

We make the following assumptions:

(H1) $\tilde{u}_0 \in H^1_0 \cap H^2$;

(H2) $f \in L^2(0, T; H^1_0)$;

(H3) $\mu \in C^0(\mathbb{R}; \mathbb{R})$ such that $\mu(0) = 0$, $\mu(z) > 0$, for all $z \in \mathbb{R}$, $z \neq 0$;

(H4) $\sigma \in C^1(\mathbb{R}; \mathbb{R})$ such that

1. $\sigma(0) = 0$, $\sigma(z) > 0$, $z\sigma'(z) > 0$, for all $z \in \mathbb{R}$, $z \neq 0$,

2. $y(\int_0^y z\sigma'(z)dz) \leq y^2\sigma(y)$ for all $y \in \mathbb{R}$.

An example of the function $\sigma$ satisfying (H4) is

$$\sigma(z) = |z|^q,$$

where $q > 1$ is a constant. It is obvious that (H4) holds, because

$$\sigma(z) = |z|^q, \quad \sigma'(z) = q|z|^{q-2}z,$$

$$\sigma(0) = 0, \quad \sigma(z) > 0, \quad z\sigma'(z) = q|z|^q > 0, \quad \forall z \in \mathbb{R}, \quad z \neq 0,$$

$$y\left(\int_0^y z\sigma'(z)dz\right) = qy\left(\int_0^y |z|^qdz\right) = qy\frac{|y|^qy}{q+1} = \frac{q}{q+1}|y|^{q+2} = \frac{q}{q+1}y^2\sigma(y) \leq y^2\sigma(y).$$

**Theorem 3.1.** Let $T > 0$ and (H1)–(H4) hold. Then, problem (3.1) has a unique weak solution $u$ satisfying

$$u \in L^\infty(0, T; H^1_0 \cap H^2), \quad u' \in L^2(0, T; H^1_0).$$  \hfill (3.4)

Furthermore, we have the estimate

$$\|u\|_{L^\infty(Q_T)} \leq \max\{\|\tilde{u}_0\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(Q_T)}\}. $$  \hfill (3.5)

Estimate (3.5) appears naturally, both physically and mathematically, from the maximum principle in the study of partial differential equation of the kind of (3.1).

**Proof.** The proof consists of several steps.

**Step 1:** The Faedo-Galerkin approximation (introduced by Lions [9]). Consider a special orthonormal basis $\{w_j\}$ on $H^1_0$:

$$w_j(x) = \sqrt{2}\sin(j\pi x), \quad j \in \mathbb{N}, \quad \text{formed by the eigenfunctions of the Laplacian } \Delta = -\frac{\partial^2}{\partial x^2}:

-\Delta w_j = \lambda_j w_j, \quad w_j \in C^\infty([0, 1]), \quad \lambda_j = (j\pi)^2, \quad j = 1, 2, \ldots

Put

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t)w_j,$$  \hfill (3.6)
where the coefficients \( c_{mj}(t) \) satisfy a system of nonlinear differential equations

\[
\langle u_m'(t), w_j \rangle + \langle u_{mx}'(t), w_{jx} \rangle + \langle u_{mx}(t) + \bar{\mu}(u_{mx}(t)), w_{jx} \rangle + \langle (1 + \sigma(u_{mx}(t)))u_m(t), w_j \rangle = \langle f(t), w_j \rangle, \quad 1 \leq j \leq m, \tag{3.7}
\]

in which

\[
u_0m = \sum_{j=1}^{m} \beta_{mj}w_j \to \tilde{u}_0 \quad \text{strongly in } H^1_0 \cap H^2. \tag{3.8}
\]

System (3.7) can be rewritten in the form

\[
c_{mi}'(t) + c_{mi}(t) + \frac{1}{1 + \lambda_i} \left( \langle \bar{\mu}(u_{mx}(t)), w_{ix} \rangle + \langle \sigma(u_{mx}(t))u_m(t), w_i \rangle \right) = \frac{1}{1 + \lambda_i} \langle f(t), w_i \rangle, \tag{3.9}
\]

\[
c_{mi}(0) = \beta_{mi}, \quad 1 \leq i \leq m.
\]

It is clear that for each \( m \) there exists a solution \( u_m(t) \) in form (3.6) which satisfies (3.7) almost everywhere on \( 0 \leq t \leq T_m \) for some \( T_m, 0 < T_m \leq T \). The following estimates allow us to take \( T_m = T \) for all \( m \).

**Step 2: A priori estimates.**

(a) First estimate. Multiplying the \( j \)th equation of (3.7) by \( c_{mj}(t) \) and summing up with respect to \( j \), afterwards, integrating with respect to the time variable from \( 0 \) to \( t \), we obtain after some rearrangements

\[
S_m(t) = S_m(0) + 2 \int_0^t \langle f(s), u_m(s) \rangle ds, \tag{3.10}
\]

where

\[
S_m(t) = \|u_m(t)\|_{H^1}^2 + 2 \int_0^t \|u_m(s)\|_{H^1}^2 ds \\
+ 2 \int_0^t \langle \mu(u_{mx}(s)), u_{mx}(s) \rangle ds + 2 \int_0^t \langle \sigma(u_{mx}(s)), u_{mx}(s) \rangle ds. \tag{3.11}
\]

By \( u_{0m} \to \tilde{u}_0 \) strongly in \( H^1_0 \cap H^2 \), we deduce

\[
S_m(0) = \|u_{0m}\|_{H^1}^2 \leq \bar{S}_0 \quad \forall m \in \mathbb{N}, \tag{3.12}
\]

where \( \bar{S}_0 \) always indicates a constant depending on \( \tilde{u}_0 \).

Note that

\[
y\bar{\mu}(y) = y \int_0^y \mu(z) dz \geq 0, \quad \forall y \in \mathbb{R}.
\]

On the other hand, we have

\[
2 \int_0^t \langle f(s), u_m(s) \rangle ds \leq \int_0^t \|f(s)\|^2 ds + \int_0^t \|u_m(s)\|^2 ds \\
\leq \int_0^T \|f(s)\|^2 ds + \frac{1}{2} S_m(t). \tag{3.13}
\]

Therefore,

\[
S_m(t) \leq 2\bar{S}_0 + 2 \int_0^T \|f(s)\|^2 ds \leq C^1_T. \tag{3.14}
\]
(b) Second estimate. Next, by replacing $w_j$ in (3.7) by $-w_{jjx}$, we obtain that
\[
\langle u_{mxx}(t), w_{jjx} \rangle + \langle \Delta u_{m}(t), \Delta w_{j} \rangle + \langle \Delta u_{m}(t), \Delta w_{j} \rangle \\
+ \langle u_{mx}(t), w_{jjx} \rangle + \langle \mu(u_{mx}(t))\Delta u_{m}(t), \Delta w_{j} \rangle \\
+ \langle \sigma'(u_{mx}(t))u_{m}(t)\Delta u_{m}(t) + \sigma(u_{mx}(t))u_{mx}(t), w_{jjx} \rangle \\
= (f_{x}(t), w_{jjx}), \quad 1 \leq j \leq m.
\]
Similar to (3.7), we have
\[
P_{m}(t) = P_{m}(0) - 2 \int_{0}^{t} \left[ \langle \sigma'(u_{mx}(s))u_{m}(s)\Delta u_{m}(s), u_{mx}(s) \rangle \\
+ \langle \sigma(u_{mx}(s)), |u_{mx}(s)|^{2} \rangle \right] ds + 2 \int_{0}^{t} (f_{x}(s), u_{mx}(s)) ds \\
= P_{m}(0) + I_{1} + I_{2},
\]
where
\[
P_{m}(t) = \|u_{mx}(t)\|^{2} + \|\Delta u_{m}(t)\|^{2} + 2 \int_{0}^{t} (\|u_{mx}(s)\|^{2} + \|\Delta u_{m}(s)\|^{2}) ds \\
+ 2 \int_{0}^{t} (\mu(u_{mx}(s)), |\Delta u_{m}(s)|^{2}) ds.
\]
From $u_{0m} \to \bar{u}_{0}$ strongly in $H_{1}^{1} \cap H^{2}$, we deduce
\[
P_{m}(0) = \|u_{mx}(0)\|^{2} + \|\Delta u_{m}(0)\|^{2} = \|u_{0mx}\|^{2} + \|\Delta u_{0m}\|^{2} \leq \tilde{P}_{0} \quad \forall m \in \mathbb{N},
\]
where $\tilde{P}_{0}$ always indicates a constant depending on $\bar{u}_{0}$.

**Estimating $I_{1}$**. Note that
\[
-2 \langle \sigma'(u_{mx}(s))u_{mx}(s)\Delta u_{m}(s), u_{m}(s) \rangle \\
= -2 \int_{0}^{1} \sigma'(u_{mx}(x, s))u_{mx}(x, s)\Delta u_{m}(x, s)u_{m}(x, s) dx \\
= -2 \int_{0}^{1} u_{m}(x, s) \frac{\partial}{\partial x} \left( \int_{0}^{u_{mx}(x, s)} z\sigma'(z)dz \right) dx \\
= -2 \left[ u_{m}(x, s) \left( \int_{0}^{u_{mx}(x, s)} z\sigma'(z)dz \right) \right]^{1}_{0} \\
- \int_{0}^{1} u_{m}(x, s) \left( \int_{0}^{u_{mx}(x, s)} z\sigma'(z)dz \right) dx \\
= 2 \int_{0}^{1} u_{mx}(x, s) \left( \int_{0}^{u_{mx}(x, s)} z\sigma'(z)dz \right) dx \\
\leq 2 \int_{0}^{1} u_{mx}^{2}(x, s) \sigma(u_{mx}(x, s)) dx \\
= 2 \sigma(u_{mx}(s)), |u_{mx}(s)|^{2},
\]
since $y(\int_{0}^{y} z\sigma'(z)dz) \leq y^{2}\sigma(y)$ for all $y \in \mathbb{R}$. Hence
\[
I_{1} = -2 \int_{0}^{t} \left[ \langle \sigma'(u_{mx}(s))u_{m}(s)\Delta u_{m}(s), u_{mx}(s) \rangle \\
+ \langle \sigma(u_{mx}(s)), |u_{mx}(s)|^{2} \rangle \right] ds \leq 0.
\]
Estimating $I_2$.

$$
I_2 = 2 \int_0^t (f_x(s), u_{mx}(s)) ds \leq \int_0^T \|f_x(s)\| \|u_{mx}(s)\| ds
$$
\begin{equation}
\leq \int_0^T \|f_x(s)\| \sqrt{S_m(s)} ds \leq \sqrt{C_T^{(1)}} \int_0^T \|f_x(s)\| ds.
\end{equation}

(3.21)

It follows from (3.16), (3.18), (3.20), (3.21) that

$$
P_m(t) \leq \bar{P}_0 + \sqrt{C_T^{(1)}} \int_0^T \|f_x(s)\| ds \leq C_T^{(2)}.
$$

(3.22)

(c) Third estimate. Multiplying the $j^{th}$ equation of (3.7) by $\sigma_j(t)$ and summing up with respect to $j$, afterwards, integrating with respect to the time variable from 0 to $t$, we obtain after some rearrangements

$$
Q_m(t) = Q_m(0) - 2 \int_0^t (\sigma(u_{mx}(s))u_m(s), u_m'(s)) ds + 2 \int_0^t (f(s), u_m'(s)) ds
$$
\begin{equation}
= Q_m(0) + J_1 + J_2,
\end{equation}

(3.23)

where

$$
Q_m(t) = \|u_m(t)\|_{H^1}^2 + 2 \int_0^t \|u_m'(s)\|_{H^1}^2 ds + 2 \int_0^1 \bar{\mu}(u_{mx}(x,t)) dx,
$$
\begin{equation}
\bar{\mu}(z) = \int_0^z \bar{\mu}(y) dy \geq 0 \quad \forall z \in \mathbb{R}.
\end{equation}

(3.24)

Estimating $Q_m(0)$. From $u_{0m} \to \bar{u}_0$ strongly in $H_0^1 \cap H^2$, we can deduce the existence of a constant $Q_0 > 0$ independent of $m$ such that

$$
Q_m(0) = \|u_{0m}\|_{H^1}^2 + 2 \int_0^1 \tilde{\mu}(u_{mx}(x)) dx \leq \tilde{Q}_0 \quad \forall m \in \mathbb{N}.
$$

(3.25)

Estimating $J_1$. By (3.22), we have

$$
\|u_{mx}(x,s)\| \leq \|u_{mx}(s)\|_{C^0([0,1])} \leq \sqrt{2} \|u_{mx}(s)\|_{H^1}
$$
\begin{equation}
\leq \sqrt{2} \sqrt{\|u_{mx}(s)\|^2 + \|\Delta u_m(s)\|^2} \leq \sqrt{2} \sqrt{2} \|\Delta u_m(s)\|^2
\end{equation}
\begin{equation}
\leq 2 \|\Delta u_m(s)\| \leq 2 \sqrt{P_m(s)} \leq 2 \sqrt{C_T^{(2)}}.
\end{equation}
Hence

$$J_1 = -2 \int_0^t (\sigma(u_{mx}(s))u_m(s), u'_m(s))ds$$

$$\leq 2 \sup_{|z| \leq 2\sqrt{C_T^{(2)}}} \sigma(z) \int_0^t \|u_m(s)\|\|u'_m(s)\|ds$$

$$\leq 2 \sup_{|z| \leq 2\sqrt{C_T^{(2)}}} \sigma(z) \int_0^t \|u_m(s)\|\|u'_m(s)\|ds$$

$$\leq 2\sqrt{C_T^{(1)}} \sup_{|z| \leq 2\sqrt{C_T^{(2)}}} \sigma(z) \int_0^t \|u'_m(s)\|ds$$

$$\leq 2TC_T^{(1)} \sup_{|z| \leq 2\sqrt{C_T^{(2)}}} \sigma(z) + \frac{1}{2} \int_0^t \|u_m(s)\|^2ds$$

$$\leq 2TC_T^{(1)} \sup_{|z| \leq 2\sqrt{C_T^{(2)}}} \sigma(z) + \frac{1}{4} Q_m(t).$$

**Estimating J_2.**

$$J_2 = 2 \int_0^t (f(s), u'_m(s))ds$$

$$\leq 2 \int_0^T \|f(s)\|^2ds + \frac{1}{2} \int_0^T \|u'_m(s)\|^2ds$$

$$\leq 2 \int_0^T \|f(s)\|^2ds + \frac{1}{4} Q_m(t).$$

Then, it follows from (3.23), (3.25)–(3.27) that

$$Q_m(t) \leq 2 \left( \bar{Q}_0 + 2TC_T^{(1)} \sup_{|z| \leq 2\sqrt{C_T^{(2)}}} \sigma(z) + \frac{1}{4} \int_0^T \|f(s)\|^2ds \right) \leq C_T^{(3)}. \quad (3.28)$$

**Step 3: Limiting process.** Thanks to (3.14), (3.22), (3.28) there exists a subsequence of \( \{u_m\} \), still denoted by \( \{u_m\} \) such that

$$u_m \rightharpoonup u \text{ in } L^\infty(0,T; H^1_0 \cap H^2) \text{ weakly},$$

$$u'_m \rightharpoonup u' \text{ in } L^2(0,T; H^1_0) \text{ weakly.} \quad (3.29)$$

Using the compactness lemma of Lions [9, p.57], and applying Fischer-Riesz theorem, from (3.29), there exists a subsequence of \( \{u_m\} \), denoted by the same symbol satisfying

$$u_m \rightharpoonup u \text{ strongly in } L^2(0,T; H^1_0) \text{ and a.e. in } Q_T,$$

$$u_{mx} \rightharpoonup u_x \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T. \quad (3.30)$$

Then, it follows from (3.30), that

$$\bar{\mu}(u_{mx}(x,t)) \rightharpoonup \bar{\mu}(u(x,t)) \text{ a.e., } (x,t) \text{ in } Q_T,$$

$$\sigma(u_{mx}(x,t))u_m(x,t) \rightharpoonup \sigma(u_x(x,t))u(x,t) \text{ a.e., } (x,t) \text{ in } Q_T. \quad (3.31)$$
On the other hand, by (3.22), we have
\[
|u_{mx}(x,t)| \leq \|u_{mx}(t)\|_{C^0([0,1])} \leq \sqrt{2}\|u_{mx}(t)\|_{H^2}
\]
\[
\leq 2\|\Delta u_m(t)\| \leq 2\sqrt{P_m(t)} \leq 2\sqrt{C_T^{(2)}};
\]
\[
|\tilde{\mu}(u_{mx}(x,t))| \leq \sup_{|z| \leq 2\sqrt{C_T^{(2)}}} |\tilde{\mu}(z)| \leq C_T;
\]
\[
|\sigma(u_{mx}(x,t))u_m(x,t)| \leq \|u_{mx}(t)\| \|\sigma(u_{mx}(x,t))\|
\leq \sqrt{C_T^{(2)}} \sup_{|z| \leq 2\sqrt{C_T^{(2)}}} |\sigma(z)| \leq C_T.
\]

Applying the dominated convergence theorem, from (3.31), (3.32) we obtain
\[
\tilde{\mu}(u_{mx}) \to \tilde{\mu}(u_x) \quad \text{strongly in } L^2(Q_T),
\]
\[
\sigma(u_{mx})u_m \to \sigma(u_x)u \quad \text{strongly in } L^2(Q_T).
\]

Passing to the limit in (3.7) by (3.8), (3.29), (3.30) and (3.33), we have \(u\) satisfying
\[
\langle u_t(t), w \rangle + \langle u_{xt}(t), w_x \rangle + \langle u_x(t) + \tilde{\mu}(u_x(t)), w_x \rangle + \langle (1 + \sigma(u_x(t)))u(t), w \rangle
= \langle f(t), w \rangle, \quad \forall w \in H^1_0,
\]
\[
u(0) = 0.
\]

Furthermore,
\[
u \in L^\infty(0, T; H^1_0 \cap H^2), \quad u' \in L^2(0, T; H^1_0).
\]

**Step 4: Uniqueness of the solution.** Let \(u\) and \(v\) be two weak solutions of (3.1) such that
\[
u, v \in L^\infty(0, T; H^1_0 \cap H^2), \quad u', v' \in L^2(0, T; H^1_0).
\]

Then \(w = u - v\) satisfies
\[
\langle w_t(t), y \rangle + \langle w_{xt}(t), y_x \rangle + \langle w_x(t), y_x \rangle + \langle w(t), y \rangle
+ \langle \tilde{\mu}(u_x(t)) - \tilde{\mu}(v_x(t)), y_x \rangle + \langle \sigma(u_x(t))u - \sigma(v_x(t))v, y \rangle = 0, \quad \forall y \in H^1_0,
\]
\[
u(0) = 0,
\]
\[
u, v \in L^\infty(0, T; H^1_0 \cap H^2), \quad u_t, v_t, w_t \in L^2(0, T; H^1_0).
\]

Take \(y = w = u - v\), in (3.30) and integrating with respect to \(t\), we obtain
\[
\rho(t) = -2 \int_0^t \langle \tilde{\mu}(u_x(s)) - \tilde{\mu}(v_x(s)), w_x(s) \rangle ds
- 2 \int_0^t \langle \sigma(u_x(s))u(s) - \sigma(v_x(s))v(s), w(s) \rangle ds
= \rho_1(t) + \rho_2(t),
\]
\[
\rho(t) = \|w(t)\|_{H^1}^2 + 2 \int_0^t \|w(s)\|_{H^1}^2 ds.
\]
Estimating $\rho_1(t)$. Using the monotonicity of the function $z \mapsto \bar{\mu}(z)$, we obtain
\[
\rho_1(t) = -2 \int_0^t \langle \bar{\mu}(u_x(s)) - \bar{\mu}(v_x(s)), w_x(s) \rangle ds \leq 0. \tag{3.39}
\]

Estimating $\rho_2(t)$. We have
\[
w = [\sigma(u_x)w + (\sigma(u_x) - \sigma(v_x))v]w \\
= \sigma(u_x)w^2 + (\sigma(u_x) - \sigma(v_x))vw \\
\geq (\sigma(u_x) - \sigma(v_x))vw.
\]  
This implies
\[
\rho_2(t) = -2 \int_0^t \langle \sigma(u_x(s))u(s) - \sigma(v_x(s))v(s), w(s) \rangle ds \\
\leq -2 \int_0^t \langle [\sigma(u_x(s)) - \sigma(v_x(s))] v(s), w(s) \rangle ds \\
\leq \int_0^t \| \sigma(u_x(s)) - \sigma(v_x(s)) \| \| v(s) \| \| w(s) \| ds \\
\leq \int_0^t \| v_x(s) \| \| w(s) \| \| w(s) \| ds.
\]

Put $M = \| u \|_{L^\infty(0,T;H^1_0 \cap H^2)} + \| v \|_{L^\infty(0,T;H^1_0 \cap H^2)}$ and $L_M = \sup_{|z| \leq M} |\sigma'(z)|$, we have
\[
|\sigma(u_x) - \sigma(v_x)| \leq L_M |w_x|. \tag{3.42}
\]

Hence
\[
\rho_2(t) \leq 2L_M \int_0^t \| w_x(s) \| \| v_x(s) \| \| w(s) \| ds \\
\leq 2ML_M \int_0^t \| w_x(s) \| \| w(s) \| ds. \tag{3.43}
\]

Then, from $\rho_1$, $\rho_2$, it follows that
\[
\rho(t) \leq ML_M \int_0^t \rho(s) ds. \tag{3.44}
\]

By Gronwall’s lemma, $\rho(t)$ leads to $\rho(t) = 0$, i.e., $w = u - v = 0$.

**Step 5: Proof of the estimate (3.5).** First, let us assume that
\[
u_0(x) \leq M, \quad \text{a.e., } x \in \Omega, \text{ and } \max\{\| \tilde{u}_0 \|_{L^\infty}, \| f \|_{L^\infty(Q_T)} \} \leq M. \tag{3.45}
\]

Then $z = u - M$ satisfies the initial and boundary value
\[
\begin{align*}
z_t - z_{xx} - \frac{\partial}{\partial x}(z_x + \bar{\mu}(z_x)) + z + (z + M)\sigma(z_x) \\
= f(x,t) - M, \quad 0 < x < 1, \quad 0 < t < T, \\
z(0,t) = z(1,t) = -M, \\
z(x,0) = \tilde{u}_0(x) - M.
\end{align*}
\]  

Multiplying equation (3.46) by \( v \in H^1_0 \), then integrating by parts with respect to variable \( x \), after some rearrangements, one has

\[
\begin{align*}
&\langle z_t(t), v \rangle + \langle z_{xt}(t), v_x \rangle + \langle z_x(t) + \bar{\mu}(z_x(t)), v_x \rangle \\
&\quad + \langle z(t) + (z(t) + M)\sigma(z_x(t)), v \rangle \\
&= \langle f(t) - M, v \rangle, \quad \text{for all } v \in H^1_0.
\end{align*}
\] (3.47)

From assumption (H1)–(H4) we deduce that the solution of the initial and boundary value problem (3.1) satisfies \( u \in L^\infty(0, T; H^1_0 \cap H^2) \), \( u' \in L^2(0, T; H^2_0) \), so that we are allowed to take \( v = z^+ = \frac{1}{2}(|z| + z) \) in (3.47). Thus, it follows that

\[
\begin{align*}
&\langle z_t(t), z^+(t) \rangle + \langle z_{xt}(t), z^+(t) \rangle + \langle z_x(t) + \bar{\mu}(z_x(t)), z^+(t) \rangle \\
&\quad + \langle z(t) + (z(t) + M)\sigma(z_x(t)), z^+(t) \rangle \\
&= \langle f(t) - M, z^+(t) \rangle.
\end{align*}
\] (3.48)

Hence

\[
\begin{align*}
&\frac{1}{2} \frac{d}{dt}(\|z^+(t)\|^2 + \|z^+_x(t)\|^2) + \|z^+_x(t)\|^2 + \|z^+(t)\|^2 \\
&= -(\bar{\mu}(z^+_x(t)), z^+_x(t)) - ((z^+(t) + M)\sigma(z^+_x(t)), z^+(t)) \\
&\quad + \langle f(t) - M, z^+(t) \rangle \leq 0,
\end{align*}
\] (3.49)

since \( M \geq \max\{\|\tilde{u}_0\|_{L^\infty}, \|f\|_{L^\infty(Q_T)}\} \) and

\[
\begin{align*}
&\langle z_t(t), z^+(t) \rangle = \int_0^1 z_t(x, t)z^+(x, t) \, dx = \int_0^1 (z^+(x, t))_t z^+(x, t) \, dx \\
&\quad = \frac{1}{2} \frac{d}{dt} \int_{0, z > 0} |z^+(x, t)|^2 \, dx = \frac{1}{2} \frac{d}{dt} \int_0^1 |z^+(x, t)|^2 \, dx \\
&\quad = \frac{1}{2} \frac{d}{dt} \|z^+(t)\|^2,
\end{align*}
\] (3.50)

and on the domain \( z > 0 \) we have \( z^+ = z, z_x = (z^+)_x \) and \( z_t = (z^+)_t \).

Integrating (3.49), we obtain

\[
\|z^+(t)\|^2 + \|z^+_x(t)\|^2 \leq \|z^+(0)\|^2 + \|z^+_x(0)\|^2.
\] (3.51)

Since \( z^+(x, 0) = (u(x, 0) - M)^+ = (\tilde{u}_0(x) - M)^+ = 0, z^+_x(x, 0) = 0 \), we obtain \( \|z^+(t)\|^2 + \|z^+_x(t)\|^2 = 0 \). Thus \( z^+ = 0 \) and \( u(x, t) \leq M \), for a.e. \((x, t) \in Q_T\).

The case \(-M \leq u_0(x)\), a.e. \( x \in \Omega \), and \( M \geq \max\{\|\tilde{u}_0\|_{L^\infty}, \|f\|_{L^\infty(Q_T)}\} \) can be dealt with, in the same manner as above, by considering \( z = u + M \) and \( z^- = \frac{1}{2}(|z| - z) \), we also obtain \( z^- = 0 \) and hence \( u(x, t) \geq -M \), for a.e. \((x, t) \in Q_T\).

From the above, one obtains \( |u(x, t)| \leq M \), a.e. \((x, t) \in Q_T\), i.e.,

\[
\|u\|_{L^\infty(Q_T)} \leq M,
\] (3.52)

for all \( M \geq \max\{\|\tilde{u}_0\|_{L^\infty}, \|f\|_{L^\infty(Q_T)}\} \). This implies (3.3). The proof is complete. \( \square \)

4. Exponential decay of solutions

This section investigates the decay of the solution of (3.1). For this purpose, we make the following assumption.

(H5) \( f \in L^2(\mathbb{R}^+; H^1_0) \) and there exist two constants \( C_0 > 0, \gamma_0 > 0 \) such that

\[
\|f(t)\| \leq C_0 e^{-\gamma_0 t}, \quad \text{for all } t \geq 0.
\]
Theorem 4.1. Assume that (H1), (H3)–(H5) hold. Then, problem (3.1) has a unique weak solution $u$ satisfying

$$u \in L^\infty(0, T; H^1_0 \cap H^2), \quad u' \in L^2(0, T; H^1_0)$$

for all $T > 0$, (4.1) and there exist positive constants $C$, $\gamma$ such that

$$\|u(t)\|_{H^1} \leq C \exp(-\gamma t) \quad \text{for all } t \geq 0.$$ (4.2)

Proof. Multiplying the $j^{th}$ equation of (3.7) by $c_m(t)$ and summing with respect to $j$, after some rearrangements, we obtain

$$\frac{d}{dt}\|u_m(t)\|_{H^1}^2 + 2\|u_m(t)\|_{H^1}^2 + 2\langle \bar{\mu}(u_{mx}(t)), u_{mx}(t) \rangle + 2\langle \sigma(u_{mx}(t)), u_{m}\rangle

= 2\langle f(t), u_m(t) \rangle.$$ (4.3)

Note that

$$2\langle f(t), u_m(t) \rangle \leq 2\|f(t)\|\|u_m(t)\| \leq 2\|f(t)\|u_m(t)\|_{H^1},$$ (4.4)

for all $\delta > 0$.

It follows from (4.3), (4.4) that

$$\frac{d}{dt}\|u_m(t)\|_{H^1}^2 + 2(1 - \delta)\|u_m(t)\|_{H^1}^2

\leq \frac{1}{2\delta}\|f(t)\|^2 \leq \frac{1}{2\delta}C_0^2 e^{-2\gamma t}, \quad \text{for all } \delta > 0.$$ (4.5)

Choose $\delta$ and $\gamma$ such that

$$0 < \delta < 1, \quad 0 < \gamma < \min\{1 - \delta, \gamma_0\}.$$ (4.6)

Then from (4.5), (4.6) we have

$$\frac{d}{dt}\|u_m(t)\|_{H^1}^2 + 2\gamma\|u_m(t)\|_{H^1}^2 \leq \frac{1}{2\delta}C_0^2 e^{-2\gamma t}.$$ (4.7)

Integrating (4.7), we obtain

$$\|u_m(t)\|_{H^1}^2 \leq \left(\|\tilde{u}_0\|_{H^1}^2 + \frac{C_0^2}{4\delta(\gamma_0 - \gamma)}\right)e^{-2\gamma t}.$$ (4.8)

Letting $m \to +\infty$ in (4.8), we obtain

$$\|u(t)\|_{H^1}^2 \leq \liminf_{m \to +\infty} \|u_m(t)\|_{H^1}^2

\leq \left(\|\tilde{u}_0\|_{H^1}^2 + \frac{C_0^2}{4\delta(\gamma_0 - \gamma)}\right)e^{-2\gamma t}, \quad \text{for all } t \geq 0.$$ (4.9)

This implies (4.2), and completes the proof. \hfill \Box

5. Existence and uniqueness of a T-periodic weak solution

In this section, we shall consider problem (1.1), (1.2), (1.4) with the constants $\alpha = \beta = \gamma = 1$,

$$u_t - u_{xxt} - (1 + \mu(u_x))u_{xx} + (1 + \sigma(u_x))u = f(x, t), \quad 0 < x < 1, \quad 0 < t < T,$n

$$u(0, t) = u(1, t) = 0,$$n

$$u(x, 0) = u(x, T).$$ (5.1)

We make the following assumptions:
Step 2: A priori estimates.

Theorem 5.2. Let $T > 0$ and (H2), (H3), (H4), (H6) hold. Then problem (5.1) has a weak solution $u$ such that

$$u \in L^\infty(0,T;H^1_0 \cap H^2) \text{ and } u' \in L^2(0,T;H^1_0).$$

Furthermore, if $\|u\|_{L^\infty(0,T;H^1_0 \cap H^2)} \leq R$, with $R \sup_{|z| \leq \sqrt{2}R} |\sigma'(z)| < 2$, then the solution is unique.

Proof. The proof consists of several steps.

Step 1: Consider the basis $\{w_j\}$ as above. Let $W_m$ be the linear space generated by $w_1, w_2, \ldots, w_m$. We consider the following problem.

Find a function $u_m(t)$ in the form (3.6) satisfying the nonlinear differential equation system (3.7) and the $T$-periodic condition

$$u_m(0) = u_m(T).$$

We consider an initial value problem given by (3.7), where $u_{0m}$ is given in $W_m$.

It is clear that for each $m$, there exists a solution $u_m(t)$ in the form (3.6) which satisfies (3.7) almost everywhere on $0 \leq t \leq T_m$ for some $T_m$, $0 < T_m \leq T$. The following a priori estimates allow us to take $T_m = T$ for all $m$.

Step 2: A priori estimates. Multiplying the $j^{th}$ equation of (3.7) by $c_{mj}(t)$ and summing with respect to $j$, we obtain

$$\frac{d}{dt}||u_m(t)||_{H^1}^2 + 2||u_m(t)||_{H^1}^2 + 2\langle \tilde{\sigma}(u_{mx}(t))u_{mx}(t), u_m(t) \rangle + 2\|\sqrt{\sigma}(u_{mx}(t))u_m(t)\|^2 = 2\langle f(t), u_m(t) \rangle.$$  

We estimate without difficulty the term $2\langle f(t), u_m(t) \rangle$ as follows

$$2\langle f(t), u_m(t) \rangle \leq \frac{1}{2\delta_1} ||f(t)||^2 + 2\delta_1 ||u_m(t)||^2 \leq \frac{1}{2\delta_1} ||f(t)||^2 + 2\delta_1 ||u_m(t)||_{H^1}^2,$$

for all $\delta_1, 0 < \delta_1 < 1$. 

Remark 5.1. The weak formulation of problem (5.1) can be given in the following manner: Find $u \in L^\infty(0,T;H^1_0 \cap H^2)$ with $u' \in L^2(0,T;H^1_0)$, such that $u$ satisfies the variational equation

$$\int_0^T \langle u'(t) + u(t), w(t) \rangle dt + \int_0^T \langle u'_x(t) + u_x(t), w_x(t) \rangle dt + \int_0^T \langle \tilde{\mu}(u_x(t)), w_x(t) \rangle dt + \int_0^T \langle \sigma(u(t), u_x(t))u(t), w(t) \rangle dt = \int_0^T \langle f(t), w(t) \rangle dt,$$  

for all $w \in L^2(0,T;H^1_0)$, $u(0) = u(T)$. 

(H6) $f$ is $T$-periodic in $t$, i.e., $f(x,0) = f(x,T)$.
Hence, from (5.5), (5.6) it follows that
\[
\frac{d}{dt} \|u_m(t)\|^2_{H^1} + 2(1 - \delta_1)\|u_m(t)\|^2_{H^1} + 2\langle \mu(u_{mx}(t)),u_{mx}(t) \rangle \\
+ 2\|\sqrt{\sigma(u_{mx}(t))}u_m(t)\|^2 \\
\leq \frac{1}{2\delta_1} \|f(t)\|^2. \tag{5.7}
\]

Next, multiplying the \(j^{th}\) equation of (3.14) by \(c_{mj}(t)\) and summing with respect to \(j\), we obtain
\[
\frac{d}{dt} \|u_{mx}(t)\|^2_{H^1} + 2\|u_{mx}(t)\|^2_{H^1} + 2\|\mu(u_{mx}(t))\|_{H^1}^2 \\
+ 2\|\sigma'(u_{mx}(t))u_m(t)\|_{H^1}^2 + 2\|\sigma(u_{mx}(t))u_{mx}(t),u_{mx}(t)\|_{H^1}^2 \\
= 2\langle f_x(t),u_{mx}(t) \rangle. \tag{5.8}
\]

Similarly, we have
\[
2\langle \sigma'(u_{mx}(t))u_m(t)\Delta u_m(t) + \sigma(u_{mx}(t))u_{mx}(t),u_{mx}(t) \rangle \\
= 2\int_0^1 u_m(x,t)u_{mx}(x,t)\sigma'(u_{mx}(x,t))\Delta u_m(x,t) \, dx \\
+ 2\int_0^1 u_{mx}^2(x,t)\sigma(u_{mx}(x,t)) \, dx \\
= 2\int_0^1 u_m(x,t)\frac{\partial}{\partial x} \left( \int_0^{u_{mx}(x,t)} y\sigma'(y) \, dy \right) \, dx \\
+ 2\int_0^1 u_{mx}^2(x,t)\sigma(u_{mx}(x,t)) \, dx \\
= -2\int_0^1 u_{mx}(x,t) \left( \int_0^{u_{mx}(x,t)} y\sigma'(y) \, dy \right) \, dx \\
+ 2\int_0^1 u_{mx}^2(x,t)\sigma(u_{mx}(x,t)) \, dx \\
= 2\int_0^1 \left[ u_{mx}^2(x,t)\sigma(u_{mx}(x,t)) - u_{mx}(x,t) \left( \int_0^{u_{mx}(x,t)} y\sigma'(y) \, dy \right) \right] \, dx \geq 0,
\]
and this implies
\[
\frac{d}{dt} \|u_{mx}(t)\|^2_{H^1} + 2(1 - \delta_1)\|u_{mx}(t)\|^2_{H^1} + 2\|\mu(u_{mx}(t))\|_{H^1}^2 \\
\leq \frac{1}{2\delta_1} \|f_x(t)\|^2, \tag{5.9}
\]
for all \(\delta_1, 0 < \delta_1 < 1\).

It follows from (5.7), (5.10) that
\[
\frac{d}{dt} \left[ \|u_m(t)\|^2_{H^1} + \|u_{mx}(t)\|^2_{H^1} \right] + 2(1 - \delta_1)(\|u_m(t)\|^2_{H^1} + \|u_{mx}(t)\|^2_{H^1}) \\
\leq \frac{1}{2\delta_1} \|f(t)\|^2_{H^1}. \tag{5.11}
\]
Integrating (5.11), we have
\[
\|u_m(t)\|^2_{H^1} + \|u_{mx}(t)\|^2_{H^1} \\
\leq \left(\|u_0\|^2_{H^1} + \|u_{0mx}\|^2_{H^1} - R^2\right)e^{-2(1-\delta_1)t} \\
+ \left(2R^2 + \frac{1}{2\delta_1}\int_0^t e^{2(1-\delta_1)s}\|f(s)\|^2_{H^1}ds\right)e^{-2(1-\delta_1)t} \\
\leq \left(\|u_0\|^2_{H^1} + \|u_{0mx}\|^2_{H^1} - R^2\right)e^{-2(1-\delta_1)t} + R^2,
\]
where \(R^2 = \sup_{0 \leq t \leq T} R_1(t)\),
\[
R_1(t) = \left\{\begin{array}{ll}
\frac{1}{2\delta_1}e^{2(1-\delta_1)t}\int_0^t e^{2(1-\delta_1)s}\|f(s)\|^2_{H^1}ds, & 0 < t \leq T, \\
\frac{1}{2\delta_1(1-\delta_1)}\|f(0)\|^2_{H^1}, & t = 0.
\end{array}\right.
\]
(5.12)

Therefore, if we choose \(u_{0m}\) such that \(\|u_0\|^2_{H^1} + \|u_{0mx}\|^2_{H^1} \leq R^2\), we obtain from (5.12) that
\[
\|u_m(t)\|^2_{H^1} + \|u_{mx}(t)\|^2_{H^1} \leq R^2, \quad \text{i.e., } T_m = T \text{ for all } m.
\]
(5.14)

Let \(\bar{B}_m(0, R)\) be a closed ball in the space \(W_m\) of linear combinations of the functions \(w_1, w_2, \ldots, w_m\), with the norm
\[
\|u_{0m}\|_* = \sqrt{\|u_0\|^2_{H^1} + \|u_{0mx}\|^2_{H^1}}.
\]
(5.15)

Let us define \(\mathcal{F}_m : \bar{B}_m(0, R) \to \bar{B}_m(0, R)\)
\[
u_{0m} \mapsto u_m(T).
\]
(5.16)

We prove that \(\mathcal{F}_m\) is continuous. Let \(u_{0m}, \bar{u}_{0m} \in \bar{B}_m(0, R)\) and let \(y_m(t) = u_m(t) - \bar{u}_m(t)\), where \(u_m(t)\) and \(\bar{u}_m(t)\) are solutions of the system (5.7) on \([0, T]\) satisfying the initial conditions \(u_m(0) = u_{0m}\) and \(\bar{u}_m(0) = \bar{u}_{0m}\), respectively. Then, \(y_m(t)\) satisfies the differential equation
\[
\mathcal{A} y_m(t) + \mathcal{B} u_m(t) \quad \text{satisfy initial condition } \quad y_m(0) = u_{0m} - \bar{u}_{0m}.
\]
(5.17)

Using the same arguments as before, we can show that
\[
\frac{d}{dt}\|y_m(t)\|^2_{H^1} + 2\|y_m(t)\|^2_{H^1} + 2(\mu(y_m(t)) - \mu(y_m(t)), y_m(t))
\]
\[
+ 2(\sigma(u_m(t))u_m(t) - \sigma(\bar{u}_m(t))\bar{u}_m(t), y_m(t)) = 0.
\]
(5.18)

On the other hand, we have
\[
\langle \mu(y_m(t)) - \mu(y_m(t)), y_m(t) \rangle \geq 0;
\]
(5.19)
\[
2(\sigma(u_m(t))u_m(t) - \sigma(\bar{u}_m(t))\bar{u}_m(t), y_m(t))
\]
\[
= 2\sqrt{\sigma(u_m(t))y_m(t)} + 2(\sigma(u_m(t)) - \sigma(\bar{u}_m(t)), \bar{u}_m(t)y_m(t)).
\]
(5.20)
Putting $K_R = \sup_{|z| \leq \sqrt{TR}} |\sigma'(z)|$, we have
\begin{align*}
2(\sigma(u_{mx}(t)) - \sigma(\tilde{u}_{mx}(t))u_m(t)) & \leq 2\|u_{mx}(t)\|\|y_m(t)\|\|\sigma(u_{mx}(t)) - \sigma(\tilde{u}_{mx}(t))\| \\
& \leq 2\tilde{K}_R\|u_{mx}(t)\|\|y_m(t)\|\|y_{mx}(t)\| \\
& \leq \tilde{K}_R\|\tilde{u}_{mx}(t)\|\|y_m(t)\|^2_{H^1} \leq R\tilde{K}_R\|y_m(t)\|^2_{H^1}.
\end{align*}
It follows from (5.18)-(5.21) that
\begin{equation}
\frac{d}{dt}\|y_m(t)\|^2_{H^1} + (2 - R\tilde{K}_R)\|y_m(t)\|^2_{H^1} \leq 0.
\end{equation}
Integrating inequality (5.22), we obtain
\begin{equation}
\|y_m(T)\|^2_{H^1} \leq e^{(R\tilde{K}_R - 2)T}\|u_{0m} - \tilde{u}_{0m}\|^2_{H^1},
\end{equation}
or
\begin{equation}
\|f_m(u_{0m}) - f_m(\tilde{u}_{0m})\|_{H^1} \leq \exp\left(\left(\frac{1}{2}R\tilde{K}_R - 1\right)T\right)\|u_{0m} - \tilde{u}_{0m}\|_{H^1}.
\end{equation}
Note that, on $W_m$, $\|v_{0m}\|_{H^1}$ and $\|v_{0m}\|_* = \sqrt{\|v_{0m}\|_{H^1}^2 + \|v_{mx}\|_{H^1}^2}$ are equivalent norms, hence, there exist two constants $D_{1m} > 0$, $D_{2m} > 0$ such that
\begin{equation}
D_{1m}\|v_{0m}\|_* \leq \|v_{0m}\|_{H^1} \leq D_{2m}\|v_{0m}\|_* \quad \text{for all } v_{0m} \in W_m.
\end{equation}
It follows from (5.23), (5.24) that
\begin{equation}
\|f_m(u_{0m}) - f_m(\tilde{u}_{0m})\|_* \leq \frac{D_{2m}}{D_{1m}}\exp\left(\left(\frac{1}{2}R\tilde{K}_R - 1\right)T\right)\|u_{0m} - \tilde{u}_{0m}\|_*
\end{equation}
for all $u_{0m}, \tilde{u}_{0m} \in W_m$.

Hence, $f_m : B_{\tilde{m}}(0, R) \rightarrow B_{\tilde{m}}(0, R)$ is continuous. Applying the fixed point theorem of Brouwer, we have (for every $m$) a function $u_{0m} \in B_{\tilde{m}}(0, R)$ such that the solution of the initial value problem (3.7) is a $T$-periodic solution of the system (3.7). This solution satisfies the inequality (5.14) a.e., in $[0, T]$ and consequently, by (5.11) we have
\begin{equation}
\|u_m(t)\|_{H^1} + \|u_{mx}(t)\|_{H^1} + 2(1 - \delta_1)\int_0^t (\|u_m(s)\|_{H^1}^2 + \|u_{mx}(s)\|_{H^1}^2) ds \leq R^2 + \frac{1}{20} \int_0^T \|f(s)\|_{H^1}^2 ds \leq C_T.
\end{equation}
On the other hand, we multiplying the $j^{th}$ equation of (3.7) by $e^{jmx}(t)$ and summing up with respect to $j$, afterwards, integrating with respect to the time variable from 0 to $T$, we obtain after some rearrangements
\begin{equation}
2\int_0^T \|u_m'(t)\|_{H^1} dt + \int_0^T \frac{d}{dt}\left(\|u_m(t)\|_{H^1}^2 + 2\int_0^1 \tilde{\mu}(u_{mx}(x, t)) dx\right) dt \\
+ 2\int_0^T \langle \sigma(u_{mx}(t))u_m(t), u_m'(t) \rangle dt \\
= 2\int_0^T \langle f(t), u_m'(t) \rangle dt,
\end{equation}
where $\tilde{\mu}(z) = \int_0^z \tilde{\mu}(y) dy \geq 0$ for all $z \in \mathbb{R}$.
From (5.4), we obtain
\[
\int_0^T \frac{d}{dt} \left[ \| u_m(t) \|_{H^1}^2 + 2 \int_0^1 \tilde{\mu}(u_{mx}(x,t)) \, dx \right] \, dt
= \| u_m(T) \|_{H^1}^2 - \| u_m(0) \|_{H^1}^2 + 2 \int_0^1 \left[ \tilde{\mu}(u_{mx}(x,T)) - \tilde{\mu}(u_{mx}(x,0)) \right] \, dx = 0.
\]
Moreover,
\[
2 \int_0^T \langle f(t), u'_m(t) \rangle \, dt \leq 2 \int_0^T \| f(t) \| \| u'_m(t) \| \, dt \\
\leq 2 \int_0^T \| f(t) \|^2 \, dt + \frac{1}{2} \int_0^T \| u'_m(t) \|^2 \, dt.
\]
Putting \( \sigma_R = \sup_{|z| \leq \sqrt{2R}} \sigma(z) \), we have
\[
2 \int_0^T \langle \sigma(u_{mx}(t)) u_m(t), u'_m(t) \rangle \, dt \\
\leq 2 \sigma_R \int_0^T \| u_m(t) \| \| u'_m(t) \| \, dt \\
\leq 2 R \sigma_R \int_0^T \| u'_m(t) \|^2 \, dt \leq 2TR^2 \sigma_R^2 + \frac{1}{2} \int_0^T \| u'_m(t) \|^2 \, dt.
\]
It follows from (5.27), (5.28), (5.29) and (5.30), that
\[
\int_0^T \| u'_m(t) \|^2_{H^1} \, dt \leq 2TR^2 \sigma_R^2 + 2 \int_0^T \| f(t) \|^2 \, dt \leq C_T,
\]
for all \( m \in \mathbb{N} \), for all \( t \in [0,T] \), where \( C_T \) always indicates a bound depending on \( T \).

**Step 3:** The limiting process. By (5.14) and (5.31) we deduce that, there exists a subsequence of \( \{u_m\} \), still denoted by \( \{u_m\} \) such that
\[
u_m \to u \quad \text{in} \quad L^\infty(0,T; H^1_0 \cap H^2)
\]
weakly*,
\[
u'_m \to u' \quad \text{in} \quad L^2(0,T; H^1_0)
\]
weakly.

From (5.4), we obtain
\[
u(0) = u(T).
\]
Using the compactness lemma of Lions [9, p.57] and applying Fischer-Riesz theorem, from (5.32), there exists a subsequence of \( \{u_m\} \), denoted by the same symbol
\[
u_m \to u \quad \text{strongly in} \quad L^2(0,T; H^1_0) \quad \text{and a.e. in} \quad Q_T,
\]
\[
u_{mx} \to u_x \quad \text{strongly in} \quad L^2(Q_T) \quad \text{and a.e. in} \quad Q_T.
\]
Applying an argument similar to the one used in the proof of Theorem 3.1, we have
\[
\tilde{\mu}(u_{mx}) \to \tilde{\mu}(u_x) \quad \text{strongly in} \quad L^2(Q_T),
\]
\[
\sigma(u_{mx})u_m \to \sigma(u_x)u \quad \text{strongly in} \quad L^2(Q_T).
\]
Denote by \( \{ \zeta_i, i = 1, 2, \ldots \} \) the orthonormal base in the real Hilbert space \( L^2(0, T) \). The set \( \{ \zeta_i w_j, i, j = 1, 2, \ldots \} \) forms an orthonormal base in \( L^2(0, T; H^1_0) \).

From (3.7), we have

\[
\int_0^T \langle u'_m(t) + u_m(t), w_j \zeta_i(t) \rangle \, dt + \int_0^T \langle u'_{mx}(t) + u_{mx}(t), w_j \zeta_i(t) \rangle \, dt \\
+ \int_0^T \langle \bar{\mu}(u_{mx}(t)), w_j \zeta_i(t) \rangle \, dt + \int_0^T \langle \sigma(u_{mx}(t))u_m(t), w_j \zeta_i(t) \rangle \, dt \tag{5.36}
\]

for all \( i, j, 1 \leq j \leq m, \, i \in \mathbb{N} \).

For \( i \) and \( j \) fixed, we deduce from (5.32) that

\[
\int_0^T \langle u'_m(t) + u_m(t), w_j \zeta_i(t) \rangle \, dt \rightarrow \int_0^T \langle u'(t) + u(t), w_j \zeta_i(t) \rangle \, dt, \tag{5.37}
\]

\[
\int_0^T \langle u'_{mx}(t) + u_{mx}(t), w_j \zeta_i(t) \rangle \, dt \rightarrow \int_0^T \langle u'_x(t) + u_x(t), w_j \zeta_i(t) \rangle \, dt.
\]

Furthermore, by (5.35), we have

\[
\int_0^T \langle \bar{\mu}(u_{mx}(t)), w_j \zeta_i(t) \rangle \, dt \rightarrow \int_0^T \langle \bar{\mu}(u_x(t)), w_j \zeta_i(t) \rangle \, dt, \tag{5.38}
\]

\[
\int_0^T \langle \sigma(u_{mx}(t))u_m(t), w_j \zeta_i(t) \rangle \, dt \rightarrow \int_0^T \langle \sigma(u_x(t))u(t), w_j \zeta_i(t) \rangle \, dt.
\]

Passing to the limit in (5.36) by (5.37), (5.38), we have

\[
\int_0^T \langle u'(t) + u(t), w_j \zeta_i(t) \rangle \, dt + \int_0^T \langle u'_x(t) + u_x(t), w_j \zeta_i(t) \rangle \, dt \\
+ \int_0^T \langle \bar{\mu}(u_x(t)), w_j \zeta_i(t) \rangle \, dt + \int_0^T \langle \sigma(u_x(t))u(t), w_j \zeta_i(t) \rangle \, dt \tag{5.39}
\]

\[
= \int_0^T \langle f(t), w_j \zeta_i(t) \rangle \, dt.
\]

This equation holds for every \( i, j \in \mathbb{N} \), i.e., the equation

\[
\int_0^T \langle u'(t) + u(t), w(t) \rangle \, dt + \int_0^T \langle u'_x(t) + u_x(t), w_x(t) \rangle \, dt \\
+ \int_0^T \langle \bar{\mu}(u_x(t)), w_x(t) \rangle \, dt + \int_0^T \langle \sigma(u_x(t))u(t), w(t) \rangle \, dt \tag{5.40}
\]

\[
= \int_0^T \langle f(t), w(t) \rangle \, dt, \quad \text{for all } w \in L^2(0, T; H^1_0),
\]

is satisfied.

**Step 4: Uniqueness of the solutions.** Let \( u \) and \( \bar{u} \) be two solutions of (5.2) such that \( \|u\|_{L^\infty(0, T; H^1_0 \cap H^2)} \leq R, \|\bar{u}\|_{L^\infty(0, T; H^1_0 \cap H^2)} \leq R \), with \( R \sup_{|z| \leq \sqrt{2R}} |\sigma(z)| < 2 \).
Then \( v = u - \bar{u} \) satisfies
\[
\begin{align*}
\int_0^T \langle v'(t) + v(t), w(t) \rangle \, dt &+ \int_0^T \langle v_x'(t) + v_x(t), w_x(t) \rangle \, dt \\
+ \int_0^T \langle \bar{\mu}(u_x(t)) - \mu(\bar{u}_x(t)), w_x(t) \rangle \, dt \\
+ \int_0^T \langle \sigma(u_x(t))u(t) - \sigma(\bar{u}_x(t))\bar{u}(t), w(t) \rangle \, dt = 0, \quad \forall w \in L^2(0, T; H^1_0), \\
v(0) = v(T), \\
v, u, \bar{u} \in L^\infty(0, T; H^1_0 \cap H^2), \quad v', u', \bar{u}' \in L^2(0, T; H^1_0).
\end{align*}
\] (5.41)

Taking \( w = v \) in (5.41), and using (5.41), we obtain
\[
\begin{align*}
\int_0^T \langle v'(t), v(t) \rangle \, dt &= \frac{1}{2} \|v(T)\|^2 - \frac{1}{2} \|v(0)\|^2 = 0; \quad (5.42) \\
\int_0^T \langle v_x'(t), v_x(t) \rangle \, dt &= \frac{1}{2} \|v_x(T)\|^2 - \frac{1}{2} \|v_x(0)\|^2 = 0; \quad (5.43) \\
\int_0^T \langle \bar{\mu}(u_x(t)) - \mu(\bar{u}_x(t)), v_x(t) \rangle \, dt &\geq 0; \quad (5.44)
\end{align*}
\] (5.45)

As for (5.21), we have
\[
\int_0^T \langle \sigma(u_x(t))u(t) - \sigma(\bar{u}_x(t))\bar{u}(t), v(t) \rangle \, dt \leq \frac{1}{2} R \bar{K}_R \int_0^T \|v(t)\|_{\dot{H}^1}^2 \, dt, \quad (5.46)
\]
with \( \bar{K}_R = \sup_{|z| \leq \sqrt{2} R} |\sigma'(z)| \). Hence
\[
\begin{align*}
\int_0^T \|v(t)\|_{\dot{H}^1}^2 \, dt &+ \int_0^T \langle \bar{\mu}(u_x(t)) - \mu(\bar{u}_x(t)), v_x(t) \rangle \, dt + \int_0^T \|\sqrt{\sigma(u_x(t))v(t)}\|^2 \, dt \\
&\leq \frac{1}{2} R \bar{K}_R \int_0^T \|v(t)\|_{\dot{H}^1}^2 \, dt.
\end{align*}
\] (5.47)

By \( \frac{1}{2} R \bar{K}_R = \frac{1}{2} R \sup_{|z| \leq \sqrt{2} R} |\sigma'(z)| < 1 \), we deduce from (5.47) that \( \int_0^T \|v(t)\|^2_{\dot{H}^1} \, dt = 0 \), i.e., \( v = u - \bar{u} = 0 \). This completes the proof. \qed

**Acknowledgments.** The authors wish to express their gratitude to the anonymous referees and the editor for their valuable comments.

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