*Electronic Journal of Differential Equations*, Vol. 2018 (2018), No. 05, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

# FRACTIONAL SCHRÖDINGER EQUATIONS WITH NEW CONDITIONS

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*Communicated by Raffaella Servadei*

Abstract. In this article, we study the nonlinear fractional Schrödinger equation

$$\begin{gathered}(-\Delta)^\alpha u + V(x)u = f(x,u)\\ u \in H^\alpha(\mathbb{R}^n,\mathbb{R}),\end{gathered}$$

where $(-\Delta)^\alpha(\alpha \in (0,1))$ stands for the fractional Laplacian of order $\alpha$, $x \in \mathbb{R}^n$, $V \in C(\mathbb{R}^n,\mathbb{R})$ may change sign and $f$ is only locally defined near the origin with respect to $u$. Under some new assumptions on $V$ and $f$, we show that the above system has infinitely many solutions near the origin. Some examples are also given to illustrate our main theoretical result.

## 1. Introduction and statement of main results

This article concerns the existence of infinitely many solutions for the fractional Schrödinger equation

$$\begin{gathered}(-\Delta)^\alpha u + V(x)u = f(x,u),\\ u \in H^\alpha(\mathbb{R}^n,\mathbb{R}),\end{gathered} \tag{1.1}$$

where $n \geq 2$, $\alpha \in (0,1)$, $x \in \mathbb{R}^n$, $V \in C(\mathbb{R}^n,\mathbb{R})$ satisfying some new conditions, and $f$ is only locally defined near the origin with respect to $u$.

Problem (1.1) is related to the existence of standing wave solutions for fractional Schrödinger equations of the form

$$i\frac{\partial\psi}{\partial t} = (-\Delta)^\alpha\psi + (V(x)+\omega)\psi - f(x,\psi), \tag{1.2}$$

where $i$ is the imaginary unit, $\alpha \in (0,1)$, $\omega$ is a constant, $(-\Delta)^\alpha$ is the fractional Laplacian operator of order $\alpha$ and $\psi : \mathbb{R}^3 \times [0,+\infty) \to \mathbb{C}$. We are interested in looking for a standing wave, namely, waves of the form

$$\psi(x,t) = e^{i\omega t}u(x),$$

where $u$ is a real-valued function, and $f$ is assumed to satisfy $f(x,e^{-i\omega t}u) = e^{-i\omega t}f(x,u)$. Clearly, $\psi(x,t)$ solves (1.2) if and only if $u(x)$ solves (1.1).

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2010 *Mathematics Subject Classification.* 35B38, 35G99.
*Key words and phrases.* Fractional Schrödinger equations; critical point theory; symmetric mountain pass theorem.

Submitted July 28, 2017. Published January 4, 2018.

The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics. It was discovered by Nick Laskin [27, 28] as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like quantumn mechanical paths. Equations involving the fractional Laplacian have attracted much attention in recent years, appear in several areas such as optimization, finance, phase transitions, stratified material, crystal dislocation, flame propagation, conservation laws, ultra-relativistic limits of quantum, material science, and water waves, see e.g. [4, 7, 14, 17] for an introduction to these topics and their applications.

When $\alpha = 1$, (1.1) becomes the classical Schrödinger equation

$$\begin{gathered} -\Delta u + V(x)u = f(x,u) \\ u \in H^1(\mathbb{R}^n, \mathbb{R}). \end{gathered} \tag{1.3}$$

There has been a a lot of studies on existence and multiplicity of solutions of problem (1.3) under various hypotheses on the potential $V(x)$ and the nonlinearity $f(x,u)$, see [3, 21, 30, 31] and the references therein. The body of literature for (1.3) is huge and we do not even try to collect here a detailed bibliography.

Nonlinear equation (1.1) involves the fractional Laplacian $(-\Delta)^\alpha$, $0 < \alpha < 1$, which is a nonlocal operator. A common approach to deal with this problem was proposed by Caffarelli and Silvestre in [9], see also [41], allowing to transform problem (1.1) into a local problem via the Dirichlet-Neumann map. That is, for $u \in H^\alpha(\mathbb{R}^n)$ one considers the problem

$$\begin{gathered} -\operatorname{div}(y^{1-2\alpha}\nabla v) = 0 \quad \text{in } \mathbb{R}^{n+1}_+ \\ v(x,0) = u, \quad \text{on } \mathbb{R}^n \end{gathered}$$

from where the fractional Laplacian is obtained as

$$(-\Delta)^\alpha u(x) = -b_\alpha \lim_{y\to 0^+} y^{1-2\alpha} v_y$$

where $b_\alpha$ is a suitable constant. With the aid of the extended techniques [9], some existence and nonexistence results for Dirichlet problem involving the fractional Laplacian on bounded domain are obtained, see e.g. [10, 44] and the references therein. Using the equivalence definition of fractional operator $(-\Delta)^\alpha$ (see Section 2), Servadei and Valdinoci [34, 35] also introduced a variational principle and studied the existence and multiplicity of solutions for non-local equations of elliptic type.

There have been many results appeared in the literature for problem (1.1). For example, Cheng [12] studied problem (1.1) when $f(x,u) = |u|^{p-1}u$ with $1 < p < \frac{4\alpha}{n} + 1$, and found the ground states under a stronger assumption on the potential $V$, i.e., $\lim_{|x|\to\infty} V(x) = \infty$. Dipierro et al. [18] studied problem (1.1) when the potential $V(x) = 1$ and $f(x,u) = |u|^{p-1}u$, with $1 < p < \frac{2n}{n-2\alpha}$; in this case, they established the existence of positive and spherically symmetric solution. Felmer et al. [21] studied a similar class of equations, in which $V(x) = 1$, and the nonlinearity satisfies suitable assumptions, using variational methods, classical positive solutions are found. Secchi [36] proved some existence results for fractional Schrödinger equations, under the assumption that the nonlinearity is either of perturbative type or satisfies the Ambrosetti-Rabinowitz condition. Recently, Teng [44] obtained infinitely many small energy solutions of (1.1) by variant of the fountain theorem in [51]. More precisely, they use the following assumptions:

(A1) $V \in C(\mathbb{R}^n, \mathbb{R})$ and $\inf_{\mathbb{R}^n} V > 0$.

(A2) For any $M > 0$ there exists $d_0 > 0$ such that
$$\lim_{|y|\to\infty} \text{meas}(\{x \in \mathbb{R}^n : |x-y| \le d_0, V(x) \le M\}) = 0,$$
where meas denotes the Lebesgue measure in $\mathbb{R}^n$.

(A3) $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), f(x,u)u \ge 0$ for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}$, and there exists a constant $\nu \in (1,2)$ such that
$$|f(x,u)| \le a(x)(1 + |u|^{\nu-1})\,\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}$$
with a positive function $a(x) \in L^{\frac{2}{2-\nu}}(\mathbb{R}^n)$.

(A4) There exists $\sigma \in [1,\nu)$ such that $\liminf_{|u|\to\infty} \frac{F(x,u)}{|u|^\sigma} \ge d > 0$ uniformly for $x \in \mathbb{R}^n$, where where $F(x,u) = \int_0^u f(x,s)ds$.

(A5) $f(x,-u) = -f(x,u)$ for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}$.

Very recently, Torres [46] studied problem (1.1) and proved the existence of at least one solutions of equation (1.1) under the assumptions:

(A6) $V(x) = \lambda v(x)$ where $\lambda > 0$ is a parameter and $v \in C(\mathbb{R}^n), v(x) \ge 0$ on $\mathbb{R}^n$;

(A7) there exists a constant $b > 0$ such that the set $\{v < b\} := \{x \in \mathbb{R}^n / v(x) < b\}$ is nonempty and has finite Lebesgue measure and $|\{v < b\}|^{\frac{2^*_\alpha - 2}{2^*_\alpha}} < \frac{1}{c_{2^*_\alpha}}$, where $c_{2^*_\alpha}$ is the Sobolev constant (see Lemma 2.1);

(A8) $f \in C(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ and there exists $\mu \in (2, 2^*)$ such that
$$0 < \mu F(x,u) \le f(x,u)u \quad \forall u \in \mathbb{R}\backslash\{0\}.$$

**Remark 1.1.** There are functions $V$ and $F$ not satisfying the corresponding assumptions of the above papers. For example:
$$V(x) = \begin{cases} ((p^2+1)^2(|x|-p) + c_0), & \text{if } p \le |x| < p + \frac{1}{p^2+1}, \\ (p^2+1) + c_0, & \text{if } p + \frac{1}{p^2+1} \le |x| < p + \frac{p^2}{p^2+1}, \\ (p^2+1)^2(p+1-|x|) + c_0, & \text{if } p + \frac{p^2}{p^2+1} \le |x| < p+1, \end{cases}$$
$$F(x,u) = \begin{cases} \cos|x||u|^s \sin\frac{1}{|u|^\varepsilon}, & \text{if } 0 < |u| < 1, \\ 0, & \text{if } u = 0, \end{cases}$$
where $p \in \mathbb{N}, c_0 \in \mathbb{R}, \varepsilon \in (0,1)$ and $s \in (1+\varepsilon, 2)$. Obviously, $F$ is locally defined near the origin.

Inspired by the above results, we investigate the situation where the potential $V$ and $F$ satisfies new assumptions different from those studied previously and covered some examples as in remark 1.1. Precisely, we suppose that

(A9) There exists a constant $a_0 > 0$ such that $V(x) + a_0 \ge 1$, and $\int_{\mathbb{R}^n} \frac{1}{V(x)+a_0} dx < \infty$.

(A10) $F \in C^1(\mathbb{R}^n \times (-\rho,\rho))$ is even, and there exists a constant $a_1 > 0$ such that
$$|f(x,u)| \le a_1, \quad \forall (x,u) \in \mathbb{R}^n \times (-\rho,\rho),$$
where $\rho > 0$.

(A11) There exist $x_0 \in \mathbb{R}^n$, two sequences of positives numbers $\varepsilon_n \to 0$, $M_n \to \infty$ as $n \to \infty$ and constants $a_2, \varepsilon, \delta > 0$ such that
$$F(x,u) \ge \varepsilon_n^2 M_n, \quad \text{for } |x - x_0| \le \delta \text{ and } |u| = \varepsilon_n$$
$$F(x,u) \ge -a_2 u^2, \quad \text{for } |x - x_0| \le \delta \text{ and } |u| \le \varepsilon.$$

Now we give our main results.

**Theorem 1.2.** *Assume that* (A9)–(A11) *are satisfied. Then, equation* (1.1) *possesses a sequence of solutions* $(u_k)$ *such that*

$$\frac{1}{2}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u_k(x)-u_k(z)|^2}{|x-z|^{n+2\alpha}}\,dz\,dx+V(x)u_k^2\Big)dx-\int_{\mathbb{R}^n}F(x,u_k)dx\to 0^-$$

*as* $k\to\infty$.

**Corollary 1.3.** *Assume that* (A9), (A10) *are satisfied and*

(A11') *there exist* $x_0\in\mathbb{R}$ *and a constant* $\delta>0$, *such that*

$$\begin{gathered}\liminf_{|u|\to 0}\inf_{|x-x_0|\le\delta}\frac{F(x,u)}{|u|^2}>-\infty,\\ \limsup_{|u|\to 0}\inf_{|x-x_0|\le\delta}\frac{F(x,u)}{|u|^2}=+\infty.\end{gathered}$$

*Then, equation* (1.1) *possesses a sequence of solutions* $(u_k)$ *such that*

$$\frac{1}{2}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u_k(x)-u_k(z)|^2}{|x-z|^{n+2\alpha}}\,dz\,dx+V(x)u_k^2\Big)dx-\int_{\mathbb{R}^n}F(x,u_k)dx\to 0^-$$

*as* $k\to\infty$.

The remainder part of this article is organized as follows. Some preliminary results are presented in Section 2. In Section 3, we give the proofs of our main results.

## 2. Variational setting and preliminaries

In this section, we recall some preliminary results which will be useful in this article. First, we will give some facts of the fractional order Sobolev spaces. For any $0<\alpha<1$, the fractional Sobolev space $H^\alpha(\mathbb{R}^n)$ is defined by

$$H^\alpha(\mathbb{R}^n)=\Big\{u\in L^2(\mathbb{R}^n):\frac{|u(x)-u(z)|}{|x-z|^{\frac{n+2\alpha}{2}}}\in L^2(\mathbb{R}^n\times\mathbb{R}^n)\Big\},$$

endowed with the natural norm

$$\|u\|_\alpha^2=\int_{\mathbb{R}^n}|u(x)|^2dx+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}}\,dz\,dx.$$

For the reader's convenience, we review the main embedding result for this class of fractional Sobolev spaces.

**Lemma 2.1** ([17])**.** *Let* $0<\alpha<1$ *such that* $2\alpha<n$. *Then there exists a constant* $c_{2_\alpha^*}$, *such that*

$$\|u\|_{L^{2_\alpha^*}(\mathbb{R}^n)}\le c_{2_\alpha^*}\|u\|_\alpha \tag{2.1}$$

*for every* $u\in H^\alpha(\mathbb{R}^n)$, *where* $2_\alpha^*=\frac{2n}{n-2\alpha}$ *is the fractional critical exponent. Moreover, the embedding* $H^\alpha(\mathbb{R}^n)\subset L^p(\mathbb{R}^n)$ *is continuous for any* $p\in[2,2_\alpha^*]$ *and is locally compact whenever* $p\in[2,2_\alpha^*)$.

**Remark 2.2.** Consider the fractional Schrödinger equation

$$\begin{gathered}(-\Delta)^\alpha u+\widehat{V}(x)u=\widehat{f}(x,u)\\ u\in H^\alpha(\mathbb{R}^n,\mathbb{R}),\end{gathered}\tag{2.2}$$

where $\widehat{V}(x)=V(x)+a_0$ and $\widehat{F}(x,u)=F(x,u)+\frac{a_0}{2}u^2$. Then (2.2) is equivalent to (1.1) and it easy to check that the hypotheses ($A9$) and (A10), (A11) still hold for $\widehat{V}$ and $\widehat{F}$ provided that those hold for $V$ and $F$. Hence, in what follows, we always assume without loss of generality that $V(x)\geq 1$ for all $x\in\mathbb{R}^n$ and $\int_{\mathbb{R}^n}\frac{1}{V(x)}dx<\infty$.

In view of Remark 2.2, we consider the space

$$\begin{aligned} H_V^\alpha(\mathbb{R}^n)=\Big\{u\in H^\alpha(\mathbb{R}^n):\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}}\,dz\,dx\\ +\int_{\mathbb{R}^n}V(x)|u(x)|^2dx<+\infty\Big\}; \end{aligned}$$

equipped with the norm

$$\|u\|_V^2=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}}\,dz\,dx+\int_{\mathbb{R}^n}V(x)|u(x)|^2dx;$$

and the inner product

$$\langle u,v\rangle_V=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{[u(x)-u(z)][v(x)-v(z)]}{|x-z|^{n+2\alpha}}\,dz\,dx+\int_{\mathbb{R}^n}V(x)u(x)v(x)dx.$$

Then $H_V^\alpha(\mathbb{R}^n)$ is a Hilbert space with this inner product.

**Lemma 2.3.** [46] *If $V$ satisfies* (A9)*, then $H_V^\alpha$ is continuously embedded in $H^\alpha(\mathbb{R})$.*

**Lemma 2.4.** *If $V$ satisfies* (A9)*, then $H_V^\alpha$ is continuously embedded in $L^1$.*

*Proof.* By (A9) and Hölder's inequality, for all $u\in H_V^\alpha$ we have

$$\begin{aligned} \int_{\mathbb{R}^n}|u|dt&=\int_{\mathbb{R}^n}|(V(x))^{-1/2}(V(x))^{1/2}u|dx\\ &\leq\int_{\mathbb{R}^n}(V(x))^{-1/2}|(V(x))^{1/2}u|dx\\ &\leq\Big(\int_{\mathbb{R}^n}(V(x))^{-1}dt\Big)^{1/2}\Big(\int_{\mathbb{R}^n}V(x)u^2dx\Big)^{1/2}\\ &\leq\Big(\int_{\mathbb{R}^n}(V(x))^{-1}dx\Big)^{1/2}\|u\|_V^2. \end{aligned} \tag{2.3}$$

□

**Lemma 2.5.** *If $V$ satisfies* (A9) *then $H_V^\alpha$ is compactly embedded in $L^1$.*

*Proof.* Let $(u_n) \subset H^\alpha_V$ be a bounded sequence such that $u_n \rightharpoonup u$ in $H^\alpha_V$. We will show that $u_n \to u$ in $L^1$. By Hölder inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |u_n - u| dx \\
&= \int_{|x|\le R} |u_n - u| dx + \int_{|x|>R} |u_n - u| dx \\
&\le 2R\Big(\int_{|x|\le R} |u_n - u|^2 dx\Big)^{1/2} + \int_{|x|>R} |(V(x))^{-1/2}(V(x))^{1/2}(u_n - u)| dx \\
&\le 2R\Big(\int_{|x|\le R} |u_n - u|^2 dx\Big)^{1/2} + \int_{|u|>R} (V(x))^{-\frac{1}{2}} |(V(x))^{1/2}(u_n - u)| dx \\
&\le 2R\Big(\int_{|x|\le R} |u_n - u|^2 dx\Big)^{1/2} \\
&\quad + \Big(\int_{|x|>R} (V(x))^{-1} dx\Big)^{1/2}\Big(\int_{|x|>R} V(x)(u_n - u)^2 dx\Big)^{1/2} \\
&\le 2R\Big(\int_{|x|\le R} |u_n - u|^2 dx\Big)^{1/2} + \Big(\int_{|x|>R} (V(x))^{-1} dx\Big)^{1/2} \|u_n - u\|_V,
\end{aligned} \tag{2.4}
$$

where $R > 0$. Since the embedding is compact on bounded domain then, by (A9) and (2.4), we have $u_n \to u$ in $L^1$. $\square$

## 3. Proofs of main results

The aim of this section is to establish the proofs of Theorem 1.2 and Corollary 1.3. For this purpose, we need to modify $F(x, u)$ for $u$ outside a neighborhood of the origin to get a globally defined $\widetilde{F}(x, u)$ as follows: Choose a constant $t_0 \in (0, \frac{\rho}{2})$ and define a cut-off function $\chi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$
\begin{gathered}
\chi(t) = \begin{cases} 1 & \text{if } 0 \le t \le t_0 \\ 0 & \text{if } t \ge 2t_0 \end{cases} \\
-\frac{2}{t_0} \le \chi'(t) < 0 \quad \text{for } t_0 < t < 2t_0.
\end{gathered} \tag{3.1}
$$

Let $\widetilde{F}(x, u) = \chi(|u|)F(x, u)$, for all$(x, u) \in \mathbb{R}^n \times \mathbb{R}$. By (3.1) and (A10) we have, for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}$,

$$
|\widetilde{F}(x, u)| \le c_1|u|, \quad |\widetilde{f}(x, u)| \le c_2. \tag{3.2}
$$

Now we consider the modified fractional Schrödinger equation

$$
\begin{gathered}
(-\Delta)^\alpha u + V(x)u = \widetilde{f}(x, u), \\
u \in H^\alpha(\mathbb{R}^n, \mathbb{R}),
\end{gathered} \tag{3.3}
$$

Define the functional $I : H^\alpha_V \to \mathbb{R}$ associated with (3.3) by

$$
\begin{aligned}
I(x) &= \frac{1}{2}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2\alpha}}\, dz\, dx + \int_{\mathbb{R}^n} V(x)|u(x)|^2 dx\Big) \\
&\quad - \int_{\mathbb{R}^n} \widetilde{F}(x, u(x)) dx \\
&= \frac{1}{2}\|u\|^2_V - \int_{\mathbb{R}^n} \widetilde{F}(x, u(x)) dx.
\end{aligned} \tag{3.4}
$$

Then, by (A9), (A10) and (3.2), we see that $I$ is a continuously Fréchet-differentiable functional defined on $H^\alpha_V$; i.e., $I \in C^1(H^\alpha_V, \mathbb{R})$. Moreover, we have

$$\begin{aligned} I'(u)v = &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{[u(x)-u(z)][v(x)-v(z)]}{|x-z|^{n+2\alpha}}\,dz\,dx + \int_{\mathbb{R}^n} V(x)u(x)v(x)dx \\ &- \int_{\mathbb{R}^n} \widetilde{f}(x,u(x))v(x)dx, \end{aligned} \tag{3.5}$$

for all $u, v \in H^\alpha_V$. According to [46], we know that in order to find solutions of (3.3), it suffices to obtain the critical points of $I$. For this purpose we recall the following definitions and results (see [26, 30]).

**Definition 3.1.** Let $E$ be a real Banach space and $\phi \in C^1(E, \mathbb{R})$.

- $\phi$ is said to satisfy the (PS) condition if any sequence $(x_k) \subset E$ for which $(\phi(x_k))$ is bounded and $\phi'(x_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in $E$.
- Set $\Sigma = \{A \subset E\backslash\{0\} : A \text{ is closed and symmetric with respect to the origin}\}$. For $A \in \Sigma$, we say genus of $A$ is $n$ (denoted by $\kappa(A) = n$), if there is an odd map $\varphi \in C(A, \mathbb{R}^n\backslash\{0\})$, and $n$ is the smallest integer with this property.

**Lemma 3.2** ([26, Theorem 1])**.** *Let $\phi$ be an even $C^1$ functional on $E$ with $\phi(0) = 0$. Suppose that $\phi$ satisfies the $(PS)$ condition and*

(1) *$\phi$ is bounded from below.*
(2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Sigma_k$ such that $\sup_{x\in A_k} \phi(x) < 0$, where $\Sigma_k = \{A \in \Sigma : \kappa(A) \geq k\}$.*

*Then either (i) or (ii) below holds.*

(i) *There exists a sequence $(x_k)$ of critical point such that $\phi(x_k) < 0$ and $\lim_{k\to\infty} x_k = 0$.*
(ii) *There exists two sequences of critical points $(x_k)$ and $(y_k)$ such that $\phi(x_k) = 0, x_k \neq 0$, $\lim_{k\to\infty} x_k = 0$, $\phi(y_k) < 0, \lim_{k\to\infty} \phi(y_k) = 0$, and $(y_k)$ converges to a non-zero limit.*

**Lemma 3.3.** *If* (A9), (A10) *are satisfied, then $I$ is bounded from below and satisfies the $(PS)$ condition.*

*Proof.* By (A10), (2.3), (3.2) and the Hölder inequality, we have, for all $u \in H^\alpha_V$,

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|_V^2 - c_3 \int_{\mathbb{R}^n} |u|dx \\ &\geq \frac{1}{2}\|u\|_V^2 - c_3\Big(\int_{\mathbb{R}^n} (V(x))^{-1}dx\Big)^{1/2}\|u\|_V. \end{aligned} \tag{3.6}$$

Then it follows that $I$ is bounded from below. Moreover, if we take $(u_n) \subset H^\alpha_V$ be a $(PS)$-sequence, then by (3.2) and (3.4), we have

$$c_4 \geq \frac{1}{2}\|u_n\|_V^2 - c_5\Big(\int_{\mathbb{R}^n} (V(x))^{-1}dx\Big)^{1/2}\|u\|_V$$

This implies that $(u_n)$ is bounded in $H^\alpha_V$. Thus there exists a subsequence $(u_{n_k})$ such that $u_{n_k} \rightharpoonup u_0$ as $k \to \infty$ for some $u_0 \in H^\alpha_V$. By Lemma 2.5, it holds that

$$u_{n_k} \to u_0 \quad \text{in } L^1 \text{ as } k \to \infty.$$

This together with (3.2) yields

$$\Big|\int_{\mathbb{R}^n}(\widetilde{f}(x,u_{n_k})-\widetilde{f}(x,u_0))(u_{n_k}-u_0)dx\Big|\le c_6\int_{\mathbb{R}^n}|u_{n_k}-u_0|dx\to 0 \tag{3.7}$$

as $k\to\infty$.

Noting that $(u_n)$ is a bounded $(PS)$-sequence, we have

$$(I'(u_{n_k})-I'(u_0))(u_{n_k}-u_0)\to 0 \text{ as } k\to\infty. \tag{3.8}$$

Combining (3.5), (3.7) and (3.8), we obtain

$$\begin{aligned}\|u_{n_k}-u_0\|_V^2&=(I'(u_{n_k})-I'(u_0))(u_{n_k}-u_0)\\&\quad+\int_{\mathbb{R}^n}(\widetilde{f}(x,u_{n_k})-\widetilde{f}(x,u_0)).(u_{n_k}-u_0)dx\to 0.\end{aligned}$$

$\square$

*Proof of Theorem 1.2.* For simplicity, we assume that $x_0=0$ in (A11). For $r>0$, let

$$D(r):=\{(x_1,x_2,x_3,\dots,x_n):0\le x_i\le r, i=1,2,3,\dots,n\}.$$

Fix $r>0$ small enough such that $D(r)\subset B(0,\delta)$, where $\delta$ is the constant given in (A11). For arbitrary $k\in\mathbb{N}$, we construct an $A_k\in\sum_k$ satisfying $\sup_{u\in A_K}I(u)<0$. Indeed, we follow the idea of dealing with elliptic problems in Kajikiya [26]. Let $m\in\mathbb{N}$ be the smallest integer such that $m^n\ge k$. We divide $D(r)$ equally into $m^n$ small cubes by planes parallel to each face of $D(r)$ and denote them by $D_i$ with $1\le i\le m^n$. We consider a cube $E_i\subset D_i$ $(i=1,2,\dots,k)$ such that $E_i$ has the same center as that of $D_i$, the faces of $E_i$ and $D_i$ are parallel and the edge of $E_i$ has length $\frac{a}{2}$. Define $\xi\in C_0^\infty(\mathbb{R},[0,1])$ such that $\xi(t)=1$ for $t\in[\frac{a}{4},\frac{3a}{4}]$, $\xi(t)=0$ for $t\in(-\infty,0]\cup[a,+\infty)$. Define

$$\zeta(x)=\xi(x_1)\xi(x_2)\xi(x_3)\dots\xi(x_n),(x_1,x_2,x_3,\dots,x_n)\in\mathbb{R}^n.$$

Then $\operatorname{supp}\zeta\subset[0,a]^n$. Now for each $1\le i\le k$, we can choose a suitable $y_i\in\mathbb{R}^n$ and define

$$\zeta_i(x)=\zeta(x-y_i),\quad \text{for all } x\in\mathbb{R}^n;$$

such that

$$\operatorname{supp}\zeta_i\subset D_i,\quad \operatorname{supp}\zeta_i\cap\operatorname{supp}\zeta_j=\emptyset\quad (i\ne j), \tag{3.9}$$

and

$$\zeta_i(t)=1,\quad \forall x\in E_i,\ 0\le\zeta_i(x)\le 1,\ \forall x\in\mathbb{R}^n$$

Set

$$\begin{gathered}\Theta_k\equiv\big\{(l_1,l_2,\dots,l_k)\in\mathbb{R}^k;\max_{1\le i\le k}|l_i|=1\big\},\\ S_k\equiv\Big\{\sum_{i=1}^k l_i\zeta_i;(l_1,l_2,\dots,l_k)\in\Theta_k\Big\}.\end{gathered} \tag{3.10}$$

Then $\Theta_k$ is homeomorphic to the unit sphere in $\mathbb{R}^k$ by an odd mapping. Thus $\kappa(\Theta_k)=k$. If we define the following odd and homeomorphic mapping: $\psi:\Theta_k\to S_k$ by

$$\psi(l_1,l_2,\dots,l_k)=\sum_{i=1}^k l_i\zeta_i,\quad \forall(l_1,l_2,\dots,l_k)\in\Theta_k,$$

Then $\kappa(S_k) = \kappa(\Theta_k) = k$. Moreover, it is evident that $S_k$ is compact and hence there is a constant $\lambda_k > 0$ such that

$$\|u\|_V \le \lambda_k, \quad \forall u \in S_k. \tag{3.11}$$

For any $s \in (0, \varepsilon)$, $u = \sum_{i=1}^k l_i\zeta_i \in S_k$ and by (3.2) and (3.4), we have

$$\begin{aligned} I(su) &\le \frac{s}{2}\|x\|_V^2 - \int_{\mathbb{R}^n} F\Big(x, s\sum_{i=1}^k l_i\zeta_i\Big)dx \\ &\le \frac{s^2\lambda_k^2}{2} - \sum_{i=1}^k \int_{D_i} F(x, sl_i\zeta_i)dx. \end{aligned} \tag{3.12}$$

By (3.10), there exists an integer $i_0 \in [1, k]$ such that $|l_{i_0}| = 1$. Then it follows that

$$\begin{aligned} \sum_{i=1}^k \int_{D_i} F(x, sl_i\zeta_i)dx &= \int_{E_{i_0}} F(x, sl_{i_0}\zeta_{i_0})dx + \int_{D_{i_0}\setminus E_{i_0}} F(x, sl_{i_0}\zeta_{i_0})dx \\ &\quad + \sum_{i\neq i_0} \int_{D_i} F(x, sl_i\zeta_i)dx. \end{aligned} \tag{3.13}$$

Noting that $|l_{i_0}| = 1$, $\zeta_{i_0} \equiv 1$ on $E_{i_0}$, and $F(x, u)$ is even in $u$, we get

$$\int_{E_{i_0}} F(x, sl_{i_0}\zeta_{i_0})dx = \int_{E_{i_0}} F(x, s)dx. \tag{3.14}$$

By (A10),

$$\int_{D_{i_0}\setminus E_{i_0}} F(x, sl_{i_0}\zeta_{i_0})dx + \sum_{i\neq i_0} \int_{D_i} F(x, sl_i\zeta_i)dx \ge -c_k s^2. \tag{3.15}$$

Here $c_k > 0$ depends only on $k$. Combining (3.11)-(3.15), one has

$$I(su) \le \frac{s^2\lambda_k^2}{2} + c_k s^2 - \int_{E_{i_0}} F(x, s)dx.$$

Substituting $s = \varepsilon_n$ and using (A11), we obtain

$$I(\varepsilon_n u) \le \varepsilon_n^2\Big(\frac{s^2\lambda_k^2}{2} + c_k - (\frac{a}{2})^2 M_n\Big).$$

Since $\varepsilon_n \to 0^+$ and $M_n \to \infty$, we choose $n_0$ large enough such that the right side of the last inequality is negative. Define

$$A_k = \{\varepsilon_{n_0} u; u \in S_k\}.$$

Then, we have

$$\kappa(A_k) = \kappa(S_k) = k \quad \text{and} \quad \sup_{x\in A_k} I(x) < 0.$$

Consequently, by Lemma 3.3, there exist a sequence of nontrivial critical points $(u_k)$ of $I$ such that $I(u_k) \le 0$ for all $k \in \mathbb{N}$ and $u_k \to 0$ in $H_V^\alpha$ as $k \to \infty$. Hence, $(u_k)$ is a sequence of solutions of (3.3). Therefore, for $k$ large enough, they are solutions of (1.1). $\square$

*Proof of Corollary 1.3.* By (A11'), there exist a constant $x_0 \in \mathbb{R}^n$, two sequences of positives numbers $\varepsilon_n \to 0$, $M_n \to \infty$ as $n \to \infty$ and constants $a_2, \varepsilon, \delta > 0$ such that

$$F(x,u) \geq \varepsilon_n^2 M_n, \quad \text{for } |x - x_0| \leq \delta \text{ and } |u| = \varepsilon_n,$$
$$F(x,u) \geq -a_2 u^2, \quad \text{for } |x - x_0| \leq \delta \text{ and } |u| \leq \varepsilon,$$

which implies the condition (A11). An easy application of Theorem 1.2 shows that Corollary 1.3 holds. This completes the proof. □

**Acknowledgments.** The author would like to thank the anonymous referees for their careful reading, critical comments, and helpful suggestions, which helped us improve the quality of this article.

## References


[1] V. Ambrosio, G. M. Figueiredo; *Ground state solutions for a fractional Schrödinger equation with critical growth*, Asymptotic Anal., 105 (2017), 159-191.

[2] T. Bartsch, Z. Wang; *Existence and multiplicity results for some superlinear elliptic problems on* $\mathbb{R}^n$, Commun. in PDE, 20 (1995), 1725-1741.

[3] T. Bartsch, Z. Tang; *Multibump solutions of nonlinear Schrödinger equations with steep potential well and indefinite potential*, Discrete Contin. Dyn. Syst., 33, (2013), 7-26.

[4] J. Bertoin; *Lévy Processes* Cambridge Tracts in Mathematics, 121, Cambridge University Press, Cambridge, 1996.

[5] A. Benhassine; *Multiplicity of solutions for nonperiodic perturbed fractional Hamiltonian equations*, Electron. J. of Differential Equations, 2017 no. 93 (2017), 1-15.

[6] A. Benhassine; *Multiple of homoclinic solutions for a perturbed dynamical systems with combined nonlinearities*, Medit. J. Math., 14, (3), 1-20.

[7] G. M. Bisci, V. D. Radulescu and R. Servadei; *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of Mathematics and its Applications, Vol. 162, Cambridge University Press, Cambridge, 400 pp., 2016.

[8] G. M. Bisci, V. D. Radulescu; *Ground state solutions of scalar field fractional Schrödinger equations*, Calculus of Variations and Partial Differential Equations, 54 (2015), 2985-3008

[9] L. Caffarelli, L. Silvestre; *An extension problems related to the fractional Laplacian*, Comm. PDE, 32 (2007), 1245C1260.2.

[10] X. Cabré, J. Tan; *Positive solutions of nonlinear problems involving the square root of the Laplacian*, Adv. Math., 224 (2010), 2052-2093.

[11] A. Capella, J. Dávila, L. Dupaigne, Y. Sire; *Regularity of radial extremal solutions for some non local semilnear equations*, Comm. PDE, 36 (2011), 1353-1384.

[12] M. Cheng; *Bound state for the fractional Schrödinger equation with undounded potential*, J. Math. Phys., 53 (2012), 043-507.

[13] G. Chen, Y. Zheng; *Concentration phenomenon for fractional nonlinear Schrödinger equations*, Comm. Pure Appl. Anal., 13 (2014), 2359-2376.

[14] R. Cont, P. Tankov; *Financial Modelling with Jump Processes*, Chapman & Hall/CRC Financ. Math. Ser., Chapman & Hall/CRC press, Boca Raton, FL, 2004.

[15] J. Dvila, M. Del Pino, J. Wei; *Concentrating standing waves for the fractional nonlinear Schrödinger equation*, J. Differential Equations, 256 (2014), 858-892.

[16] J. Dvila, M. Del Pino, S. Dipierro, E. Valdinoci; *Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum*, Anal. PDE, 8 (2015), 1165-1235.

[17] E. Di Nezza, G. Patalluci, E. Valdinoci; *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. math., 136 (2012), 521-573.

[18] S. Dipierro, G. Palatucci, E. Valdinoci; *Existence and symmetry results for a schrödinger type problem involving the fractional laplacian*, Matematiche (Catania), 68 (2013) 201-216.

[19] J. Dong and M.Xu; *Some solutions to the space fractional Schrödinger equation using momentum representation method*, J. Math. Phys., 48 (2007), 072105.

[20] M. Fall, F. Mahmoudi, E. Valdinoci; *Ground states and concentration phenomena for the fractional Schrödinger equation*, Nonlinearity, 28 (2015), 1937-1961.

[21] P. Felmer, A. Quaas, J. Tan; *Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh Sect. A., 142 (2012), 1237-1262.
[22] P. Felmer and C. Torres; *Non-linear Schrödinger equation with non-local regional diffusion*, Calc. Var., 54 (2015), 75-98.
[23] P. Felmer, C. Torres; *Radial symmetry of ground state for a fractional nonlinear Schrödinger equation*, Comm. Pure and Applied Ana., 13 (2014), 2395-2406.
[24] G. M. Figueiredo, G. Siciliano; *A multiplicity result via Ljusternick-Schnirelmann category and Morse theory for a fractional Schrödinger equation in* $\mathbb{R}^N$, Nonlinear Differ. Equ. Appl. (2016) 23: 12.
[25] X. Guo, M. Xu; *Some physical applications of fractional Schrödinger equation*, J. Math. Phys., 47 (2006), 082104.
[26] R. Kajikiya; *A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations*, J. Funct Anal 225 (2005), 352-370
[27] N. Laskin; *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A, 268 (2000), 298-305.
[28] N. Laskin; *Fractional Schrödinger equation*, Phys. Rev. E, 66 (2002), 056108.
[29] E. de Oliveira, F. Costa, J. Vaz; *The fractional Schrödinger equation for delta potentials*, J. Math. Phys., 51 (2012), 123-517.
[30] P. Rabinowitz; *Minimax method in critical point theory with applications to differential equations*, CBMS Amer. Math. Soc., 65, 1986.
[31] P. Rabinowitz; *On a class of nonlinear Schrödinguer equations*, ZAMP, 43 (1992), 270-291.
[32] V. D. Radulescu, M. Xiang, B. Zhang; *Existence of solutions for perturbed fractional p-Laplacian equations*, J. of Differential Equations 260 (2016), 1392-1413
[33] V. D. Radulescu, M. Xiang, B. Zhang; *Existence of solutions for a bi-nonlocal fractional p-Kirchhoff type problem*, Computers and Mathematics with Applications 71 (2016), 255-266
[34] R. Servadei, E. Valdinoci; *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst., 33 (2013), 2105-2137.
[35] R. Servadei, E. Valdinoci; *A Brezis-Nirenberg result for non-local critical equations in low dimension*, Commun. Pure Appl. Anal., 12 (2013) 2445-2464.
[36] S. Secchi; *Perturbation results for some nonlinear equations involving fractional operators*, 08 2012, http://arxiv.org/abs/1208.2644.
[37] S. Secchi; *Ground state solutions for nonlinear fractional Schrödinger equations in* $\mathbb{R}^n$, J. Math. Phys., 54 (2013), 031501.
[38] S. Secchi; *On fractional Schrödinger equations in* $\mathbb{R}^N$ *without the Ambrosetti-Rabinowitz condition*. Topological Methods in Nonlinear Analysis. 47(1) (2016), 19-41.
[39] J. Seok; *Spike-layer solutions to nonlinear fractional Schröding equations with almost optimal nonlinearities*, Electron. J. Differential Equations, 2015 no. 196 (2015), 1-19.
[40] X. D. Shang, J. H. Zhang; *Concentrating solutions of nonlinear fractional Schrödinger equation with potentials*, J. Differ. Equ., 258(4) (2015), 1106-1128.
[41] L. Silvestre; *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math., 60 (2007), 67-112.
[42] J. Tan; *The Brezis-Nirenberg type problem involving the square root of the Laplacian*, Calc. Var. Partial Differential Equations, 42 (2011), 21-41.
[43] X. H. Tang; *Infinitely many solutions for semilinear Schrödinger equations with sign-changing potential and nonlinearity*, J. Math. Anal. Appl., 401 (2013) 407-415.
[44] K. M. Teng; *Multiple solutions for a class of fractional Schrödinger equations in* $\mathbb{R}^n$, Nonlinear Anal. Real World Appl., 71 (2015), 4927-4934.
[45] K. M. Teng, X. M. He; *Ground state solutions for fractional Schrödinger equations with critical Sobolev exponent*, Commun. Pure Apple. Anal., 16 (2016), 991-1008.
[46] C. Torres; *Existence and concentration of solutions for a non-linear fractional Schrödinger with steep potential well*, Com. Pure and Appl. Anal., 15 (2016), 535-547.
[47] Y. H. wei, X. F. Su; *Multiplicity of solutions for non-local elliptic equations driven by the fractional Laplacian*, Calc. Var., 52 (2015), 95-124.
[48] L. Wang, B. Zhang ; *Infinitely many solutions for Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian and critical exponent*, Electronic J. of Differential Equations, 2016 no. 339 (2016), 1-18.
[49] J. J. Zhang, J. M. do O, M. Squassina; *Fractional Schrödinger-Poisson systems with a general subcritical or critical nonlinearity*, Adv. Nonlinear Stud. 16 (1) (2016), 15-30.

[50] J. Zhang, W. Jiang; *Existence and concentration of solutions for a fractional Schrödinger equations with sublinear nonlinearity*, arXiv:1502.02221v1.
[51] W. Zou; *Variant fountain theorems and their applications*, Manuscripta Math., 104 (2001), 343-358.
[52] W. Zou, M. Schechter; *Critical Point Theory and Its Applications*, Springer, New York, 2006.
[53] X. Zhang, B. L. Zhang, D. Repovš; *Existence and symmetry of solutions for critical fractional Schrödinger equations with bounded potentials*. Nonlinear Analysis, 142 (2016), 48-68.

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