Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 116, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

INITIAL BOUNDARY VALUE PROBLEM FOR A MIXED PSEUDO-PARABOLIC p-LAPLACIAN TYPE EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. We consider the initial boundary value problem for a mixed pseudo-parabolic p-Laplacian type equation with logarithmic nonlinearity. Constructing a family of potential wells and using the logarithmic Sobolev inequality, we establish the existence of global weak solutions. we consider two cases: global boundedness and blowing-up at ∞. Moreover, we discuss the asymptotic behavior of solutions and give some decay estimates and growth estimates.

1. Introduction

In this article we study the following initial-boundary value problem for a nonlinear evolution equation with logarithmic source

$$\begin{gathered}
u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) - k\triangle u_t = |u|^{p-2}u\log|u|, \quad \Omega\times(0,T), \\
u(x,t)=0, \quad \partial\Omega\times(0,T), \\
u(x,0)=u_0(x), \quad \Omega\,,
\end{gathered} \tag{1.1}$$

where $1<p<2$, $u_0\in H_0^1(\Omega)$, $T\in(0,+\infty]$, $k\geq 0$, $\Omega\subset\mathbb{R}^n (n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$.

Problem (1.1) is a mixed pseudo-parabolic p-Laplacian type equation, whose abstract form was first considered by Showalter [18], and sometimes referred to as Showalter equation [1]. When $k=0$, (1.1) is the classical fast diffusive p-Laplacian, which appears to be relevant in the theory of non-Newtonian fluids. When $k>0$, (1.1) belongs to the pseudo-parabolic equations, which are characterized by the occurrence of first-order partial derivative in time of the highest order term [19]. These equations arise from a variety of important physical processes, such as the flows of fluids through fissured rock [3], nonlinear dispersive long waves [4], the heat conduction involving two temperatures [8], the aggregation of populations [9], etc. Particularly, (1.1) is from shearing flows of incompressible simple fluids [2]. The quantity $|\nabla u|^{p-2}\nabla u + k\nabla u_t$ can be viewed as approximation to the stress functional in such a flow, and $k\nabla u_t$ can be interpreted as viscous relaxation effects. On the other hand, when considering the influence of many factors, such as the

2010 *Mathematics Subject Classification.* 35K58, 35K35, 35B40.
Key words and phrases. Pseudo-parabolic; p-Laplacian; logarithmic nonlinearity; long time behavior.

Submitted December 26, 2017. Published May 14, 2018.

molecular and ion effects, the nonlinear term $\nabla(|\nabla u|^{p-2}\nabla u)$ appears to replace Δu in pseudo-parabolic models.

Let us introduce the research on the asymptotic behavior of solutions that related to our work. We mainly review the following three aspects.

(i) For the fast diffusive p-Laplacian equations, Jin et al [23] considered the initial boundary value problem of the equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = u^q,$$

with $0 < p < 2$ and $q > 0$. They determined both the critical extinction exponent $q_0 = p - 1$ and the critical blow-up exponent $q_c = 1$. Lately, Qu et al [16] and Li et al [13] extended the critical exponent results to the sign-changing solutions for p-Laplacian equations with nonlocal source $|u|^q - \frac{1}{|\Omega|}\int_\Omega |u|^q dx$.

(ii) For the pseudo-parabolic equation

$$u_t - \Delta u_t - \Delta u = u^q, \tag{1.2}$$

Cao et al [5] studied the Cauchy problem of (1.2) and obtained the complete Fujita type result with showing $q_c = 1 + \frac{2}{n}$. For the initial boundary value problem of (1.2), via the potential well method, Xu et al [22] also confirmed the Fujita exponent $q_c = \infty$ $(n = 1, 2)$ and $q_c = \frac{n+2}{n-2}$ $(n \geq 3)$ with bounded initial energy. Lately, Chen et al. [7] carried out the research on pseudo-parabolic equations with logarithmic source

$$u_t - \Delta u_t - \Delta u = u \log |u|, \tag{1.3}$$

and found the blowing-up at ∞ of the solutions, which with [22] reveal that the polynomial nonlinearity is an important condition for the solutions to be blow-up in finite time.

(iii) Recently, Le et al [12] investigated (1.1) with $p > 2$. Owing to the slow diffusion, there exist both global existence and blowing-up in finite time of the weak solutions, under the same conditions in [7]. Moreover, Le et al gave the large time decay of the global weak solutions.

In this article, we would like to reveal the effect from fast diffusive, pseudo-parabolic viscosity and logarithmic nonlinearity on the asymptotic behavior of solutions. First, different from the case $p > 2$, we prove that the weak solutions of (1.1) are global and can not blow up in finite time. This means that the fast diffusion is dominant, and the logarithmic source is not strong enough to cause blowing-up in finite time. Next, similar to [7], we find the sufficient conditions to divide the global boundedness and blowing-up at ∞ of the weak solutions (Theorems 4.1 and 5.1). Moreover, we derive some decay estimates of the global bounded solutions, namely Theorem 4.2, as while as some growth estimates of the unbounded solutions, namely Theorem 5.3. From Theorem 4.2, the global bounded solutions of the 1-D case decay exponentially, which is the same as the case $p = 2$, while different from the algebraical decay of the case $p > 2$. Theorem 4.2 also tells us that the upper bound of the decay rate are proportional to k, which seems that the pseudo-parabolic viscosity slows down the decay. From Theorem 5.3 and Theorem 2.3, the weak solutions that blow up at ∞ grow algebraically. Theorem 5.3 also indicates that the lower bound of growth estimates is smaller than that of the case $p = 2$, which is caused by the fast diffusion.

Here we exploit the potential well method which was proposed by Sattinger et al [17]. Liu et al [14, 15] generalized and improved the method by introducing a family of potential wells which include the known potential well as a special case.

Nowadays, it is one of the most useful method for proving global existence and nonexistence of solutions, and vacuum isolating of solutions for parabolic equations [6, 21].

This article is organized as follows. In Section 2, we prove the global existence and uniqueness of the weak solution. Section 3 gives some preliminary lemmas of the potential wells. In Section 4, we treat the global bounded case and the decay estimates. Section 5 is devoted to the blow-up at ∞ and the growth estimates.

2. Global existence and uniqueness

We start this section with the definition of the weak solutions. Set

$$E = \left\{u \in C(0, T; H_0^1(\Omega)); u_t \in L^2(0, T; H_0^1(\Omega))\right\}.$$

Definition 2.1. A function $u(x,t)$ is said to be a weak solution of (1.1), if $u \in E$, $u(x,0) = u_0(x) \in H_0^1(\Omega)$, it holds

$$(u_t, \varphi)_2 + (|\nabla u|^{p-2}\nabla u, \nabla\varphi)_2 + k(\nabla u_t, \nabla\varphi)_2 = (|u|^{p-2}u \log|u|, \varphi)_2, \tag{2.1}$$

for any $\varphi \in H_0^1(\Omega)$, and for a.e. $t \in (0,T)$, where $(\cdot,\cdot)_2$ means the inner product of $L^2(\Omega)$.

Lemma 2.2 (Imbedding inequality). *For any function $u \in W_0^{1,q}(\Omega)$, we have the inequality*

$$\|u\|_p \le C(p, q, n, \Omega)\|\nabla u\|_q,$$

for all $1 \le p \le q^*$, where $q^* = \frac{nq}{n-q}$ if $n > q$ and $q^* = \infty$ if n=q.

Theorem 2.3 (Global existence and uniqueness). *Assume that $u_0(x) \in H_0^1(\Omega)$. Then for any $T > 0$, the problem* (1.1) *admits a unique weak solution.*

Proof. Here we use the Galerkin approximation method to prove the existence of the global weak solutions for (1.1).

Step 1: Approximation problem. Let $\{w_j(x)\}$ be the orthogonal basis in $H_0^1(\Omega)$, which is also orthogonal in $L^2(\Omega)$. We look for the approximate solutions of the following form

$$u^m(x,t) = \sum_{j=1}^m g_j^m(t) w_j(x), \quad m = 1, 2, ...,$$

where the coefficients $g_j^m(t) = (u^m, w_j)_2$, satisfy the system of ODEs

$$\begin{gathered}
\begin{aligned}
&(u_t^m, w_j)_2 + (|\nabla u^m|^{p-2}\nabla u^m, \nabla w_j)_2 + k(\nabla u_t^m, \nabla w_j)_2 \\
&= (|u^m|^{p-2}u^m \log|u^m|, w_j)_2,
\end{aligned} \\
u_0^m(x) = \sum_{j=1}^m g_j^m(0) w_j(x) \to u_0, \quad \text{in } H_0^1(\Omega),
\end{gathered} \tag{2.2}$$

for $j = 1, 2, \dots, m$. The standard theory of ODEs, e.g. Peano's theorem, yields that $g_j^m(t) \in C^1[0,\infty)$. Thus $u^m \in C^1([0,\infty); H_0^1(\Omega))$.

Step 2: A priori estimates. We need some a priori estimates of the approximate solutions u^m. Multiplying the first equality of (2.2) by $g_j^m(t)$ and summing for j, we have

$$\frac{1}{2}\frac{d}{dt}\|u^m\|_2^2 + \frac{k}{2}\frac{d}{dt}\|\nabla u^m\|_2^2 + \|\nabla u^m\|_p^p = \int_\Omega |u^m|^p \log|u^m| dx. \tag{2.3}$$

Via a direct calculation and Lemma 2.2, it holds

$$\int_\Omega |u^m|^p \log |u^m| dx \le \frac{1}{e\alpha_0} \int_\Omega |u^m|^{p+\alpha_0} dx \le \frac{1}{e\alpha_0} \Big(\int_\Omega |\nabla u^m|^2 dx \Big)^{\frac{p+\alpha_0}{2}}, \tag{2.4}$$

where α_0 satisfies $1 \le p + \alpha_0 < 2$, e.g. we can choose $\alpha_0 = \frac{2-p}{2}$. Substituting (2.4) into (2.3), we can deduce that

$$\frac{d}{dt}\|u^m\|_2^2 + k\frac{d}{dt}\|\nabla u^m\|_2^2 dx \le \frac{2}{e\alpha_0 k^{(p+\alpha_0)/2}} \left(\|u^m\|_2^2 + k\|\nabla u^m\|_2^2\right)^{\frac{p+\alpha_0}{2}},$$

which implies

$$\begin{aligned} &\|u^m\|_2^2 + k\|\nabla u^m\|_2^2 \\ &\le \Big(\frac{2(1-\frac{p+\alpha_0}{2})t}{e\alpha_0 k^{(p+\alpha_0)/2}} + \left(\|u_0^m\|_2^2 + k\|\nabla u_0^m\|_2^2\right)^{1-\frac{p+\alpha_0}{2}} \Big)^{\frac{1}{1-\frac{p+\alpha_0}{2}}}. \end{aligned} \tag{2.5}$$

Multiplying the first equality of (2.2) by $\frac{d}{dt} g_j^m(t)$, summing for j, and integrating with respect to time from 0 to t, we obtain

$$\begin{aligned} &\int_0^t \|u_\tau^m\|_2^2 d\tau + k\int_0^t \|\nabla u_\tau^m\|_2^2 d\tau + \frac{1}{p}\|\nabla u^m\|_p^p + \frac{1}{p^2}\|u^m\|_p^p \\ &= \frac{1}{p}\|\nabla u_0^m\|_p^p - \frac{1}{p}\int_\Omega |u_0^m|^p \log |u_0^m| dx + \frac{1}{p^2}\|u_0^m\|_p^p + \frac{1}{p}\int_\Omega |u^m|^p \log |u^m| dx, \end{aligned} \tag{2.6}$$

On the one hand, the convergence of $u_0^m(x)$ gives

$$\frac{1}{p}\|\nabla u_0^m\|_p^p - \frac{1}{p}\int_\Omega |u_0^m|^p \log |u_0^m| dx + \frac{1}{p^2}\|u_0^m\|_p^p \le C(u_0),$$

for sufficiently large m, with

$$C(u_0) = \frac{1}{p}\|\nabla u_0\|_p^p - \frac{1}{p}\int_\Omega |u_0|^p \log |u_0| dx + \frac{1}{p^2}\|u_0\|_p^p + 1.$$

On the other hand, (2.4) and (2.5) tell us that

$$\frac{1}{p}\int_\Omega |u^m|^p \log |u^m| dx \le C(u_0, t)$$

with

$$C(u_0,t) = \frac{1}{pe\alpha_0 k^{(p+\alpha_0)/2}} \Big(\frac{2(1-\frac{p+\alpha_0}{2})t}{e\alpha_0 k^{(p+\alpha_0)/2}} + \left(\|u_0^m\|_2^2 + k\|\nabla u_0^m\|_2^2\right)^{1-\frac{p+\alpha_0}{2}} \Big)^{\frac{1}{\frac{2}{p+\alpha_0}-1}}.$$

Substituting the above two inequalities into (2.6), we obtain

$$\int_0^t \|u_\tau^m\|_2^2 d\tau + k\int_0^t \|\nabla u_\tau^m\|_2^2 d\tau + \frac{1}{p}\|\nabla u^m\|_p^p + \frac{1}{p^2}\|u^m\|_p^p \le C(u_0) + C(u_0,t). \tag{2.7}$$

Step 3: Passing to the limit. Therefore, from (2.5) and (2.7), for any $T > 0$, there exist $u \in L^\infty(0,T; H_0^1(\Omega))$ and a subsequence of u^m, which is still denoted by itself, such that when sending $m \to \infty$,

$$\begin{gathered} u^m \to u \quad \text{weak}\star \text{ in } L^\infty(0,T; H_0^1(\Omega)) \text{ and a.e. in } \Omega \times [0,T), \\ u_t^m \to u_t \quad \text{weakly in } L^2(0,T; H_0^1(\Omega)), \\ |\nabla u^m|^{p-2}\nabla u^m \to \chi \quad \text{weak}\star \text{ in } L^\infty(0,T; L^{\frac{p}{p-1}}(\Omega)). \end{gathered}$$

Since the convergence of u^m and u_t^m, it follows from Aubin-Lions compactness theorem that

$$u^m \to u \quad \text{strongly in } C(0,T;L^2(\Omega)),$$

which implies

$$|u^m|^{p-2}u^m \log|u^m| \to |u|^{p-2}u\log|u| \quad \text{a.e. in } \Omega\times[0,T).$$

For j fixed, we can pass to the limit in (2.2) to get

$$(u_t, w_j)_2 + (\chi, \nabla w_j)_2 + k(\nabla u_t, \nabla w_j)_2 = (|u|^{p-2}u\log|u|, w_j)_2.$$

Then for any $\varphi \in H_0^1(\Omega)$, it holds

$$(u_t, \varphi)_2 + (\chi, \nabla\varphi)_2 + k(\nabla u_t, \nabla\varphi)_2 = (|u|^{p-2}u\log|u|, \varphi)_2. \tag{2.8}$$

We only need to prove that $\chi = |\nabla u|^{p-2}\nabla u$ in the weak sense, namely

$$(\chi, \nabla\varphi)_2 = (|\nabla u|^{p-2}\nabla u, \nabla\varphi)_2, \quad \forall\varphi \in H_0^1(\Omega). \tag{2.9}$$

In fact, for any $v \in L^\infty(0,T;W_0^{1,p}(\Omega))$, $\psi \in H_0^1(\Omega)$, $0 \le \psi \le 1$, we have

$$\int_\Omega \psi\left(|\nabla u^m|^{p-2}\nabla u^m - |\nabla v|^{p-2}\nabla v\right)\nabla(u^m - v)dx \ge 0,$$

namely

$$\begin{aligned}
&\int_\Omega \psi|\nabla u^m|^{p-2}|\nabla u^m|^2 dx - \int_\Omega \psi|\nabla u^m|^{p-2}\nabla u^m\nabla v dx\\
&-\int_\Omega \psi|\nabla v|^{p-2}\nabla v\nabla(u^m - v)dx \ge 0.
\end{aligned}$$

Letting $m \to \infty$ in the above equation and noticing that

$$\begin{aligned}
&\int_\Omega \psi|\nabla u^m|^{p-2}|\nabla u^m|^2 dx\\
&= -\int_\Omega \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m)u^m\psi dx - \int_\Omega |\nabla u^m|^{p-2}\nabla u^m u^m\nabla\psi dx\\
&= -\int_\Omega u_t^m u^m\psi dx - k\int_\Omega \nabla u_t^m\nabla u^m\psi dx - k\int_\Omega \nabla u_t^m u^m\nabla\psi dx\\
&\quad + \int_\Omega |u^m|^p\log|u^m|\psi dx - \int_\Omega |\nabla u^m|^{p-2}\nabla u^m u^m\nabla\psi dx,
\end{aligned}$$

we have

$$\begin{aligned}
&-\int_\Omega u_t u\psi dx - k\int_\Omega \nabla u_t\nabla u\psi dx - k\int_\Omega \nabla u_t u\nabla\psi dx + \int_\Omega |u|^p\log|u|\psi dx\\
&-\int_\Omega \chi u\nabla\psi dx - \int_\Omega \psi\chi\nabla v dx - \int_\Omega \psi|\nabla v|^{p-2}\nabla v\nabla(u-v)dx \ge 0.
\end{aligned} \tag{2.10}$$

Choosing $\varphi = u\psi$ in (2.8), we obtain

$$\begin{aligned}
&\int_\Omega u_t u\psi dx + \int_\Omega \chi\nabla u\psi dx + \int_\Omega \chi\nabla\psi u dx\\
&+ k\int_\Omega \nabla u_t\nabla u\psi dx + k\int_\Omega \nabla u_t u\nabla\psi dx\\
&= \int_\Omega |u|^p\log|u|\psi dx.
\end{aligned} \tag{2.11}$$

Combining (2.11) with (2.10), we obtain

$$\int_\Omega \psi\left(\chi - |\nabla v|^{p-2}\nabla v\right)\nabla(u-v)dx \geq 0.$$

Choosing $v = u - \lambda\varphi$, $\lambda \geq 0$, $\varphi \in H_0^1(\Omega)$ in the above inequality, we arrive at

$$\int_\Omega \psi\left(\chi - |\nabla(u-\lambda\varphi)|^{p-2}\nabla(u-\lambda\varphi)\right)\nabla\varphi dx \geq 0.$$

Taking $\lambda \to 0$, we have

$$\int_\Omega \psi\left(\chi - |\nabla u|^{p-2}\nabla u\right)\nabla\varphi dx \geq 0, \quad \forall \varphi \in H_0^1(\Omega).$$

Obviously, if we choose $\lambda \leq 0$, we can deduce the similar inequality replacing "$\geq$" by "$\leq$". Hence, (2.9) holds. On the other hand, from (2.2) we obtain $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$. Thus u is a global weak solution of (1.1).

Step 4: Uniqueness. Suppose (1.1) admits two weak solutions u_1 and u_2. Set $w = u_1 - u_2$, then w satisfies

$$\begin{gathered} w_t - \operatorname{div}((p-1)|\nabla\overline{w}|^{p-2}\nabla w) - k\Delta w_t = ((p-1)\log|\tilde{w}|+1)|\tilde{w}|^{p-2}w, \quad \Omega\times(0,T), \\ w(x,t) = 0, \quad \partial\Omega\times(0,T), \\ w(x,0) = 0, \quad \Omega\,, \end{gathered} \tag{2.12}$$

where $\overline{w} = \theta_1 u_1 + (1-\theta_1)u_2$, $\tilde{w} = \theta_2 u_1 + (1-\theta_2)u_2$ with $\theta_1, \theta_2 \in [0,1]$.

Multiplying (2.12) by w and integrating on Ω, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 dx + \int_\Omega (p-1)|\nabla\overline{w}|^{p-2}|\nabla w|^2 dx + \frac{k}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx \\ &= \int_\Omega ((p-1)\log|\tilde{w}|+1)|\tilde{w}|^{p-2}w^2 dx. \end{aligned}$$

For any $t \in (0,T)$, integrating both side of the above equation on $(0,t)$ and noticing that $w(x,0) = 0$, we can get

$$\frac{1}{2}\int_\Omega w^2 dx + \frac{k}{2}\int_\Omega |\nabla w|^2 dx \leq \int_0^t \int_\Omega ((p-1)\log|\tilde{w}|+1)|\tilde{w}|^{p-2}w^2 dx d\tau.$$

In fact, since when $1 < p < 2$, it holds

$$\lim_{f\to+\infty}((p-1)\log f+1)f^{p-2} = 0, \quad \lim_{f\to 0^+}((p-1)\log f+1)f^{p-2} < 0;$$

thus $((p-1)\log f+1)f^{p-2} \leq C$ with $f = e^{\frac{2p-3}{(2-p)(p-1)}}$ as the maximum point, and $((p-1)\log f+1)f^{p-2} < 0$ with $0 < f < e^{-\frac{1}{p-1}}$. Thus we can find a positive constant C independent of u_1 and u_2, such that

$$\frac{1}{2}\int_\Omega w^2 dx + \frac{k}{2}\int_\Omega |\nabla w|^2 dx \leq C\int_0^t \int_\Omega w^2 dx d\tau.$$

It follows from Gronwall's inequality that

$$\int_\Omega w^2 dx = 0, \quad \text{a.e. } (0,t).$$

Thus $w = 0$ a.e in $\Omega \times (0,T)$. $\square$

3. Potential wells

We define the following two functionals on $H_0^1(\Omega)$:

$$
\begin{aligned}
J(u) &= \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \log|u| dx + \frac{1}{p^2}\|u\|_p^p, \\
I(u) &= \|\nabla u\|_p^p - \int_\Omega |u|^p \log|u| dx.
\end{aligned}
\tag{3.1}
$$

It is obvious that

$$
J(u) = \frac{1}{p} I(u) + \frac{1}{p^2}\|u\|_p^p. \tag{3.2}
$$

Remark 3.1. Since $u \in E$ and $1 < p < 2$, we can use the Hölder inequality and Lemma 2.2 to derive that

$$
\begin{gathered}
\|u\|_p + \|\nabla u\|_p \le C(p,\Omega)(\|u\|_2 + \|\nabla u\|_2), \\
\int_\Omega |u|^p \log|u| dx \le \frac{1}{e\alpha}\|\nabla u\|_2^{p+\alpha},
\end{gathered}
$$

where α satisfies $1 \le p + \alpha < 2^*$, which imply that $J(u)$ and $I(u)$ are well-defined in $H_0^1(\Omega)$ and $W_0^{1,p}(\Omega)$. Further, similar to the Step 4 of Theorem 2.3, one can prove that

$$
u \mapsto \int_\Omega |u|^p \log|u| dx
$$

is continuous from $H_0^1(\Omega)$ to $\mathbb{R}$. It follows that $J(u)$ and $I(u)$ are continuous.

Let

$$
d = \inf\{\sup_{\lambda \ge 0} J(\lambda u) | u \in H_0^1(\Omega), \|\nabla u\|_p^p \ne 0\}, \tag{3.3}
$$

and

$$
N = \{u \in H_0^1(\Omega) | I(u) = 0, \|\nabla u\|_p^p \ne 0\}.
$$

Then Lemma 3.3 and Lemma 3.5 below tell us that

$$
d = \inf_{u \in N} J(u) \ge M = \frac{1}{p^2}\Big(\frac{p^2 e}{n\mathcal{L}_p}\Big)^{n/p},
$$

where $\mathcal{L}_p$ can be found in (3.9). Thus we can define

$$
\begin{gathered}
W = \{u \in H_0^1(\Omega) | I(u) > 0, J(u) < d\} \cup \{0\}, \\
V = \{u \in H_0^1(\Omega) | I(u) < 0,\ J(u) < d\}.
\end{gathered}
$$

For $\delta > 0$, we introduce

$$
I_\delta(u) = \delta\|\nabla u\|_p^p - \int_\Omega |u|^p \log|u| dx, \tag{3.4}
$$

$$
N_\delta = \{u \in H_0^1(\Omega) | I_\delta(u) = 0, \|\nabla u\|_p^p \ne 0\}, \tag{3.5}
$$

$$
d(\delta) = \inf_{u \in N_\delta} J(u), \tag{3.6}
$$

$$
W_\delta = \{u \in H_0^1(\Omega) | I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\}, \tag{3.7}
$$

$$
V_\delta = \{u \in H_0^1(\Omega) | I_\delta(u) < 0, J(u) < d(\delta)\}. \tag{3.8}
$$

To handle the logarithmic nonlinearity $|u|^{p-2}u\log|u|$, we need the following L^p logarithmic Sobolev inequality

Lemma 3.2 ([11, 10])**.** *For any $u \in W^{1,p}(\mathbb{R}^n)$ with $p \in (1, +\infty)$, $u \neq 0$, and any $\mu > 0$,*
$$p \int_{\mathbb{R}^n} |u|^p \log(\frac{|u|}{\|u\|_p}) dx + \frac{n}{p} \log(\frac{p\mu e}{n\mathcal{L}_p}) \int_{\mathbb{R}^n} |u|^p dx \le \mu \int_{\mathbb{R}^n} |\nabla u|^p dx,$$
where
$$\mathcal{L}_p = \frac{p}{n} (\frac{p-1}{e})^{p-1} \pi^{-\frac{p}{2}} \Big[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\frac{p-1}{p}+1)}\Big]^{p/n}. \tag{3.9}$$

For $u \in W^{1,p}(\Omega)$, we can define $u = 0$ for $x \in \mathbb{R}^n \setminus \Omega$, such that $u \in W^{1,p}(\mathbb{R}^n)$. Thus it holds the L^p logarithmic Sobolev inequality for bounded domain Ω
$$p \int_{\Omega} |u|^p \log(\frac{|u|}{\|u\|_p}) dx + \frac{n}{p} \log(\frac{p\mu e}{n\mathcal{L}_p}) \int_{\Omega} |u|^p dx \le \mu \int_{\Omega} |\nabla u|^p dx. \tag{3.10}$$

Lemmas 3.3, 3.4, 3.5 and 3.6 are similar to [7, Lemmas 2.1, 2.2, 2.3 and 2.4], so we omit most of their proofs.

Lemma 3.3. *Assume $\lambda > 0$, $u \in H_0^1(\Omega)$ and $\|u\|_p \neq 0$, then we have*

1. *$J(\lambda u)$ strictly increases on $0 < \lambda \le \lambda^*$, strictly decreases on $\lambda^* \le \lambda < \infty$ and takes the maximum at $\lambda = \lambda^*$. Further $\lim_{\lambda \to 0} J(\lambda u) = 0$, and $\lim_{\lambda \to +\infty} J(\lambda u) = -\infty$;*
2. *$I(\lambda u) > 0$ on $0 < \lambda < \lambda^*$, $I(\lambda^* u) = 0$ and $I(\lambda u) < 0$ on $\lambda^* < \lambda < \infty$, where*
$$\lambda^* = \exp\{\frac{\|\nabla u\|_p^p - \int_\Omega |u|^p \log |u| dx}{\|u\|_p^p}\}.$$

Lemma 3.4. *Let $u \in W_0^{1,p}(\Omega)$ and $\|u\|_p \neq 0$. Then we have*

1. *if $0 < \|\nabla u\|_p \le r(\delta)$, then $I_\delta(u) \ge 0$;*
2. *if $I_\delta(u) < 0$, then $\|\nabla u\|_p > r(\delta)$;*
3. *if $I_\delta(u) = 0$, then $\|\nabla u\|_p \ge r(\delta)$,*

where $r(\delta) = \lambda_1^{1/p} (\frac{p^2 \delta e}{n\mathcal{L}_p})^{\frac{n}{p^2}}$, and λ_1 is the first eigenvalue of the problem
$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = \lambda |u|^{p-2} u, \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{gathered}$$

Proof. (1) Using the L^p Sobolev logarithmic inequality (3.10), for any $\mu > 0$, we have
$$I_\delta(u) \ge (\delta - \frac{\mu}{p}) \|\nabla u\|_p^p + (\frac{n}{p^2} \log(\frac{p\mu e}{n\mathcal{L}_p}) - \log \|u\|_p) \|u\|_p^p. \tag{3.11}$$
Taking $\mu = p\delta$ in (3.11), we obtain that
$$I_\delta(u) \ge (\frac{n}{p^2} \log(\frac{p^2 \delta e}{n\mathcal{L}_p}) - \log \|u\|_p) \|u\|_p^p. \tag{3.12}$$
By the Poincaré inequality, if $0 < \|\nabla u\|_p \le r(\delta)$, then $0 < \|u\|_p \le \lambda_1^{-\frac{1}{p}} \|\nabla u\|_p \le (\frac{p^2 \delta e}{n\mathcal{L}_p})^{\frac{n}{p^2}}$. Thus $I_\delta(u) \ge 0$.

The proof for (2) and (3) is similar to that of [7, Lemma 2.2], so we omit it here. $\square$

Lemma 3.5. *For $d(\delta)$ in (3.6), we have*

1. *$d(\delta) \ge \frac{1}{p}(1-\delta) r^p(\delta) + \frac{1}{p^2} (\frac{p^2 \delta e}{n\mathcal{L}_p})^{n/p}$. In particular, $d(1) \ge \frac{1}{p^2} (\frac{p^2 e}{n\mathcal{L}_p})^{n/p} =: M$;*

(2) *there exists a unique* b, $b \in (1, 1 + \frac{1}{p\lambda_1}]$ *such that* $d(b) = 0$, *and* $d(\delta) > 0$ *for* $1 \le \delta < b$;

(3) $d(\delta)$ *is strictly increasing on* $0 < \delta \le 1$, *decreasing on* $1 \le \delta \le b$, *and takes the maximum* $d = d(1)$ *at* $\delta = 1$.

Now, we can define

$$d_0 = \lim_{\delta \to 0^+} d(\delta), \tag{3.13}$$

where $d_0 \ge 0$ from Lemma 3.5.

Lemma 3.6. *Let* $d_0 < J(u) < d$ *for some* $u \in H_0^1(\Omega)$, *and* $\delta_1 < 1 < \delta_2$ *are the two roots of the equation* $d(\delta) = J(u)$. *Then the sign of* $I_\delta(u)$ *is unchangeable for* $\delta_1 < \delta < \delta_2$.

In what follows, we prove that when $0 < J(u_0) < d$, W_δ and V_δ are the invariant sets of (1.1). The discussion is divided into two parts: $J(u_0)$ being in the monotonous interval of $d(\delta)$, and $J(u_0)$ being in the non-monotonous interval of $d(\delta)$.

Proposition 3.7. *Assume* $u_0 \in H_0^1(\Omega)$, u *is a weak solution of* (1.1) *with* $J(u_0) = \sigma$. *Then we have the following results.*

(1) *If* $0 < \sigma \le d_0$, *then there exists a unique* $\bar{\delta} \in (1, b)$ *such that* $d(\bar{\delta}) = \sigma$, *where* b *is the constant in Lemma* 3.5 *(2). Furthermore, if* $I(u_0) > 0$, *then* $u \in W_\delta$ *for any* $1 \le \delta < \bar{\delta}$; *else if* $I(u_0) < 0$, *then* $u \in V_\delta$ *for any* $1 \le \delta < \bar{\delta}$.

(2) *If* $d_0 < \sigma < d$, *then there exists* δ_1 *and* δ_2 *such that* $\delta_1 < 1 < \delta_2$ *and* $d(\delta_1) = d(\delta_2) = \sigma$. *Furthermore, if* $I(u_0) > 0$, *then* $u \in W_\delta$ *for any* $\delta_1 < \delta < \delta_2$; *else if* $I(u_0) < 0$, *then* $u \in V_\delta$ *for any* $\delta_1 < \delta < \delta_2$.

Proof. Case 1. $0 < J(u_0) = \sigma \le d_0$, namely $J(u_0)$ is in the monotonous interval of $d(\delta)$. From Lemma 3.5, there exists a unique $\bar{\delta} \in (1, b)$ such that $d(\bar{\delta}) = \sigma$. For any $\delta \in [1, \bar{\delta})$, we have

$$I_\delta(u_0) = (\delta - 1)\|\nabla u_0\|_p^p + I(u_0) \ge I(u_0), \quad J(u_0) = \sigma = d(\bar{\delta}) < d(\delta). \tag{3.14}$$

Multiplying both sides of (1.1) by u_t and integrating on $\Omega \times [0, t]$, it holds

$$\int_0^t (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau + J(u) = J(u_0) = d(\bar{\delta}) < d(\delta), \tag{3.15}$$

for all $t \in (0, T)$ and all $\delta \in [1, \bar{\delta})$, where T is the maximal existence time.

If $I(u_0) > 0$, then (3.14) means that $u_0 \in W_\delta$ for $\delta \in [1, \bar{\delta})$. We assert that $u \in W_\delta$ for $t \in (0, T)$ and $\delta \in [1, \bar{\delta})$. If it is false, then there exists $\delta^* \in [1, \bar{\delta})$ and $t_0 \in (0, T)$, such that $u \in W_{\delta^*}$ for $t \in (0, t_0)$, but $u(x, t_0) \in \partial W_{\delta^*}$, namely

$$I_{\delta^*}(u(t_0)) = 0, \quad \|\nabla u(t_0)\|_p^p \ne 0, \quad \text{or} \quad J(u(t_0)) = d(\delta^*).$$

In fact, (3.15) shows that $J(u(t_0)) \le J(u_0) < d(\delta^*)$, which implies $I_{\delta^*}(u(t_0)) = 0$ and $\|\nabla u(t_0)\|_p^p \ne 0$, namely $u(x, t_0) \in N_{\delta^*}$. Thus from the definition of $d(\delta^*)$, we have $J(u(t_0)) \ge d(\delta^*)$, which is a contradiction.

Next, we prove that if $I(u_0) < 0$, then $u_0 \in V_\delta$ for $\delta \in [1, \bar{\delta})$, and $u \in V_\delta$ for $t \in (0, T)$ and $\delta \in [1, \bar{\delta})$. If the assertion of u_0 is false, then (3.14) shows that there exists $\delta_* \in [1, \bar{\delta})$ being the first number such that $u_0 \in V_\delta$ for $\delta \in [1, \delta_*)$ and $u_0 \in \partial V_{\delta_*}$, namely

$$I_{\delta_*}(u_0) = 0, \quad \text{or} \quad J(u_0) = d(\delta_*).$$

Since $J(u_0)$ is in the strictly decreasing interval of $d(\delta)$, then $J(u_0) = d(\bar{\delta}) < d(\delta_*)$, which indicates that $I_{\delta_*}(u_0) = 0$. Since $I_\delta(u_0) < 0$ for $\delta \in [1, \delta_*)$, then Lemma 3.4 (2) gives $\|\nabla u_0\|_p > r(\delta) > 0$, which indicates that $u_0 \in N_{\delta_*}$. By the definition of $d(\delta_*)$, we have $J(u_0) = d(\bar{\delta}) \geq d(\delta_*)$, which is contradict with the monotonicity of $d(\delta)$. If the assertion of u is false, then there exists $\delta_*^* \in [1, \bar{\delta})$ and $t_0 \in (0, T)$, such that $u \in V_{\delta_*^*}$ for $t \in (0, t_0)$, but $u(x, t_0) \in \partial V_{\delta_*^*}$, namely

$$I_{\delta_*^*}(u(t_0)) = 0, \quad \text{or} \quad J(u(t_0)) = d(\delta_*^*).$$

In fact, (3.15) shows that $J(u(t_0)) \leq J(u_0) < d(\delta_*^*)$, which implies $I_{\delta_*^*}(u(t_0)) = 0$. If $I_{\delta_*^*}(u(t_0)) = 0$, then from Lemma 3.4 (3), $\|\nabla u(t_0)\|_p \geq r(\delta)$, namely $u(x, t_0) \in N_{\delta_*^*}$. Thus from the definition of $d(\delta_*^*)$, we have $J(u(t_0)) \geq d(\delta_*^*)$, which is a contradiction.

Case 2. $d_0 < J(u_0) = \sigma < d$, namely $J(u_0)$ is in the non-monotonous interval of $d(\delta)$. From Lemma 3.5, there exist $\delta_1 < 1 < \delta_2$ being two roots of $d(\delta) = \sigma$, and $d_0 < J(u_0) = d(\delta_1) = d(\delta_2) < d(\delta)$ for $\delta \in (\delta_1, \delta_2)$.

If $I(u_0) > 0$, then from Lemma 3.6, the sign of $I_\delta(u)$ is unchangeable for $\delta_1 < \delta < \delta_2$. Thus we have $I_\delta(u_0) > 0$ for $\delta \in (\delta_1, \delta_2)$. Therefore, $u_0 \in W_\delta$ for $\delta \in (\delta_1, \delta_2)$. The proof of $u \in W_\delta$ is similar to that in Case 1.

If $I(u_0) < 0$, also from Lemma 3.6, we have $I_\delta(u_0) < 0$ for $\delta \in (\delta_1, \delta_2)$, which with $J(u_0) < d(\delta)$ for $\delta \in (\delta_1, \delta_2)$, imply that $u_0 \in V_\delta$ for $\delta \in (\delta_1, \delta_2)$. The proof of $u \in V_\delta$ is similar to that in Case 1. $\square$

Proposition 3.8. *Assume $u_0 \in H_0^1(\Omega)$ with $u_0 \not\equiv 0$, $J(u_0) = d$, u is a weak solution of* (1.1). *If $I(u_0) > 0$, then $I(u(t)) \geq 0$ for all $0 < t < T$; if $I(u_0) < 0$, then $I(u(t)) < 0$ for all $0 \leq t < T$, where T is the maximal existence time of u.*

Proof. We prove the proposition by contradiction. When $I(u_0) > 0$, if there exists $t_1 \in (0, T)$ such that $I(u(t_1)) < 0$, then we can find $t_0 \in (0, t_1)$ being the first point satisfying $I(u) = 0$, namely

$$I(u(t_0)) = 0, \quad \text{and} \quad I(u(t)) > 0 \quad \text{for all } 0 < t < t_0.$$

Thus $\int_0^t (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau > 0$ for $0 < t < t_0$. Otherwise $u_t = 0$ and $\nabla u_t = 0$ a.e. $(x, t) \in \Omega \times (0, t_0)$, which are contradict with the fact $I(u) = -\int_\Omega u_t u dx - k\int_\Omega \nabla u_t \cdot \nabla u dx > 0$ for $0 < t < t_0$. Thus

$$J(u(t)) = J(u_0) - \int_0^t (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau < d, \quad \text{for all } 0 < t \leq t_0. \tag{3.16}$$

Also $I(u(t_0)) = 0$ imply that $u(x, t_0) = 0$ or $\|\nabla u(t_0)\|_p^p \geq r(1) \neq 0$. If $u(x, t_0) = 0$, then from the uniqueness of solutions, $u(x, t) = 0$ for $t > t_0$, which is a contradiction, since $I(u(t_1)) < 0$. If $\|\nabla u(t_0)\|_p^p \neq 0$, then by the definition of $d(\delta)$, we have $J(u(t_0)) \geq d$, which is contradict with (3.16).

When $I(u_0) < 0$, if there exists $t_1 \in (0, T)$ such that $I(u(t_1)) = 0$, and $I(u(t)) < 0$ for all $0 < t < t_1$. Similar to the proof of (3.16), we have

$$J(u(t)) = J(u_0) - \int_0^t (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau < d, \quad \text{for all } 0 < t \leq t_1. \tag{3.17}$$

Also from Lemma 3.4 and $I(u(t)) < 0$ for all $0 \leq t < t_1$, then $\|\nabla u(t_0)\|_p^p \geq r(1) \neq 0$. By the definition of $d(\delta)$, we have $J(u(t_0)) \geq d$, which is contradict with (3.17). $\square$

4. Global boundedness and decay estimation

In this section, we treat the globally bounded case, especially including the decay estimates. First we need to point out that if u is a solution of (1.1) with $J(u_0) \le d$, $I(u_0) \ge 0$, and there exists $t_2 > 0$ such that $\|\nabla u(t_2)\|_p = 0$, then from the uniqueness of the solution, $u = 0$ for all $t \ge t_2$. So in what follows, we do not consider this type of solutions.

Theorem 4.1. *When $J(u_0) \le d$ and $I(u_0) \ge 0$, the weak solution of* (1.1) *is globally bounded.*

Step 1: $J(u_0) < d$. Actually, we only need to focus on the case $0 < J(u_0) < d$ & $I(u_0) > 0$, irrespectively of other cases. The reasons are that the case $J(u_0) < 0$ & $I(u_0) \ge 0$ is contradict with (3.2); the case $0 < J(u_0) < d$ & $I(u_0) = 0$ is contradict with the definition of d; if $J(u_0) = 0$ and $I(u_0) \ge 0$, then $u_0 \equiv 0$, which is a trivial case.

Multiplying the first equation of (1.1) by u_t and integrating with respect to time from 0 to t, we obtain

$$\int_0^t \|u_\tau\|_2^2 d\tau + k\int_0^t \|\nabla u_\tau\|_2^2 d\tau + J(u(t)) = J(u(0)) < d, \quad \text{for } t > 0. \tag{4.1}$$

We assert that $u(x,t) \in W$ for any $t > 0$. If it is false, then there exists $t_0 > 0$ such that $u(x,t_0) \in \partial W$, then

$$I(u(t_0)) = 0, \|\nabla u(t_0)\|_p \neq 0, \quad \text{or} \quad J(u(t_0)) = d.$$

On the one hand, (4.1) indicates that $J(u(t_0)) = d$ is not true. On the other hand, if $I(u(t_0)) = 0, \|\nabla u(t_0)\|_p \neq 0$, then by the definition of d, we have $J(u(t_0)) \ge d$, which is also contradict with (4.1). Thus we have $u(x,t) \in W$, which with (3.2) deduce that

$$\|u\|_p^p < p^2 d. \tag{4.2}$$

Taking $\mu = \frac{p}{2}$ in (3.10), we have

$$\begin{aligned}
\|\nabla u\|_p^p &= I(u) + \int_\Omega |u|^p \log|u| dx \\
&= 2I(u) + 2\int_\Omega |u|^p \log|u| dx - \|\nabla u\|_p^p \\
&\le 2I(u) + 2\|u\|_p^p \log\|u\|_p - \frac{2n}{p^2}\log(\frac{p^2 e}{2n\mathcal{L}_p})\|u\|_p^p \\
&= 2pJ(u) + (2\log\|u\|_p - \frac{2}{p} - \frac{2n}{p^2}\log(\frac{p^2 e}{2n\mathcal{L}_p}))\|u\|_p^p \\
&\le Cd.
\end{aligned} \tag{4.3}$$

Also, (4.1) implies

$$\int_0^t \|u_\tau\|_2^2 d\tau + k\int_0^t \|\nabla u_\tau\|_2^2 d\tau < d. \tag{4.4}$$

From (4.2), (4.3) and (4.4), we have

$$\int_0^t \|u_\tau\|_2^2 d\tau + k\int_0^t \|\nabla u_\tau\|_2^2 d\tau + \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p^2}\|u\|_p^p \le \big(2 + \frac{C}{p}\big)d. \tag{4.5}$$

Multiplying the first equation of (1.1) by u, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2dx + \frac{k}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2dx + I(u) = 0 \tag{4.6}$$

Combining (4.6) and the fact that $u(x,t) \in W$ for any $t > 0$, we find that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2dx + \frac{k}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2dx < 0,$$

which means that

$$\|u\|_2^2 + \|\nabla u\|_2^2 \le C(\|u_0\|_2^2 + \|\nabla u_0\|_2^2). \tag{4.7}$$

Thus (4.5) and (4.7) show that u is globally bounded in E.

Step 2: $J(u_0) = d$**.** Let $\mu_m = 1 - \frac{1}{m}$ and $u_{m0} = \mu_m u_0$ for $m \ge 2$. We consider the following problem:

$$\begin{gathered} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) - k\triangle u_t = |u|^{p-2}u\log|u|, \quad \Omega\times(0,T),\\ u(x,t) = 0, \quad \partial\Omega\times(0,T),\\ u(x,0) = u_{m0}(x), \quad \Omega. \end{gathered} \tag{4.8}$$

We assert $J(u_{m0}) < d$ and $I(u_{m0}) > 0$. If $\|u_0\|_p = 0$, then from (3.2) and $J(u_0) = d$, we have $I(u_0) = pJ(u_0) = pd$. Thus $I(u_{m0}) = \mu_m^p I(u_0) = \mu_m^p pd > 0$, $J(u_{m0}) = \mu_m^p J(u_0) = \mu_m^p d < d$. If $\|u_0\|_p \ne 0$, then from $I(u_0) \ge 0$ and Lemma 3.3, we have $\lambda^* \ge 1$. We can also deduce that $I(u_{m0}) = I(\mu_m u_0) > 0$, and $J(u_{m0}) = J(\mu_m u_0) < J(u_0) = d$.

Using the similar arguments as in Theorem 2.3 and Step 1, (4.8) admits a unique global bounded weak solution $u_m \in E$. Since the initial data $u_{m0}(x) \to u_0$ strongly in $H_0^1(\Omega)$, then via a standard procedure, $u_m \to u$ strongly in E. Thus u is globally bounded in E.

Theorem 4.2. *Let $u = u(x,t)$ be the global bounded weak solution in Theorem 4.1.*

(1) If $J(u_0) < M$ and $I(u_0) \ge 0$, then we have

$$\lim_{t\to\infty}(\|u\|_p^p + k\|\nabla u\|_p^p) = 0. \tag{4.9}$$

Furthermore, when $n = 1$, there exists time $t_\beta > 0$ such that

$$\|u(t)\|_2^2 + k\|\nabla u(t)\|_2^2 \le (\|u(t_\beta)\|_2^2 + k\|\nabla u(t_\beta)\|_2^2)e^{\frac{1}{2}-C\alpha_1 t}, \quad \text{for all} \quad t \ge t_\beta,$$

where

$$\alpha_1 = \min\{\frac{1}{k}(1-\frac{\mu}{p}), \frac{n}{p^2}\log(\frac{p\mu e}{n\mathcal{L}_p}) - \frac{1}{p}\log(p^2J(u_0))\} > 0,$$

for any $\mu \in ([p^2J(u_0)]^{p/n}\frac{n\mathcal{L}_p}{pe}, p)$ and $\mathcal{L}_p$ is (3.9).

(2) If $J(u_0) = M$ and $I(u_0) > 0$, then

$$\lim_{t\to\infty}(\|u\|_p^p + k\|\nabla u\|_p^p) = 0.$$

Furthermore, when $n = 1$, there exists time $t_\gamma > 0$, such that

$$\|u(t)\|_2^2 + k\|\nabla u(t)\|_2^2 \le (\|u(t_\gamma)\|_2^2 + k\|\nabla u(t_\gamma)\|_2^2)e^{\frac{1}{2}-C\alpha_2 t}, \quad \text{for all } t \ge t_\gamma,$$

where

$$\alpha_2 = \min\{\frac{1}{k}(1-\frac{\mu}{p}), \frac{n}{p^2}\log(\frac{p\mu e}{n\mathcal{L}_p}) - \frac{1}{p}\log(p^2(M-\gamma))\} > 0,$$

for any $\mu \in ([p^2(M-\gamma)]^{p/n}\frac{n\mathcal{L}_p}{pe}, p)$ and $\mathcal{L}_p$ is (3.9).

Remark 4.3. When $p > 2$, under similar conditions as in Theorem 4.2, the global bounded solutions decay algebraically [12]. However, if $p < 2$, Theorem 4.2 shows that the global bounded solutions decay exponentially, which is the same as the results in [7] for $p = 2$. Further, Theorem 4.2 tells us that the upper bound of the decay rate $e^{-\alpha_1 t}$ and $e^{-\alpha_2 t}$ are proportional to k, which seems that the pseudo-parabolic viscosity slows down the decay.

To prove the theorem, we need to introduce the following two lemmas.

Lemma 4.4 ([7, Lemma 3.1])**.** *Let $y(t) : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function. Assume that there is a constant $A > 0$ such that*
$$\int_t^{+\infty} y(s)ds \le Ay(t), \quad 0 \le t < +\infty.$$
Then $y(t) \le y(0)e^{1-\frac{t}{A}}$, for all $t > 0$.

Lemma 4.5 ([20, Prop. 6.2.3])**.** *Assume that a is a positive constant, $g(t), h(t) \in C^1([a,\infty))$, $h(t) \ge 0$ and $g(t)$ is bounded blow. If there exists a positive b and C, such that*
$$g'(t) \le -bh(t), \quad h'(t) \le C, \quad t \in [a,\infty),$$
then $\lim_{t\to\infty} h(t) = 0$.

Proof. **Case 1. Decay estimates for $J(u_0) < M$.** Let $u = u(x,t)$ be the global bounded solution of (1.1) with $J(u_0) < M \le d$ and $I(u_0) \ge 0$. As in the proof for Theorem 4.1, we only need to discuss the case $0 < J(u_0) < M$ and $I(u_0) > 0$. Proposition 3.7 reveals that $u \in W_\delta$ for $1 \le \delta < \bar{\delta}$ or $\delta_1 < \delta < \delta_2$ with $\delta_1 < 1 < \delta_2$ and particularly $I(u) > 0$. Then from (3.2) and (3.15), we have
$$\|u\|_p^p < p^2 J(u) \le p^2 J(u_0) < p^2 M. \tag{4.10}$$
Because $J(u_0) < M = \frac{1}{p^2}(\frac{p^2 e}{n\mathcal{L}_p})^{n/p}$, for $\mu \in ([p^2 J(u_0)]^{p/n}\frac{n\mathcal{L}_p}{pe}, p)$, we obtain the following inequality from (3.10) and (4.10),
$$\begin{aligned} I(u) &\ge \|\nabla u\|_p^p - \|u\|_p^p \log\|u\|_p + \frac{n}{p^2}\log(\frac{p\mu e}{n\mathcal{L}_p})\|u\|_p^p - \frac{\mu}{p}\|\nabla u\|_p^p \\ &\ge (1-\frac{\mu}{p})\|\nabla u\|_p^p + (\frac{n}{p^2}\log(\frac{p\mu e}{n\mathcal{L}_p}) - \frac{1}{p}\log(p^2 J(u_0)))\|u\|_p^p \\ &\ge \alpha_1(\|u\|_p^p + k\|\nabla u\|_p^p), \end{aligned} \tag{4.11}$$
where
$$\alpha_1 = \min\{\frac{1}{k}(1-\frac{\mu}{p}), \frac{n}{p^2}\log(\frac{p\mu e}{n\mathcal{L}_p}) - \frac{1}{p}\log(p^2 J(u_0))\} > 0.$$
Combining (4.11) with
$$I(u) = -\frac{1}{2}\frac{d}{dt}\|u\|_2^2 - \frac{k}{2}\frac{d}{dt}\|\nabla u\|_2^2,$$
it holds
$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \frac{k}{2}\frac{d}{dt}\|\nabla u\|_2^2 \le -\alpha_1(\|u\|_p^p + k\|\nabla u\|_p^p). \tag{4.12}$$
Next we first prove that $\|u\|_p^p + k\|\nabla u\|_p^p$ decays to 0 as $t \to \infty$. For this purpose, Lemma 4.5 is useful. Set
$$g(t) = \|u\|_2^2 + k\|\nabla u\|_2^2, \quad h(t) = \|u\|_p^p + k\|\nabla u\|_p^p.$$

Then it is sufficient to prove $h'(t) \le C$. Multiplying the first equation of (1.1) by u_t and using the Young inequality, we can obtain

$$\begin{aligned} &\int_\Omega |u_t|^2 dx + k\int_\Omega |\nabla u_t|^2 dx + \frac{d}{dt}\int_\Omega \frac{|\nabla u|^p}{p} dx \\ &\le \frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\int_\Omega |u|^{2p-2}(\log|u|)^2 dx. \end{aligned} \tag{4.13}$$

Since

$$\lim_{f\to+\infty} f^{-\alpha}\log f = 0, \quad \lim_{f\to 0^+} f^{\alpha}\log f = 0, \quad \text{for} \quad 0<\alpha<1,$$

then we can deduce that

$$\int_\Omega |u|^{2p-2}(\log|u|)^2 dx \le C\int_\Omega |u|^2 dx + C,$$

which with (4.13) and (4.7) indicate that

$$\int_\Omega |u_t|^2 dx + k\int_\Omega |\nabla u_t|^2 dx + \frac{d}{dt}\int_\Omega |\nabla u|^p dx \le C.$$

Thus we find that

$$\begin{aligned} h'(t) &= \int_\Omega p|u|^{p-2}uu_t dx + \frac{d}{dt}\int_\Omega k|\nabla u|^p dx \\ &\le \frac{1}{2}\int_\Omega p^2|u|^{2p-2} dx + \frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{d}{dt}\int_\Omega k|\nabla u|^p dx \le C. \end{aligned}$$

Then from Lemma 4.5 and (4.12), we can prove (4.9).

Next, we deal with the decay estimates of the solutions for the 1-Dimensional case. On the one hand, (4.9) and the Sobolev imbedding inequality imply that

$$|u|_{0;\Omega} = \sup_\Omega |u| \to 0, \quad \text{as } t\to\infty. \tag{4.14}$$

On the other hand, multiplying the first equation of (1.1) by Δu and integrating on Ω, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (\frac{1}{2}|\nabla u|^2 + \frac{k}{2}|\Delta u|^2) dx + (p-1)\int_\Omega |\nabla u|^{p-2}|\Delta u|^2 dx \\ &= \int_\Omega |u|^{p-2}((p-1)\log|u|+1)|\nabla u|^2 dx, \end{aligned}$$

which with (4.14) indicate that there exists a $t_\beta > 0$, such that

$$|u|_{0;\Omega} < e^{-\frac{1}{p-1}} \quad \text{and} \quad \int_\Omega |\Delta u|^2 dx \le C, \quad \text{for } t \ge t_\beta.$$

Using the Sobolev imbedding inequality again, we have that

$$|\nabla u|_{0;\Omega} = \sup_\Omega |\nabla u| < C. \tag{4.15}$$

Substituting (4.14) and (4.15) into (4.12) gives

$$\begin{aligned} \frac{d}{dt}\|u\|_2^2 + \frac{d}{dt}k\|\nabla u\|_2^2 &\le -2\alpha_1(\|u\|_p^p + k\|\nabla u\|_p^p) \\ &= -2\alpha_1(\int_\Omega |u|^2|u|^{p-2} dx + k\int_\Omega |\nabla u|^2|\nabla u|^{p-2} dx) \\ &\le -2C\alpha_1(\|u\|_2^2 + k\|\nabla u\|_2^2). \end{aligned}$$

Integrating the above inequality from t to T with $t \geq t_\beta$, we have

$$\begin{aligned}\int_t^T (\|u\|_2^2 + k\|\nabla u\|_2^2) ds &\leq \frac{1}{2C\alpha_1}(\|u(t)\|_2^2 + k\|\nabla u(t)\|_2^2 - (\|u(T)\|_2^2 + k\|\nabla u(T)\|_2^2)) \\ &\leq \frac{1}{2C\alpha_1}(\|u(t)\|_2^2 + k\|\nabla u(t)\|_2^2).\end{aligned}$$

Let $T \to \infty$ and from Lemma 4.4, we can find

$$\|u(t)\|_2^2 + k\|\nabla u(t)\|_2^2 \leq (\|u(t_\beta)\|_2^2 + k\|\nabla u(t_\beta)\|_2^2) e^{\frac{1}{2} - C\alpha_1 t},$$

for all $t \geq t_\beta$.

Case 2. Decay estimates for $J(u_0) = M$. Let $u = u(x,t)$ be the global bounded solution of the problem (1.1) with $J(u_0) = M \leq d$ and $I(u_0) > 0$. From Propositions 3.7 and 3.8, we know that

$$I(u) = -(u_t, u) - k(\nabla u_t, \nabla u) \geq 0, \quad \text{for all } t > 0, \tag{4.16}$$

and there exists a $t_0 > 0$, such that

$$I(u(t_0)) = 0, \quad \text{and} \quad I(u(t)) > 0, \quad \text{for } 0 < t < t_0,$$

which implies

$$\int_0^t (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau > 0, \quad 0 < t < t_0.$$

Thus we can choose some time $0 < t_\gamma < t_0$, such that

$$\int_0^{t_\gamma} (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau = \gamma,$$

where γ is a sufficiently small positive number. If we take t_γ as the initial time, then we have

$$I(u(t_\gamma)) > 0,$$

$$J(u(t_\gamma)) = J(u_0) - \int_0^{t_\gamma} (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2) d\tau = M - \gamma < M,$$

which is the same as Case 1. Similar to the proof for Case 1, we can choose t_γ large enough such that

$$\|u(t)\|_2^2 + k\|\nabla u(t)\|_2^2 \leq (\|u(t_\gamma)\|_2^2 + k\|\nabla u(t_\gamma)\|_2^2) e^{\frac{1}{2} - C\alpha_2 t}, \quad \text{for all } t \geq t_\gamma,$$

where

$$\alpha_2 = \min\Big\{\frac{1}{k}(1 - \frac{\mu}{p}),\ \frac{n}{p^2}\log(\frac{p\mu e}{nL_p}) - \frac{1}{p}\log(p^2(M-\gamma))\Big\} > 0,$$

for all $\mu \in ([p^2(M-\gamma)]^{p/n}\frac{n\mathcal{L}_p}{pe}, p)$. $\square$

5. Blow-up at $+\infty$ and growth estimation

Actually, the estimation (2.5) in Theorem 2.3 tells us that the solution of (1.1) would not blow up at any finite time $T > 0$. However, in this section, we prove that the solution may blow up at $+\infty$ and further give some growth estimates of the solution.

Theorem 5.1. *When $J(u_0) \le d$ and $I(u_0) < 0$, then the weak solution of* (1.1) *blows up at $+\infty$, namely*

$$\lim_{t\to+\infty}(\|u\|_2^2 + k\|\nabla u\|_2^2) = +\infty.$$

Remark 5.2. Under the similar conditions, when $p > 2$, the weak solutions blow up in finite time [12]. However, when $p \le 2$, the weak solutions blow up at ∞.

Proof. **Step 1: $J(u_0) < d$.** From Proposition 3.7, we obtain for all $t \ge 0$, $u \in V_\delta$ for any $1 \le \delta < \bar{\delta}$ or $\delta_1 < \delta < \delta_2$ with $\delta_1 < 1 < \delta_2$. Then by $I_\delta(u) < 0$ and Lemma 3.4, we obtain $\|\nabla u\|_p^p > r^p(\delta) = \lambda_1(\frac{p^2\delta e}{n\mathcal{L}_p})^{n/p}$ for all $t \ge 0$. Set

$$G(t) = \int_0^t (\|u\|_2^2 + k\|\nabla u\|_2^2)d\tau.$$

A simple calculation indicates that

$$\begin{aligned} G''(t) &= -2I(u) = 2(\delta-1)\|\nabla u\|_p^p - 2I_\delta(u) \\ &> 2(\delta-1)\|\nabla u\|_p^p \\ &> 2(\delta-1)r^p(\delta), \quad \text{for all } t \ge 0. \end{aligned}$$

Thus setting $\delta > 1$, we can have

$$G'(t) = G'(0) + \int_0^t G''(\tau)d\tau > 2(\delta-1)\lambda_1(\frac{p^2\delta e}{n\mathcal{L}_p})^{n/p}t, \quad \text{for all } t \ge 0, \tag{5.1}$$

namely

$$\|u(t)\|_2^2 + k\|\nabla u(t)\|_2^2 > 2(\delta-1)\lambda_1(\frac{p^2\delta e}{n\mathcal{L}_p})^{n/p}t, \quad \text{for all } t > 0,$$

where $\delta > 1$ in Proposition 3.7, λ_1 can be found in Lemma 3.4 and $\mathcal{L}_p$ is (3.9). This means that the weak solution u will blow up at $+\infty$.

Step 2: $J(u_0) = d$. From Proposition 3.8, we know $I(u) = -(u_t, u) - k(\nabla u_t, \nabla u) < 0$ for $t \ge 0$, and then $\int_0^t (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2)d\tau$ is strictly positive for $t > 0$. For any sufficiently small positive number t_1, we have

$$J(u(t_1)) = J(u_0) - \int_0^{t_1}(\|u\|_2^2 + k\|\nabla u_\tau\|_2^2)d\tau < d.$$

If we take $t = t_1$ as the initial time, then similar to Step 1, we can obtain that the weak solution u blows up at $+\infty$. □

Theorem 5.3. *Let $u = u(x,t)$ be the weak solution in Theorem 5.1. If $J(u_0) \le M$ and $I(u_0) < 0$, then for any $\alpha_3 \in (0,1)$, there exist $t_{\alpha_3} > 0$ such that*

$$\|u\|_2^2 + k\|\nabla u\|_2^2 \ge C_{\alpha_3}(t - t_{\alpha_3})^{\frac{1}{1-\frac{p\alpha_3}{2}} - 1}, \quad \textit{for all } t \ge t_{\alpha_3}, \tag{5.2}$$

where

$$C_{\alpha_3} = ((1 - \frac{p\alpha_3}{2})G^{-\frac{p\alpha_3}{2}}(t_{\alpha_3})G'(t_{\alpha_3}))^{\frac{1}{1-\frac{p\alpha_3}{2}}}$$

with $G(t) = \int_0^t (\|u\|_2^2 + k\|\nabla u\|_2^2)d\tau$.

Remark 5.4. From (5.2) and (2.5), the weak solutions that blow up at ∞ grow algebraically. (5.2) also indicates that the lower bound of growth estimates is smaller than that of the case $p = 2$, which is caused by fast diffusion.

Proof. Let $u = u(x,t)$ be the weak solution of (1.1) with $J(u_0) \le M$ and $I(u_0) < 0$. Then Propositions 3.7 and 3.8 tell us that $u \in V$ and $I(u) < 0$ for all $t \ge 0$. Taking $\mu = p$ in (3.10) and noticing $I(u) < 0$, we can obtain

$$\|u\|_p^p \ge (\frac{p^2 e}{n\mathcal{L}_p})^{n/p} = p^2 M, \quad \text{for all } t \ge 0, \tag{5.3}$$

which also implies $\|u\|_2^2 > 0$ for all $0 \le t < T$. Thus

$$G'(t) = \|u\|_2^2 + k\|\nabla u\|_2^2 > 0 \quad \text{and} \quad G''(t) = -2I(u) > 0, \quad \text{for all } t \ge 0.$$

Furthermore, from (5.3), we obtain

$$\begin{aligned} G''(t) &= -2I(u) = -2pJ(u) + \frac{2}{p}\|u\|_p^p \\ &= -2pJ(u_0) + 2p\int_0^t (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2)d\tau + \frac{2}{p}\|u\|_p^p \\ &\ge 2p(M - J(u_0)) + 2p\int_0^t (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2)d\tau, \quad \text{for all } t \ge 0. \end{aligned} \tag{5.4}$$

Since

$$\begin{aligned} \Big(\int_0^t ((u_\tau, u)_2 + k(\nabla u_\tau, \nabla u)_2)d\tau\Big)^2 &= \frac{1}{4}(\int_0^t \frac{d}{d\tau}(\|u\|_2^2 + k\|\nabla u\|_2^2)d\tau)^2 \\ &= \frac{1}{4}(G'(t) - G'(0))^2 \\ &= \frac{1}{4}(G'^2(t) - 2G'(t)G'(0) + G'^2(0)), \end{aligned} \tag{5.5}$$

then combining (5.4) and (5.5), and using the Hölder inequality, we can calculate

$$\begin{aligned} &G(t)G''(t) - \frac{p}{2}G'^2(t) \\ &\ge 2p(M - J(u_0))G(t) + 2p\int_0^t (\|u_\tau\|_2^2 + k\|\nabla u_\tau\|_2^2)d\tau \int_0^t (\|u\|_2^2 + k\|\nabla u\|_2^2)d\tau \\ &\quad - 2p(\int_0^t ((u_\tau, u)_2 + k(\nabla u_\tau, \nabla u)_2)d\tau)^2 - pG'(t)(\|u_0\|_2^2 + k\|\nabla u_0\|_2^2) \\ &\quad + \frac{p}{2}(\|u_0\|_2^2 + k\|\nabla u_0\|_2^2)^2 \\ &\ge 2p(M - J(u_0))G(t) - pG'(t)(\|u_0\|_2^2 + k\|\nabla u_0\|_2^2) \\ &\ge -p(\|u_0\|_2^2 + k\|\nabla u_0\|_2^2)G'(t). \end{aligned} \tag{5.6}$$

Then for $0 < \alpha_3 < 1$, we obtain

$$G(t)G''(t) - \frac{p\alpha_3}{2}G'^2(t) \ge (1 - \alpha_3)\frac{p}{2}G'^2(t) - p(\|u_0\|_2^2 + k\|\nabla u_0\|_2^2)G'(t).$$

From (5.1), there exists $t_1 > 0$ such that $G'(t)$ is large enough and

$$G(t)G''(t) - \frac{p\alpha_3}{2}G'^2(t) > 0, \quad \text{for all } t \ge t_1. \tag{5.7}$$

Then we have

$$(G^{1-\frac{p\alpha_3}{2}}(t))' = (1 - \frac{p\alpha_3}{2})G^{-\frac{p\alpha_3}{2}}(t)G'(t),$$

$$(G^{1-\frac{p\alpha_3}{2}}(t))'' = (1 - \frac{p\alpha_3}{2})G^{-\frac{p\alpha_3}{2}-1}(t)(G(t)G''(t) - \frac{p\alpha_3}{2}G'^2(t)) > 0.$$

Now, we take $t_{\alpha_3} \geq t_1$ satisfying $G(t_{\alpha_3}) > 0$. Then for $t \geq t_{\alpha_3}$,

$$\begin{aligned} G(t) &= (G^{1-\frac{p\alpha_3}{2}}(t))^{\frac{1}{1-\frac{p\alpha_3}{2}}} \\ &= (G^{1-\frac{p\alpha_3}{2}}(t_{\alpha_3}) + \int_{t_{\alpha_3}}^{t}(1-\frac{p\alpha_3}{2})G^{-\frac{p\alpha_3}{2}}(\tau)G'(\tau)d\tau)^{\frac{1}{1-\frac{p\alpha_3}{2}}} \\ &\geq C_{\alpha_3}(t-t_{\alpha_3})^{\frac{1}{1-\frac{p\alpha_3}{2}}} \end{aligned} \tag{5.8}$$

with

$$C_{\alpha_3} = ((1-\frac{p\alpha_3}{2})G^{-\frac{p\alpha_3}{2}}(t_{\alpha_3})G'(t_{\alpha_3}))^{\frac{1}{1-\frac{p\alpha_3}{2}}}.$$

Since $G''(t) > 0$ for all $t \geq 0$, then we have $\int_0^t G'(\tau)d\tau \leq tG'(t)$, namely

$$t(\|u\|_2^2 + k\|\nabla u\|_2^2) \geq G(t), \quad \text{for all } t \geq 0,$$

which combining with (5.8) we deduce that for $0 < \alpha_3 < 1$ and $t \geq t_{\alpha_3}$,

$$\|u\|_2^2 + k\|\nabla u\|_2^2 \geq C_{\alpha_3}(t-t_{\alpha_3})^{\frac{1}{1-\frac{p\alpha_3}{2}}-1}.$$

$\square$

Acknowledgments. This research was supported by the National Natural Science Foundation of China (11571062, 11671155), and by the Fundamental Research Funds for the Central Universities (DUT16LK01)

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