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TWO SOLUTIONS FOR NONHOMOGENEOUS KLEIN-GORDON-MAXWELL SYSTEM WITH SIGN-CHANGING POTENTIAL

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Abstract. In this article, we study the nonhomogeneous Klein-Gordon-Maxwell system

$$\begin{gathered} -\Delta u+\lambda V(x)u-K(x)(2\omega+\phi)\phi u=f(x,u)+h(x), \quad x\in\mathbb{R}^3,\\ \Delta\phi=K(x)(\omega+\phi)u^2, \quad x\in\mathbb{R}^3, \end{gathered}$$

where $\omega>0$ is a constant and $\lambda>0$ is a parameter. Using the Linking theorem and Ekeland's variational principle in critical point theory, we prove the existence of multiple solutions, under suitable assumptions that allow a sign-changing potential.

1. Introduction and statement of main results

In this article, we study the nonhomogeneous Klein-Gordon-Maxwell system

$$\begin{gathered} -\Delta u+\lambda V(x)u-K(x)(2\omega+\phi)\phi u=f(x,u)+h(x), \quad x\in\mathbb{R}^3,\\ \Delta\phi=K(x)(\omega+\phi)u^2, \quad x\in\mathbb{R}^3, \end{gathered} \tag{1.1}$$

where $\omega>0$ is a constant and $\lambda\ge 1$ is a parameter, $V\in C(\mathbb{R}^3,\mathbb{R})$ and $f\in C(\mathbb{R}^3\times\mathbb{R},\mathbb{R})$. By using the Linking Theorem and the Ekeland's variational principle in critical point theory, we obtain the multiple solutions for (1.1). Here, the potential V is allowed to be a sign-changing function. Such system was firstly studied by Benci and Fortunato [5] as a model which describes nonlinear Klein-Gordon fields in three dimensional space interacting with the electrostatic field. For more details on the physical aspects of the problem we refer the readers to see [6] and the references therein.

When $h\equiv 0$, that is the homogeneous case, has been widely studied in recent years. In 2002, Benci and Fortunato [6] considered the Klein-Gordon-Maxwell system

$$\begin{gathered} -\Delta u+[m^2-(\omega+\phi)^2]\phi u=f(x,u), \quad x\in\mathbb{R}^3,\\ \Delta\phi=(\omega+\phi)u^2, \quad x\in\mathbb{R}^3, \end{gathered} \tag{1.2}$$

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for the power of nonlinearity $f(x,u)=|u|^{q-2}u$, where ω and m are constants. By using a version of the mountain pass theorem, they proved that (1.2) has infinitely many radially symmetric solutions under $|m|>|\omega|$ and $4<q<6$. D'Aprile and Mugnai [15] sutdied the case $2<q<4$ assuming $\sqrt{\frac{q-2}{2}}m>\omega>0$. Later, the authors in [3] gave a small improvement with $2<q<4$. Azzollini and Pomponio [2] obtained the existence of a ground state solution for (1.2) under one of the conditions

(i) $4\le q<6$ and $m>\omega$;
(ii) $2<q<4$ and $m\sqrt{q-2}>\omega\sqrt{6-q}$.

Soon afterwards, it is improved by Wang [26]. Motivated by the methods of Benci and Fortunato, Cassani [9] considered (1.2) for the critical case by adding a lower order perturbation

$$\begin{gathered}-\Delta u+[m^2-(\omega+\phi)^2]\phi u=\mu|u|^{q-2}u+|u|^{2^*-2}u,\quad x\in\mathbb{R}^3,\\ \Delta\phi=(\omega+\phi)u^2,\quad x\in\mathbb{R}^3,\end{gathered}\tag{1.3}$$

where $\mu>0$ and $2^*=6$. He showed that (1.3) has at least a radially symmetric solution under one of the following conditions:

(i) $4<q<6, |m|>|\omega|$ and $\mu>0$;
(ii) $q=4, |m|>|\omega|$ and μ is sufficiently large.

Which was improved by the result in [10] provided one of the following conditions is satisfied:

(i) $4<q<6, |m|>|\omega|>0$ and $\mu>0$;
(ii) $q=4, |m|>|\omega|>0$ and μ is sufficiently large;
(iii) $2<q<4, |m|\sqrt{\frac{q-2}{2}}>|\omega|>0$ and μ is sufficiently large.

Subsequently, Wang [25] generalized the result of [10]. Recently, the authors in [11] proved the existence of positive ground state solutions for the problem (1.3) with a periodic potential V; that is,

$$\begin{gathered}-\Delta u+V(x)u+[m^2-(\omega+\phi)^2]\phi u=\mu|u|^{q-2}u+|u|^{2^*-2}u,\quad x\in\mathbb{R}^3,\\ \Delta\phi=(\omega+\phi)u^2,\quad x\in\mathbb{R}^3.\end{gathered}$$

In [20], Georgiev and Visciglia introduced a system like homogeneous (1.1) with potentials and $\lambda=1$, however they considered a small external Coulomb potential in the corresponding Lagrangian density. Cunha [13] considered the existence of positive ground state solutions for (1.1) with periodic potential $V(x)$. Other related results about homogeneous Klein-Gordon-Maxwell system can be found in [14, 16, 17, 18, 21, 22] and other equations with sign-changing potential see [30].

Next, we consider the nonhomogeneous case, that is $h\not\equiv 0$. Chen and Song [12] proved that (1.1), with $\lambda=1$ and $K(x)\equiv 1$, has two nontrivial solutions if $f(x,t)$ satisfies the local (AR) condition:

There exist $\mu>2$ and $r_0>0$ such that $\mathcal{G}(x,t):=\frac{1}{\mu}f(x,t)t-F(x,t)\ge 0$ for every $x\in\mathbb{R}^3$ and $|t|\ge r_0$, where $F(x,t)=\int_0^t f(x,s)ds$.

Xu and Chen [29] studied the existence and multiplicity of solutions for system (1.1) for the pure power of nonlinearity with $f(x,u)=|u|^{q-2}u$, $\lambda=1$, and $K(x)\equiv 1$. They also assumed that $V(x)\equiv 1$ and $h(x)$ is radially symmetric.

Motivated by the above works, we consider system (1.1) with more general potential $V(x)$ and $f(x,u)$. We make the following assumptions:

- (A1) There is $b > 0$ such that $\operatorname{meas}\{x \in \mathbb{R}^3 : V(x) \le b\} < +\infty$, where meas denotes the Lebesgue measures;
- (A2) $V \in C(\mathbb{R}^3, \mathbb{R})$ and V is bounded below;
- (A3) $\Omega = \operatorname{int} V^{-1}(0)$ is nonempty and has smooth boundary and $\overline{\Omega} = V^{-1}(0)$.

This type of hypotheses was first introduced by Bartsch and Wang [4] in the study of a nonlinear Schrödinger equation and the potential $V(x)$ satisfying (A1)–(A3) is referred as the steep well potential.

Under assumptions (A1)–(A2) and some more generic 4-superlinear conditions on the continuous function $f(x,u)$, we prove the existence of multiple solutions of problem (1.1) when $\lambda > 0$ large by using the variation method.

- (A4) $F(x,u) = \int_0^u f(x,s)ds \ge 0$ for all (x,u) and $f(x,u) = o(u)$ uniformly in x as $u \to 0$, $|f(x,u)| \le C(|u| + |u|^q)$, $q < 6$ for all (x,u);
- (A5) $F(x,u)/u^4 \to +\infty$ as $|u| \to +\infty$ uniformly in x;
- (A6) $\mathcal{F}(x,u) := \frac{1}{4} f(x,u)u - F(x,u) \ge 0$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$;
- (A7) There exist $a_1, L_1 > 0$ and $\tau \in (3/2, 2)$ such that
$$|f(x,u)|^\tau \le a_1 \mathcal{F}(x,u)|u|^\tau, \quad \text{for all } x \in \mathbb{R}^3 \text{ and } |u| \ge L_1;$$
- (A8) $K(x) \in L^3(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ and $K(x) \ge 0$ is not identically zero for a.e. $x \in \mathbb{R}^3$;
- (A9) $h(x) \in L^2(\mathbb{R}^3)$ and $h(x) \ge 0$ for a.e. $x \in \mathbb{R}^3$.

Remark 1.1. It follows from (A6) and (A7) that $|f(x,u)|^\tau \le \frac{a_1}{4}|f(x,u)||u|^{\tau+1}$ for large u. Thus, by (A4), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that
$$|f(x,u)| \le \varepsilon|u| + C_\varepsilon |u|^{q-1}, \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1.4}$$
$$|F(x,u)| \le \varepsilon|u|^2 + C_\varepsilon |u|^q, \quad \forall (x,u) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1.5}$$
where $q = 2\tau/(\tau - 1) \in (4, 2^*)$ and $2^* = 6$ is the critical exponent for the Sobolev embedding in dimension 3.

Remark 1.2. It is not difficult to find out functions f satisfying (A4)–(A7), for example,
$$f(x,t) = g(x)t^3\Big(2\ln(1+t^2) + \frac{t^2}{1+t^2}\Big), \quad \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R},$$
where g is a is a continuous bounded function with $\inf_{x\in\mathbb{R}^3} g(x) > 0$.

Before stating our main results, we give some notation. For $1 \le s \le +\infty$ and $\Omega \subset \mathbb{R}^3$, $L^s(\Omega)$ denotes a Lebesgue space; the norm in $L^s(\Omega)$ is denoted by $|u|_{s,\Omega}$, where Ω is a proper subset of $\mathbb{R}^3$, by $|\cdot|_s$ when $\Omega = \mathbb{R}^3$. Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm
$$\|u\|_D^2 := \|u\|_{D^{1,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx.$$

The usual Sobolev space $H^1(\mathbb{R}^3)$ is endowed with the standard product and norm
$$(u,v)_{H^1} = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)\, dx; \quad \|u\|_{H^1}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)\, dx.$$

The best Sobolev constant $\bar{S}$ for the Sobolev embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is

$$\bar{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|_D}{|u|_6}.$$

For $r > 0$ and $z \in \mathbb{R}^3, B_r(z)$ denotes the ball of radius r centered at z.

We denote by " $\rightharpoonup$ " the weak convergence and by " $\to$ " strong convergence. Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again $\{u_n\}$. We use $o(1)$ to denote any quantity which tends to zero when $n \to \infty$. The letters d_i, C, C_i will be used to denote various positive constants which may vary from line to line and are not essential to the problem. Now we can state our main results.

Theorem 1.3. *Assume that* (A1)–(A2), (A4)–(A9) *are satisfied. If* $V(x) < 0$ *for some* $x \in \mathbb{R}^3$*, then for each* $k \in \mathbb{N}$*, there exist* $\lambda_k > k, b_k > 0$ *and* $\eta_k > 0$ *such that* (1.1) *has at least two nontrivial solutions for every* $\lambda = \lambda_k, |K|_\infty < b_k (or |K|_3 < b_k)$ *and* $|h|_2 \le \eta_k$*.*

Theorem 1.4. *Assume that* (A1)–(A9) *are satisfied. If* $V^{-1}(0)$ *has nonempty interior, then there exist* $\Lambda > 0, b_\lambda > 0$ *and* $\eta_\lambda > 0$ *such that problem* (1.1) *has at least two nontrivial solutions for every* $\lambda > \Lambda, |h|_2 \le \eta_\lambda$ *and* $|K|_\infty < b_\lambda (or |K|_3 < b_\lambda)$*.*

If $V \ge 0$, we remove the restriction of the norm of K and we have the following theorem.

Theorem 1.5. *Assume* $V \ge 0$*,* (A1)–(A9) *are satisfied. If* $V^{-1}(0)$ *has nonempty interior* Ω *and* $h \ne 0$*, then there exist* $\Lambda_* > 0$ *and* $\eta > 0$ *such that* (1.1) *has at least two nontrivial solutions for every* $\lambda > \Lambda_*$ *and* $|h|_2 \le \eta$*.*

To obtain our main results, we have to overcome some difficulties in using variational method. The main difficulty consists in the lack of compactness of the Sobolev embedding $H^1(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$, $p \in (2, 6)$. Since we assume that the potential is not radially symmetric, we cannot use the usual way to recover compactness, for example, restricting in the subspace $H^1_r(\mathbb{R}^3)$ of radially symmetric functions. To recover the compactness, we borrow some ideas used in [4, 19] and establish the parameter dependent compactness conditions.

To the best of our knowledge, our theorems are the first results about the existence of multiple solutions for the nonhomogeneous Klein-Gordon-Maxwell equations on $\mathbb{R}^3$ with general nonlinear term and sign-changing potential. As it is pointed out in [13], many technical difficulties arise due to the presence of a nonlocal term ϕ, which is not homogeneous as it is in the Schrödinger-Poisson systems. In other words, the adaptation of the ideas to the procedure of our problem is not trivial at all, because of the presence of the nonlocal term ϕ_u. Hence, a more careful analysis of the interaction between the couple (u, ϕ) is required.

The paper is organized as follows. We introduce the variational setting and the compactness conditions in Section 2. In Section 3, we give the proofs of main results.

2. Variational setting and compactness condition

By [3], we know that the signs of ω are not relevant to the existence of solutions, so we assume that $\omega > 0$. In this section, we firstly give the variational setting of problem (1.1) and then establish the compactness conditions.

Let $V(x) = V^+(x) - V^-(x)$, where $V^\pm = \max\{\pm V(x), 0\}$. Let

$$E = \Big\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^2 + V^+(x)u^2\,dx < \infty\Big\}$$

be equipped with the inner product $(u, v) = \int_{\mathbb{R}^3}(\nabla u\nabla v + V^+(x)uv)\,dx$ and the norm $\|u\| = (u, u)^{1/2}$. For $\lambda > 0$, we also need the following inner product and norm

$$(u, v)_\lambda = \int_{\mathbb{R}^3}(\nabla u\nabla v + \lambda V^+(x)uv)\,dx, \quad \|u\|_\lambda = (u, u)_\lambda^{1/2}.$$

It is clear $\|u\| \le \|u\|_\lambda$ for $\lambda \ge 1$. Set $E_\lambda = (E, \|\cdot\|_\lambda)$. It follows from the Poincaré inequality and (A1)–(A2), we know that the embedding $E_\lambda \hookrightarrow H^1(\mathbb{R}^3)$ is continuous, and therefore, for $s \in [2, 6]$, there exists $d_s > 0$(independent of $\lambda \ge 1$) such that

$$|u|_s \le d_s\|u\|_\lambda,\ \forall u \in E_\lambda. \tag{2.1}$$

Let

$$F_\lambda = \{u \in E_\lambda : \operatorname{supp} u \subset V^{-1}([0, \infty))\},$$

and $F_\lambda^\perp$ denote the orthogonal complement of F_λ in E_λ. Clearly, $F_\lambda = E_\lambda$ if $V \ge 0$, otherwise $F_\lambda^\perp \ne \{0\}$. Define

$$A_\lambda := -\Delta + \lambda V,$$

then A_λ is formally self-adjoint in $L^2(\mathbb{R}^3)$ and the associated bilinear form

$$a_\lambda(u, v) = \int_{\mathbb{R}^3}(\nabla u\nabla v + \lambda V(x)uv)\,dx$$

is continuous in E_λ. As in [19], for fixed $\lambda > 0$, we consider the eigenvalue problem

$$-\Delta u + \lambda V^+(x)u = \mu\lambda V^-(x)u, \quad u \in F_\lambda^\perp. \tag{2.2}$$

By (A1)–(A2), we know that the quadratic form $u \mapsto \int_{\mathbb{R}^3} \lambda V^-(x)u^2\,dx$ is weakly continuous. Hence following [28, Theorems 4.45 and 4.46], we can deduce the following proposition, which is the spectral theorem for compact self-adjoint operators jointly with the Courant-Fischer minimax characterization of eigenvalues.

Proposition 2.1. *Suppose that* (A1), (A2) *hold, then for any fixed* $\lambda > 0$, *the eigenvalue problem* (2.2) *has a sequence of positive eigenvalues* $\{\mu_j(\lambda)\}$, *which may be characterized by*

$$\mu_j(\lambda) = \inf_{\dim M \ge j, M \subset F_\lambda^\perp} \sup\Big\{\|u\|_\lambda^2 : u \in M, \int_{\mathbb{R}^3} \lambda V^-(x)u^2\,dx = 1\Big\},$$

for $j = 1, 2, 3, \dots$. *Furthermore,* $\mu_1(\lambda) \le \mu_2(\lambda) \le \dots \le \mu_j(\lambda) \to +\infty$ *as* $j \to +\infty$, *and the corresponding eigenfunctions* $\{e_j(\lambda)\}$, *which may be be chosen so that* $(e_i(\lambda), e_j(\lambda))_\lambda = \delta_{ij}$, *are a basis of* $F_\lambda^\perp$.

Next, we give some properties for the eigenvalues $\{\mu_j(\lambda)\}$ defined above.

Proposition 2.2 ([19])**.** *Assume that* (A1)–(A2) *hold and* $V^- \not\equiv \{0\}$. *Then, for each fixed* $j \in \mathbb{N}$,

(i) $\mu_j(\lambda) \to 0$ *as* $\lambda \to +\infty$;
(ii) $\mu_j(\lambda)$ *is a non-increasing continuous function of* λ.

Remark 2.3. By Proposition 2.2, there exists $\Lambda_0 > 0$ such that $\mu_1(\lambda) \le 1$ for all $\lambda > \Lambda_0$.

Denote

$$E_\lambda^- := span\{e_j(\lambda) : \mu_j(\lambda) \le 1\} \quad \text{and} \quad E_\lambda^+ := span\{e_j(\lambda) : \mu_j(\lambda) > 1\}.$$

Then $E_\lambda = E_\lambda^- \oplus E_\lambda^+ \oplus F_\lambda$ is an orthogonal decomposition. The quadratic form a_λ is negative semidefinite on E_λ^-, positive definite on $E_\lambda^+ \oplus F_\lambda$ and it is easy to see that $a_\lambda(u,v) = 0$ if u, v are in different subspaces of the above decomposition of E_λ.

From Remark 2.3, we have that $\dim E_\lambda^- \ge 1$ when $\lambda > \Lambda_0$. Moreover, since $\mu_j(\lambda) \to +\infty$ as $j \to +\infty$, $\dim E_\lambda^- < +\infty$ for every fixed $\lambda > 0$.

System (1.1) has a variational structure. In fact, we consider the functional $\mathcal{J}_\lambda : E_\lambda \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\begin{aligned}
\mathcal{J}_\lambda(u,\phi) &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x)u^2)\,dx - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^3} K(x)(2\omega+\phi)\phi u^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} h(x)u\,dx.
\end{aligned}$$

The solutions $(u,\phi) \in E_\lambda \times D^{1,2}(\mathbb{R}^3)$ of system (1.1) are the critical points of $\mathcal{J}_\lambda$. By using the reduction method described in [7], we are led to the study of a new functional $I_\lambda(u)$ ($I_\lambda(u)$ is defined in (2.3)). We need the following technical result.

Proposition 2.4. *Let $K(x)$ satisfy the condition* (A8). *Then for any $u \in E_\lambda$, there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ which satisfies*

$$\Delta\phi = K(x)(\phi+\omega)u^2 \quad \text{in } \mathbb{R}^3.$$

Moreover, the map $\Phi : u \in E_\lambda \mapsto \phi_u \in D^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and

- (i) $-\omega \le \phi_u \le 0$ *on the set* $\{x \in \mathbb{R}^3 | u(x) \ne 0\}$;
- (ii) $\|\phi_u\|_D \le C_1|K|_3\|u\|_\lambda^2$ *and* $\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx \le C_2|K|_3^2\|u\|_\lambda^4$, *if* $K \in L^3(\mathbb{R}^3)$;
- (iii) $\|\phi_u\|_D \le C_3|K|_\infty\|u\|_\lambda^2$ *and* $\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx \le C_4|K|_\infty^2\|u\|_\lambda^4$, *if* $K \in L^\infty(\mathbb{R}^3)$.

Proof. Let $K(x) \in L^3(\mathbb{R}^3)$, $u \in E_\lambda$ and define the bilinear form

$$L(w_1,w_2) \in D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} [\nabla w_1 \nabla w_2 + K(x)u^2 w_1 w_2]\,dx \in \mathbb{R}.$$

It is easy to see that L is well defined. Moreover, since $K(x) \ge 0, L(w_1,w_1) \ge \|w_1\|_D^2$. Furthermore, since $K(x) \in L^3(\mathbb{R}^3)$, by the Hölder inequality, we obtain that

$$\begin{aligned}
L(w_1,w_2) &= \int_{\mathbb{R}^3} [\nabla w_1 \nabla w_2 + K(x)u^2 w_1 w_2]\,dx \\
&\le \|w_1\|_D\|w_2\|_D + |K|_3|u|_6^2|w_1|_6|w_2|_6 \\
&\le \|w_1\|_D\|w_2\|_D + d_6^2\bar{S}^{-2}|K|_3\|u\|_\lambda^2\|w_1\|_D\|w_2\|_D \\
&= (1 + d_6^2\bar{S}^{-2}|K|_3\|u\|_\lambda^2)\|w_1\|_D\|w_2\|_D.
\end{aligned}$$

Hence L defines an inner product, equivalent to the standard inner product in $D^{1,2}(\mathbb{R}^3)$. Moreover $E_\lambda \subset L^4(\mathbb{R}^3)$ and then

$$\Big|\int_{\mathbb{R}^3} \omega K(x)u^2 w_1\,dx\Big| \le \bar{S}^{-1}\omega|K|_3|u|_4^2\|w_1\|_D.$$

Therefore, the linear map

$$w_1 \in D^{1,2}(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} -\omega K(x) u^2 w_1 \, dx \in \mathbb{R}$$

is continuous. Hence, by the Lax-Milgram theorem, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} [\nabla \phi_u \nabla w_1 + K(x) u^2 \phi_u w_1] \, dx = \int_{\mathbb{R}^3} -\omega K(x) u^2 w_1 \, dx, \quad \forall w_1 \in D^{1,2}(\mathbb{R}^3),$$

ϕ_u is the unique solution of $\Delta \phi = K(x)(\phi + \omega) u^2$.

For the case $K \in L^\infty(\mathbb{R}^3)$ is similar to [24, Lemma 3.1], we omit it here.

(i) Arguing by contradiction, we assume that there exists an open subset $\Omega \subset \mathbb{R}^3$ satisfying

$$\phi_u < -\omega.$$

Then, for ϕ_u a solution of $\Delta \phi = K(x)(\phi + \omega) u^2$, we have

$$-\Delta(\phi_u + \omega) + K(x)(\phi_u + \omega) u^2 = -\Delta \phi_u + K(x) u^2 \phi_u + \omega K(x) u^2 = 0.$$

Set $\varphi = \phi_u + \omega$, we obtain that

$$-\Delta \varphi + K(x) \varphi u^2 = 0 \ in \ \Omega, \quad \varphi = 0 \quad \text{on } \partial\Omega.$$

Then $\varphi = 0$ contradicts $\phi_u < -\omega$.

An analogous argument shows that $\phi \leq 0$.

(ii) Since ϕ_u solves the equation $\Delta \phi = K(x)(\phi + \omega) u^2$, $K \in L^3(\mathbb{R}^3)$ and $K(x) \geq 0$, we have

$$\begin{aligned} \|\phi_u\|_D^2 &\leq -\int_{\mathbb{R}^3} (K(x) \phi_u^2 u^2 + \omega K(x) u^2 \phi_u) \, dx \\ &\leq -\int_{\mathbb{R}^3} \omega K(x) u^2 \phi_u \, dx \\ &\leq \omega |K|_3 |u|_4^2 |\phi_u|_6 \\ &\leq \omega \bar{S}^{-1} d_4^2 |K|_3 \|u\|_\lambda^2 \|\phi_u\|_D. \end{aligned}$$

Hence $\|\phi_u\|_D \leq C_1 |K|_3 \|u\|_\lambda^2$, where $C_1 = \omega \bar{S}^{-1} d_4^2$.

For the second inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} K(x) \phi_u u^2 \, dx &\leq |K|_3 |\phi_u|_6 |u|_4^2 \\ &\leq \bar{S}^{-1} d_4^2 |K|_3 \|\phi_u\|_D \|u\|_\lambda^2 \\ &\leq \omega \bar{S}^{-2} d_4^4 |K|_3^2 \|u\|_\lambda^4 \\ &\leq C_2 |K|_3^2 \|u\|_\lambda^4, \end{aligned}$$

where $C_2 = \omega \bar{S}^{-2} d_4^4$.

(iii) Again by ϕ_u solving the equation $\Delta \phi = K(x)(\phi + \omega) u^2$, $K \in L^\infty(\mathbb{R}^3)$ and $K(x) \geq 0$, we have

$$\begin{aligned} \|\phi_u\|_D^2 &\leq -\int_{\mathbb{R}^3} (K(x) \phi_u^2 u^2 + \omega K(x) u^2 \phi_u) \, dx \\ &\leq -\int_{\mathbb{R}^3} \omega K(x) u^2 \phi_u \, dx \\ &\leq \omega |K|_\infty |u|_{12/5}^2 |\phi_u|_6 \end{aligned}$$

$$\le \omega \bar{S}^{-1} d_{12/5}^2 |K|_\infty \|u\|_\lambda^2 \|\phi_u\|_D.$$

Hence $\|\phi_u\|_D \le C_3 |K|_\infty \|u\|_\lambda^2$, where $C_3 = \omega \bar{S}^{-1} d_{12/5}^2$.

For the second inequality,

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx &\le |K|_\infty |\phi_u|_6 |u|_{12/5}^2 \\ &\le \bar{S}^{-1} d_{12/5}^2 |K|_\infty \|\phi_u\|_D \|u\|_\lambda^2 \\ &\le \omega \bar{S}^{-2} d_{12/5}^4 |K|_\infty^2 \|u\|_\lambda^4 \\ &\le C_4 |K|_\infty^2 \|u\|_\lambda^4,\end{aligned}$$

where $C_4 = \omega \bar{S}^{-2} d_{12/5}^4$. The proof is complete. □

Remark 2.5. By the proof of Proposition 2.4, we can know that the condition (A8) can be replaced by

(A8') $K(x) \in L^{q_1}(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$ and $K(x) \ge 0$ is not identically zero for a.e. $x \in \mathbb{R}^3$, where $q_1 \ge 3$.

Multiplying $-\triangle \phi_u + K(x)\phi_u u^2 = -\omega K(x) u^2$ by ϕ_u and integration by parts, we obtain

$$\int_{\mathbb{R}^3} (|\nabla \phi_u|^2 + K(x)\phi_u^2 u^2)\,dx = -\int_{\mathbb{R}^3} \omega K(x)\phi_u u^2\,dx.$$

By above equality and the definition of $\mathcal{J}_\lambda$, we obtain a C^1 functional $I_\lambda : E_\lambda \to \mathbb{R}$ given by

$$\begin{aligned} I_\lambda(u) &= \mathcal{J}_\lambda(u,\phi_u) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x) u^2)\,dx - \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla \phi_u|^2 + K(x)\phi_u^2 u^2)\,dx \\ &\quad - \int_{\mathbb{R}^3} \omega K(x)\phi_u u^2\,dx - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} h(x) u\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x) u^2)\,dx - \frac{1}{2}\int_{\mathbb{R}^3} K(x)\omega \phi_u u^2\,dx \\ &\quad - \int_{\mathbb{R}^3} F(x,u)\,dx - \int_{\mathbb{R}^3} h(x) u\,dx\,. \end{aligned} \tag{2.3}$$

Its Gateaux derivative is

$$\begin{aligned}\langle I_\lambda'(u), v\rangle &= \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \lambda V(x) uv)\,dx - \int_{\mathbb{R}^3} K(x)(2\omega + \phi_u)\phi_u uv\,dx \\ &\quad - \int_{\mathbb{R}^3} f(x,u) v\,dx - \int_{\mathbb{R}^3} h(x) v\,dx\end{aligned}$$

for all $v \in E_\lambda$. Here we use the fact that $\phi_u = (\triangle - K(x)u^2)^{-1}[\omega K(x) u^2]$. Set

$$M(u) = \int_{\mathbb{R}^3} -\omega K(x)\phi_u u^2\,dx.$$

Now we give some properties of the functional M. Its derivative M' possesses the BL-splitting property, which is similar to Brezis-Lieb Lemma [8].

Proposition 2.6. *Let $K \in L^\infty(\mathbb{R}^3) \cup L^3(\mathbb{R}^3)$. If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^3$, then*

(i) *$\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ and $M(u) \le \liminf_{n\to\infty} M(u_n)$;*

(ii) $M(u_n - u) = M(u_n) - M(u) + o(1)$;
(iii) $M'(u_n - u) = M'(u_n) - M'(u) + o(1)$ *in* $H^{-1}(\mathbb{R}^3)$.

Proof. (i) A straight forward adaption of [33, Lemma 2.1]. The proof of (ii) and (iii) have been given in [30, 32] for $N(u) = \int_{\mathbb{R}^3} K(x)\phi_u u^2\,dx$, and it is easy to see that the conclusions remain valid for $M(u)$. The proof is complete. □

Next, we investigate the compactness conditions for the functional I_λ. Recall that a C^1 functional J satisfies (PS) condition at level c if any sequence $\{u_n\} \subset E$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ has a convergent subsequence; and such sequence is called a $(PS)_c$ sequence. We only consider the case $K \in L^\infty(\mathbb{R}^3)$, the other case $K \in L^3(\mathbb{R}^3)$ is similar.

Lemma 2.7. *Suppose that* (A1)–(A2), (A4)–(A9) *are satisfied. Then every* $(PS)_c$ *sequence of* I_λ *is bounded in* E_λ *for each* $c \in \mathbb{R}$.

Proof. Let $\{u_n\} \subset E_\lambda$ be a $(PS)_c$ sequence of I_λ. Suppose by contradiction that

$$I_\lambda(u_n) \to c, \quad I'_\lambda(u_n) \to 0, \quad \|u_n\|_\lambda \to \infty \tag{2.4}$$

as $n \to \infty$ after passing to a subsequence. Take $w_n := u_n/\|u_n\|_\lambda$. Then $\|w_n\|_\lambda = 1, w_n \rightharpoonup w$ in E_λ and $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^3$.

We first consider the case $w = 0$. By (2.4), (A6), Proposition 2.4 and the fact $w_n \to 0$ in $L^2(\{x \in \mathbb{R}^3 : V(x) < 0\})$, we obtain

$$\begin{aligned}
o(1) &= \frac{1}{\|u_n\|_\lambda^2}\Big(I_\lambda(u_n) - \frac{1}{4}\langle I'_\lambda(u_n), u_n\rangle\Big)\\
&= \frac{1}{4}\|w_n\|_\lambda^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} V^-(x)w_n^2\,dx + \frac{1}{4\|u_n\|_\lambda^2}\int_{\mathbb{R}^3} K(x)\phi_{u_n}^2 u_n^2\,dx\\
&\quad + \frac{1}{\|u_n\|_\lambda^2}\int_{\mathbb{R}^3} \mathcal{F}(x,u_n)\,dx - \frac{3}{4\|u_n\|_\lambda^2}\int_{\mathbb{R}^3} h(x)u_n\,dx\\
&\ge \frac{1}{4} - \frac{\lambda}{4}|V^-|_\infty \int_{\operatorname{supp} V^-} w_n^2\,dx - \frac{3}{4}|h|_2 d_2 \frac{1}{\|u_n\|_\lambda}\\
&= \frac{1}{4} + o(1),
\end{aligned}$$

which is a contradiction.

If $w \neq 0$, then $\Omega_1 := \{x \in \mathbb{R}^3 : w(x) \neq 0\}$ has positive Lebesgue measure. For $x \in \Omega_1$, one has $|u_n(x)| \to \infty$ as $n \to \infty$, and then, by (A5),

$$\frac{F(x,u_n(x))}{u_n^4(x)} w_n^4(x) \to +\infty \quad \text{as } n \to \infty,$$

which, jointly with Fatou's lemma, shows that

$$\int_{\Omega_1} \frac{F(x,u_n)}{u_n^4} w_n^4\,dx \to +\infty \quad \text{as } n \to \infty. \tag{2.5}$$

Combining this with (A4), (2.3), the first limit of (2.4), (A8), (A9) and Proposition 2.4 (ii), we obtain

$$\frac{C_4}{2}|K|_\infty \omega \ge \limsup_{n\to\infty} \int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|_\lambda^4}\,dx \ge \limsup_{n\to\infty} \int_{\Omega_1} \frac{F(x,u_n)}{u_n^4} w_n^4\,dx = +\infty.$$

This is impossible. Hence $\{u_n\}$ is bounded in E_λ.

For the case $K \in L^3(\mathbb{R}^3)$, we can use the Cauchy-Schwarz inequality and the boundedness of ϕ_{u_n} to get the result. □

Lemma 2.8. *Suppose that (A1), (A2), (A8), (A9) and* (1.4) *hold. If* $u_n \rightharpoonup u$ *in* E_λ, $u_n(x) \to u(x)$ *a.e. in* $\mathbb{R}^3$, *and we denote* $w_n := u_n - u$, *then*

$$I_\lambda(u_n) = I_\lambda(w_n) + I_\lambda(u) + o(1), \tag{2.6}$$

$$\langle I'_\lambda(u_n), \varphi\rangle = \langle I'_\lambda(w_n), \varphi\rangle + \langle I'_\lambda(u), \varphi\rangle + o(1), \quad \textit{uniformly for all } \varphi \in E_\lambda \tag{2.7}$$

as $n \to \infty$. *In particular, if* $I_\lambda(u_n) \to c(\in \mathbb{R})$ *and* $I'_\lambda(u_n) \to 0$ *in* E^*_λ *(the dual space of* E_λ*), then* $I'_\lambda(u) = 0$ *and*

$$\begin{gathered} I_\lambda(w_n) \to c - I_\lambda(u), \\ \langle I'_\lambda(w_n), \varphi\rangle \to 0, \quad \textit{uniformly for all } \varphi \in E_\lambda \end{gathered} \tag{2.8}$$

after passing to a subsequence.

Proof. Since $u_n \rightharpoonup u$ in E_λ, we have $(u_n - u, u)_\lambda \to 0$ as $n \to \infty$, which implies that

$$\|u_n\|^2_\lambda = (w_n + u, w_n + u)_\lambda = \|w_n\|^2_\lambda + \|u\|^2_\lambda + o(1). \tag{2.9}$$

By (A1), the Hölder inequality and $w_n \rightharpoonup 0$, we have

$$\Big|\int_{\mathbb{R}^3} V^-(x) w_n u \, dx\Big| = \Big|\int_{\operatorname{supp} V^-} V^- w_n u \, dx\Big| \le |V^-|_\infty \Big(\int_{\operatorname{supp} V^-} w_n^2 \, dx\Big)^{1/2} |u|_2 \to 0$$

as $n \to \infty$. Thus

$$\int_{\mathbb{R}^3} V^-(x) u_n^2 \, dx = \int_{\mathbb{R}^3} V^-(x) w_n^2 \, dx + \int_{\mathbb{R}^3} V^-(x) u^2 \, dx + o(1).$$

By Proposition 2.6 (ii), we have

$$M(u_n) = M(w_n) + M(u) + o(1).$$

Since $h \in L^2(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} h(x) u_n \, dx = \int_{\mathbb{R}^3} h(x) w_n \, dx + \int_{\mathbb{R}^3} h(x) u \, dx,$$

therefore, to prove (2.6) and (2.7), it suffices to check that

$$\int_{\mathbb{R}^3} (F(x, u_n) - F(x, w_n) - F(x, u)) \, dx = o(1), \tag{2.10}$$

$$\sup_{\|\phi\|_\lambda = 1} \int_{\mathbb{R}^3} (f(x, u_n) - f(x, w_n) - f(x, u)) \phi \, dx = o(1). \tag{2.11}$$

We prove (2.10) firstly. Inspired by [1], we observe that

$$F(x, u_n) - F(x, u_n - u) = -\int_0^1 \Big(\frac{d}{dt} F(x, u_n - tu)\Big) dt = \int_0^1 f(x, u_n - tu) u dt.$$

and hence, by (1.4), we obtain

$$|F(x, u_n) - F(x, u_n - u)| \le \varepsilon_1 |u_n||u| + \varepsilon_1 |u|^2 + C_{\varepsilon_1} |u_n|^{p-1} |u| + C_{\varepsilon_1} |u|^p,$$

where $\varepsilon_1, C_{\varepsilon_1} > 0$ and $p \in (4, 6)$. Therefore, for each $\varepsilon > 0$, and the Young inequality, we obtain

$$|F(x, u_n) - F(x, w_n) - F(x, u)| \le C[\varepsilon |u_n|^2 + C_\varepsilon |u|^2 + \varepsilon |u_n|^p + C_\varepsilon |u|^p].$$

Next, we consider the function f_n given by

$$f_n(x) := \max\{|F(x, u_n) - F(x, w_n) - F(x, u)| - C\varepsilon(|u_n|^2 + |u_n|^p), 0\}.$$

Then $0 \le f_n(x) \le CC_\varepsilon(|u|^2+|u|^p) \in L^1(\mathbb{R}^3)$. Moreover, by the Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}^3} f_n(x)\,dx \to 0 \quad \text{as } n \to \infty, \tag{2.12}$$

since $u_n \to u$ a.e. in $\mathbb{R}^3$. By the definition of f_n, it follows that

$$|F(x,u_n) - F(x,w_n) - F(x,u)| \le f_n(x) + C\varepsilon(|u_n|^2 + |u_n|^p).$$

Combining this with (2.12) and (1.5), shows that

$$\int_{\mathbb{R}^3} |F(x,u_n) - F(x,w_n) - F(x,u)|\,dx \le C\varepsilon$$

for n sufficiently large. It implies that

$$\int_{\mathbb{R}^3} [F(x,u_n) - F(x,w_n) - F(x,u)]\,dx = o(1).$$

The prove of (2.11) is similar to [31, Lemma 4.7], we omit here. Now, we check that $I'_\lambda(u) = 0$. In fact, for each $\psi \in C_0^\infty(\mathbb{R}^3)$, we have

$$(u_n - u, \psi)_\lambda \to 0 \quad \text{as } n \to \infty. \tag{2.13}$$

and

$$\Big|\int_{\mathbb{R}^3} V^-(x)(u_n-u)\psi\,dx\Big| \le |V^-|_\infty\Big(\int_{\operatorname{supp}\psi}(u_n-u)^2\,dx\Big)^{1/2}|\psi|_2 \to 0 \tag{2.14}$$

as $n \to \infty$, since $u_n \to u$ in $L^2_{\rm loc}(\mathbb{R}^3)$. By Proposition 2.6 (i), $u_n \rightharpoonup u$ in E_λ yields $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$. So

$$\phi_{u_n} \rightharpoonup \phi_u \quad \text{in } L^6(\mathbb{R}^3).$$

For every $\psi \in C_0^\infty(\mathbb{R}^3)$ and Proposition 2.6 (ii), we obtain

$$\int_{\mathbb{R}^3} 2\omega K(x)\phi_{u_n}u_n\psi\,dx = \int_{\mathbb{R}^3} 2\omega K(x)\phi_{w_n}w_n\psi\,dx + \int_{\mathbb{R}^3} 2\omega K(x)\phi_u u\psi\,dx + o(1).$$

Now we need to prove

$$\int_{\mathbb{R}^3} K(x)\phi_{u_n}^2u_n\psi\,dx = \int_{\mathbb{R}^3} K(x)\phi_{w_n}^2w_n\psi\,dx + \int_{\mathbb{R}^3} K(x)\phi_u^2u\psi\,dx + o(1).$$

By $u_n \to u$ in $L^s_{\rm loc}(\mathbb{R}^3), 1 \le s < 6$; $\phi_{u_n} \to \phi_u$ in $L^s_{\rm loc}(\mathbb{R}^3), 1 \le s < 6$, the boundedness of (ϕ_{u_n}) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} K(x)(\phi_{u_n}^2u_n - \phi_u^2u)\psi\,dx \\
&= \int_{\mathbb{R}^3} K(x)\phi_{u_n}^2(u_n-u)\psi\,dx + \int_{\mathbb{R}^3} K(x)(\phi_{u_n}^2-\phi_u^2)u\psi\,dx \\
&\le C|K|_\infty\|\nabla\phi_{u_n}\|^2\Big(\int_{\Omega_\psi}|u_n-u|^{3/2}\,dx\Big)^{2/3} \\
&\quad + |K|_\infty\int_{\Omega_\psi}(\phi_{u_n}^2-\phi_u^2)u\psi\,dx \to 0,
\end{aligned} \tag{2.15}$$

as $n \to \infty$, here Ω_ψ is the support set of ψ.

Furthermore, by the dominated convergence theorem and (1.4), we have

$$\int_{\mathbb{R}^3}[f(x,u_n) - f(x,u)]\psi\,dx = \int_{\Omega_\psi}[f(x,u_n) - f(x,u)]\psi\,dx = o(1).$$

Since $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$ and $h \in L^2(\mathbb{R}^3)$, we obtain $\int_{\mathbb{R}^3} h(u_n - u)\,dx = o(1)$. This jointly with (2.13), (2.14), (2.15) and the dominated convergence theorem, shows that

$$\langle I'_\lambda(u), \psi\rangle = \lim_{n\to\infty} \langle I'_\lambda(u_n), \psi\rangle = 0, \quad \forall \psi \in C_0^\infty(\mathbb{R}^3).$$

Hence $I'_\lambda(u) = 0$. Combining with (2.6)-(2.7) and Proposition 2.6 (iii), we obtain (2.8). The proof is complete. $\square$

Lemma 2.9. *Assume that $V \ge 0$, and* (A1)–(A2), (A4)–(A9), *hold. Then, for any $M > 0$, there is $\Lambda = \Lambda(M) > 0$ such that I_λ satisfies $(PS)_c$ condition for all $c < M$ and $\lambda > \Lambda$.*

Proof. Let $\{u_n\} \subset E_\lambda$ be a $(PS)_c$ sequence with $c < M$. By Lemma 2.7, we know that $\{u_n\}$ is bounded in E_λ, and there exists $C > 0$ such that $\|u_n\|_\lambda \le C$. Therefore, up to a subsequence, we can assume that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } E_\lambda; \\ u_n \to u \quad \text{in } L^s_{\rm loc}(\mathbb{R}^3)\ (1 \le s < 2^*); \\ u_n(x) \to u(x) \quad \text{a.e. } x \in \mathbb{R}^3. \end{gathered} \tag{2.16}$$

Now we can show that $u_n \to u$ in E_λ for $\lambda > 0$ large. Denote $w_n := u_n - u$, then $w_n \rightharpoonup 0$ in E_λ. According to Lemma 2.8 and the fact (2.8) holds uniformly for all $\varphi \in E_\lambda$, we have $I'_\lambda(u) = 0$, and

$$I_\lambda(w_n) \to c - I_\lambda(u), \quad I'_\lambda(w_n) \to 0 \quad \text{as } n \to \infty. \tag{2.17}$$

According to $V \ge 0$ and (A6), we obtain

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle \\ &= \frac{1}{4}\|u\|_\lambda^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^2 u^2\,dx + \int_{\mathbb{R}^3} \mathcal{F}(x,u)\,dx - \frac{3}{4}\int_{\mathbb{R}^3} hu\,dx \\ &= \Phi_\lambda(u) - \frac{3}{4}\int_{\mathbb{R}^3} hu\,dx, \end{aligned}$$

here

$$\Phi_\lambda(u) = \frac{1}{4}\|u\|_\lambda^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^2 u^2\,dx + \int_{\mathbb{R}^3} \mathcal{F}(x,u)\,dx \ge 0.$$

Again by (2.17), (2.16) and Proposition 2.4 (i), we have

$$\begin{aligned} &\frac{1}{4}\|w_n\|_\lambda^2 + \int_{\mathbb{R}^3} \mathcal{F}(x,w_n)\,dx \\ &= I_\lambda(w_n) - \frac{1}{4}\langle I'_\lambda(w_n), w_n\rangle + \frac{3}{4}\int_{\mathbb{R}^3} hw_n\,dx + o(1) \\ &\le c - I_\lambda(u) + o(1) \\ &= c - \Big[\Phi_\lambda(u) - \frac{3}{4}\int_{\mathbb{R}^3} hu\,dx\Big] + \frac{3}{4}\int_{\mathbb{R}^3} hw_n\,dx + o(1) \\ &= c - \Phi_\lambda(u) + \frac{3}{4}\int_{\mathbb{R}^3} hu\,dx + o(1) \\ &\le M + \widetilde{M} + o(1). \end{aligned} \tag{2.18}$$

Here we use the fact $c < M$ and

$$\frac{3}{4}|h|_2|u|_2 \le \frac{3}{4}|h|_2 d_2\|u\|_\lambda \le \frac{3}{4}|h|_2 d_2 \liminf_{n\to\infty}\|u_n\|_\lambda \le |h|_2 d_2 C \le \widetilde{M},$$

where $\widetilde{M}$ is a positive constant independent of λ. Hence

$$\int_{\mathbb{R}^3} \mathcal{F}(x,w_n)\,dx \le M + \widetilde{M} + o(1). \tag{2.19}$$

Because $V(x) < b$ on a set of finite measure and $w_n \rightharpoonup 0$, we obtain

$$|w_n|_2^2 \le \frac{1}{\lambda b}\int_{V\ge b} \lambda V^+(x) w_n^2\,dx + \int_{V<b} w_n^2\,dx \le \frac{1}{\lambda b}\|w_n\|_\lambda^2 + o(1). \tag{2.20}$$

For $2 < s < 2^*$, by the Hölder and Sobolev inequality and (2.20), we have

$$\begin{aligned}
|w_n|_s^s &= \int_{\mathbb{R}^3} |w_n|^s\,dx \\
&\le \Big(\int_{\mathbb{R}^3} |w_n|^2\,dx\Big)^{\frac{6-s}{s}} \Big(\int_{\mathbb{R}^3} |w_n|^6\,dx\Big)^{\frac{9s-18}{s}} \\
&\le \Big[\frac{1}{\lambda b}\int_{\mathbb{R}^3} \big(|\nabla w_n|^2 + \lambda V^+ w_n^2\big)\,dx\Big]^{\frac{6-s}{s}} \Big(\bar{S}^{-6}\Big[\int_{\mathbb{R}^3} |\nabla w_n|^2\,dx\Big]^3\Big)^{\frac{9s-18}{s}} \\
&\quad + o(1) \\
&\le \big(\frac{1}{\lambda b}\big)^{\frac{6-s}{4}} \bar{S}^{-\frac{3(s-2)}{2}} \|w_n\|_\lambda^s + o(1).
\end{aligned} \tag{2.21}$$

According to (A4), for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(x,t)| \le \varepsilon|t|$ for all $x \in \mathbb{R}^3$ and $|t| \le \delta$, and (A7) is satisfied for $|t| \ge \delta$ (with the same τ but possibly larger than a_1). Hence we have that

$$\int_{|w_n|\le\delta} f(x,w_n)w_n\,dx \le \varepsilon\int_{|w_n|\le\delta} w_n^2\,dx \le \frac{\varepsilon}{\lambda b}\|w_n\|_\lambda^2 + o(1), \tag{2.22}$$

and

$$\begin{aligned}
\int_{|w_n|\ge\delta} f(x,w_n)w_n\,dx &\le \Big(\int_{|w_n|\ge\delta} |\frac{f(x,w_n)}{w_n}|^\tau\,dx\Big)^{1/\tau} |w_n|_s^2 \\
&\le \Big(\int_{|w_n|\ge\delta} a_1\mathcal{F}(x,w_n)\,dx\Big)^{1/\tau} |w_n|_s^2 \\
&\le [a_1(M+\widetilde{M})]^{1/\tau} \bar{S}^{-\frac{3(2s-4)}{2s}} \big(\frac{1}{\lambda b}\big)^\theta \|w_n\|_\lambda^2 + o(1)
\end{aligned} \tag{2.23}$$

by (A7), (2.19), (2.21) with $s = 2\tau/(\tau-1)$ and the Hölder inequality, where $\theta = \frac{6-s}{2s} > 0$.

Since $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^3)$ and $h \in L^2(\mathbb{R}^3)$, we obtain that

$$\int_{\mathbb{R}^3} h(u_n - u)\,dx \to 0 \quad \text{as } n\to\infty. \tag{2.24}$$

Therefore, by (2.22), (2.23), (2.24) and Proposition 2.4 (i), we have

$$\begin{aligned} o(1) &= \langle I'_\lambda(w_n), w_n\rangle \\ &\geq \|w_n\|_\lambda^2 - \int_{\mathbb{R}^3} K(x)(2\omega+\phi_{w_n})\phi_{w_n} w_n^2\,dx - \int_{\mathbb{R}^3} f(x,w_n)w_n\,dx \\ &\quad - \int_{\mathbb{R}^3} h w_n\,dx \\ &\geq \Big[1-\frac{\varepsilon}{\lambda b} - [a_1(M+\widetilde{M})]^{1/\tau}\bar{S}^{-\frac{3(2s-4)}{2s}}\Big(\frac{1}{\lambda b}\Big)^{\theta}\Big]\|w_n\|_\lambda^2 + o(1). \end{aligned} \tag{2.25}$$

So, there exists $\Lambda = \Lambda(M) > 0$ such that $w_n \to 0$ in E_λ when $\lambda > \Lambda$. Since $w_n = u_n - u$, it follows that $u_n \to u$ in E_λ. This completes the proof. ☐

Lemma 2.10. *Assume* (A1)–(A2), (A4)–(A9) *hold. Let* $\{u_n\}$ *be a* $(PS)_c$ *sequence of* I_λ *with level* $c > 0$. *Then for any* $M > 0$, *there is* $\Lambda = \Lambda(M) > 0$ *such that, up to a subsequence,* $u_n \rightharpoonup u$ *in* E_λ *with* u *being a nontrivial critical point of* I_λ *and satisfying* $I_\lambda(u) \leq c$ *for all* $c < M$ *and* $\lambda > \Lambda$.

Proof. We modify the proof of Lemma 2.9. By Lemma 2.8, we obtain

$$I'_\lambda(u) = 0, \quad I_\lambda(w_n) \to c - I_\lambda(u), \quad I'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.26}$$

However, since V is allowed to be sign-changing and the appearance of nonlinear term h, from

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle \\ &= \frac{1}{4}\|u\|_\lambda^2 - \frac{\lambda}{4}\int_{\mathbb{R}^3} V^-(x)u^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u^2 u^2\,dx \\ &\quad + \int_{\mathbb{R}^3} \mathcal{F}(x,u)\,dx - \frac{3}{4}\int_{\mathbb{R}^3} hu\,dx, \end{aligned}$$

we cannot deduce that $I_\lambda(u) \geq 0$. We consider two possibilities:

(i) $I_\lambda(u) < 0$;
(ii) $I_\lambda(u) \geq 0$.

If $I_\lambda(u) < 0$, then $u \neq 0$ is nontrivial and the proof is done. If $I_\lambda(u) \geq 0$, following the argument in the proof of Lemma 2.9 step by step, we can get $u_n \to u$ in E_λ. Indeed, by (A1) and $w_n \to 0$ in $L^2(\{x \in \mathbb{R}^3 : V(x) < b\})$, we obtain

$$\Big|\int_{\mathbb{R}^3} V^-(x)w_n^2(x)\,dx\Big| \leq |V^-|_\infty \int_{\operatorname{supp} V^-} w_n^2\,dx = o(1),$$

which jointly this with (2.26) and Proposition 2.4 (i), we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \mathcal{F}(x,w_n)\,dx \\ &= I_\lambda(w_n) - \frac{1}{4}\langle I'_\lambda(w_n), w_n\rangle - \frac{1}{4}\|w_n\|_\lambda^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{R}^3} \lambda V^-(x)w_n^2\,dx - \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{w_n}^2 w_n^2\,dx + \frac{3}{4}\int_{\mathbb{R}^3} hw_n\,dx \\ &\leq c - I_\lambda(u) + o(1) \leq M + o(1). \end{aligned}$$

It follows that (2.23), (2.24) and (2.25) remain valid. Therefore $u_n \to u$ in E_λ and $I_\lambda(u) = c(> 0)$. The proof is complete. ☐

3. Proofs of main results

If V is sign-changing, we first verify that the functional I_λ has the linking geometry to apply the following linking theorem [23].

Proposition 3.1. *Let $E = E_1 \oplus E_2$ be a Banach space with $\dim E_2 < \infty$, $\Phi \in C^1(E, \mathbb{R}^3)$. If there exist $R > \rho > 0, \alpha > 0$ and $e_0 \in E_1$ such that*

$$\alpha := \inf \Phi(E_1 \cap S_\rho) > \sup \Phi(\partial Q)$$

where $S_\rho = \{u \in E : \|u\| = \rho\}$, $Q = \{u = v + te_0 : v \in E_2, t \geq 0, \|u\| \leq R\}$. Then Φ has a $(PS)_c$ sequence with $c \in [\alpha, \sup \Phi(Q)]$.

In our paper, we use Proposition 3.1 with $E_1 = E_\lambda^+ \oplus F_\lambda$ and $E_2 = E_\lambda^-$. By Proposition 2.2, $\mu_j(\lambda) \to 0$ as $\lambda \to \infty$ for every fixed j. By Remark 2.3, there is $\Lambda_1 > 0$ such that $E_\lambda^- \neq \emptyset$ and E_λ^- is finite dimensional for $\lambda > \Lambda_1$. Now we can investigate the linking structure of the functional I_λ.

Lemma 3.2. *Assume that* (A1)–(A2), (A8), (A9) *and* (1.4) *with $p \in (4, 2^*)$ are satisfied. Then, for each $\lambda > \Lambda_1$ (the constant given in Remark 2.3), there exist $\alpha_\lambda, \rho_\lambda$ and $\eta_\lambda > 0$ such that*

$$I_\lambda(u) \geq \alpha_\lambda \quad \text{for all } u \in E_\lambda^+ \oplus F_\lambda \text{ with } \|u\|_\lambda = \rho_\lambda \text{ and } |h|_2 < \eta_\lambda. \tag{3.1}$$

Furthermore, if $V \geq 0$, we can choose $\alpha, \rho, \eta > 0$ independent of λ.

Proof. For any $u \in E_\lambda^+ \oplus F_\lambda$, writing $u = u_1 + u_2$ with $u_1 \in E_\lambda^+$ and $u_2 \in F_\lambda$. Clearly, $(u_1, u_2)_\lambda = 0$, and

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x) u^2)\,dx = \int_{\mathbb{R}^3} (|\nabla u_1|^2 + \lambda V(x) u_1^2)\,dx + \|u_2\|_\lambda^2. \tag{3.2}$$

By Proposition 2.1, we obtain that $\mu_j(\lambda) \to +\infty$ as $j \to +\infty$ for each fixed $\lambda > \Lambda_1$. So there is a positive integer n_λ such that $\mu_j(\lambda) \leq 1$ for $j \leq n_\lambda$ and $\mu_j(\lambda) > 1$ for $j > n_\lambda + 1$. For $u_1 \in E_\lambda^+$, we set $u_1 = \Sigma_{j=n_\lambda+1}^\infty \mu_j(\lambda) e_j(\lambda)$. Thus

$$\begin{aligned} \int_{\mathbb{R}^3} (|\nabla u_1|^2 + \lambda V(x) u_1^2)\,dx &= \|u_1\|_\lambda^2 - \int_{\mathbb{R}^3} \lambda V^-(x) u_1^2\,dx \\ &\geq \Big(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\Big) \|u_1\|_\lambda^2. \end{aligned} \tag{3.3}$$

By using (2.1), (3.2), (3.3) and $-\omega \leq \phi_u \leq 0$ on the set $\{x \in \mathbb{R}^3 | u(x) \neq 0\}$, we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\Big(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\Big)\|u\|_\lambda^2 - \varepsilon |u|_2^2 - C_\varepsilon |u|_p^p - |h|_2 |u|_2 \\ &\geq \frac{1}{2}\Big(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\Big)\|u\|_\lambda^2 - \varepsilon d_2^2 \|u\|_\lambda^2 - C_\varepsilon d_p^p \|u\|_\lambda^p - d_2 |h|_2 \|u\|_\lambda \\ &\geq \|u\|_\lambda \Big\{ \Big[\frac{1}{2}\Big(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}\Big) - \varepsilon d_2^2\Big] \|u\|_\lambda - C_\varepsilon d_p^p \|u\|_\lambda^{p-1} - d_2 |h|_2 \Big\}. \end{aligned}$$

Let $g(t) = [\frac{1}{2}(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)} - \varepsilon d_2^2] t - C_\varepsilon d_p^p t^{p-1}$, for $t > 0$, $p \in (4, 6)$ there exists

$$\rho(\lambda) = \Big[\frac{\frac{1}{2}(1 - \frac{1}{\mu_{n_\lambda+1}(\lambda)}) - \varepsilon d_2^2}{C_\varepsilon d_p^p (p-1)}\Big]^{\frac{1}{p-2}}$$

such that $\max_{t\geq 0} g(t) = g(\rho(\lambda)) > 0$. From above inequality, $I_\lambda(u) \mid_{\|u\|_\lambda=\rho(\lambda)} > 0$ for all $|h|_2 < \eta_\lambda := \frac{g(\rho(\lambda))}{2d_2}$. Of course, $\rho(\lambda)$ can be chosen small enough, we can obtain the same result: there exists $\alpha_\lambda > 0$, such that $I_\lambda(u) \geq \alpha_\lambda$, here $\|u\|_\lambda = \rho_\lambda$.

If $V \geq 0$, since $E_\lambda = F_\lambda$, and

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + \lambda V(x)u^2)\,dx = \|u\|_\lambda^2,$$

we can choose $\alpha, \rho, \eta > 0$ (independent of λ) such that (3.1) holds. □

Lemma 3.3. *Suppose that* (A1), (A2), (A4), (A5), (A8), (A9) *are satisfied. Then, for any finite dimensional subspace $\tilde{E}_\lambda \subset E_\lambda$, it holds*

$$I_\lambda(u) \to -\infty \quad as\ \|u\|_\lambda \to \infty, \quad u \in \tilde{E}_\lambda.$$

Proof. Arguing indirectly, we can assume that there is a sequence $(u_n) \subset \tilde{E}_\lambda$ with $\|u_n\|_\lambda \to \infty$ such that

$$-\infty < \inf_n I_\lambda(u_n). \tag{3.4}$$

Take $v_n := u_n/\|u_n\|_\lambda$. Since $\dim \tilde{E}_\lambda < +\infty$, there exists $v \in \tilde{E}_\lambda \setminus \{0\}$ such that

$$v_n \to v \text{ in } \tilde{E}_\lambda, \quad v_n(x) \to v(x) \text{ a.e. } x \in \mathbb{R}^3$$

after passing to a subsequence. If $v(x) \neq 0$, then $|u_n(x)| \to +\infty$ as $n \to \infty$, and hence by (A5), we obtain that

$$\frac{F(x, u_n(x))}{u_n^4(x)} v_n^4(x) \to +\infty \quad as\ n \to \infty,$$

which jointly this with (A4), (2.3), Proposition 2.4 (ii) and Fatou's lemma, we obtain

$$\begin{aligned}
\frac{I_\lambda(u_n)}{\|u_n\|_\lambda^4} &\leq \frac{1}{2\|u_n\|_\lambda^2} + \frac{C_4\omega}{2}|K|_\infty - \int_{\mathbb{R}^3} \frac{F(x,u_n)}{\|u_n\|_\lambda^4}\,dx - \int_{\mathbb{R}^3} h(x)\frac{u_n}{\|u_n\|_\lambda^4}\,dx \\
&\leq \frac{1}{2\|u_n\|_\lambda^2} + \frac{C_4\omega}{2}|K|_\infty - \Big(\int_{v=0} + \int_{v\neq 0}\Big)\frac{F(x,u_n)}{u_n^4} v_n^4\,dx + \frac{|h|_2 d_2}{\|u_n\|_\lambda^3} \\
&\leq \frac{1}{2\|u_n\|_\lambda^2} + \frac{C_4\omega}{2}|K|_\infty - \int_{v\neq 0} \frac{F(x,u_n)}{u_n^4} v_n^4\,dx + \frac{|h|_2 d_2}{\|u_n\|_\lambda^3} \\
&\to -\infty.
\end{aligned}$$

This contradicts (3.4). □

Lemma 3.4. *Suppose that* (A1), (A2), (A4), (A5) (A8), (A9) *and are satisfied. If $V(x) < 0$ for some x, then for each $k \in \mathbb{N}$, there exist $\lambda_k > k, b_k > 0, w_k \in E_{\lambda_k}^+ \oplus F_{\lambda_k}, R_{\lambda_k} > \rho_{\lambda_k}$ (ρ_{λ_k} is the constant given in Lemma 3.2), and $\eta_k > 0$ such that, for $|h|_2 < \eta_k$, $|K|_\infty < b_k$(or$|K|_3 < b_k$),*

(a) $\sup I_{\lambda_k}(\partial Q_k) \leq 0$*;*
(b) $\sup I_{\lambda_k}(Q_k)$ *is bounded above by a constant independent of λ_k,*

where $Q_k := \{u = v + tw_k : v \in E_{\lambda_k}^-, t \geq 0, \|u\|_{\lambda_k} \leq R_{\lambda_k}\}$.

Proof. We adapt an argument from Ding and Szulkin [19]. For each $k \in \mathbb{N}$, since $\mu_j(k) \to +\infty$ as $j \to \infty$, there exists $j_k \in \mathbb{N}$ such that $\mu_{j_k}(k) > 1$. By Proposition 2.2, there exists $\lambda_k > k$ such that

$$1 < \mu_{j_k}(\lambda_k) < 1 + \frac{1}{\lambda_k}.$$

Taking $w_k := e_{j_k}(\lambda_k)$ be an eigenfunction of $\mu_{j_k}(\lambda_k)$, then $w_k \in E^+_{\lambda_k}$ as $\mu_{j_k}(\lambda_k) > 1$. Because $\dim E^-_{\lambda_k} \oplus \mathbb{R}w_k < +\infty$, it follows directly from Lemma 3.3 that (a) holds with $R_{\lambda_k} > 0$ large enough.

According to (A5), for each $\tilde\eta > |V^-|_\infty$, there is $r_{\tilde\eta} > 0$ such that $F(x,t) \ge \frac{1}{2}\tilde\eta t^2$ if $|t| \ge r_{\tilde\eta}$. For $u = v + w \in E^-_{\lambda_k} \oplus \mathbb{R}w_k$, we have

$$\int_{\mathbb{R}^3} V^-(x)u^2\,dx = \int_{\mathbb{R}^3} V^-(x)v^2\,dx + \int_{\mathbb{R}^3} V^-(x)w^2\,dx$$

by the orthogonality of $E^-_{\lambda_k}$ and $\mathbb{R}w_k$. Therefore, by Proposition 2.4 (ii), we obtain

$$\begin{aligned} I_{\lambda_k}(u) &\le \frac{1}{2}\int_{\mathbb{R}^3} \big(|\nabla w|^2 + \lambda_k V(x)w^2\big)\,dx - \frac{1}{2}\int_{\mathbb{R}^3} K(x)\omega\phi_u u^2\,dx \\ &\quad - \int_{\operatorname{supp} V^-} F(x,u)\,dx - \int_{\mathbb{R}^3} hu\,dx \\ &\le \frac{1}{2}[\mu_{j_k}(\lambda_k) - 1]\lambda_k \int_{\mathbb{R}^3} V^-(x)w^2\,dx - \frac{1}{2}\int_{\operatorname{supp} V^-} \tilde\eta u^2\,dx \\ &\quad + \frac{C_4\omega}{2}|K|_\infty \|u\|^4_{\lambda_k} + d_2|h|_2\|u\|_{\lambda_k} - \int_{\operatorname{supp} V^-, |u| \le r_{\tilde\eta}} \Big(F(x,u) - \frac{1}{2}\tilde\eta u^2\Big)\,dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^3} V^-(x)w^2\,dx - \frac{\tilde\eta}{2|V^-|_\infty}\int_{\mathbb{R}^3} V^-(x)w^2\,dx + C_{\tilde\eta} + \frac{C_4\omega}{2}|K|^2_\infty R^4_{\lambda_k} \\ &\quad + d_2|h|_2 R_{\lambda_k} \\ &\le C_{\tilde\eta} + 1 \end{aligned}$$

for $u = v + w \in E^-_{\lambda_k} \oplus \mathbb{R}w_k$ with $\|u\|_{\lambda_k} \le R_{\lambda_k}$, $|K|_\infty < b_k := (C_4\omega R^4_{\lambda_k})^{-1/2}$, C_4 is defined in Proposition 2.4 (iii) and $|h|_2 < \eta_k := \frac{1}{2d_2R_{\lambda_k}}$, where $C_{\tilde\eta}$ depends on $\tilde\eta$ but not λ_k.

If $K \in L^3(\mathbb{R}^3)$, by the Hölder inequality, we obtain that

$$\begin{aligned} \Big|\int_{\mathbb{R}^3} K(x)\omega\phi_u u^2\,dx\Big| &\le \omega|K|_3|\phi_u|_6|u|^2_4 \\ &\le \omega|K|_3\bar S^{-1}\|\phi_u\|_D d_4^2\|u\|^2_\lambda \\ &\le C_1|K|_3\|u\|^4_\lambda \\ &\le C_1|K|_3 R^4_{\lambda_k}. \end{aligned}$$

for $|K|_3 < b_k := (C_1R^4_{\lambda_k})^{-1}$. □

Lemma 3.5. *Suppose that* (A1), (A2), (A4), (A5) (A8), (A9) *are satisfied. If* $\Omega := intV^{-1}(0)$ *is nonempty, then, for each* $\lambda > \Lambda_1$ *(is the constant given in Remark 2.3), there exist* $w \in E^+_\lambda \oplus F_\lambda, R_\lambda > 0, b_\lambda > 0$ *and* $\eta_\lambda > 0$ *such that for* $|h|_2 < \eta_\lambda, |K|_\infty < b_\lambda or(|K|_3 < b_\lambda)$,

- (a) $\sup I_\lambda(\partial Q) \le 0$;
- (b) $\sup I_\lambda(Q)$ *is bounded above by a constant independent of* λ,

where $Q := \{u = v + tw : v \in E^-_\lambda, t \ge 0, \|u\|_\lambda \le R_\lambda\}$.

Proof. Choose $e_0 \in C_0^\infty(\Omega) \setminus \{0\}$, then $e_0 \in F_\lambda$. By Lemma 3.3, there is $R_\lambda > 0$ large such that $I_\lambda(u) \le 0$ where $u \in E^-_\lambda \oplus \mathbb{R}e_0$ and $\|u\|_\lambda \ge R_\lambda$.

For $u = v + w \in E_\lambda^- \oplus \mathbb{R}e_0$, we have

$$\begin{aligned} I_\lambda(u) &\le \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^3} K(x)\omega\phi_u u^2\,dx - \int_\Omega F(x,u)\,dx - \int_{\mathbb{R}^3} hu\,dx \\ &\le \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w|^2\,dx - \frac{\tilde\eta}{2}\int_\Omega u^2\,dx - \int_{\Omega,|u|\le r_{\tilde\eta}} \Big(F(x,u) - \frac{\tilde\eta}{2}u^2\Big)\,dx \\ &\quad + \frac{C_4\omega}{2}|K|_\infty \|u\|_\lambda^4 + |h|_2 d_2 \|u\|_\lambda \\ &\le \frac{1}{2}\int_{\mathbb{R}^3} |\nabla w|^2\,dx - \frac{\tilde\eta}{2}\int_\Omega u^2\,dx + C_{\tilde\eta} + \frac{C_4\omega}{2}|K|_\infty^2 \|u\|_\lambda^4 + |h|_2 d_2 \|u\|_\lambda. \end{aligned} \tag{3.5}$$

Observing that $w \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla w|^2\,dx &= \int_\Omega (-\Delta w)u\,dx \le |\Delta w|_2 |u|_{2,\Omega} \\ &\le c_0 |\nabla w|_2 |u|_{2,\Omega} \le \frac{c_0^2}{2\tilde\eta}|\nabla w|_2^2 + \frac{\tilde\eta}{2}|u|_{2,\Omega}^2, \end{aligned} \tag{3.6}$$

where c_0 is a constant depending on e_0. Choosing $\tilde\eta > c_0^2$, we have $|\nabla w|_2^2 \le \tilde\eta |u|_{2,\Omega}^2$, and it follows from (3.5) that

$$I_\lambda(u) \le C_{\tilde\eta} + \frac{C_4\omega}{2}|K|_\infty R_\lambda^4 + |h|_2 d_2 R_\lambda \le C_{\tilde\eta} + 1$$

for all $u \in E_\lambda^- \oplus \mathbb{R}e_0$ with $\|u\|_\lambda \le R_\lambda, |h|_2 < \eta_\lambda := \frac{1}{2d_2R_\lambda}$ and $|K|_\infty < b_\lambda := (C_4\omega R_\lambda^4)^{-1/2}$, where $C_{\tilde\eta}$ depends on $\tilde\eta$ but not λ.

If $K \in L^3(\mathbb{R}^3)$, by the Hölder inequality, obtain

$$\begin{aligned} \Big|\int_{\mathbb{R}^3} K(x)\omega\phi_u u^2\,dx\Big| &\le \omega|K|_3|\phi_u|_6|u|_4^2 \le \omega|K|_3\bar S^{-1}\|\phi_u\|_D d_4^2\|u\|_\lambda^2 \\ &\le C_1|K|_3\|u\|_\lambda^4 \le C_1|K|_3R_\lambda^4. \end{aligned}$$

for $|K|_3 < b_\lambda := \bar S(C_1 d_4^2\omega R_\lambda^4)^{-1}$. □

Now we are in a position to prove our main results.

Proof of Theorem 1.3. It is divided into two steps.

Step 1 There exists a function $u_\lambda \in E_\lambda$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) < 0$. Since $h \in L^2(\mathbb{R}^3)$ and $h \ge 0 (\not\equiv 0)$, we can choose a function $\psi \in E_\lambda$ such that

$$\int_{\mathbb{R}^3} h(x)\psi(x)\,dx > 0.$$

Hence, by $-\omega \le \phi_u \le 0$ we obtain

$$\begin{aligned} I_\lambda(t\psi) &= \frac{t^2}{2}\|\psi\|_\lambda^2 - \frac{\lambda t^2}{2}\int_{\mathbb{R}^3} V^-(x)\psi^2\,dx - \frac{1}{2}\int_{\mathbb{R}^3} K(x)\omega\phi_{t\psi}(t\psi)^2\,dx \\ &\quad - \int_{\mathbb{R}^3} F(x,t\psi)\,dx - t\int_{\mathbb{R}^3} h(x)\psi\,dx \\ &\le \frac{t^2}{2}\|\psi\|_\lambda^2 + \frac{t^2}{2}\int_{\mathbb{R}^3}\omega^2\psi^2\,dx + \frac{t^4}{4}C_1\|\psi\|_\lambda^4 - t\int_{\mathbb{R}^3} h(x)\psi\,dx \\ &< 0 \quad \text{for } t > 0 \text{ small enough.} \end{aligned}$$

Thus, there exists u_λ small enough such that $I_\lambda(u_\lambda) < 0$. By Lemma 3.3, we have

$$c_{0,\lambda} = \inf\{I_\lambda(u) : u \in \overline{B}_{\rho_\lambda}\} < 0,$$

where $\rho_\lambda > 0$ is given by Lemma 3.2, $B_{\rho_\lambda} = \{u \in E_\lambda : \|u\|_\lambda < \rho_\lambda\}$. By the Ekeland's variational principle, there exists a minimizing sequence $\{u_{n,\lambda}\} \subset \overline{B}_{\rho_\lambda}$ such that

$$c_{0,\lambda} \leq I_\lambda(u_{n,\lambda}) < c_{0,\lambda} + \frac{1}{n_\lambda},$$
$$I_\lambda(w_\lambda) \geq I_\lambda(u_{n,\lambda}) - \frac{1}{n_\lambda}\|w_\lambda - u_{n,\lambda}\|_\lambda$$

for all $w_\lambda \in \overline{B}_{\rho_\lambda}$. Therefore, $\{u_{n,\lambda}\}$ is a bounded Palais-Smale sequence of I_λ. Then, by a standard procedure, Lemmas 2.8 and 2.9 imply that there is a function $u_\lambda \in E_\lambda$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_{0,\lambda} < 0$.

If $V \geq 0$, we can show that $\rho_\lambda, c_{0,\lambda}, u_{0,\lambda}$ are independent of λ.

Step 2 There exists a function $\widetilde{u}_\lambda \in E_\lambda$ such that $I'_\lambda(\widetilde{u}_\lambda) = 0$ and $I_\lambda(\widetilde{u}_\lambda) > 0$. It follows from Lemmas 3.2, 3.4 and Proposition 3.1 that, for each $k \in \mathbb{N}, \lambda = \lambda_k$ and $|h|_2 < \eta_k$, I_{λ_k} has a $(PS)_c$ sequence with $c \in [\alpha_{\lambda_k}, \sup I_{\lambda_k}(Q_k)]$. Setting $M := \sup I_{\lambda_k}(Q_k)$, then I_{λ_k} has a nontrivial critical point according to Lemmas 2.7, 2.10 and Proposition 3.1. Hence there exists a function $\widetilde{u}_\lambda \in E_\lambda$ such that $I'_\lambda(\widetilde{u}_\lambda) = 0$ and $I_\lambda(\widetilde{u}_\lambda) = c \geq \alpha_{\lambda_k} > 0$. The proof is complete. $\square$

Proof of Theorem 1.4. The first solution is similar to the first solution of Theorem 1.3. The second solution follows from Lemmas 2.7, 2.10, 3.2 and 3.5, and Proposition 3.1. The proof is complete. $\square$

Proof of Theorem 1.5. It is divided into two steps.

Step 1 There exists a function $u_0 \in E_\lambda$ such that $I'_\lambda(u_0) = 0$ and $I_\lambda(u_0) < 0$. In the proof of Theorem 1.3, we can choose $c_0 = c_{0,\lambda}, B_\rho = B_{\rho,\lambda}$, then by the Ekeland's variational principle, there exists a sequence $\{u_n\} \subset \overline{B}_\rho$ such that

$$c_0 \leq I_\lambda(u_n) < c_0 + \frac{1}{n},$$
$$I_\lambda(w) \geq I_\lambda(u_n) - \frac{1}{n}\|w - u_n\|_\lambda$$

for all $w \in \overline{B}_\rho$. Then by a standard procedure, we can show that $\{u_n\}$ is a bounded Palais-Smale sequence of I_λ. Therefore Lemmas 2.8 and 2.9 imply that there exists a function $u_0 \in E_\lambda$ such that $I'_\lambda(u_0) = 0$ and $I_\lambda(u_0) = c_0 < 0$.

Step 2 There exists a function $\widetilde{u}_\lambda \in E_\lambda$ such that $I'_\lambda(\widetilde{u}_\lambda) = 0$ and $I_\lambda(\widetilde{u}_\lambda) > 0$. Since we suppose $V \geq 0$, the functional I_λ has mountain pass geometry and the existence of nontrivial solutions can be obtained by mountain pass theorem [23, 27, 34]. Indeed, by Lemma 3.2, there exist constants $\alpha, \rho, \eta > 0$ (independent of λ) such that, for each $\lambda > \Lambda_0$,

$$I_\lambda(u) \geq \alpha \quad \text{for } u \in E_\lambda \text{ with } \|u\|_\lambda = \rho \text{ and } |h|_2 < \eta.$$

Take $e \in C_0^\infty(\Omega) \setminus \{0\}$, by (A4), (A5) and Fatou's lemma, we obtain

$$\begin{aligned}\frac{I_\lambda(te)}{t^4} \leq \frac{1}{2t^2}\int_\Omega |\nabla e|^2\,dx - \frac{1}{2t^2}\int_\Omega K(x)\omega^2 e^2\,dx - \int_{\{x\in\Omega: e(x)\neq 0\}} \frac{F(x,te)}{(te)^4} e^4\,dx \\ - t^{-3}\int_\Omega he\,dx \to -\infty\end{aligned}$$

as $t \to +\infty$, which yields that $I_\lambda(te) < 0$ for $t > 0$ large. Clearly, there is $C > 0$ (independent of λ) such that

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) \le \sup_{t\ge 0} I_\lambda(te_0) \le C$$

where $\Gamma = \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0, \|\gamma(1)\|_\lambda \ge \rho, I_\lambda(\gamma(1)) < 0\}$. By the Mountain pass theorem and Lemma 2.9, we obtain a nontrivial critical point $\tilde{u}_\lambda$ of I_λ with $I_\lambda(\tilde{u}_\lambda) \in [\alpha, C]$ for λ large. The proof is complete. $\square$

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References

[1] C. O. Alves, M. A. S. Souto, S. H. M. Soares; *Schrödinger-Poisson equations without Ambrosetti-Rabinowitz condition*, J. Math. Anal. Appl. 377 (2011) 584-592.

[2] A. Azzollini, A. Pomponio; *Ground state solutions for the nonlinear Klein-Gordon-Maxwell equations*, Topol. Methods Nonlinear Anal. 35 (2010) 33-42.

[3] A. Azzollini, L. Pisani, A. Pomponio; *Improved estimates and a limit case for the electrostatic Klein-Gordon-Maxwell system*, Proc. Roy. Soc. Edinb. Sect. A 141 (2011) 449-463.

[4] T. Bartsch, Z. Q. Wang; *Existence and multiplicity results for some superlinear elliptic problem on $\mathbb{R}^N$*, Comm. Partial Differ. Equa. 20 (1995) 1725-1741.

[5] V. Benci, D. Fortunato; *The nonlinear Klein-Gordon equation coupled with the Maxwell equations*, Nonlinear Anal. 47 (2001) 6065-6072.

[6] V. Benci, D. Fortunato; *Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations*, Rev. Math. Phys. 14 (2002) 409-420.

[7] V. Benci, D. Fortunato, A. Masiello, L. Pisani; *Solitons and the electromagnetic field*, Math. Z. 232 (1) (1999) 73-102.

[8] H. Brézis, L. Nirenberg; *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Commun. Pure Appl. Math. 36 (1983) 437-477.

[9] D. Cassani; *Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell's equations*, Nonlinear Anal. 58 (2004) 733-747.

[10] P. C. Carrião, P. L. Cunha, O. H. Miyagaki; *Existence results for the Klein-Gordon-Maxwell equations in higher dimensions with critical exponents*, Commun. Pure Appl. Anal. 10 (2) (2011) 709-718.

[11] P. C. Carrião, P. L. Cunha, O. H. Miyagaki; *Positive ground state solutions for the critical Klein-Gordon-Maxwell system with potentials*, Nonlinear Anal. 75 (10) (2012) 4068-4078.

[12] S. J. Chen, S. Z. Song; *Multiple solutions for nonhomogeneous Klein-Gordon-Maxwell equations on $\mathbb{R}^3$*, Nonlinear Analysis: Real World Applications 22 (2015) 259-271.

[13] P. Cunha; *Subcritical and supercritical Klein-Gordon-Maxwell equtions without Ambrosetti-Rabinowitz condition*, Differ. Integral Equat. 27(3/4) (2014) 387-399.

[14] T. D'Aprile, D. Mugnai; *Non-existence results for the coupled Klein-Gordon-Maxwell equations*, Adv. Nonlinear Stud. 4 (2004) 307-322.

[15] T. D'Aprile, D. Mugnai; *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schröinger-Maxwell equations*, Proc. Roy. Soc. Edinb. Sect. A 134 (2004) 893-906.

[16] P. d'Avenia, L. Pisani, G. Siciliano; *Dirichlet and Neumann problems for Klein-Gordon-Maxwell systems*, Nonlinear Anal. 71(12) (2009) 1985-1995.

[17] P. d'Avenia, L. Pisani, G. Siciliano; *Klein-Gordon-Maxwell system in a bounded domain*, Discrete Contin. Dyn. Syst. 26(1) (2010) 135-159.

[18] L. Ding, L. Li; *Infinitely many standing wave solutions for the nonlinear Klein-Gordon-Maxwell system with sign-changing potential*, Computers and Mathematics with Applications 68 (2014) 589-595.

[19] Y. H. Ding, A. Szulkin; *Bound states for semilinear Schrödinger equations with sign-changing potential*, Calc. Var. Partial Differ. Equ. 29 (2007), 397-419.

[20] V. Georgiev, N. Visciglia; *Solitary waves for Klein-Gordon-Maxwell system with external Coulomb potential*, J. Math. Pures Appl. 9 (2005) 957-983.

[21] X. M. He; *Multiplicity of solutions for a nonlinear Klein-Gordon-Maxwell system*, Acta Appl. Math. 130 (2014) 237-250.

[22] L. Li, C. L. Tang; *Infinitely many solutions for a nonlinear Klein-Gordon-Maxwell system*, Nonlinear Anal. 110 (2014) 157-169.

[23] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math. 65, American Mathematical Society, Providence, 1986.

[24] G. Vaira; *Semiclassical states for the nonlinear Klein-Gordon-Maxwell system*, J. Pure and Applied Mathematics: Advances and Aplications 4(1), (2010) 59-95.

[25] F. Z. Wang; *Solitary waves for the Klein-Gordon-Maxwell system with critical exponent*, Nonlinear Anal. 74 (2011) 827-835.

[26] F. Z. Wang; *Ground-state solutions for the electrostatic nonlinear Klein-Gordon-Maxwell system*, Nonlinear Anal. 74 (14) (2011), 4796-4803.

[27] M. Willem; *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser Boston Inc., Boston, MA, 1996.

[28] M. Willem; *Analyse Harmonique Réelle*. Hermann, Paris, 1995.

[29] L. P. Xu, H. B. Chen; *Existence and multiplicity of solutions for nonhomogenous Klein-Gordon-Maxwell equations*, Elec. Diff. Equat. 102 (2015) 1-12.

[30] Y. W. Ye, C. L. Tang; *Existence and multiplicity of solutions for Schrödinger-Poisson equations with sign-changing potential*, Calc. Var. 53 (2015), 383-411.

[31] J. J. Zhang, W. M. Zou; *Solutions concentrating around the saddle points of the potential for critical Schrödinger equations*, Calc. Var. 54 (2015), 4119-4142.

[32] L. G. Zhao, F. K. Zhao; *On the existence of solutions for the Schrödinger-Poisson equations*, J. Math. Anal. Appl. 346(1) (2008), 155-169.

[33] L. G. Zhao, F. K. Zhao; *Positive solutions for Schrödinger-Poisson equations with a critical exponent*, Nonlinear Anal. 70 (2009), 2150-2164.

[34] W. M. Zou, M. Schechter; *Critical Point Theory and Its Applications*, Springer, New York, 2006.

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