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MULTIPLICITY OF SOLUTIONS FOR A PERTURBED FRACTIONAL SCHRÖDINGER EQUATION INVOLVING OSCILLATORY TERMS

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Abstract. In this article we study the perturbed fractional Schrödinger equation involving oscillatory terms

$$\begin{gathered}(-\Delta)^{\alpha}u+u=Q(x)\Big(f(u)+\epsilon g(u)\Big),\quad x\in\mathbb{R}^N\\ u\geq 0,\end{gathered}$$

where $\alpha\in(0,1)$ and $N>2\alpha$, $(-\Delta)^{\alpha}$ stands for the fractional Laplacian, $Q:\mathbb{R}^N\to\mathbb{R}^N$ is a radial, positive potential, $f\in C([0,\infty),\mathbb{R})$ oscillates near the origin or at infinity and $g\in C([0,\infty),\mathbb{R})$ with $g(0)=0$. By using the variational method and the principle of symmetric criticality for non-smooth Szulkin-type functionals, we establish that: (1) the unperturbed problem, i.e. with $\epsilon=0$ has infinitely many solutions; (2) the number of distinct solutions becomes greater and greater when $|\epsilon|$ is smaller and smaller. Moreover, various properties of the solutions are also described in terms of the L^{∞}- and $H^{\alpha}(\mathbb{R}^N)$-norms.

1. Introduction

In this paper we consider the multiplicity of positive solutions for the fractional Schrödinger equation

$$\begin{gathered}(-\Delta)^{\alpha}u+u=Q(x)\Big(f(u)+\epsilon g(u)\Big),\quad x\in\mathbb{R}^N,\\ u\geq 0,\end{gathered}\tag{1.1}$$

where $\alpha\in(0,1)$, $N>2\alpha$, $(-\Delta)^{\alpha}$ stands for the fractional Laplacian, $Q:\mathbb{R}^N\to\mathbb{R}^N$ ia a radial, positive potential, $f:[0+\infty)\to\mathbb{R}^N$ is a continuous nonlinearity which oscillates near the origin or at infinity and $g:[0,\infty)\to R$ is an continuous function with $g(0)=0$.

In the local case, that is, when $\alpha=1$, the arbitrarily many solutions for the perturbed elliptic problem (1.1) involving oscillatory terms, for the case $N\geq 2$, has been studied in [5]. Kristály [5] first proved the unperturbed problem (1.1) with $\epsilon=0$ in (1.1), has infinitely many distinct solutions. Then, he proved that the number of distinct solutions for the perturbed problem (1.1) becomes greater and greater when $|\epsilon|$ is smaller and smaller.

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In the nonlocal case, that is, when $\alpha \in (0,1)$, to the best of our knowledge, there are no studies for the fractional nonlinear equation (1.1), maybe because technique developed for local case cannot be adapted immediately, c.f. [14]. Motivated by [5], we establish the multiplicity of positive solutions for (1.1). Because of the nonlocal nature of the fractional Laplacian, we would like to point out that some estimates in [5] cannot be obtained directly when $\alpha \in (0,1)$. In this paper, we will overcome these difficulties by more careful estimates for the energy functional associated with the auxiliary problem, see proof of Theorem 2.1. Another novelty is the truncation function $\omega_s(x)$ in [5] will be replaced by a more general function.

Throughout this paper, we always assume

- (A1) $Q : \mathbb{R}^N \to \mathbb{R}^N$ is a positive, continuous, radially symmetric potential such that $Q \in L^p(\mathbb{R}^N)$ for every $p \in [1,2]$.

We recall that, for any $\alpha \in (0,1)$, the fractional Laplacian $(-\Delta)^\alpha u$ of a function $u : \mathbb{R}^N \to \mathbb{R}^N$, with sufficient decay, is defined by

$$\mathcal{F}((-\Delta)^\alpha u)(\xi) = |\xi|^{2\alpha}\mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N,$$

where $\mathcal{F}$ denotes the Fourier transform,

$$\mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi\cdot x}\phi(x)dx \equiv \widehat{\phi}(\xi),$$

for function ϕ in the Schwartz class. $(-\Delta)^\alpha u$ can also be computed by the following singular integral:

$$(-\Delta)^\alpha u = c_{N,\alpha}\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}dy,$$

here P.V. is the principal value and $c_{N,\alpha}$ ia a normalization constant.

The fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ is defined by

$$H^\alpha(\mathbb{R}^N) = \big\{u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{2}+\alpha}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\big\},$$

endowed with the norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} u^2dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy\Big)^{1/2},$$

where the term

$$[u]_{H^\alpha(\mathbb{R}^N)} = \|(-\Delta)^{\alpha/2}u\|_{L^2(\mathbb{R}^N)} := \Big(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy\Big)^{1/2}$$

is the so-called *Gagliardo semi-norm* of u.

For $N > 2\alpha$, from [4] we know that there exists a constant $C = C(N,\alpha) > 0$ such that

$$\|u\|_{L^{2^*_\alpha}(\mathbb{R}^N)} \le C\|u\|_{H^\alpha(\mathbb{R}^N)}$$

for every $u \in H^\alpha(\mathbb{R}^N)$, where $2^*_\alpha = \frac{2N}{N-2\alpha}$ is the fractional critical exponent. Moreover, the embedding $H^\alpha(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ is continuous for any $q \in [2, 2^*_\alpha]$, and is locally compact whenever $q \in [2, 2^*_\alpha)$. For the basic properties of the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$, we refer to [4, 8, 11, 14].

Let $f \in C([0,\infty),\mathbb{R})$ and $F(t) = \int_0^t f(\tau)d\tau$, $t \ge 0$. We assume:

- (A2) $-\infty < \liminf_{t\to 0^+} \frac{F(t)}{t^2} \le \limsup_{t\to 0^+} \frac{F(t)}{t^2} = +\infty$.

(A3) There exists a sequence $(t_i)_i \subset (0,\infty)$ converging to 0 such that $f(t_i) < 0$ for every $i \in N$.

Remark 1.1. (1) Assumptions (A2) and (A3) imply an oscillatory behavior of f near the origin.

(2) Let $\alpha, \beta, \gamma \in \mathbb{R}^N$ such that $0 < \alpha < 1 < \alpha + \beta$, and $\gamma \in (0,1)$. Then, the function $f \in C([0,\infty),\mathbb{R})$ defined by $f(0) = 0$ and $f(s) = s^\alpha(\gamma + \sin s^{-\beta})$ satisfies (A2) and (A3), respectively.

For the unperturbed problem (1.1) with $\epsilon = 0$, we have the following result.

Theorem 1.2. *Assume* (A1) *holds and* $f \in C([0,\infty),\mathbb{R})$ *satisfying* (A2) *and* (A3). *Then there exists a sequence* $\{u_i^0\}_i \subset H^\alpha(\mathbb{R}^N)$ *of distinct, radially symmetric weak solutions of* (1.1) *with* $\epsilon = 0$ *such that*

$$\lim_{i\to\infty} \|u_i^0\|_{L^\infty} = \lim_{i\to\infty} \|u_i^0\| = 0. \tag{1.2}$$

For the perturbed problem (1.1), one has the weaker result.

Theorem 1.3. *Assume* (A1) *holds,* $f \in C([0,\infty),\mathbb{R})$ *satisfying* (A2) *and* (A3) *and* $g \in C([0,\infty),\mathbb{R})$ *with* $g(0) = 0$. *Then, for every* $k \in N$, *there exists* $\epsilon_k^0 > 0$ *such that* (1.1) *has at least* k *distinct, radially symmetric weak solutions in* $H^\alpha(\mathbb{R}^N)$ *whenever* $\epsilon \in [-\epsilon_k^0, \epsilon_k^0]$. *Moreover, if this* k *solutions are denoted by* $u_{i,\epsilon}^0 \in H^\alpha(\mathbb{R}^N)$, $i = 1,\dots,k$, *then*

$$\|u_{i,\epsilon}^0\|_{L^\infty} < \frac{1}{i} \quad \text{and} \quad \|u_{i,\epsilon}^0\| < \frac{1}{i} \quad \text{for any } i = 1,\dots,k. \tag{1.3}$$

Remark 1.4. Note (1.2) and (1.3) are in a perfect concordance. Moreover, the perturbed and unperturbed ones are equivalent in the sense that they are deducible from each other. Clearly, the perturbed problem contains the unperturbed one by choosing $g = 0$. Conversely, exploiting the behavior of certain sequences which appear in the proof of Theorem 1.2, we can show that for every $k \in N$, there exists $\epsilon_k^0 > 0$ such that the perturbed problem has at least k distinct solutions in $H^\alpha(\mathbb{R}^N)$ whenever $\epsilon \in [-\epsilon_k^0, \epsilon_k^0]$.

Next, we will state the counterparts of Theorems 1.2 and 1.3 when f oscillates at infinity. We assume:

(A4) $-\infty < \liminf_{t\to\infty} \frac{F(t)}{t^2} \le \limsup_{t\to\infty} \frac{F(t)}{t^2} = +\infty$.

(A5) There exists a sequence $(t_i)_i \subset (0,\infty)$ converging to $+\infty$ such that $f(t_i) < 0$ for every $i \in N$.

Remark 1.5. (1) The assumptions (A4) and (A5) imply an oscillatory behavior of f at infinity.

(2) Let $\alpha, \beta, \gamma \in \mathbb{R}^N$ such that $\alpha > 1$, $|\alpha - \beta| < 1$, and $\gamma \in (0,1)$. Then, the function $f \in C([0,\infty),\mathbb{R})$ defined by $f(s) = s^\alpha(\gamma + \sin s^{-\beta})$ satisfies (A4) and (A5), respectively.

For problem (1.1) with $\epsilon = 0$, we have the counterpart of Theorem 1.2.

Theorem 1.6. *Assume* (A1) *holds and* $f \in C([0,\infty),\mathbb{R})$ *satisfying* (A4), (A5) *and* $f(0) = 0$. *Then there exists a sequence* $\{u_i^\infty\}_i \subset H^\alpha(\mathbb{R}^N)$ *of radially symmetric weak solutions of* (1.1) *with* $\epsilon = 0$ *such that*

$$\lim_{i\to\infty} \|u_i^\infty\|_{L^\infty} = \infty. \tag{1.4}$$

Remark 1.7. Beside of (A4) and (A5), no further growth condition is assumed on the nonlinear tern at infinity. Actually, this is why we cannot give $H^\alpha(\mathbb{R}^N)$-norm estimates for the solutions in Theorem 1.6. If we further assume that f satisfies the following growth condition at infinity, i.e., there exists $q \in (1, \frac{2^*_\alpha}{2})$ and $C > 0$ such that

$$|f(t)| \le C(1 + t^{q-1}) \quad \text{for all } t \in [0, \infty). \tag{1.5}$$

Then, we have

$$\lim_{i\to\infty} \|u_i^\infty\| = \infty. \tag{1.6}$$

It is easy to see that (1.5) and the right side of (A4) imply $q > 2$. Thus, (1.6) is possible for the lower dimensions $N = 1, 2, 3$ and adding some restriction for α, that is, when $4\alpha > N > 2\alpha$, since $2 < \frac{2^*_\alpha}{2}$. In fact, for (1.6) holds, we need to further assume that $\alpha \in (\frac{1}{4}, \frac{1}{2})$, if $N = 1$; $\alpha \in (\frac{1}{2}, 1)$, if $N = 2$; $\alpha \in (\frac{3}{4}, 1)$, if $N = 3$. Another way to guarantee (1.6) is to complete assumption (A1) by allowing for instance $Q \in L^\infty(\mathbb{R}^N)$ and (1.5) with $q \in (2, 2^*_\alpha)$.

For problem (1.1), we also have the counterpart of Theorem 1.3.

Theorem 1.8. *Assume* (A1) *holds,* $f \in C([0,\infty), \mathbb{R})$ *satisfying* (A4), (A5) *with* $f(0) = 0$, *and* $g \in C([0,\infty), \mathbb{R})$ *with* $g(0) = 0$. *Then, for every* $k \in N$, *there exists* $\epsilon_k^\infty > 0$ *such that* (1.1) *has at least* k *distinct, radially symmetric weak solutions in* $H^\alpha(\mathbb{R}^N)$ *whenever* $\epsilon \in [-\epsilon_k^\infty, \epsilon_k^\infty]$. *Moreover, for this* k *solutions are denoted by* $u_{i,\epsilon}^\infty \in H^\alpha(\mathbb{R}^N)$, $i = 1, \dots, k$, *we have*

$$\|u_{i,\epsilon}^\infty\|_{L^\infty} > i - 1 \quad \textit{for } i = 1, \dots, k. \tag{1.7}$$

Remark 1.9. Equations (1.4) and (1.7) are also in concordance. Moreover, if both functions f and g verify (1.5) with $q \in (2, \frac{2^*_\alpha}{2})$ and if $N = 1$, $\alpha \in (\frac{1}{4}, \frac{1}{2})$, if $N = 2$, $\alpha \in (\frac{1}{2}, 1)$, if $N = 3$, $\alpha \in (\frac{3}{4}, 1)$, then besides of (1.7), whenever $\epsilon \in [-\epsilon_k^\infty, \epsilon_k^\infty]$, we also have

$$\|u_{i,\epsilon}^\infty\| > i - 1 \quad \text{for } i = 1, \dots, k. \tag{1.8}$$

In recent years, the study of the various nonlinear equations or systems involving fractional Laplacian has received considerable attention. These problems mainly arise in fractional quantum mechanics [6, 7], physics and chemistry [9], obstacle problems [16], optimization and finance [3] and so on. The literature on non-local fractional Laplacian operators and their application to differential equations is quite large, we refer the interested reader to [1, 2, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20] and the references therein.

The rest of this article is organized as follows. In Section 2, we present an auxiliary result which is important for our problem. In Section 3 we prove Theorems 1.2 and 1.3. In Section 4 we prove Theorems 1.6 and 1.8, Remarks 1.7 and 1.9.

2. An auxiliary result

In this section we consider the generic problem

$$\begin{gathered} (-\Delta)^\alpha u + u = Q(x)h(u), \quad x \in \mathbb{R}^N \\ u \ge 0, \end{gathered} \tag{2.1}$$

Beside of the assumption (A1), we further assume that

(A6) $h : [0, +\infty) \to \mathbb{R}^N$ is a continuous, bounded function such that $h(0) = 0$;

(A7) There are $0 < a < b$ such that $h(s) \leq 0$ for all $s \in [a, b]$.

By assumption (A6), we may put $h(s) = 0$ for $s \leq 0$. The energy functional J_h on $H^\alpha(\mathbb{R}^N)$ associated with problem (2.1) is

$$J_h(u) := \frac{1}{2}\int_{\mathbb{R}^N} \Big(|(-\Delta)^{\alpha/2}u|^2 + u^2\Big)dx - \int_{\mathbb{R}^N} Q(x)H(u)\,dx, \tag{2.2}$$

where $H(u) = \int_0^u h(s)ds$. By the mean value theorem and Hölder inequality, for any $u \in H^\alpha(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} Q(x)H(u)\,dx \leq \int_{\mathbb{R}^N} Q(x)|H(u)|\,dx \leq M_h\|Q\|_{L^2}\|u\|_{L^2} < \infty,$$

where $M_h = \sup_{s\in\mathbb{R}} |h(s)|$, so the functional J_h is well defined. Moreover, by the assumptions (A1), (A6) and Lebesgue dominated convergence theorem, $J \in C^1(H^\alpha(\mathbb{R}^N), \mathbb{R})$ and its critical points are the solutions of problem (2.1).

Now, we denote by $H^\alpha_{\rm rad}(\mathbb{R}^N)$ radial functions in $H^\alpha(\mathbb{R}^N)$, and let

$$R_h = J_h\big|_{H^\alpha_{\rm rad}(\mathbb{R}^N)},$$

i.e., the restriction of J_h to $H^\alpha_{\rm rad}(\mathbb{R}^N)$. Moreover, for $b \in \mathbb{R}^+$, we denote

$$W^b = \{u \in H^\alpha(\mathbb{R}^N) : \|u\|_{L^\infty} \leq b\} \quad \text{and} \quad W^b_{\rm rad} = W^b \cap H^\alpha_{\rm rad}(\mathbb{R}^N).$$

Now we state the main result of this section.

Theorem 2.1. *Assume that* (A1), (A6), (A7) *hold. Then*

(i) *The functional R_h is bounded from below on $W^b_{\rm rad}$ and attains its infimum at $u_h \in W^b_{\rm rad}$.*

(ii) *$u_h \in [0, a]$ for a.e. $x \in \mathbb{R}^N$.*

(iii) *u_h is a radial weak solution of problem* (2.1).

Proof. (i) For any $u \in H^\alpha_{\rm rad}(\mathbb{R}^N)$, by (A1) and (A6), we have

$$\begin{aligned} R_h(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} Q(x)H(u)dx \\ &\geq \frac{1}{2}\|u\|^2 - M_h\|Q\|_{L^2}\|u\|_{L^2} \\ &\geq \frac{1}{2}\|u\|^2 - M_h\|Q\|_{L^2}\|u\| \\ &\geq -\frac{1}{2}M_h^2\|Q\|^2_{L^2}, \end{aligned}$$

so the functional R_h is bounded from below on $W^b_{\rm rad}$. Now we prove that it attains infimum at $u_h \in W^b_{\rm rad}$. Noting that $W^b_{\rm rad}$ is convex and closed, so it is weakly closed. By the above inequality, the functional R_h is coercive, so we only need to show that the functional R_h is sequentially weakly lower semicontinuous. Since $u \mapsto \|u\|$ is sequentially weakly lower semicontinuous, it is enough to show that $u \mapsto \int_{\mathbb{R}^N} Q(x)H(u)$ is sequentially weakly continuous. Arguing by contraction, suppose that for a sequence $\{u_n\}_n \subset H^\alpha_{\rm rad}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u \in H^\alpha_{\rm rad}(\mathbb{R}^N)$, there exists a number $\epsilon_0 > 0$ such that

$$0 < \epsilon_0 \leq \Big|\int_{\mathbb{R}^N} Q(x)H(u_n) - \int_{\mathbb{R}^N} Q(x)H(u)\Big| \quad \text{for all } n \in \mathbb{N}.$$

By [8], we can see that $H^\alpha_{\rm rad}(\mathbb{R}^N)$ is compactly embedded into $L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*_\alpha)$, so $u_n \to u$ in $L^q(\mathbb{R}^N)$. By the mean value theorem and Hölder inequality, we have

$$\begin{aligned}
0 < \epsilon_0 &\le \Big| \int_{\mathbb{R}^N} Q(x)H(u_n) - \int_{\mathbb{R}^N} Q(x)H(u) \Big| \le M_h \int_{\mathbb{R}^N} Q(x)|u_n - u| dx \\
&\le M_h \|Q\|_{L^{\frac{q}{q-1}}} \|u_n - u\|_{L^q},
\end{aligned}$$

this is a contradiction and the proof part (i) is complete.

(ii) Let $A = \{x \in \mathbb{R}^N : u_h(x) \notin [0,a]\}$ and suppose that $|A| > 0$, where $|A|$ denotes the Lebesgue measure of the set A. Define the function $\gamma : \mathbb{R} \to \mathbb{R}^N$ by $\gamma(s) = \min(s_+, a)$, where $s_+ = \max(s, 0)$, then γ ia a Lipschitz function and $\gamma(0) = 0$. Set $\omega = \gamma \circ u_h$, it is clear that ω is radial, $0 \le \omega \le a$ for a.e. $x \in \mathbb{R}^N$ and $\omega \in H^\alpha(\mathbb{R}^N)$.

Now we define the sets

$$A_1 = \{x \in A : u_h(x) < 0\}, \quad A_2 = \{x \in A : u_h(x) > a\}.$$

Then $A = A_1 \cup A_2$, and we have that $\omega(x) = u_h(x)$ for all $x \in \mathbb{R}^N \setminus A$, $\omega(x) = 0$ for all $x \in A_1$, and $\omega(x) = a$ for all $x \in A_2$. Thus,

$$\begin{aligned}
&R_h(\omega) - R_h(u_h) \\
&= \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\omega(x) - \omega(y)|^2}{|x-y|^{N+2\alpha}} \,dx\,dy - \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_h(x) - u_h(y)|^2}{|x-y|^{N+2\alpha}} \,dx\,dy \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} (\omega^2 - u_h^2) dx + \int_{\mathbb{R}^N} Q(x)\big(H(\omega) - H(u_h)\big) dx \\
&= \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\omega(x) - \omega(y)|^2}{|x-y|^{N+2\alpha}} \,dx\,dy - \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_h(x) - u_h(y)|^2}{|x-y|^{N+2\alpha}} \,dx\,dy \\
&\quad + \frac{1}{2} \int_A (\omega^2 - u_h^2) dx + \int_A Q(x)\Big(H(\omega) - H(u_h)\Big) dx.
\end{aligned}$$

If $x \in A_1$ and $y \in A_1$, then

$$|\omega(x) - \omega(y)| = 0 \le |u_h(x) - u_h(y)|. \tag{2.3}$$

If $x \in A_1$ and $y \in A_2$, then

$$|\omega(x) - \omega(y)| = a \le |u_h(x) - u_h(y)|. \tag{2.4}$$

If $x \in A_1$ and $y \in \mathbb{R}^N \setminus A$, then

$$|\omega(x) - \omega(y)| = u_h(y) \le |u_h(x) - u_h(y)|. \tag{2.5}$$

If $x \in A_2$, $y \in A_2$, then

$$|\omega(x) - \omega(y)| = 0 \le |u_h(x) - u_h(y)|. \tag{2.6}$$

If $x \in A_2$, $y \in \mathbb{R}^N \setminus A$, then

$$|\omega(x) - \omega(y)| = |a - u_h(y)| \le |u_h(x) - u_h(y)|. \tag{2.7}$$

If $x \in \mathbb{R}^N \setminus A$, $y \in \mathbb{R}^N \setminus A$, then

$$|\omega(x) - \omega(y)| = |u_h(x) - u_h(y)|. \tag{2.8}$$

From (2.3)-(2.8), for any $x, y \in \mathbb{R}^N$, one has

$$|\omega(x) - \omega(y)| \le |u_h(x) - u_h(y)|,$$

so
$$\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy-\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u_h(x)-u_h(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy\le 0.$$
Note that
$$\int_A(\omega^2-u_h^2)dx=-\int_{A_1}u_h^2dx+\int_{A_2}(a^2-u_h^2)dx\le 0.$$
Since $h(s)=0$ for all $s\le 0$, we have
$$\int_{A_1}Q(x)\Big(H(\omega)-H(u_h)\Big)dx=0.$$
By the mean value theorem, for a.e. $x\in A_2$, there exists $\theta(x)\in[a,u_h(x)]\subseteq[a,b]$ such that
$$H(\omega(x))-H(u_h(x))=H(a)-H(u_h(x))=h(\theta(x))(a-u_h(x)).$$
By (A7), we have
$$\int_{A_2}Q(x)\Big(H(\omega)-H(u_h)\Big)dx\le 0.$$
So $R_h(\omega)-R_h(u_h)\le 0$. Moreover $R_h(\omega)-R_h(u_h)\ge 0$ according to the definition of u_h. Thus
$$\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\omega(x)-\omega(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy-\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u_h(x)-u_h(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy=0,$$
$$\frac{1}{2}\int_A(\omega^2-u_h^2)dx=0,\quad \int_A Q(x)\Big(H(\omega)-H(u_h)\Big)dx=0.$$
In particular,
$$\int_{A_1}u_h^2dx=\int_{A_2}(a^2-u_h^2)dx=0,$$
this implies that meas(A) should be zero and this is a contradiction with the assumption.

(iii) By [17], we have
$$R_h'(u_h)(\omega-u_h)\ge 0\quad\text{for every }\omega\in W^b. \tag{2.9}$$
where we use a non-smooth symmetric critical principle for the Szulkin-type functional.

Now we prove that u_h is a weak solution of (2.1), that is, for all $v\in H^\alpha(\mathbb{R}^N)$,
$$\int_{\mathbb{R}^N}(-\Delta)^{\alpha/2}u_h(-\Delta)^{\alpha/2}v+u_hv\,dx=\int_{\mathbb{R}^N}Q(x)h(u_h)v\,dx.$$
By (2.9), for all $\omega\in W^b$, it follows that
$$\begin{aligned}&\int_{\mathbb{R}^N}(-\Delta)^{\alpha/2}u_h(-\Delta)^{\alpha/2}(\omega-u_h)+u_h(\omega-u_h)dx\\&-\int_{\mathbb{R}^N}Q(x)h(u_h)(\omega-u_h)\,dx\ge 0.\end{aligned}\tag{2.10}$$
Define the function $\gamma(s)=\operatorname{sgn}(s)\min(|s|,b)$, and fix $\epsilon>0$ and $v\in H^\alpha(\mathbb{R}^N)$ arbitrarily. Since γ is Lipschitz and $\gamma(0)=0$, $\omega_\gamma=\gamma\circ(u_h+\epsilon v)\in H^\alpha(\mathbb{R}^N)$. The

explicit expression of the truncation function ω_γ is

$$\omega_\gamma(x)=\begin{cases}-b & \text{if } x\in\{u_h+\epsilon v<-b\},\\ u_h(x)+\epsilon v(x) & \text{if } x\in\{-b\le u_h+\epsilon v<b\},\\ b & \text{if } x\in\{u_h+\epsilon v\ge b\},\end{cases}$$

thus $\omega_\gamma\in W^b$. Taking $\omega=\omega_\gamma$ as a test function in (2.10), we obtain

$$\begin{aligned}
0&\le \int_{\mathbb{R}^N}(-\Delta)^{\alpha/2}u_h(-\Delta)^{\alpha/2}(\omega_\gamma-u_h)+u_h(\omega_\gamma-u_h)dx\\
&\quad-\int_{\mathbb{R}^N}Q(x)h(u_h)(\omega_\gamma-u_h)\,dx\\
&=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u_h(x)-u_h(y))(\omega_\gamma(x)-u_h(x)-\omega_\gamma(y)+u_h(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\quad+\int_{\mathbb{R}^N}u_h(\omega_\gamma-u_h)dx-\int_{\mathbb{R}^N}Q(x)h(u_h)(\omega_\gamma-u_h)\,dx\\
&=\iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v<-b\}}\frac{-|u_h(x)-u_h(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\quad+\iint_{\{u_h+\epsilon v<-b\}\times\{-b\le u_h+\epsilon v<b\}}\frac{(u_h(x)-u_h(y))(-b-u_h(x)-\epsilon v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\quad+\iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v\ge b\}}\frac{(u_h(x)-u_h(y))(-2b-u_h(x)+u_h(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\quad+\iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v<-b\}}\frac{(u_h(x)-u_h(y))(\epsilon v(x)+b+u_h(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\quad+\epsilon\iint_{\{-b\le u_h+\epsilon v<b\}\times\{-b\le u_h+\epsilon v<b\}}\frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\quad+\iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v\ge b\}}\frac{(u_h(x)-u_h(y))(\epsilon v(x)-b+u_h(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\quad+\iint_{\{u_h+\epsilon v\ge b\}\times\{u_h+\epsilon v<-b\}}\frac{(u_h(x)-u_h(y))(b-u_h(x)-\epsilon v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\quad+\iint_{\{u_h+\epsilon v\ge b\}\times\{-b\le u_h+\epsilon v<b\}}\frac{(u_h(x)-u_h(y))(b-u_h(x)-\epsilon v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\quad+\iint_{\{u_h+\epsilon v\ge b\}\times\{u_h+\epsilon v\ge b\}}\frac{(u_h(x)-u_h(y))(u_h(y)-u_h(x))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\quad-\int_{\{u_h+\epsilon v<-b\}}[(b+u_h)u_h-Q(x)h(u_h)(b+u_h)]dx\\
&\quad+\epsilon\int_{\{-b\le u_h+\epsilon v<b\}}[u_hv-Q(x)h(u_h)v]dx\\
&\quad-\int_{\{u_h+\epsilon v\ge b\}}[(u_h-b)u_h+Q(x)h(u_h)(u_h-b)]dx.
\end{aligned}$$

After a suitable rearrangement of the above terms, we obtain that

$$0\le\epsilon\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy+\epsilon\int_{\mathbb{R}^N}u_hv\,dx$$

$$
\begin{aligned}
&- \epsilon \int_{\mathbb{R}^N} Q(x)h(u_h)v\,dx \\
&+ \iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v<-b\}} \frac{-|u_h(x)-u_h(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&- \epsilon \iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v<-b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&+ \iint_{\{u_h+\epsilon v<-b\}\times\{-b\le u_h+\epsilon v<b\}} \frac{(u_h(x)-u_h(y))(-b-u_h(x)-\epsilon v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&- \epsilon \iint_{\{u_h+\epsilon v<-b\}\times\{-b\le u_h+\epsilon v<b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&+ \iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v\ge b\}} \frac{(u_h(x)-u_h(y))(-2b-u_h(x)+u_h(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&- \epsilon \iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v\ge b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&+ \iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v<-b\}} \frac{(u_h(x)-u_h(y))(\epsilon v(x)+b+u_h(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&- \epsilon \iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v<-b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&+ \iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v\ge b\}} \frac{(u_h(x)-u_h(y))(\epsilon v(x)-b+u_h(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&- \epsilon \iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v\ge b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&+ \iint_{\{u_h+\epsilon v\ge b\}\times\{u_h+\epsilon v<-b\}} \frac{(u_h(x)-u_h(y))(b-u_h(x)-\epsilon v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&- \epsilon \iint_{\{u_h+\epsilon v\ge b\}\times\{u_h+\epsilon v<-b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&+ \iint_{\{u_h+\epsilon v\ge b\}\times\{-b\le u_h+\epsilon v<b\}} \frac{(u_h(x)-u_h(y))(b-u_h(x)-\epsilon v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&- \epsilon \iint_{\{u_h+\epsilon v\ge b\}\times\{-b\le u_h+\epsilon v<b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&+ \iint_{\{u_h+\epsilon v\ge b\}\times\{u_h+\epsilon v\ge b\}} \frac{(u_h(x)-u_h(y))(u_h(y)-u_h(x))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&- \epsilon \iint_{\{u_h+\epsilon v\ge b\}\times\{u_h+\epsilon v\ge b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&+ \int_{\{u_h+\epsilon v<-b\}} [Q(x)h(u_h)-u_h](b+u_h+\epsilon v)dx \\
&+ \int_{\{u_h+\epsilon v\ge b\}} [Q(x)h(u_h)-u_h](-b+u_h+\epsilon v)dx.
\end{aligned}
$$

By the direct computation, one has

$$
\begin{aligned}
&\iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v<-b\}} \frac{-|u_h(x)-u_h(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&-\epsilon\iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v<-b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\le \epsilon\iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v<-b\}} \frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy,
\end{aligned}
$$

$$
\begin{aligned}
&\iint_{\{u_h+\epsilon v<-b\}\times\{-b\le u_h+\epsilon v<b\}} \frac{(u_h(x)-u_h(y))(-b-u_h(x)-\epsilon v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&-\epsilon\iint_{\{u_h+\epsilon v<-b\}\times\{-b\le u_h+\epsilon v<b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\le \epsilon\iint_{\{u_h+\epsilon v<-b\}\times\{-b\le u_h+\epsilon v<b\}} \frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy,
\end{aligned}
$$

$$
\begin{aligned}
&\iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v\ge b\}} \frac{(u_h(x)-u_h(y))(-2b-u_h(x)+u_h(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&-\epsilon\iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v\ge b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\le \epsilon\iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v\ge b\}} \frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy,
\end{aligned}
$$

$$
\begin{aligned}
&\iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v<-b\}} \frac{(u_h(x)-u_h(y))(\epsilon v(x)+b+u_h(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&-\epsilon\iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v<-b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\le \epsilon\iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v<-b\}} \frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy,
\end{aligned}
$$

$$
\begin{aligned}
&\iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v\ge b\}} \frac{(u_h(x)-u_h(y))(\epsilon v(x)-b+u_h(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&-\epsilon\iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v\ge b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\le \epsilon\iint_{\{-b\le u_h+\epsilon v<b\}\times\{u_h+\epsilon v\ge b\}} \frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy,
\end{aligned}
$$

$$
\begin{aligned}
&\iint_{\{u_h+\epsilon v\ge b\}\times\{u_h+\epsilon v<-b\}} \frac{(u_h(x)-u_h(y))(b-u_h(x)-\epsilon v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&-\epsilon\iint_{\{u_h+\epsilon v\ge b\}\times\{u_h+\epsilon v<-b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\le \epsilon\iint_{\{u_h+\epsilon v\ge b\}\times\{u_h+\epsilon v<-b\}} \frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy,
\end{aligned}
$$

$$
\begin{aligned}
&\iint_{\{u_h+\epsilon v\geq b\}\times\{-b\leq u_h+\epsilon v<b\}} \frac{(u_h(x)-u_h(y))(b-u_h(x)-\epsilon v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&-\epsilon\iint_{\{u_h+\epsilon v\geq b\}\times\{-b\leq u_h+\epsilon v<b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\leq \epsilon\iint_{\{u_h+\epsilon v\geq b\}\times\{-b\leq u_h+\epsilon v<b\}} \frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy,
\end{aligned}
$$

and

$$
\begin{aligned}
&\iint_{\{u_h+\epsilon v\geq b\}\times\{u_h+\epsilon v\geq b\}} \frac{(u_h(x)-u_h(y))(u_h(y)-u_h(x))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&-\epsilon\iint_{\{u_h+\epsilon v\geq b\}\times\{u_h+\epsilon v\geq b\}} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&\leq \epsilon\iint_{\{u_h+\epsilon v\geq b\}\times\{u_h+\epsilon v\geq b\}} \frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy.
\end{aligned}
$$

Moreover, from $u_h\in[0,a]\subset[-b,b]$ for a.e. $x\in\mathbb{R}^N$, one has

$$
\begin{aligned}
&\int_{\{u_h+\epsilon v<-b\}}\Big(Q(x)h(u_h)-u_h\Big)(b+u_h+\epsilon v)\,dx\\
&\leq -\epsilon\int_{\{u_h+\epsilon v<-b\}}\Big(M_hQ(x)+u_h(x)\Big)v(x)dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\{u_h+\epsilon v\geq b\}}\Big(Q(x)h(u_h)-u_h\Big)(-b+u_h+\epsilon v)dx\\
&\leq \epsilon M_h\int_{\{u_h+\epsilon v\geq b\}}Q(x)v(x)dx.
\end{aligned}
$$

Using the above the estimates and dividing by $\epsilon>0$, we obtain

$$
\begin{aligned}
0\leq&\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy+\int_{\mathbb{R}^N}u_hv\,dx\\
&-\int_{\mathbb{R}^N}Q(x)h(u_h)v\,dx\\
&+\iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v<-b\}}\frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&+\iint_{\{u_h+\epsilon v<-b\}\times\{-b\leq u_h+\epsilon v<b\}}\frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&+\iint_{\{u_h+\epsilon v<-b\}\times\{u_h+\epsilon v\geq b\}}\frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&+\iint_{\{-b\leq u_h+\epsilon v<b\}\times\{u_h+\epsilon v<-b\}}\frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&+\iint_{\{-b\leq u_h+\epsilon v<b\}\times\{u_h+\epsilon v\geq b\}}\frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy\\
&+\iint_{\{u_h+\epsilon v\geq b\}\times\{u_h+\epsilon v<-b\}}\frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy
\end{aligned}
$$

$$
\begin{aligned}
&+ \iint_{\{u_h+\epsilon v\geq b\}\times\{-b\leq u_h+\epsilon v<b\}} \frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&+ \iint_{\{u_h+\epsilon v\geq b\}\times\{u_h+\epsilon v\geq b\}} \frac{|u_h(x)-u_h(y)||v(x)-v(y)|}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&- \int_{\{u_h+\epsilon v<-b\}} \Big(M_hQ(x)+u_h(x)\Big)v(x)dx \\
&+ M_h\int_{\{u_h+\epsilon v\geq b\}} Q(x)v(x)dx.
\end{aligned}
$$

Letting $\epsilon \to 0^+$, we have

$$
\operatorname{meas}(\{u_h+\epsilon v<-b\})\to 0 \quad \text{and} \quad \operatorname{meas}(\{u_h+\epsilon v\geq b\})\to 0,
$$

respectively. So, it follows that

$$
\begin{aligned}
0 \leq & \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u_h(x)-u_h(y))(v(x)-v(y))}{|x-y|^{N+2\alpha}}\,dx\,dy \\
&+ \int_{\mathbb{R}^N} u_h v\,dx - \int_{\mathbb{R}^N} Q(x)h(u_h)v\,dx.
\end{aligned}
$$

Using $-v$ instead of v, we also have the above inequality. So, u_h is a weak solution of (2.1). The proof is complete. □

Fix $\sigma\in(0,1)$ and $\rho>0$, for any $t>0$ we introduce the function

$$
\omega_\sigma^t(x) := \begin{cases} 0 & \text{if } x\in\mathbb{R}^N\setminus B_\rho, \\ \frac{t}{(1-\sigma)\rho}(\rho-|x|) & \text{if } x\in B_\rho\setminus B_{\sigma\rho}, \\ t & \text{if } x\in B_{\sigma\rho}, \end{cases} \tag{2.11}
$$

where B_r denotes the N-dimensional ball with center 0 and radius $r>0$. It is clear that $\omega_\sigma^t(x)$ is radial. Later we will show that $\omega_\sigma^t(x)\in H^\alpha(\mathbb{R}^N)$. To prove the main theorems in this paper, we need a important estimate for the norm of $\omega_\alpha^t(x)$. For this, we set

$$
\nu_0 := 1+\frac{1}{\lambda_1}, \quad \lambda_1 = \inf_{u\in H_0^1(B_\rho)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(B_\rho)}}{\|u\|^2_{L^2(B_\rho)}}. \tag{2.12}
$$

Proposition 2.2. *Let $\sigma\in(0,1)$, $\rho>0$ and $t>0$. Let ω_σ^s be the function given in (2.11), S_{N-2} be the Lebesgue measure of the unit sphere in $\mathbb{R}^{N-1}$, and*

$$
\Gamma(t) := \int_0^{+\infty} z^{t-1}e^{-z}dz, \quad t>0,
$$

be the usual Gamma function. Then $\omega_\sigma^t\in H^\alpha(\mathbb{R}^N)$, and one has

$$
\begin{aligned}
\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\omega_\sigma^t(x)-\omega_\sigma^t(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy &< \frac{t^2}{(1-\sigma)^2}\cdot\frac{\pi^{N/2}\rho^{N-2}(1-\sigma^N)}{\Gamma(1+\frac{N}{2})}\kappa_1\kappa_2 \\
&= K(\rho,\sigma),
\end{aligned} \tag{2.13}
$$

where

$$
\kappa_1 := \begin{cases} 2\nu_0 & \text{if } N=1, \\ (\pi+\frac{4}{1+2\alpha})\nu_0 & \text{if } N=2,\,, \\ S_{n-2}(\frac{\pi}{2}+\frac{2}{1+2\alpha})\nu_0 & \text{if } N\geq 3, \end{cases}
$$

$$\kappa_2 := \frac{1}{2(1-\alpha)} + \frac{2}{\alpha},$$

where ν_0 *is given in* (2.12).

Proof. The proof can be found in [11], for the sake of completeness, we give it here. Computing the standard seminorm of the function ω_σ^t in $H^1(\mathbb{R}^N)$, one has

$$\begin{aligned}
[\omega_\sigma^t]^2_{H^1(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |\nabla \omega_\sigma^t(x)|^2 dx \\
&= \int_{B_\rho \setminus B_{\sigma\rho}} \frac{t^2}{(1-\sigma)^2\rho^2} dx \\
&= \frac{t^2}{(1-\sigma)^2\rho^2}(|B_\rho| - |B_{\sigma\rho}|) \\
&= \frac{t^2}{(1-\sigma)^2} \frac{\pi^{N/2}\rho^{N-2}(1-\sigma^N)}{\Gamma(1+\frac{N}{2})}.
\end{aligned} \tag{2.14}$$

Since $\omega_\sigma^t \in H_0^1(B_\rho)$, by [11, Proposition 1.1(b)], it follows that $\omega_\sigma^t \in W^{\alpha,2}(B_\rho)$. Moreover, the boundary ∂B_ρ is Lipschitz, by [11, Lemma 1.3], we have that $\omega_\sigma^t \in H^\alpha(\mathbb{R}^N)$.

Hence, since $\alpha \in (0,1)$, [11, Corollary 1.15] yields

$$\begin{aligned}
[\omega_\sigma^s]^2_{H^\alpha(\mathbb{R}^N)} &\le 2\Big(\int_{\mathbb{R}^N} \frac{1-\cos x_1}{|x|^{N+2\alpha}} dx\Big) \int_{\mathbb{R}^N} |\xi|^{2\alpha} ||\mathcal{F}\omega_\sigma^t(\xi)|^2 d\xi \\
&< 2\Big(\int_{\mathbb{R}^N} \frac{1-\cos x_1}{|x|^{N+2\alpha}} dx\Big) \int_{\mathbb{R}^N} (1+|\xi|^2|)|\mathcal{F}\omega_\sigma^t(\xi)|^2 d\xi.
\end{aligned} \tag{2.15}$$

Now, by standard arguments on the Fourier transform and Poincaré inequality, we have

$$\int_{\mathbb{R}^N} (1+|\xi|^2|)|\mathcal{F}\omega_\sigma^t(\xi)|^2 d\xi \le \nu_0 [\omega_\sigma^t]^2_{H^1(\mathbb{R}^N)}. \tag{2.16}$$

Moreover, by Parseval-Plancherel identity, it follows that

$$\omega_\sigma^s \in L^2(\mathbb{R}^N) \quad \text{if and only if} \quad \mathcal{F}\omega_\sigma^t \in L^2(\mathbb{R}^N)$$

and

$$\|\omega_\sigma^t\|^2_{L^2(\mathbb{R}^N)} = \|\mathcal{F}\omega_\sigma^t\|^2_{L^2(\mathbb{R}^N)}. \tag{2.17}$$

Moreover,

$$|\omega_\sigma^t| \in L^2(\mathbb{R}^N) \quad \text{if and only if} \quad |\xi|\mathcal{F}\omega_\sigma^t \in L^2(\mathbb{R}^N),$$

and

$$\|\nabla\omega_\sigma^t\|^2_{L^2(\mathbb{R}^N)} = \|\,|\xi|\mathcal{F}\omega_\sigma^t\|^2_{L^2(\mathbb{R}^N)}. \tag{2.18}$$

By (2.17) and (2.18), we have

$$\int_{\mathbb{R}^N} (1+|\xi|^2|)|\mathcal{F}\omega_\sigma^t(\xi)|^2 d\xi = \|\omega_\sigma^t\|^2_{L^2(\mathbb{R}^N)} + \|\nabla\omega_\sigma^t\|^2_{L^2(\mathbb{R}^N)}.$$

Hence, by (2.12) and the definition of λ_1, we have that the inequality (2.16) is a direct consequence of above equality, taking into account that $\omega_\sigma^t \in H_0^1(B_\rho)$.

Then, by (2.15) and (2.16), it follows that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\omega_\sigma^t(x) - \omega_\sigma^t(y)|^2}{|x-y|^{N+2\alpha}}\, dx\, dy < 2\Big(\int_{\mathbb{R}^N} \frac{1-\cos x_1}{|x|^{N+2\alpha}} dx\Big)[\omega_\sigma^t]^2_{H^1(\mathbb{R}^N)}.$$

Finally. from (2.14) and [11, Remark 1.13], we can get the inequality (2.13). $\square$

3. Proofs of Theorems 1.2 and 1.3

By (A3) and the continuity of f and g, we may fix the positive sequences $\{a_i\}_i$, $\{b_i\}_i$ and $\{\epsilon_i\}_i$ such that $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = 0$ and for all $i \in N$,

$$b_{i+1} < a_i < t_i < b_i < 1, \tag{3.1}$$

$$f(t) + \epsilon g(t) \le 0 \quad \text{for all } t \in [a_i, b_i] \text{ and } \epsilon \in [-\epsilon_i, \epsilon_i]. \tag{3.2}$$

For every $i \in N$, we define the truncation function $f_i, g_i : [0, +\infty) \to \mathbb{R}^N$ by

$$f_i(t) = f(\min(t, b_i)) \quad \text{and} \quad g_i(t) = g(\min(t, b_i)). \tag{3.3}$$

By (A2) and (A3), it is easy to see that $f(0) = 0$. Since $f_i(0) = g_i(0) = 0$, we may extend continuously the function f_i and g_i to $\mathbb{R}^N$, taking 0 for negative arguments. For every $t \in \mathbb{R}^N$ and $i \in N$, let $F_i(t) = \int_0^t f_i(\tau)d\tau$ and $G_i(t) = \int_0^t g_i(\tau)d\tau$.

For every $i \in N$ and $\epsilon \in [-\epsilon_i, \epsilon_i]$, the function $h^0_{i,\epsilon} := f_i + \epsilon g_i$ is a continuous, bounded function with $h^0_{i,\epsilon}(0) = 0$. By (3.2) and (3.3), we have $h^0_{i,\epsilon}(t) \le 0$ for all $t \in [a_i, b_i]$. Thus, by Theorem 2.1, for every $i \in N$ and $\epsilon \in [-\epsilon_i, \epsilon_i]$, the problem

$$\begin{gathered} (-\Delta)^\alpha u + u = Q(x)h^0_{i,\epsilon}(u), \quad x \in \mathbb{R}^N \\ u \ge 0, \end{gathered} \tag{3.4}$$

has a radially symmetric weak solution $u^0_{i,\epsilon} \in H^\alpha(\mathbb{R}^N)$ with

$$u^0_{i,\epsilon} \in [0, a_i] \quad \text{for a.e. } x \in \mathbb{R}^N, \tag{3.5}$$

$$u^0_{i,\epsilon} \text{ is the minimizer of the functional } R^\epsilon_i \text{ on } W^{b_i}_{\rm rad}, \tag{3.6}$$

where R^ϵ_i is the functional associated with problem (3.4), and

$$R^\epsilon_i(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} Q(x)\Big(F_i(u) + \epsilon G_i(u)\Big)dx, \quad u \in H^\alpha_{\rm rad}(\mathbb{R}^N).$$

By (3.3) and (3.5), $u^0_{i,\epsilon}$ is a weak solution not only for (3.4) but also for problem (1.1).

Proof of Theorem 1.2. As an abbreviation, for every $i \in N$, write $u^0_i = u^0_{i,0}$ and $R_i = R^0_i$. According the observation for problem (3.4), we only need to show that there are infinitely many distinct elements in the sequence $\{u^0_{i,0}\}_i$ verifying (1.2).

We first prove that

$$R_i(u^0_i) < 0 \quad \text{for all } i \in N, \tag{3.7}$$

$$\lim_{i\to\infty} R_i(u^0_i) = 0\,. \tag{3.8}$$

The left side of (A2) implies that there exist $l_0 > 0$ and $\delta \in (0, b_1)$ such that

$$F(t) \ge -l_0 t^2 \quad \text{for all } t \in (0, \delta). \tag{3.9}$$

Let L_0 be large enough such that

$$\frac{1}{2}K(\rho,\sigma)l_0\|Q\|_{L^1} < L_0(\sigma\rho)^N\omega_N \min B_{\sigma\rho}Q. \tag{3.10}$$

where $\rho > 0$ and $K(\rho, \sigma)$ come from (2.13). The right side of (A2) implies there is a sequence $\{\widetilde{t}_i\} \subset (0, \delta)$ such that $\widetilde{t}_i \le a_i$ and $F(\widetilde{t}_i) > L_0\widetilde{t}_i^2$ for all $i \in N$. Let $i \in N$

fixed and $\omega_\sigma^{\widetilde{t}_i} \in H^\alpha_{\rm rad}(\mathbb{R}^N)$ be the function from (2.11) corresponding to $\widetilde{t}_i > 0$. Then $\omega_\sigma^{\widetilde{t}_i} \in W^{b_i}_{\rm rad}$, and by (2.13) and (3.9), we have

$$
\begin{aligned}
R_i(\omega_\sigma^{\widetilde{t}_i}) &= \frac{1}{2}\|\omega_\sigma^{\widetilde{t}_i}\|^2 - \int_{\mathbb{R}^N} Q(x)F_i(\omega_\sigma^{\widetilde{t}_i})dx \\
&= \frac{1}{2}\|\omega_\sigma^{\widetilde{t}_i}\|^2 - F(\widetilde{t}_i)\int_{B_{\sigma\rho}} Q(x)dx - \int_{B_\rho\setminus B_{\sigma\rho}} Q(x)F_i(\omega_\sigma^{\widetilde{t}_i})dx \\
&\le \Big[\frac{1}{2}K(\rho,\sigma) - L_0(\sigma\rho)^N\omega_N \min_{B_{\sigma\rho}} Q + l_0\|Q\|_{L^1}\Big]\widetilde{t}_i^2
\end{aligned}
$$

So, by (3.10), one has

$$
R_i(u_i^0) = \min_{W^{b_i}_{rad}} R_i \le R_i(\omega_\sigma^{\widetilde{t}_i}) < 0 \tag{3.11}
$$

which proves (3.7). Now we prove the limit (3.8). For every $i \in N$, by the mean value theorem, one has

$$
R_i(u_i^0) \ge -\int_{\mathbb{R}^N} Q(x)F_i(u_i^0)dx \ge -\|Q\|_{L^1}\max_{t\in[0,1]}|f(t)|a_i.
$$

Because of $\lim_{i\to\infty} a_i = 0$, the above inequality and (3.11) leads to the limit (3.8).

By (3.3) and (3.5), we know that

$$
R_i(u_i^0) = R_1(u_i^0) \quad \text{for all } i \in N.
$$

Moreover, by (3.7) and (3.8), we know that the sequence $\{u_i^0\}$ are the infinitely many distinct radially symmetric weak solutions of problem (1.1).

Finally, we prove (1.2). It is clear that $\|u_i^0\|_{L^\infty} \le a_i$ for all $i \in N$ by (3.5). Since $\lim_{i\to\infty} a_i = 0$, the first limit holds. For the second limit, by (3.11), (3.1), (3.3) and (3.5), for $i \in N$, one has

$$
\frac{1}{2}\|u_i^0\|^2 \le \|Q\|_{L^1}\max_{t\in[0,1]}|f(t)|a_i \to 0, \quad \text{as } i \to \infty.
$$

The proof is complete. □

Proof of Theorem 1.3. By the observation for problem (3.4), to prove this theorem, we only need to prove that for every $k \in N$, there are at least k distinct elements $u_{i,\epsilon}^0$ verifying (1.3) when ϵ belongs to a certain interval around the origin.

Let $\{\theta_i\}_i$ be a sequence such that $\theta_i < 0$ for all $i \in N$ and $\lim_{i\to\infty}\theta_i = 0$. By (3.8) and (3.11), we have $\lim_{i\to\infty} R_i(\omega_\sigma^{\widetilde{t}_i}) = 0$. Thus, up to a subsequence, we may assume that the sequence $\{(\theta_i, R_i(u_i^0), R_i(\omega_\sigma^{\widetilde{t}_i}), a_i)\}_i \subset R^4$ which converges to $(0,0,0,0)$ has the property that for all $i \in N$,

$$
\theta_i < R_i(u_i^0) \le R_i(\omega_\sigma^{\widetilde{t}_i}) < \theta_{i+1}, \tag{3.12}
$$

$$
a_i < \min\Big\{\frac{1}{i}, \frac{1}{2i^2\|Q\|_{L^1}(\max_{[0,1]}|f| + \max_{[0,1]}|g| + 1)}\Big\}. \tag{3.13}
$$

Denote

$$
\epsilon_i' = \frac{\theta_{i+1} - R_i(\omega_\sigma^{\widetilde{t}_i})}{\|Q\|_{L^1}(\max_{[0,1]}|g| + 1)} \quad \text{and} \quad \epsilon_i'' = \frac{R_i(u_i^0) - \theta_i}{\|Q\|_{L^1}(\max_{[0,1]}|g| + 1)} \quad \forall i \in N.
$$

and fix $k \in N$. By (3.12),

$$
\epsilon_k^0 = \min\{1, \epsilon_1, \dots, \epsilon_k, \epsilon_1', \dots, \epsilon_k', \epsilon_1'', \dots, \epsilon_k''\} > 0.
$$

Then for every $i \in \{1,\dots,k\}$ and $\epsilon \in [-\epsilon_k^0, \epsilon_k^0]$, it follows from (3.1), (3.6) and the choice of ϵ_i' that

$$\begin{aligned} R_i^\epsilon(u_{i,\epsilon}^0) &\le R_i^\epsilon(\omega_\sigma^{\widetilde{t}_i}) \\ &= R_i(\omega_\sigma^{\widetilde{t}_i}) - \epsilon \int_{\mathbb{R}^N} Q(x) G_i(\omega_\sigma^{\widetilde{t}_i}) dx < \theta_{i+1}. \end{aligned}$$

Meanwhile, we also have from (3.1), (3.6) for $\epsilon = 0$ and the choice of ϵ_i'',

$$\begin{aligned} R_i^\epsilon(u_{i,\epsilon}^0) &= R_i(u_{i,\epsilon}^0) - \epsilon \int_{\mathbb{R}^N} Q(x) G_i(u_{i,\epsilon}^0) dx \\ &\ge R_i(u_i^0) - \epsilon \int_{\mathbb{R}^N} Q(x) G_i(u_{i,\epsilon}^0) dx > \theta_i. \end{aligned}$$

Thus, for every $i \in \{1,\dots,k\}$ and $\epsilon \in [-\epsilon_k^0, \epsilon_k^0]$, one has

$$\theta_i < R_i^\epsilon(u_{i,\epsilon}^0) < \theta_{i+1},$$

so

$$R_1^\epsilon(u_{1,\epsilon}^0) < \dots < R_k^\epsilon(u_{k,\epsilon}^0).$$

But $u_{i,\epsilon}^0 \in W_{\rm rad}^{b_1}$ for every $i \in \{1,\dots,k\}$, so $R_i^\epsilon(u_{i,\epsilon}^0) = R_1^\epsilon(u_{i,\epsilon}^0)$ from (3.3). Thus for every $\epsilon \in [-\epsilon_k^0, \epsilon_k^0]$, we obtain

$$R_1^\epsilon(u_{1,\epsilon}^0) < \dots < R_1^\epsilon(u_{k,\epsilon}^0),$$

this shows that the elements $u_{1,\epsilon}^0, \dots, u_{k,\epsilon}^0$ are distinct whenever $\epsilon \in [-\epsilon_k^0, \epsilon_k^0]$.

Now we show that (1.3). By (3.5) and (3.13), the left limit of (1.3) holds. For the right limit of (1.3), it is easy to see that for every $i \in \{1,\dots,k\}$ and $\epsilon \in [-\epsilon_k^0, \epsilon_k^0]$,

$$R_1^\epsilon(u_{i,\epsilon}^0) = R_i^\epsilon(u_{i,\epsilon}^0) < \theta_{i+1} < 0.$$

Thus, for every $i \in \{1,\dots,k\}$ and $\epsilon \in [-\epsilon_k^0, \epsilon_k^0]$, by the mean value theorem, (3.1), (3.5), (3.13) and $\epsilon_k^0 \le 1$, one has

$$\begin{aligned} \frac{1}{2}\|u_{i,\epsilon}^0\|^2 &< \int_{\mathbb{R}^N} Q(x)\Big(F_i(u_{i,\epsilon}^0) + \epsilon G_i(u_{i,\epsilon}^0)\Big) \\ &\le \|Q\|_{L^1}\Big(\max_{[0,1]}|f| + \max_{[0,1]}|g|\Big) a_i \\ &\le \frac{1}{2i^2}, \end{aligned}$$

which completes the proof. $\square$

4. Proofs of Theorems 1.6 and 1.8

The left-hand side of (A4) implies the existence of $l_\infty > 0$ and $\delta > 0$ such that

$$F(t) \ge -l_\infty t^2 \quad \text{for all } t > \delta. \tag{4.1}$$

Let L_∞ be large enough such that

$$\frac{1}{2}K(\rho,\sigma) + l_\infty \|Q\|_{L^1} < L_\infty (\sigma\rho)^N \omega_N \min_{B_{\sigma\rho}} Q, \tag{4.2}$$

where $\rho > 0$ and $K(\rho,\sigma)$ comes from (2.13). The right side of (A4) implies there is a sequence $\{\widetilde{t}_i\}_i \subset (0,\infty)$ such that $\lim_{i\to\infty} \widetilde{t}_i = \infty$ and

$$F(\widetilde{t}_i) > L_\infty \widetilde{t}_i^2 \quad \text{for all } i \in N. \tag{4.3}$$

Since $\lim_{i\to\infty}\widetilde{t}_i=\infty$, we may fix a subsequence $\{\widetilde{t}_{m_i}\}_i$ of $\{\widetilde{t}_i\}_i$ such that $\widetilde{t}_i\le\widetilde{t}_{m_i}$ for all $i\in N$. Moreover, since f and g are continuous, we may fix the positive sequences $\{a_i\}_i$, $\{b_i\}_i$ and $\{\epsilon_i\}_i$ such that $\lim_{i\to\infty}a_i=\lim_{i\to\infty}b_i=+\infty$ and for all $i\in N$,

$$a_i<\widetilde{t}_{m_i}<b_i<a_{i+1}, \tag{4.4}$$

$$f(t)+\epsilon g(t)\le 0 \quad \text{for all } t\in[a_i,b_i] \text{ and } \epsilon\in[-\epsilon_i,\epsilon_i]. \tag{4.5}$$

For every $i\in N$, we define the truncation function $f_i,g_i:[0,+\infty)\to\mathbb{R}^N$ as in (3.3). Since $f_i(0)=g_i(0)=0$, so we may extend continuously the function f_i and g_i to $\mathbb{R}^N$, taking 0 for negative arguments. For every $t\in\mathbb{R}^N$ and $i\in N$, let $F_i(t)=\int_0^t f_i(\tau)d\tau$ and $G_i(t)=\int_0^t g_i(\tau)d\tau$.

For every $i\in N$ and $\epsilon\in[-\epsilon_i,\epsilon_i]$, the function $h^{\infty}_{i,\epsilon}:=f_i+\epsilon g_i$ is a continuous, bounded function with $h^{\infty}_{i,\epsilon}(0)=0$. By (4.5) and (3.3), we have $h^{\infty}_{i,\epsilon}(t)\le 0$ for all $t\in[a_i,b_i]$. Thus, by Theorem 2.1, for every $i\in N$ and $\epsilon\in[-\epsilon_i,\epsilon_i]$, the problem

$$\begin{gathered}(-\Delta)^{\alpha}u+u=Q(x)h^{\infty}_{i,\epsilon}(u),\quad x\in\mathbb{R}^N\\ u\ge 0,\end{gathered} \tag{4.6}$$

has a radially symmetric weak solution $u^{\infty}_{i,\epsilon}\in H^{\alpha}(\mathbb{R}^N)$ with

$$u^{\infty}_{i,\epsilon}\in[0,a_i]\quad\text{for a.e. } x\in\mathbb{R}^N, \tag{4.7}$$

$$u^{\infty}_{i,\epsilon}\quad\text{is the minimizer of the functional } R_i^{\epsilon} \text{ on } W^{b_i}_{\rm rad}, \tag{4.8}$$

where R_i^{ϵ} is defined as in section 4. By (3.3) and (4.7), $u^{\infty}_{i,\epsilon}$ is a weak solution not only for (4.6) but also for problem (1.1).

Proof of Theorem 1.6. As in proof of Theorem 1.2, for every $i\in N$, write $u_i^{\infty}=u^{\infty}_{i,0}$ and $R_i=R_i^0$. According the observation for problem (4.6), we only need to show that there are infinitely many distinct elements in the sequence $\{u^{\infty}_{i,0}\}_i$ verifying (1.4). We prove that

$$\lim_{i\to\infty}R_i(u_i^{\infty})=-\infty\,. \tag{4.9}$$

Let $i\in N$ be fixed and $\omega_{\sigma}^{\widetilde{t}_i}\in H^{\alpha}_{\rm rad}(\mathbb{R}^N)$ be the function from (2.11) corresponding to $\widetilde{t}_i>0$. Then $\omega_{\sigma}^{\widetilde{t}_i}\in W^{b_i}_{\rm rad}$, and by (2.13), (4.1) and (4.3), we have

$$\begin{aligned}
R_i(\omega_{\sigma}^{\widetilde{t}_i})&=\frac12\|\omega_{\sigma}^{\widetilde{t}_i}\|^2-\int_{\mathbb{R}^N}Q(x)F_i(\omega_{\sigma}^{\widetilde{t}_i})dx\\
&=\frac12\|\omega_{\sigma}^{\widetilde{s}_i}\|^2-F(\widetilde{s}_i)\int_{B_{\sigma\rho}}Q(x)dx-\int_{(B_{\rho}\setminus B_{\sigma\rho})\cap\{\omega_{\sigma}^{\widetilde{t}_i}>\delta\}}Q(x)F_i(\omega_{\sigma}^{\widetilde{t}_i})dx\\
&\quad-\int_{(B_{\rho}\setminus B_{\sigma\rho})\cap\{\omega_{\sigma}^{\widetilde{t}_i}\le\delta\}}Q(x)F_i(\omega_{\sigma}^{\widetilde{t}_i})dx\\
&\le\Big[\frac12K(\rho,\sigma)-L_{\infty}(\sigma\rho)^N\omega_N\min_{B_{\sigma\rho}}Q+l_{\infty}\|Q\|_{L^1}\Big]\widetilde{t}_i^2+\|Q\|_{L^1}\max_{t\in[0,\delta]}|F(t)|.
\end{aligned}$$

Since $\lim_{i\to\infty}\widetilde{t}_i=\infty$ and (4.2), one has $\lim_{i\to\infty}R_i(\omega_{\sigma}^{\widetilde{t}_i})=-\infty$. But $R_i(u_i^{\infty})\le R_i(\omega_{\sigma}^{\widetilde{t}_i})$ for all $i\in N$, which proves (4.9).

Suppose that in the sequence $\{u_i^{\infty}\}_i$ there are only finitely many distinct elements, denote $\{u_1^{\infty},\dots,u_{i_0}^{\infty}\}$ for some $i_0\in N$. Thus the sequence $\{R_i(u_i^{\infty})\}_i$ reduces mostly to the finite set which contradicts (4.9).

Now we prove the limit (1.4). Argument by contradiction. Assume that there exists a subsequence $\{u_{k_i}^\infty\}_i$ of $\{u_i^\infty\}_i$ such that for all $i \in N$, $\|u_{k_i}^\infty\|_{L^\infty} \le M$, for some $M > 0$. In particular, $\{u_{k_i}^\infty\}_i \subset W_{\rm rad}^{b_l}$ for some $l \in N$. Thus for every $k_i \ge l$, one has

$$\begin{aligned} R_l(u_l^\infty) &= \min_{W_{\rm rad}^{b_l}} R_l = \min_{W_{\rm rad}^{b_l}} R_{k_i} \\ &\ge \min_{W_{\rm rad}^{b_{k_i}}} R_{k_i} = R_{k_i}(u_{k_i}^\infty) \\ &\ge \min_{W_{\rm rad}^{b_l}} R_{k_i} \\ &= R_l(u_l^\infty). \end{aligned}$$

As a consequence,

$$R_{k_i}(u_{k_i}^\infty) = R_l(u_l^\infty) \quad \text{for all } i \in N. \tag{4.10}$$

Moreover the sequence $\{R_i(u_i^\infty)\}_i$ is non-increasing. By (3.3) and (4.8), for all $i \in N$, we have

$$R_{i+1}(u_{i+1}^\infty) = \min_{W_{\rm rad}^{b_{i+1}}} R_{i+1} \le \min_{W_{\rm rad}^{b_i}} R_{i+1} = \min_{W_{\rm rad}^{b_i}} R_i = R_i(u_i^\infty).$$

Combining this latter fact with (4.10), one can find a number $i_0 \in N$ such that $R_i(u_i^\infty) = R_l(u_l^\infty)$ for all $i \ge i_0$, this is a contraction with (4.9). The proof is complete. $\square$

Proof of Remark 1.5. Assume that (1.5) holds for f with $q \in (2, \frac{2_\alpha^*}{2})$. Assume by contradiction that there exists a subsequence $\{u_{k_i}^\infty\}_i$ of $\{u_i^\infty\}_i$ such that for all $i \in N$, we have $\|u_{k_i}^\infty\| \le \tilde{M}$ for some $\tilde{M} > 0$. Fix $\beta \in [2q, 2_\alpha^*)$. By (1.5) and the mean value theorem, one has

$$\begin{aligned} \Big|\int_{\mathbb{R}^N} Q(x) F_{k_i}(u_{k_i}^\infty) dx\Big| &\le C\Big(\|Q\|_{L^2} \|u_{k_i}^\infty\|_{L^2} + \|Q\|_{L^{\frac{\beta}{\beta-q}}} \|u_{k_i}^\infty\|_{L^\beta}^q\Big) \\ &\le C_1\Big(\|Q\|_{L^2} \tilde{M} + \|Q\|_{L^{\frac{\beta}{\beta-q}}} \tilde{M}^q\Big) < \infty. \end{aligned}$$

Thus, the sequence $\{R_{k_i}(u_{k_i}^\infty)\}_i$ is bounded. Since the sequence $\{R_{k_i}(u_{k_i}^\infty)\}_i$ is non-increasing, it will be bounded, this contradicts (4.9). $\square$

Proof of Theorem 1.8. By the observation for problem (4.6), in order to prove this theorem, we only need to prove that for every $k \in N$, there are at least k distinct elements $u_{i,\epsilon}^\infty$ verifying (1.7) when ϵ belongs to a certain interval around the origin.

Let $\{\theta_i\}_i$ be a sequence such that $\theta_i < 0$ for all $i \in N$ and $\lim_{i\to\infty} \theta_i = -\infty$. By the proof of Theorem 1.6, up to a subsequence, we may assume that the sequence $\{(\theta_i, R_i(u_i^\infty), R_i(\omega_\sigma^{\tilde{t}_i}), a_i)\}_i \subset \mathbb{R}^4$ which converges to $(-\infty, -\infty, -\infty, \infty)$ has the property that for all $i \in N$,

$$\theta_{i+1} < R_i(u_i^\infty) \le R_i(\omega_\sigma^{\tilde{t}_i}) < \theta_i, \tag{4.11}$$

$$a_i \ge i. \tag{4.12}$$

Denote

$$\epsilon_i' = \frac{\theta_i - R_i(\omega_\sigma^{\tilde{t}_i})}{\|Q\|_{L^1}(\max_{[0,b_i]} |g| + 1) b_i} \quad \text{and} \quad \epsilon_i'' = \frac{R_i(u_i^\infty) - \theta_{i+1}}{\|Q\|_{L^1}(\max_{[0,b_i]} |g| + 1) b_i} \quad i \in N.$$

and fix $k \in N$. By (4.11), one has

$$\epsilon_k^\infty = \min\{1, \epsilon_1, \dots, \epsilon_k, \epsilon_1', \dots, \epsilon_k', \epsilon_1'', \dots, \epsilon_k''\} > 0.$$

Then for every $i \in \{1, \dots, k\}$ and $\epsilon \in [-\epsilon_k^0, \epsilon_k^0]$, it follows from (4.4), (4.8), the choice of ϵ_i' and $\widetilde{t}_i \le \widetilde{t}_{m_i}$ that

$$\begin{aligned} R_i^\epsilon(u_{i,\epsilon}^\infty) &\le R_i^\epsilon(\omega_\sigma^{\widetilde{s}_i}) \\ &= R_i(\omega_\sigma^{\widetilde{s}_i}) - \epsilon \int_{\mathbb{R}^N} Q(x) G_i(\omega_\sigma^{\widetilde{s}_i}) dx < \theta_i. \end{aligned}$$

Meanwhile, from (4.8) for $\epsilon = 0$, (4.4), the choice of ϵ_i'' and $\widetilde{t}_i \le \widetilde{t}_{m_i}$ we have

$$\begin{aligned} R_i^\epsilon(u_{i,\epsilon}^\infty) &= R_i(u_{i,\epsilon}^\infty) - \epsilon \int_{\mathbb{R}^N} Q(x) G_i(u_{i,\epsilon}^\infty) dx \\ &\ge R_i(u_i^\infty) - \epsilon \int_{\mathbb{R}^N} Q(x) G_i(u_{i,\epsilon}^\infty) dx > \theta_{i+1}. \end{aligned}$$

Thus, for every $i \in \{1, \dots, k\}$ and $\epsilon \in [-\epsilon_k^\infty, \epsilon_k^\infty]$, one has

$$\theta_{i+1} < R_i^\epsilon(u_{i,\epsilon}^\infty) < \theta_i, \tag{4.13}$$

so

$$R_k^\epsilon(u_{k,\epsilon}^\infty) < \dots < R_1^\epsilon(u_{1,\epsilon}^0) < 0. \tag{4.14}$$

But $u_{i,\epsilon}^\infty \in W_{\rm rad}^{b_k}$ for every $i \in \{1, \dots, k\}$ by (4.4), so $R_i^\epsilon(u_{i,\epsilon}^\infty) = R_1^\epsilon(u_{i,\epsilon}^\infty)$ by (3.3). Thus for every $\epsilon \in [-\epsilon_k^0, \epsilon_k^0]$, it follows from (4.14) that

$$R_k^\epsilon(u_{k,\epsilon}^\infty) < \dots < R_k^\epsilon(u_{1,\epsilon}^\infty) < 0,$$

this shows that the elements $u_{1,\epsilon}^\infty, \dots, u_{k,\epsilon}^\infty$ are distinct whenever $\epsilon \in [-\epsilon_k^\infty, \epsilon_k^\infty]$.

Now we prove (1.7). Fix $\epsilon \in [-\epsilon_k^\infty, \epsilon_k^\infty]$. First, since $R_1^\epsilon(u_{1,\epsilon}^\infty) < 0 = R_1^\epsilon(0)$, then $\|u_{1,\epsilon}^\infty\|_{L^\infty} > 0$ which proves (1.7) for $i = 1$. We further prove that

$$\|u_{i,\epsilon}^\infty\|_{L^\infty} > a_{i-1} \quad \text{for all } i \in \{2, \dots, k\}. \tag{4.15}$$

Argument by contradiction. Assume that there exists an element $i_0 \in \{2, \dots, k\}$ such that $\|u_{i_0,\epsilon}^\infty\|_{L^\infty} \le a_{i_0-1}$. Since $a_{i_0-1} < b_{i_0-1}$, then $u_{i_0,\epsilon}^\infty \in W_{\rm rad}^{b_{i_0-1}}$. So, by (3.3) and (4.8), it follows that

$$R_{i_0-1}^\epsilon(u_{i_0-1,\epsilon}^\infty) = \min_{W_{\rm rad}^{b_{i_0-1}}} R_{i_0-1}^\epsilon \le R_{i_0-1}^\epsilon(u_{i_0,\epsilon}^\infty) = R_{i_0}^\epsilon(u_{i_0,\epsilon}^\infty),$$

which contradicts (4.14). Thus, (4.15) holds. By (4.15) and (4.12), we can complete the proof. $\square$

Proof of Remark 1.7. Assume that both functions f and g satisfy (1.5) with $q \in (2, \frac{2_\alpha^*}{2})$. Fix $\beta \in (2q, 2_\alpha^*)$. We may assume that the sequence $\{\theta_i\}_i$ from (4.11) satisfies

$$\theta_i < -2C_1(1 + |\epsilon|)\Big(\|Q\|_{L^2}(i_0 - 1) + \|Q\|_{L^{\frac{\beta}{\beta-q}}}(i_0 - 1)^q\Big) \quad \text{for all } i \in N,$$

where $C_1 > 0$. Fix $\epsilon \in [-\epsilon_k^\infty, \epsilon_k^\infty]$ and assume that $\|u_{i_0,\epsilon}^\infty\| \le i_0 - 1$ for some $i_0 \in \{1, \dots, k\}$. Then, we have

$$\begin{aligned} \frac{1}{2}\|u_{i_0,\epsilon}^\infty\|^2 &= R_{i_0}^\epsilon(u_{i_0,\epsilon}^\infty) + \int_{\mathbb{R}^N} Q(x)\Big(F_{i_0}(u_{i_0,\epsilon}^\infty) + G_{i_0}(u_{i_0,\epsilon}^\infty)\Big)dx \\ &\le \theta_{i_0} + C_1(1 + |\epsilon|)\Big(\|Q\|_{L^2}\|u_{i_0,\epsilon}^\infty\| + \|Q\|_{L^{\frac{\beta}{\beta-q}}}\|u_{k_i}^\infty\|^q\Big) \end{aligned}$$

$$\leq \theta_{i_0} + 2C_1\Big(\|Q\|_{L^2}(i_0-1) + \|Q\|_{L^{\frac{\beta}{\beta-q}}}(i_0-1)^q\Big) < 0.$$

This is a contradiction. Therefore (1.8) holds. □

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