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ASYMMETRIC ROBIN BOUNDARY-VALUE PROBLEMS WITH p-LAPLACIAN AND INDEFINITE POTENTIAL

SALVATORE A. MARANO, NIKOLAOS S. PAPAGEORGIOU

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ABSTRACT. Four nontrivial smooth solutions to a Robin boundary-value problem with p-Laplacian, indefinite potential, and asymmetric nonlinearity superlinear at $+\infty$ are obtained, all with sign information. The semilinear case is also investigated, producing a nonzero fifth solution. Our proofs use variational methods, truncation techniques, and Morse theory.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with a C^2-boundary $\partial\Omega$, let $a \in L^\infty(\Omega)$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(\cdot, 0) = 0$. Consider the Robin problem

$$\begin{gathered} -\Delta_p u + a(x)|u|^{p-2}u = f(x,u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(x)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $1 < p < +\infty$, Δ_p indicates the p-Laplacian, $\frac{\partial u}{\partial n_p} := |\nabla u|^{p-2}\nabla u \cdot n$, with n being the outward unit normal vector to $\partial\Omega$, and $\beta \in C^{0,\alpha}(\partial\Omega, \mathbb{R}_0^+)$. We say that $u \in W^{1,p}(\Omega)$ is a (weak) solution of (1.1) provided

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx + \int_{\partial\Omega} \beta|u|^{p-2}uv\,d\sigma + \int_\Omega a|u|^{p-2}uv\,dx = \int_\Omega f(x,u)v\,dx$$

for all $v \in W^{1,p}(\Omega)$.

This paper studies the existence of multiple solutions to (1.1) when

- the potential function $x \mapsto a(x)$ is indefinite, i.e., sign changing, and
- the reaction term $(x,t) \mapsto f(x,t)$ exhibits an asymmetric behaviour as t goes from $-\infty$ to $+\infty$.

For $(x, \xi) \in \Omega \times \mathbb{R}$, we define

$$F(x,\xi) := \int_0^\xi f(x,\tau)d\tau, \quad H(x,\xi) := f(x,\xi)\xi - pF(x,\xi)\,. \tag{1.2}$$

Roughly speaking, our assumptions on the rate of f at infinity are the following.

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(1) $\lim_{\xi\to+\infty} F(x,\xi)\xi^{-p} = +\infty$ uniformly in $x \in \Omega$ and there exists $c_1 > 0$ such that
$$H(x,\xi_1) \le H(x,\xi_2) + c_1 \quad \text{whenever} \quad 0 \le \xi_1 \le \xi_2.$$
(2) For appropriate $c_2 \in \mathbb{R}$ one has
$$c_2 \le \liminf_{t\to-\infty} \frac{f(x,t)}{|t|^{p-2}t} \le \limsup_{t\to-\infty} \frac{f(x,t)}{|t|^{p-2}t} \le \hat{\lambda}_1\,, \quad \lim_{\xi\to-\infty} H(x,\xi) = +\infty$$
uniformly in $x \in \Omega$.

Here $\hat{\lambda}_n$ denotes the n^{th}-eigenvalue of the problem
$$-\Delta_p u + a(x)|u|^{p-2}u = \lambda|u|^{p-2}u \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n_p} + \beta(x)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega\,. \tag{1.3}$$
It should be noted that a possible interaction (resonance) with $\hat{\lambda}_1$ is allowed and that $f(x,\cdot)$ grows $(p-1)$-super-linearly near $+\infty$. Nevertheless, contrary to most previous works, we do not need here the stronger unilateral Ambrosetti-Rabinowitz condition.

Under (1), (2), and some additional hypotheses, one of which forces a p-concave behaviour of $t \mapsto f(x,t)$ at zero, there are four C^1-solutions to (1.1), two positive, one negative, and the remaining nodal; see Section 3. If $p := 2$ then (1.1) becomes
$$\begin{gathered} -\Delta u + a(x)u = f(x,u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.4}$$
As in [6, 14], the assumptions on a and β can be significantly relaxed. However, we obtain five nontrivial smooth solutions; cf. Theorem 4.4.

The adopted approach exploits variational methods, truncation techniques, and results from Morse theory. Regularity is a standard matter, unless $p := 2$, in which case [24, Lemmas 5.1, 5.2] are employed.

Problem (1.4) has been widely investigated under various points of view; see, for instance, [6, 14] and the references given there. On the contrary, the equation
$$-\Delta_p u + a(x)|u|^{p-2}u = f(x,u) \quad \text{in} \quad \Omega\,,$$
with Dirichlet, Neumann, or Robin boundary conditions, did not receive much attention when $p \ne 2$, a sign-changing potential appears, and $t \mapsto f(x,t)$ is asymmetric. Actually, we can only mention [16], where the Dirichlet problem is studied, [18], dealing with symmetric reactions and Neumann boundary conditions, [4, 9], devoted to $(p-1)$-super-linear reactions. The situation looks somewhat different if $a \equiv 0$; vide, e.g., [8, 15, 20, 21] and their bibliographies.

2. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. Given a set $V \subseteq X$, write $\overline{V}$ for the closure of V, ∂V for the boundary of V, and $\text{int}_X(V)$ or simply $\text{int}(V)$, when no confusion can arise, for the interior of V. If $x \in X$ and $\delta > 0$ then
$$B_\delta(x) := \{z \in X : \|z - x\| < \delta\}\,.$$
The symbol $(X^*, \|\cdot\|_{X^*})$ denotes the dual space of X, $\langle\cdot,\cdot\rangle$ indicates the duality pairing between X and X^*, while $x_n \to x$ (respectively, $x_n \rightharpoonup x$) in X means 'the sequence $\{x_n\}$ converges strongly (respectively, weakly) in X'.

We say that $\Phi : X \to \mathbb{R}$ is coercive if

$$\lim_{\|x\|\to+\infty} \Phi(x) = +\infty.$$

A function Φ is called weakly sequentially lower semi-continuous when

$$x_n \rightharpoonup x \quad \text{in } X \implies \Phi(x) \le \liminf_{n\to\infty} \Phi(x_n).$$

Let $\Phi \in C^1(X)$. The classical Cerami compactness condition for Φ reads as follows.

(C) Every sequence $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ is bounded and

$$\lim_{n\to+\infty}(1 + \|x_n\|)\|\Phi'(x_n)\|_{X^*} = 0$$

has a convergent subsequence.

For $c \in \mathbb{R}$, we define

$$\Phi^c := \{x \in X : \Phi(x) \le c\}, \quad K_c(\Phi) := K(\Phi) \cap \Phi^{-1}(c),$$

where, as usual, $K(\Phi)$ denotes the critical set of Φ, i.e.,

$$K(\Phi) := \{x \in X : \Phi'(x) = 0\}.$$

We say that $A : X \to X^*$ is of type $(\mathrm{S})_+$ if

$$x_n \rightharpoonup x \quad \text{in } X, \quad \limsup_{n\to+\infty}\langle A(x_n), x_n - x\rangle \le 0 \implies x_n \to x.$$

Given a topological pair (A, B) fulfilling $B \subset A \subseteq X$, the symbol $H_q(A, B)$, $q \in \mathbb{N}_0$, indicates the q^{th}-relative singular homology group of (A, B) with integer coefficients. If $x_0 \in K_c(\Phi)$ is an isolated point of $K(\Phi)$ then

$$C_q(\Phi, x_0) := H_q(\Phi^c \cap V, \Phi^c \cap V \setminus \{x_0\}), \quad q \in \mathbb{N}_0,$$

are the critical groups of Φ at x_0. Here, V stands for any neighborhood of x_0 such that $K(\Phi) \cap \Phi^c \cap V = \{x_0\}$. By excision, this definition does not depend on the choice of V. Suppose Φ satisfies condition (C), $\Phi\lfloor_{K(\Phi)}$ is bounded below, and $c < \inf_{x\in K(\Phi)} \Phi(x)$. Put

$$C_q(\Phi, \infty) := H_q(X, \Phi^c), \quad q \in \mathbb{N}_0.$$

The second deformation lemma [10, Theorem 5.1.33] implies that this definition does not depend on the choice of c. If $K(\Phi)$ is finite, then setting

$$M(t, x) := \sum_{q=0}^{+\infty} \operatorname{rank} C_q(\Phi, x)t^q, \quad P(t, \infty) := \sum_{q=0}^{+\infty} \operatorname{rank} C_q(\Phi, \infty)t^q$$

for $(t, x) \in \mathbb{R} \times K(\Phi)$, the following Morse relation holds

$$\sum_{x\in K(\Phi)} M(t, x) = P(t, \infty) + (1 + t)Q(t), \tag{2.1}$$

where $Q(t)$ denotes a formal series with nonnegative integer coefficients; see for instance [17, Theorem 6.62].

Now, let X be a Hilbert space, let $x \in K(\Phi)$, and let Φ be C^2 in a neighborhood of x. If $\Phi''(x)$ turns out to be invertible, then x is called non-degenerate. The Morse index d of x is the supremum of the dimensions of the vector subspaces of X on which $\Phi''(x)$ turns out to be negative definite. When x is non-degenerate and with Morse index d one has

$$C_q(\Phi, x) = \delta_{q,d}\mathbb{Z}, \quad q \in \mathbb{N}_0. \tag{2.2}$$

The monograph [17] represents a general reference on the subject.

Throughout this article, Ω denotes a bounded domain of the real Euclidean N-space $(\mathbb{R}^N, |\cdot|)$ whose boundary $\partial\Omega$ is C^2 while $n(x)$ indicates the outward unit normal vector to $\partial\Omega$ at its point x. On $\partial\Omega$ we will employ the $(N-1)$-dimensional Hausdorff measure σ. The symbol m stands for the Lebesgue measure, $p \in (1, +\infty)$, $p' := p/(p-1)$, $\|\cdot\|_q$ with $q \geq 1$ is the usual norm of $L^q(\Omega)$, $X := W^{1,p}(\Omega)$, and

$$\|u\| := \left(\|\nabla u\|_p^p + \|u\|_p^p\right)^{1/p}, \quad u \in X.$$

Write p^* for the critical exponent of the Sobolev embedding $W^{1,p}(\Omega) \subseteq L^q(\Omega)$. Recall that $p^* = Np/(N-p)$ if $p < N$, $p^* = +\infty$ otherwise, and the embedding turns out to be compact whenever $1 \leq q < p^*$.

Given $t \in \mathbb{R}$, $u, v : \Omega \to \mathbb{R}$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$, define

$$t^{\pm} := \max\{\pm t, 0\}, \quad u^{\pm}(x) := u(x)^{\pm}, \quad N_f(u)(x) := f(x, u(x)).$$

$u \leq v$ (respectively, $u < v$, etc.) means $u(x) \leq v(x)$ (respectively, $u(x) < v(x)$, etc.) for almost every $x \in \Omega$. If u, v belong to a function space, say Y, then we set

$$[u, v] := \{w \in Y : u \leq w \leq v\}, \quad Y_+ := \{w \in Y : w \geq 0\}.$$

Putting $C_+ := C^1(\overline{\Omega})_+$, $\operatorname{int}(C_+) := \operatorname{int}_{C^1(\overline{\Omega})}(C_+)$, $D_+ := \operatorname{int}_{C^0(\overline{\Omega})}(C_+)$, and

$$\hat{C}_+ := \left\{u \in C_+ : u(x) > 0 \ \forall x \in \Omega, \quad \frac{\partial u}{\partial n}\Big|_{\partial\Omega \cap u^{-1}(0)} < 0 \text{ if } \partial\Omega \cap u^{-1}(0) \neq \emptyset\right\},$$

one evidently has $D_+ = \{u \in C_+ : u(x) > 0 \ \forall x \in \overline{\Omega}\}$ as well as

$$D_+ \subseteq \hat{C}_+ \subseteq \operatorname{int}(C_+).$$

Let $A_p : X \to X^*$ be the nonlinear operator stemming from the negative p-Laplacian Δ_p, i.e.,

$$\langle A_p(u), v\rangle := \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x)\, dx \quad \forall u, v \in X.$$

A standard argument [17, Proposition 2.72] ensures that A_p is of type $(\mathrm{S})_+$.

Remark 2.1. Given $u \in X$, $w \in L^{p'}(\Omega)$, and $\beta \in C^{0,\alpha}(\partial\Omega, \mathbb{R}_0^+)$, the assertion

$$\langle A_p(u), v\rangle + \int_{\partial\Omega} \beta(x)|u(x)|^{p-2}u(x)v(x)d\sigma = \int_\Omega w(x)v(x)dx, \quad v \in X,$$

is equivalent to

$$-\Delta_p u = w \text{ in } \quad \Omega, \quad \frac{\partial u}{\partial n_p} + \beta(x)|u|^{p-2}u = 0 \text{ on } \quad \partial\Omega.$$

This easily stems from the nonlinear Green's identity [10, Theorem 2.4.54]; see for instance the proof of [19, Proposition 3].

We shall employ some facts about the spectrum of the operator

$$u \mapsto -\Delta_p u + a(x)|u|^{p-2}u$$

in X with homogeneous Robin boundary conditions. So, consider the eigenvalue problem (1.3), where, henceforth,

$$a \in L^\infty(\Omega) \text{ and } \beta \in C^{0,\alpha}(\partial\Omega, \mathbb{R}_0^+) \text{ with } \alpha \in (0,1). \tag{2.3}$$

Define

$$\mathcal{E}(u) := \|\nabla u\|_p^p + \int_\Omega a(x)|u(x)|^p dx + \int_{\partial\Omega} \beta(x)|u(x)|^p d\sigma \quad \forall u \in X. \tag{2.4}$$

The Liusternik-Schnirelman theory provides a strictly increasing sequence $\{\hat{\lambda}_n\}$ of eigenvalues for (1.3). Denote by $E(\hat{\lambda}_n)$ the eigenspace corresponding to $\hat{\lambda}_n$. As in [18, 19], one has

$$\hat{\lambda}_1 \text{ is isolated and simple. Further, } \hat{\lambda}_1 = \inf_{u\in X\setminus\{0\}} \frac{\mathcal{E}(u)}{\|u\|_p^p}. \tag{2.5}$$

$$\text{There exists an } L^p\text{-normalized eigenfunction } \hat{u}_1 \in D_+ \text{ associated with } \hat{\lambda}_1. \tag{2.6}$$

Let $p := 2$. It is known [6, 14] that $H^1(\Omega) = \overline{\oplus_{n=1}^\infty E(\hat{\lambda}_n)}$ and that, for any $n \geq 2$,

$$\hat{\lambda}_n = \inf\big\{\frac{\mathcal{E}(u)}{\|u\|_2^2} : u \in \hat{H}_n,\, u \neq 0\big\} = \sup\big\{\frac{\mathcal{E}(u)}{\|u\|_2^2} : u \in \bar{H}_n,\, u \neq 0\big\}, \tag{2.7}$$

where

$$\bar{H}_m := \oplus_{n=1}^m E(\hat{\lambda}_n), \quad \hat{H}_m := \oplus_{n=m}^\infty E(\hat{\lambda}_n).$$

3. Existence results

To avoid unnecessary technicalities, for every $x \in \Omega$' will take the place of 'for almost every $x \in \Omega$' while $c_1, c_2, \dots$ indicate positive constants arising from the context.

Henceforth, $f : \Omega\times\mathbb{R} \to \mathbb{R}$ denotes a Carathéodory function such that $f(\cdot, 0) = 0$. Let F and H be given by (1.2). We shall make the following assumptions.

(A1) There exist $a_1 \in L^\infty(\Omega)$ and $r \in (p, p^*)$ such that

$$|f(x,t)| \leq a_1(x)(1 + |t|^{r-1}) \quad \forall (x,t) \in \Omega\times\mathbb{R}.$$

(A2) $\lim_{\xi\to+\infty} F(x,\xi)\xi^{-p} = +\infty$ uniformly in $x \in \Omega$. Moreover, for appropriate $a_2 \in L^1(\Omega)_+$,

$$0 \leq \xi_1 \leq \xi_2 \implies H(x,\xi_1) \leq H(x,\xi_2) + a_2(x) \quad \forall x \in \Omega. \tag{3.1}$$

(A3) There exists $\bar{u} \in D_+$ fulfilling

$$\frac{\partial \bar{u}}{\partial n}\Big|_{\partial\Omega} < 0, \quad \Delta_p \bar{u} \in L^{p'}(\Omega), \quad \langle A_p(\bar{u}), v\rangle \geq 0 \quad \forall v \in W^{1,p}(\Omega)_+,$$

and $\operatorname{ess\,sup}_{x\in\Omega}[f(x,\bar{u}(x)) - a(x)\bar{u}(x)^{p-1}] < 0$.

(A4) For some $a_3 \in L^\infty(\Omega)$ one has

$$a_3(x) \leq \liminf_{t\to-\infty} \frac{f(x,t)}{|t|^{p-2}t} \leq \limsup_{t\to-\infty} \frac{f(x,t)}{|t|^{p-2}t} \leq \hat{\lambda}_1, \quad \lim_{\xi\to-\infty} H(x,\xi) = +\infty$$

uniformly with respect to $x \in \Omega$.

(A5) There exist $q \in (1,p)$ and $\delta_1 > 0$ satisfying

$$0 < f(x,\xi)\xi \leq qF(x,\xi) \quad \text{in } \Omega\times([-\delta_1,\delta_1]\setminus\{0\})$$

as well as $\operatorname{ess\,inf}_{x\in\Omega} F(x,\delta_1) > 0$.

(A6) To every $\rho > 0$ there corresponds $\mu_\rho > 0$ such that $t \mapsto f(x,t) + \mu_\rho t^{p-1}$ is nondecreasing on $[0,\rho]$ for all $x \in \Omega$.

Remark 3.1. The assumption $\lim_{\xi\to+\infty} F(x,\xi)\xi^{-p} = +\infty$ is weaker than the unilateral Ambrosetti-Rabinowitz condition below.

(AR) For appropriate $\theta > p$ and $M > 0$ one has $\operatorname{ess\,inf}_{x\in\Omega} F(x,M) > 0$ and
$$0 < \theta F(x,\xi) \le f(x,\xi)\xi \quad \text{in } \Omega \times [M,+\infty).$$

A standard example is $f(x,t) := t^{p-1}\log t$, $t \ge M > 1$.

Remark 3.2. Property (3.1) has been thoroughly investigated in [11, Lemma 2.4]. Among other things, this result ensures that (A2) forces $\lim_{t\to+\infty} f(x,t)t^{-p+1} = +\infty$, i.e., $f(x,\cdot)$ turns out to be $(p-1)$-super-linear at $+\infty$.

Remark 3.3. Assumption (A3) implies $\Delta_p \bar{u} \le 0$. Indeed, via the nonlinear Green's identity [10, Theorem 2.4.54] we get
$$\int_\Omega v(x)\,\Delta_p\bar{u}(x)\,dx = -\langle A_p(\bar{u}), v\rangle + \langle \frac{\partial \bar{u}}{\partial n_p}, v\rangle_{\partial\Omega} \le 0 \quad \forall v \in W^{1,p}(\Omega)_+\,.$$
Here, $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the duality pairing between $W^{-\frac{1}{p'},p'}(\partial\Omega)$ and $W^{\frac{1}{p'},p}(\partial\Omega)$. Moreover,
$$\langle A_p(\overline{u}), v\rangle + \int_\Omega a(x)\overline{u}(x)^{p-1}v(x)\,dx \ge \int_\Omega f(x,\overline{u}(x))v(x)\,dx, \quad v \in W^{1,p}(\Omega)_+,$$
whence $\overline{u}$ is a super-solution of (1.1).

Remark 3.4. Reasoning as in [6, Lemma 3.1] shows that (A4) entails
$$\lim_{\xi\to-\infty}[\hat{\lambda}_1|\xi|^p - pF(x,\xi)] = +\infty \quad \text{uniformly with respect to } x \in \Omega\,.$$
Problem (1.1) is thus coercive in the negative direction, and direct methods can be used to find a negative solution.

Remark 3.5. After integration, (A5) easily leads to
$$\theta|\xi|^q \le F(x,\xi) \quad \forall (x,\xi) \in \Omega \times [-\delta_1,\delta_1], \tag{3.2}$$
with suitable $\theta > 0$. Consequently, $f(x,\cdot)$ exhibits a concave behaviour at zero.

We start by pointing out some auxiliary results.

Proposition 3.6. *Suppose* $0 \le a$. *If* $h_i \in L^\infty(\Omega)$, $u_i \in C^1(\overline{\Omega})$, $i = 1,2$, *fulfill*

- $-\Delta_p u_i + a(x)|u_i|^{p-2}u_i = h_i$ *in* Ω,
- $\operatorname{ess\,inf}_{x\in K}[h_2(x) - h_1(x)] > 0$ *for any compact set* $K \subseteq \Omega$,
- $u_1 \le u_2$ *and* $\frac{\partial u_2}{\partial n} < 0$ *on* $\partial\Omega$,

then $u_2 - u_1 \in \hat{C}_+$.

Proof. Recall that $a \in L^\infty(\Omega)$. The first conclusion, namely $u_2(x) - u_1(x) > 0$ for all $x \in \Omega$, is achieved arguing exactly as in the proof of [3, Proposition 2.6], while the other directly follows from [22, Theorem 5.5.1]. □

Proposition 3.7. *Let* (A3) *and* (A6) *be satisfied. Then each nontrivial solution* $\tilde{u} \in [0,\bar{u}]$ *to* (1.1) *lies in* $\operatorname{int}(C_+) \cap (\bar{u} - \hat{C}_+)$.

Proof. Standard regularity arguments ensure that $\tilde{u} \in C_+ \setminus \{0\}$. Fix
$$\rho := \|\bar{u}\|_\infty \ge \|\tilde{u}\|_\infty > 0.$$
Assumption (A6) provides $\mu_\rho > \|a\|_\infty$ fulfilling
$$-\Delta_p\tilde{u}(x) + (a(x)+\mu_\rho)\tilde{u}(x)^{p-1} = f(x,\tilde{u}(x)) + \mu_\rho\tilde{u}(x)^{p-1} \ge 0 \quad \text{a.e. in} \quad \Omega\,.$$

Therefore, by [23, Theorem 5], $\tilde{u} \in \hat{C}_+ \subseteq \operatorname{int}(C_+)$. Next, define $u_\delta := \tilde{u} + \delta$, where $\delta > 0$. Since

$$\begin{aligned}
-\Delta_p \tilde{u} + (a+\mu_\rho)\tilde{u}^{p-1} &\le -\Delta_p \tilde{u} + (a+\mu_\rho)u_\delta^{p-1} \\
&= -\Delta_p \tilde{u} + (a+\mu_\rho)\tilde{u}^{p-1} + o(\delta) \\
&= f(x,\tilde{u}) + \mu_\rho \tilde{u}^{p-1} + o(\delta),
\end{aligned}$$

using (A6) and (A3), with appropriate $c_1 > 0$, we obtain

$$\begin{aligned}
-\Delta_p \tilde{u} + (a+\mu_\rho)\tilde{u}^{p-1} &\le f(x,\bar{u}) + \mu_\rho \bar{u}^{p-1} + o(\delta) \\
&\le (a+\mu_\rho)\bar{u}^{p-1} - c_1 + o(\delta) \\
&\le (a+\mu_\rho)\bar{u}^{p-1} - \frac{c_1}{2} \\
&\le -\Delta_p \bar{u} + (a+\mu_\rho)\bar{u}^{p-1} - \frac{c_1}{2},
\end{aligned}$$

for any $\delta > 0$ small enough, because $\Delta_p \bar{u} \le 0$; cf. Remark 3.3. Proposition 3.6 now gives $\bar{u} - \tilde{u} \in \hat{C}_+$, as desired. □

To simplify notation, write $X := W^{1,p}(\Omega)$. The energy functional $\varphi : X \to \mathbb{R}$ stemming from problem (1.1) is

$$\varphi(u) := \frac{1}{p}\mathcal{E}(u) - \int_\Omega F(x,u(x))\,dx, \quad u \in X, \tag{3.3}$$

with $\mathcal{E}$ and F given by (2.4) and (1.2), respectively. One clearly has $\varphi \in C^1(X)$.

Proposition 3.8. *Under* (2.3)*,* (A1)*,* (A2)*, and* (A4)*, the functional* φ *satisfies condition* (C)*.*

The proof is rather technical but standard (see, e.g., [14, Proposition 3.2]). So, we omit it.

Henceforth $\bar{a}$ *will denote a real constant strictly greater than* $\|a\|_\infty$.

3.1. **Positive solutions.** Truncation-perturbation techniques and minimization methods produce a first positive solution whenever (A3) is assumed.

Theorem 3.9. *Let* (2.3)*,* (A1)*,* (A3)*,* (A5)*, and* (A6) *be fulfilled. Then* (1.1) *has a positive solution* $u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}([0,\bar{u}])$*. Moreover,* u_0 *turns out to be a local minimizer of* φ*.*

Proof. For $x \in \Omega$ and $t, \xi \in \mathbb{R}$, we define

$$\begin{gathered}
\bar{f}(x,t) := \begin{cases} f(x,t^+) + \bar{a}(t^+)^{p-1} & \text{if } t^+ \le \bar{u}(x), \\ f(x,\bar{u}(x)) + \bar{a}\bar{u}(x)^{p-1} & \text{otherwise,} \end{cases} \\
\bar{F}(x,\xi) := \int_0^\xi \bar{f}(x,t)\,dt.
\end{gathered} \tag{3.4}$$

It is evident that the corresponding functional

$$\bar{\varphi}(u) := \frac{1}{p}\left(\mathcal{E}(u) + \bar{a}\|u\|_p^p\right) - \int_\Omega \bar{F}(x,u(x))\,dx, \quad u \in X,$$

belongs to $C^1(X)$. A standard argument, which exploits Sobolev's embedding theorem besides the compactness of the trace operator, ensures that $\bar{\varphi}$ is weakly sequentially lower semi-continuous. Since, by (2.3), the choice of $\bar{a}$, and (3.4), it is coercive, we have

$$\inf_{u\in X} \bar{\varphi}(u) = \bar{\varphi}(u_0) \tag{3.5}$$

for some $u_0 \in X$. Set $\delta := \min\{\delta_1, \min_{x\in\overline{\Omega}} \bar{u}(x)\}$, where δ_1 is as in (A5). If $\tau \in (0,1)$ complies with $\tau\hat{u}_1 \le \delta$, then

$$\bar{\varphi}(\tau\hat{u}_1) \le \frac{\tau^p}{p}\mathcal{E}(\hat{u}_1) - \theta\tau^q\|\hat{u}_1\|_q^q = \tau^q\Big(\frac{\tau^{p-q}}{p}\hat{\lambda}_1 - \theta\|\hat{u}_1\|_q^q\Big)$$

thanks to (3.4), (3.2), and (2.6). Thus, for τ small enough, $\bar{\varphi}(\tau\hat{u}_1) < 0$, which entails

$$\bar{\varphi}(u_0) < 0 = \bar{\varphi}(0).$$

Consequently, $u_0 \ne 0$. Through (3.5) we get $\bar{\varphi}'(u_0) = 0$, namely

$$\langle A_p(u_0), v\rangle + \int_\Omega (a+\bar{a})|u_0|^{p-2}u_0 v\,dx + \int_{\partial\Omega} \beta|u_0|^{p-2}u_0 v\,d\sigma = \int_\Omega \bar{f}(x,u_0)v\,dx, \tag{3.6}$$

for $v \in X$. Using (3.4) and (3.6) written for $v := -u_0^-$ produces

$$\min\{1, \bar{a} - \|a\|_\infty\}\,\|u_0^-\|^p \le \mathcal{E}(u_0^-) + \bar{a}\|u_0^-\|_p^p = 0,$$

whence $u_0 \ge 0$. Now, choose $v := (u_0 - \bar{u})^+$ in (3.6) and observe that

$$\begin{aligned}
&\int_\Omega \bar{f}(x,u_0)(u_0-\bar{u})^+dx \\
&= \int_\Omega [f(x,\bar{u}) + \bar{a}\bar{u}^{p-1}](u_0-\bar{u})^+dx \\
&\le \int_\Omega (a+\bar{a})\bar{u}^{p-1}(u_0-\bar{u})^+dx + \int_{\partial\Omega} \beta u_0^{p-1}(u_0-\bar{u})^+d\sigma
\end{aligned}$$

because of (3.4), (A3), and (2.3). This yields

$$\langle A_p(u_0) - A_p(\bar{u}), (u_0-\bar{u})^+\rangle + (\bar{a} - \|a\|_\infty)\int_\Omega (u_0^{p-1} - \bar{u}^{p-1})(u_0-\bar{u})^+dx \le 0,$$

i.e., $u_0 \le \bar{u}$. Therefore, both $u_0 \in [0,\bar{u}] \setminus \{0\}$ and u_0 solves problem (1.1), so that, due to Proposition 3.7, $u_0 \in \operatorname{int}(C_+) \cap (\bar{u} - \hat{C}_+)$, which implies $u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}([0,\bar{u}])$. Finally, since

$$\varphi\lfloor_{[0,\bar{u}]} = \bar{\varphi}\lfloor_{[0,\bar{u}]},$$

Equation (3.5), combined with [19, Proposition 3], ensures that u_0 is a local minimizer for φ. □

Critical point arguments produce a second positive solution.

Theorem 3.10. *If* (2.3), (A1)–(A3), (A5)–(A6) *hold, then* (1.1) *possesses a solution* $u_1 \in \operatorname{int}(C_+) \setminus \{u_0\}$ *such that* $u_0 \le u_1$.

Proof. For $x \in \Omega$ and $t, \xi \in \mathbb{R}$, we define

$$\begin{gathered}
f_0(x,t) := \begin{cases} f(x,u_0(x)) + \bar{a}u_0(x)^{p-1} & \text{if } t \le u_0(x), \\ f(x,t) + \bar{a}t^{p-1} & \text{otherwise,} \end{cases} \\
F_0(x,\xi) := \int_0^\xi f_0(x,t)\,dt.
\end{gathered} \tag{3.7}$$

It is evident that the corresponding truncated functional

$$\varphi_0(u) := \frac{1}{p}\left(\mathcal{E}(u) + \bar{a}\|u\|_p^p\right) - \int_\Omega F_0(x,u(x))\,dx, \quad u \in X, \tag{3.8}$$

belongs to $C^1(X)$ also. A standard argument, which exploits Sobolev's embedding theorem and the compactness of the trace operator, ensures that φ_0 is weakly sequentially lower semi-continuous.

Claim 1: φ_0 satisfies condition (C). Let $\{u_n\}$ be a sequence in X be such that

$$|\varphi_0(u_n)| \le c_1 \quad \forall n \in \mathbb{N}, \tag{3.9}$$

$$\lim_{n\to+\infty}(1+\|u_n\|)\|\varphi_0'(u_n)\|_{X^*} = 0. \tag{3.10}$$

Through (3.10) one has

$$\begin{aligned} &\Big|\langle A_p(u_n), w\rangle + \int_{\partial\Omega} \beta|u_n|^{p-2}u_n w\,d\sigma \\ &+ \int_\Omega (a+\bar{a})|u_n|^{p-2}u_n w\,dx - \int_\Omega f_0(x,u_n)w\,dx\Big| \\ &\le \frac{\varepsilon_n\|w\|}{1+\|u_n\|} \quad \forall w \in X, \end{aligned} \tag{3.11}$$

where $\varepsilon_n \to 0^+$. We first show that $\{u_n\}$ is bounded. This evidently happens once the same holds for both $\{u_n^-\}$ and $\{u_n^+\}$. By (3.7), choosing $w := -u_n^-$ in (3.11) easily yields

$$\mathcal{E}(u_n^-) + \bar{a}\|u_n^-\|_p^p \le c_2.$$

From (2.3) and the choice of $\bar{a}$ it thus follows $\|u_n^-\| \le c_3$. As n was arbitrary, the sequence $\{u_n^-\}$ turns out to be bounded. So, in particular, on account of (3.9),

$$\mathcal{E}(u_n^+) + \bar{a}\|u_n^+\|_p^p - p\int_\Omega F_0(x,u_n^+(x))\,dx \le c_4 \quad \forall n \in \mathbb{N}.$$

Since

$$\int_\Omega F_0(x,u_n^+)\,dx = \int_\Omega [F_0(x,u_n^+) - F_0(x,u_0)]dx + \int_\Omega [f(x,u_0) + \bar{a}u_0^{p-1}]u_0\,dx,$$

an easy computation shows that

$$\mathcal{E}(u_n^+) - p\int_\Omega F(x,u_n^+(x))\,dx \le c_5, \quad n \in \mathbb{N}. \tag{3.12}$$

Now, (3.11) written with $w := u_n^+$ furnishes

$$\begin{aligned} &-\mathcal{E}(u_n^+) - \bar{a}\|u_n^+\|_p^p + \int_{\Omega_1}[f(x,u_0) + \bar{a}u_0^{p-1}]u_n^+dx + \int_{\Omega_2}[f(x,u_n^+) + \bar{a}(u_n^+)^{p-1}]u_n^+dx \\ &\le \varepsilon_n, \end{aligned}$$

where $\Omega_1 := \{x \in \Omega : 0 \le u_n(x) \le u_0(x)\}$ and $\Omega_2 := \{x \in \Omega : u_n(x) > u_0(x)\}$. Hence,

$$-\mathcal{E}(u_n^+) + \int_\Omega f(x,u_n^+)u_n^+dx \le c_6. \tag{3.13}$$

Inequalities (3.12)–(3.13) lead to

$$\int_\Omega H(x,u_n^+(x))\,dx \le c_7 \quad \forall n \in \mathbb{N}.$$

Via the same arguments used in the proof (Claim 1) of [14, Proposition 3.2], with 2 replaced by p, we achieve $\|u_n^+\| \le c_8$. Therefore, $\{u_n\} \subseteq X$ is bounded. As before, and along a subsequence when necessary, one has $u_n \to u$ in X.

Claim 2: $K(\varphi_0) \subseteq \{u \in X : u_0 \le u\}$. If $u \in K(\varphi_0)$ then

$$\langle A_p(u), v\rangle + \int_\Omega (a+\bar{a})|u|^{p-2}uv\,dx + \int_{\partial\Omega} \beta|u|^{p-2}uv\,d\sigma = \int_\Omega f_0(x,u)v\,dx,$$

for all $v \in X$. Letting $v := (u_0 - u)^+$ and recalling that u_0 solves (1.1) yields

$$\begin{aligned}&\langle A_p(u_0) - A_p(u), (u_0-u)^+\rangle + \int_\Omega (a+\bar{a})(u_0^{p-1} - |u|^{p-2}u)(u_0-u)^+ dx\\ &+ \int_{\partial\Omega} \beta(u_0^{p-1} - |u|^{p-2}u)(u_0-u)^+ d\sigma = 0.\end{aligned}$$

By (2.3) this entails

$$\langle A_p(u_0) - A_p(u), (u_0-u)^+\rangle + \int_\Omega (a+\bar{a})(u_0^{p-1} - |u|^{p-2}u)(u_0-u)^+ dx \le 0,$$

whence $u_0 \le u$, because $\bar{a} > \|a\|_\infty$.

We may evidently assume

$$K(\varphi_0) \cap [0,\bar{u}] = \{u_0\}, \tag{3.14}$$

otherwise, thanks to Claim 2, there would exist $u_1 \in K(\varphi_0) \cap [u_0,\bar{u}] \setminus \{u_0\}$, i.e., a second solution of (1.1). Moreover, Proposition 3.7 would give $u_1 \in \operatorname{int}(C_+) \cap (\bar{u} - \hat{C}_+)$, and the conclusion follows.

For every $x \in \Omega$, $t, \xi \in \mathbb{R}$, we put

$$\bar{f}_0(x,t) := \begin{cases} f_0(x,t) & \text{if } t \le \bar{u}(x), \\ f_0(x,\bar{u}(x)) & \text{otherwise}, \end{cases} \quad \bar{F}_0(x,\xi) := \int_0^\xi \bar{f}_0(x,t)\,dt. \tag{3.15}$$

The associated truncated functional

$$\bar{\varphi}_0(u) := \frac{1}{p}\left(\mathcal{E}(u) + \bar{a}\|u\|_p^p\right) - \int_\Omega \bar{F}_0(x,u(x))\,dx, \quad u \in X,$$

belongs to $C^1(X)$ and is coercive. A standard argument, based on the Sobolev embedding theorem and the compactness of the trace operator, ensures that $\bar{\varphi}_0$ is weakly sequentially lower semi-continuous. So,

$$\inf_{u\in X} \bar{\varphi}_0(u) = \bar{\varphi}_0(\bar{u}_0) \tag{3.16}$$

for some $\bar{u}_0 \in X$. Since, like in the proof of Theorem 3.9, one has $K(\bar{\varphi}_0) \subseteq [u_0,\bar{u}]$, (3.14)–(3.16) produce $\bar{u}_0 = u_0$. Observe now that

$$\bar{\varphi}_0\lfloor_{[0,\bar{u}]} = \varphi_0\lfloor_{[0,\bar{u}]}$$

while, by Theorem 3.9, $u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}([0,\bar{u}])$. Thus, due to [19, Proposition 3], u_0 is a local minimizer for φ_0. Without loss of generality, suppose u_0 isolated in $K(\varphi_0)$, or else (1.1) would possess infinitely many solutions bigger that u_0; cf. Claim 2 and (3.7). The same reasoning made in the proof of [1, Proposition 29] provides here $\rho > 0$ fulfilling

$$\varphi_0(u_0) < \inf_{u\in\partial B_\rho(u_0)} \varphi_0(u).$$

From (3.7) and (A2) it easily follows that

$$\lim_{\tau\to+\infty}\varphi_0(\tau\hat{u}_1)=-\infty.$$

Claim 1 guarantees that condition (C) holds for φ_0. Hence, the mountain-pass theorem gives a point $u_1\in K(\varphi_0)\setminus\{u_0\}$. Obviously, $u_0\le u_1$ by Claim 2 and u_1 solves (1.1). Through the regularity arguments used above we then achieve $u_1\in C^1(\overline{\Omega})$. It remains to check that $u_1\in\operatorname{int}(C_+)$, which can be performed arguing as in the proof of Proposition 3.7. □

3.2. **Negative solutions.** The minimization method yields a negative solution whenever (A4) is assumed.

Theorem 3.11. *Let* (2.3), (A1), (A4), *and* (A5) *be satisfied. Then* (1.1) *possesses a solution* $u_2\in-\operatorname{int}(C_+)$.

Proof. For $x\in\Omega$ and $t,\xi\in\mathbb{R}$, we define

$$\tilde{f}(x,t):=\begin{cases}f(x,t)+\bar{a}|t|^{p-2}t & \text{if } t\le 0,\\ 0 & \text{otherwise,}\end{cases}\qquad \tilde{F}(x,\xi):=\int_0^\xi\tilde{f}(x,t)\,dt.$$

It is evident that the corresponding functional

$$\tilde{\varphi}(u):=\frac{1}{p}\left(\mathcal{E}(u)+\bar{a}\|u\|_p^p\right)-\int_\Omega\tilde{F}(x,u(x))\,dx,\quad u\in X,$$

belongs to $C^1(X)$. A standard reasoning, which exploits Sobolev's embedding theorem besides the compactness of the trace operator, ensures that $\tilde{\varphi}$ turns out to be weakly sequentially lower semi-continuous. Moreover, $\tilde{\varphi}$ is coercive. Indeed, if

$$\|u_n\|\to+\infty\quad\text{and}\quad\tilde{\varphi}(u_n)\le c_1\quad\forall n\in\mathbb{N},\tag{3.17}$$

then

$$\begin{aligned}&\frac{1}{p}\mathcal{E}(u_n^-)-\int_\Omega F(x,-u_n^-(x))\,dx\\ &\le\frac{1}{p}\min\{1,\bar{a}-\|a\|_\infty\}\|u_n^+\|^p+\frac{1}{p}\mathcal{E}(u_n^-)-\int_\Omega F(x,-u_n^-(x))\,dx\\ &\le\frac{1}{p}\left(\mathcal{E}(u_n)+\bar{a}\|u_n\|_p^p\right)-\int_\Omega\tilde{F}(x,-u_n^-(x))\,dx\le c_1\,,\quad n\in\mathbb{N}.\end{aligned}\tag{3.18}$$

Suppose $\|u_n^-\|\to+\infty$ and write $w_n:=\|u_n^-\|^{-1}u_n^-$. From $\|w_n\|=1$ it follows, along a subsequence when necessary,

$$w_n\rightharpoonup w\text{ in }X,\quad w_n\to w\text{ in }L^p(\Omega)\text{ and in }L^p(\partial\Omega),\quad w\ge 0.\tag{3.19}$$

Through (3.18) one has

$$\frac{1}{p}\mathcal{E}(w_n)-\frac{1}{\|u_n^-\|^p}\int_\Omega F(x,-u_n^-(x))\,dx\le\frac{c_1}{\|u_n^-\|^p}\quad\forall n\in\mathbb{N}\tag{3.20}$$

while by (A1) the sequence $\{\|u_n^-\|^{-p}N_F(-u_n^-)\}\subseteq L^1(\Omega)$ is uniformly integrable. Using the arguments made in the proof of [1, Proposition 14], besides (A4), we thus obtain a function $\theta\in L^\infty(\Omega)$ such that $-c_2\le\theta\le\hat{\lambda}_1/p$ and

$$\frac{1}{\|u_n^-\|^p}N_F(-u_n^-)\rightharpoonup\frac{1}{p}\theta w^p\quad\text{in }L^1(\Omega).\tag{3.21}$$

Thanks to (3.19)–(3.20) this implies, as $n \to +\infty$,

$$\mathcal{E}(w) \le \int_\Omega \theta(x) w(x)^p dx. \tag{3.22}$$

If $\theta \neq \hat{\lambda}_1$, then [18, Lemma 4.11] forces $w = 0$. From (3.19)–(3.21) it follows $\|w_n\| \to 0$. However, this is impossible. So, suppose $\theta = \hat{\lambda}_1$. Gathering (3.22) and (p_2) together leads to $w = t\hat{u}_1$ for some $t \ge 0$. The above reasoning shows that $t > 0$. Hence, $w \in \operatorname{int}(C_+)$. By the definition of $\{w_n\}$ we actually have $u_n^-(x) \to +\infty$ for every $x \in \Omega$. Since (A4) easily yields

$$\lim_{\xi \to -\infty} [\hat{\lambda}_1 |\xi|^p - pF(x,\xi)] = +\infty \quad \text{uniformly in } x \in \Omega$$

(cf. Remark 3.4), Fatou's lemma gives

$$\lim_{n \to +\infty} \int_\Omega [\hat{\lambda}_1 (u_n^-)^p - pF(x, -u_n^-(x))] dx = +\infty. \tag{3.23}$$

On the other hand, via (3.18), besides (2.5), we get

$$\int_\Omega [\hat{\lambda}_1 u_n^-(x)^p - pF(x, -u_n^-(x))] dx \le pc_1 \quad \forall n \in \mathbb{N},$$

against (3.23). Therefore, the sequence $\{u_n^-\} \subseteq X$ is bounded. Using (3.18) again one sees that $\{u_n^+\}$ enjoys the same property, which contradicts (3.17).

Let $u_2 \in X$ satisfy

$$\inf_{u \in X} \tilde{\varphi}(u) = \tilde{\varphi}(u_2).$$

Arguing as in the proof of Theorem 3.9 we achieve $u_2 \le 0$ and $u_2 \neq 0$. So, u_2 solves problem (1.1) and belongs to $(-C_+) \setminus \{0\}$ by standard nonlinear regularity results. Finally, (A1) and (A4) provide $\tilde{\mu} > \|a\|_\infty$ such that

$$f(x,t) + \tilde{\mu}|t|^{p-2}t \le 0, \quad (x,t) \in \Omega \times \mathbb{R}_0^- .$$

Consequently,

$$\Delta_p(-u_2) + (a + \tilde{\mu})|u_2|^{p-2}u_2 = f(x,u_2) + \tilde{\mu}|u_2|^{p-2}u_2 \le 0,$$

whence

$$\Delta_p(-u_2) \le (a + \tilde{\mu})(-u_2)^{p-1} \quad \text{in} \quad \Omega.$$

Through [23, Theorem 5] this implies $-u_2 \in \operatorname{int}(C_+)$, as desired. □

3.3. **Extremal constant-sign and nodal solutions.** The following stronger version of (A5) will be used.

(A5') There exist $q \in (1,p)$, $a_4 > 0$, and $\delta_1 > 0$ such that

$$a_4 |\xi|^q \le f(x,\xi)\xi \le qF(x,\xi) \quad \forall (x,\xi) \in \Omega \times [-\delta_1, \delta_1].$$

It plays a crucial role in getting useful information on the critical groups of φ at zero. Precisely, the result below, whose proof is analogous to that of [21, Proposition 4.1] (cf. also [12, Theorem 3.6]), holds.

Lemma 3.12. *Suppose* (2.3), (A1), (A5') *hold and* $K(\varphi)$ *is a finite set. Then* $C_k(\varphi, 0) = 0$ *for all* $k \in \mathbb{N}_0$.

Combining (A1) with (A5') we obtain

$$f(x,t)t \geq a_4|t|^q - a_5|t|^r \quad \text{in } \Omega\times\mathbb{R} \tag{3.24}$$

for an appropriate $a_5 > 0$. Consider the auxiliary problem

$$\begin{gathered} -\Delta_p u + a(x)|u|^{p-2}u = a_4|u|^{q-2}u - a_5|u|^{r-2}u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(x)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.25}$$

Note that if u is a solution then $-u$ also solves this problem.

Lemma 3.13. *If* (2.3) *holds then* (3.25) *admits a unique positive solution* $u_+ \in \operatorname{int}(C_+)$.

Proof. The C^1-functional $\psi : X \to \mathbb{R}$ given by

$$\psi(u) := \frac{1}{p}\left(\mathcal{E}(u) + \bar{a}\|u^-\|_p^p\right) - \frac{a_4}{q}\|u^+\|_q^q + \frac{a_5}{r}\|u^+\|_r^r\,, \quad u\in X,$$

is coercive. Indeed, recalling that $\beta \geq 0$, $\bar{a} \geq \|a\|_\infty$, and $q < p < r$, we have

$$\begin{aligned} \psi(u) &= \frac{1}{p}\mathcal{E}(u^+) + \frac{a_5}{r}\|u^+\|_r^r - \frac{a_4}{q}\|u^+\|_q^q + \frac{1}{p}\left(\mathcal{E}(u^-) + \bar{a}\|u^-\|_p^p\right) \\ &\geq \frac{1}{p}\|\nabla u^+\|_p^p + c_1\|u^+\|_p^r - c_2\left(\|u^+\|_p^p + 1\right) + c_3\|u^-\|^p \\ &= \frac{1}{p}\|\nabla u^+\|_p^p + \|u^+\|_p^p\left(c_1\|u^+\|_p^{r-p} - c_2\right) + c_3\|u^-\|^p - c_2 \\ &\geq c_4\|u\|^p - c_5\,. \end{aligned}$$

Since ψ is weakly sequentially lower semi-continuous also, there exists $u_+ \in X$ fulfilling

$$\psi(u_+) = \inf_{u\in X}\psi(u).$$

Moreover, $u_+ \neq 0$ because $\psi(t) < 0$ for any $t > 0$ small enough. As in the proof of Theorem 3.9 we next get $u_+ \geq 0$. Hence, by standard nonlinear regularity results, $u_+ \in C_+ \setminus \{0\}$. The conclusion $u_+ \in \operatorname{int}(C_+)$ easily derives from

$$\Delta_p u_+ \leq \left(\|a\|_\infty + a_5\|u_+\|_\infty^{r-p}\right)u_+^{p-1} \leq c_6 u_+^{p-1};$$

cf. [23, Theorem 5]. Let us now come to uniqueness. Suppose $\hat{u} \in \operatorname{int}(C_+)$ is another solution of (3.25). For $u \in L^1(\Omega)$, we put

$$J(u) := \begin{cases} \frac{1}{p}\left(\|\nabla u^{1/p}\|_p^p + \int_{\partial\Omega} au\,d\sigma\right) & \text{if } u\geq 0,\ u^{1/p}\in X, \\ +\infty & \text{otherwise.} \end{cases}$$

[7, Lemma 1] ensures that $J : L^1(\Omega) \to \mathbb{R}\cup\{+\infty\}$ is proper, convex, and lower semi-continuous. A simple computation, chiefly based on [10, Theorem 2.4.54], yields

$$J'(u_+^p)(v) = \frac{1}{p}\int_\Omega \frac{-\Delta_p u_+}{u_+^{p-1}}\,v\,dx\,, \quad J'(\hat{u}^p)(v) = \frac{1}{p}\int_\Omega \frac{-\Delta_p \hat{u}}{\hat{u}^{p-1}}\,v\,dx \quad \forall v\in C^1(\overline{\Omega}),$$

while the monotonicity of J' leads to

$$\int_\Omega \left(\frac{-\Delta_p u_+}{u_+^{p-1}} - \frac{-\Delta_p\hat{u}}{\hat{u}^{p-1}}\right)\left(u_+^p - \hat{u}^p\right)dx \geq 0.$$

Therefore,

$$\int_\Omega \Big[a_4\Big(\frac{1}{u_+^{p-q}} - \frac{1}{\hat{u}^{p-q}}\Big) - a_5(u_+^{r-p} - \hat{u}^{r-p})\Big]\left(u_+^p - \hat{u}^p\right) dx \geq 0,$$

which implies $u_+ = \hat{u}$, because $q < p < r$. □

Remark 3.14. Recall that when u is a solution, so is $-u$. Then $u_- := -u_+$ represents the unique negative solution of (3.25).

We define

$$\Sigma_+ := \{u \in X \setminus \{0\} : 0 \leq u,\ u \text{ solves (1.1)}\},$$
$$\Sigma_- := \{u \in X \setminus \{0\} : u \leq 0,\ u \text{ solves (1.1)}\}.$$

We already know (see Sections 3.1–3.2) that these sets are both nonempty and that

$$\Sigma_+, -\Sigma_- \subseteq \operatorname{int}(C_+).$$

Moreover, Σ_+ (resp., Σ_-) turns out to be downward (resp., upward) directed, as a standard argument shows; see for instance [8, Lemmas 4.2–4.3].

Lemma 3.15. *Under assumptions* (A1)–(A4), (A5'), *and* (A6) *one has*

$$u_+ \leq u \quad \forall u \in \Sigma_+\,, \quad u \leq u_- \quad \forall u \in \Sigma_-\,.$$

Proof. Pick $u \in \Sigma_+$. For $x \in \Omega$, $t, \xi \in \mathbb{R}$, we define

$$g(x,t) := \begin{cases} a_4(t^+)^{q-1} - a_5(t^+)^{r-1} & \text{if } t^+ \leq u(x), \\ a_4 u(x)^{q-1} - a_5 u(x)^{r-1} + \bar{a}u(x)^{p-1} & \text{otherwise,} \end{cases}$$

$$G(x,\xi) := \int_0^\xi g(x,t)\, dt\,.$$

Evidently, the functional

$$\psi_+(w) := \frac{1}{p}\left(\mathcal{E}(w) + \bar{a}\|w\|_p^p\right) - \int_\Omega G(x, w(x))\, dx\,, \quad w \in X,$$

is C^1, weakly sequentially lower semi-continuous, and coercive. So, there exists $w_0 \in X$ such that

$$\psi_+(w_0) = \inf_{w \in X} \psi_+(w).$$

From $q < p < r$ it follows $\psi_+(w_0) < 0 = \psi_+(0)$, whence $w_0 \neq 0$. Via (3.24), reasoning as in the proof of Theorem 3.9, we arrive at

$$w_0 \in [0, u] \cap \operatorname{int}(C_+). \tag{3.26}$$

So, w_0 turns out to be a positive solution of (3.25). By Lemma 3.13 one has $w_0 = u_+$, and (3.26) then yields $u_+ \leq u$. Analogously, $u \leq u_-$ for all $u \in \Sigma_-$. □

Theorem 3.16. *Let* (2.3), (A1)–(A4), (A5'), (A6) *be satisfied. Then* (1.1) *possesses a smallest positive solution* u_* *and a biggest negative solution* v_**. Further,* $-v_*, u_* \in \operatorname{int}(C_+)$*.*

Proof. Recall that Σ_+ is downward directed. The same arguments employed to establish [2, Proposition 8] yield

(1) $\inf \Sigma_+ = \inf_{n \in \mathbb{N}} u_n = u_*$ for some $\{u_n\} \subseteq \Sigma_+$, $u_* \in X$;
(2) $u_n \to u_*$ in X and in $L^p(\partial\Omega)$.

Hence, the function u_* solves (1.1). Through Lemma 3.15 we next obtain $u_+ \le u_*$, namely $u_* \in \Sigma_+ \subseteq \operatorname{int}(C_+)$. Finally, 1) ensures that u_* is minimal. A similar proof gives a function v_* with the asserted properties. □

Next, for every $x \in \Omega$ and $t, \xi \in \mathbb{R}$, we define

$$\begin{gathered}\hat{f}(x,t) := \begin{cases} f(x, v_*(x)) + \bar{a}|v_*(x)|^{p-2}v_*(x) & \text{if } t < v_*(x), \\ f(x,t) + \bar{a}|t|^{p-2}t & \text{if } v_*(x) \le t \le u_*(x), \\ f(x, u_*(x)) + \bar{a}u_*(x)^{p-1} & \text{if } t > u_*(x), \end{cases} \\ \hat{f}_\pm(x,t) := \hat{f}(x, t^\pm), \\ \hat{F}(x,\xi) := \int_0^\xi \hat{f}(x,t)dt, \quad \hat{F}_\pm(x,\xi) := \int_0^\xi \hat{f}_\pm(x,t)\,dt. \end{gathered} \tag{3.27}$$

It is evident that the corresponding truncated functionals

$$\begin{gathered} \hat{\varphi}(u) := \frac{1}{p}\left(\mathcal{E}(u) + \bar{a}\|u\|_p^p\right) - \int_\Omega \hat{F}(x, u(x))\,dx, \quad u \in X, \\ \hat{\varphi}_\pm(u) := \frac{1}{p}\left(\mathcal{E}(u) + \bar{a}\|u\|_p^p\right) - \int_\Omega \hat{F}_\pm(x, u(x))\,dx, \quad u \in X, \end{gathered} \tag{3.28}$$

belong to $C^1(X)$. Moreover, by construction, one has

$$K(\hat{\varphi}) \subseteq [v_*, u_*], \quad K(\hat{\varphi}_-) = \{0, v_*\}, \quad K(\hat{\varphi}_+) = \{0, u_*\}; \tag{3.29}$$

see, e.g., [15, Lemma 3.1].

Theorem 3.17. *If* (2.3), (A1)–(A4), (A5'), (A6) *hold, then* (1.1) *possesses a nodal solution* $u_3 \in [v_*, u_*] \cap C^1(\overline{\Omega})$.

Proof. X compactly embeds in $L^p(\Omega)$ while the Nemitskii operator $N_{\hat{f}_+}$ turns out to be continuous on $L^p(\Omega)$. Thus, a standard argument ensures that $\hat{\varphi}_+$ is weakly sequentially lower semi-continuous. Since, on account of (3.27), it is coercive, we obtain

$$\inf_{u \in X} \hat{\varphi}_+(u) = \hat{\varphi}_+(u_0)$$

for some $u_0 \in X$. Reasoning as in the proof of Theorem 3.9 produces $u_0 \in \operatorname{int}(C_+)$ and, by (3.29), $u_0 = u_*$. Since $\hat{\varphi}\lfloor_{C_+} = \hat{\varphi}_+\lfloor_{C_+}$, the function u_* turns out to be a $C^1(\overline{\Omega})$-local minimizer for $\hat{\varphi}$. Now, [19, Proposition 3] guarantees that the same remains true with X in place of $C^1(\overline{\Omega})$. A similar argument applies to v_*. Consequently, u_*, v_* are local minimizer for $\hat{\varphi}$.

We may assume $K(\hat{\varphi})$ finite, otherwise infinitely many nodal solutions do exist by (3.29). Let $\hat{\varphi}(v_*) \le \hat{\varphi}(u_*)$ (the other case is analogous). Without loss of generality, the local minimizer u_* for $\hat{\varphi}$ can be supposed proper. Thus, there exists $\rho \in (0, \|u_* - v_*\|)$ such that

$$\hat{\varphi}(u_*) < c_\rho := \inf_{u \in \partial B_\rho(u_*)} \hat{\varphi}(u). \tag{3.30}$$

Moreover, $\hat{\varphi}$ fulfills condition (C) because, by (3.27), it is coercive; vide for instance [13, Proposition 2.2]. So, the mountain-pass theorem yields a point $u_3 \in X$ complying with $\hat{\varphi}'(u_3) = 0$ and

$$c_\rho \le \hat{\varphi}(u_3) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \hat{\varphi}(\gamma(t)), \tag{3.31}$$

where

$$\Gamma := \{\gamma \in C^0([0,1], X) : \gamma(0) = v_*,\ \gamma(1) = u_*\}\,.$$

Obviously, u_3 solves (1.1). Through (3.30)–(3.31), besides (3.29), we get

$$u_3 \in [v_*, u_*] \setminus \{v_*, u_*\},$$

while standard regularity arguments yield $u_3 \in C^1(\overline{\Omega})$. The proof is thus completed once one verifies that $u_3 \neq 0$. This will follow from

$$C_1(\hat{\varphi}, 0) = 0\,, \tag{3.32}$$

because $C_1(\hat{\varphi}, u_3) \neq 0$ by [17, Corollary 6.81]. We claim that

$$C_k(\hat{\varphi}, 0) = C_k(\varphi, 0) \quad \forall k \in \mathbb{N}_0. \tag{3.33}$$

Indeed, consider the homotopy

$$h(t,u) := (1-t)\hat{\varphi}(u) + t\varphi(u)\,, \quad (t,u) \in [0,1] \times X.$$

If there exist $\{t_n\} \subseteq [0,1]$ and $\{u_n\} \subseteq X$ satisfying

$$t_n \to t, \quad u_n \to 0, \quad u_m \neq u_n \quad \text{for } m \neq n, \quad h'_u(t, u_n) = 0\ \forall n \in \mathbb{N} \tag{3.34}$$

then the same arguments of [20, Proposition 7] give $\|u_n\|_\infty \le c_1$. By regularity, the sequence $\{u_n\}$ is bounded in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, whence $u_n \to 0$ in $C^1(\overline{\Omega})$. Thus, $u_n \in [v_*, u_*]$ provided n is large enough, and (3.27), (3.29), besides (3.34), lead to $u_n \in K(\hat{\varphi})$. However, this contradicts the assumption $K(\hat{\varphi})$ finite. Now, [5, Theorem 5.2] directly yields (3.33). Combining (3.33) with Lemma 3.12 we finally arrive at (3.32), as desired. □

If $f(x,\cdot)$ exhibits a $(p-1)$-linear behavior at zero then the problem's geometry changes, and another technical approach is necessary. We will use the hypothesis

(A5") There exist $a_6 > \hat{\lambda}_2$ and $a_7 > 0$ such that

$$a_6 \le \liminf_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} \le \limsup_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} \le a_7$$

uniformly in $x \in \Omega$.

Via (A1) and (A5") one has

$$f(x,t)t \ge a_8|t|^p - a_9|t|^r, \quad (x,t) \in \Omega \times \mathbb{R},$$

for appropriate $a_8 > \hat{\lambda}_2$, $a_9 > 0$. Consider the auxiliary problem

$$\begin{gathered} -\Delta_p u + a(x)|u|^{p-2}u = a_8|u|^{p-2}u - a_9|u|^{r-2}u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(x)|u|^{p-2}u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.35}$$

Note that if u is a solution then $-u$ also solves this problem. Reasoning as above we see that:

- Problem (3.35) admits a unique positive solution $u_+ \in \mathrm{int}(C_+)$.
- $u_- := -u_+$ represents the unique negative solution of (3.35).
- Under assumptions (A1)–(A4), (A5"), (A6) and (2.3), problem (1.1) possesses both a smallest positive solution u_* and a biggest negative solution v_*. Further, $-v_*, u_* \in \mathrm{int}(C_+)$.

Now, the same arguments used in the proof of [15, Theorem 3.3] yield the following result.

Theorem 3.18. *Let* (2.3), (A1)–(A4), (A5”), *and* (A6) *be satisfied. Then* (1.1) *admits a nodal solution* $u_3 \in [v_*, u_*] \cap C^1(\overline{\Omega})$.

3.4. **Existence of at least four nontrivial solutions.** Gathering the results in Sections 3.1–3.3 we directly obtain the next one.

Theorem 3.19. *If* (2.3), (A1)–(A4), (A5’)–(A6) *hold, then* (1.1) *possesses at least four solutions* $u_0, u_1 \in \operatorname{int}(C_+)$, $u_2 \in -\operatorname{int}(C_+)$, *and* $u_3 \in [u_2, u_0] \cap C^1(\overline{\Omega})$ *nodal. Moreover,* $u_0 \le u_1$.

Remark 3.20. Hypothesis (A5’) can be substituted by (A5”) without changing the conclusion.

4. Semilinear case

From now on we shall assume $p = 2$. Then the regularity results of [24] allow to weaken (2.3) as follow, see [6, 14],

$$a \in L^s(\Omega) \text{ for some } s > N,\ a^+ \in L^\infty(\Omega), \quad \beta \in W^{1,\infty}(\partial\Omega), \text{ and } \beta \ge 0. \tag{4.1}$$

Further, the energy functional φ given by (3.3) fulfills condition (C) once (4.1), (A1), (A2), and (A4) hold; see Proposition 3.8.

Lemma 4.1. *Under assumptions* (4.1), (A1), *and*

(A7) $\hat{\lambda}_m t^2 \le f(x,t)t \le \hat{\lambda}_{m+1} t^2$ *in* $\Omega \times [-\delta_2, \delta_2]$, *with appropriate* $m \in \mathbb{N}$, $\delta_2 > 0$,

one has

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \forall k \in \mathbb{N}_0\,,$$

where $d_m := \dim(\bar{H}_m)$, *provided* φ *satisfies* (*C*) *and* $0 \in K(\varphi)$ *is isolated.*

Proof. It is similar to that of [6, Lemma 3.3]. So, we only sketch the main points. Pick a $\theta \in (\hat{\lambda}_m, \hat{\lambda}_{m+1})$ and define

$$\psi(u) := \frac{1}{2}\left(\mathcal{E}(u) - \theta\|u\|_2^2\right), \quad u \in X\,.$$

Thanks to (A7), zero is a non-degenerate critical point of ψ having Morse index d_m, which entails

$$C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \forall k \in \mathbb{N}_0\,;$$

see (2.2). Now, recall that every $v \in X$ admits a unique sum decomposition $v = \bar{v} + \hat{v}$, with $\bar{v} \in \bar{H}_m$, $\hat{v} \in \widehat{H}_{m+1}$. If $u \in C^1(\overline{\Omega})$ and $0 < \|u\|_{C^1(\overline{\Omega})} < \delta_2$ then

$$\langle \varphi'(u), \hat{u} - \bar{u} \rangle = \mathcal{E}(\hat{u}) - \mathcal{E}(\bar{u}) - \int_\Omega f(x,u)(\hat{u} - \bar{u})\,dx\,. \tag{4.2}$$

By (A7) again, one arrives at

$$f(x,u)(\hat{u} - \bar{u}) = \frac{f(x,u)}{u} u(\hat{u} - \bar{u}) \le \begin{cases} \hat{\lambda}_{m+1}(\hat{u}^2 - \bar{u}^2) & \text{if } u(\hat{u} - \bar{u}) \ge 0, \\ -\hat{\lambda}_m(\bar{u}^2 - \hat{u}^2) & \text{otherwise.} \end{cases}$$

Hence,

$$f(x,u(x))(\hat{u}(x) - \bar{u}(x)) \le \hat{\lambda}_{m+1}\hat{u}(x)^2 - \hat{\lambda}_m \bar{u}(x)^2 \quad \text{in } \Omega\,. \tag{4.3}$$

From (4.2), (4.3), and (2.7) it follows that

$$\langle \varphi'(u), \hat{u} - \bar{u} \rangle \ge \mathcal{E}(\hat{u}) - \hat{\lambda}_{m+1}\|\hat{u}\|_2^2 - [\mathcal{E}(\bar{u}) - \hat{\lambda}_m\|\bar{u}\|_2^2] \ge 0\,.$$

Using [6, Lemma 2.2] we obtain

$$\langle \psi'(u), \hat{u} - \bar{u} \rangle = \mathcal{E}(\hat{u}) - \theta \|\hat{u}\|_2^2 - [\mathcal{E}(\bar{u}) - \theta \|\bar{u}\|_2^2] \geq c_1 \|u\|^2$$

for some $c_1 > 0$. Therefore, the homotopy

$$h(t, v) := (1-t)\varphi(v) + t\psi(v), \quad (t, v) \in [0,1] \times X$$

fulfills the inequality

$$\langle h'_v(t, u), \hat{u} - \bar{u} \rangle \geq t c_1 \|u\|^2 \quad \forall t \in [0,1],$$

and [5, Theorem 5.2] can be applied. By that result $C_k(\varphi, 0) = C_k(\psi, 0)$, which completes the proof. □

The same arguments made in [20, Proposition 15] yield the next result.

Lemma 4.2. *Assume* (4.1), (A1), *and* (A2) *hold. If* φ *satisfies* (C) *and is bounded below on* $K(\varphi)$, *then* $C_k(\varphi, \infty) = 0$ *for all* $k \in \mathbb{N}_0$.

The condition below will take the place of (A1).

(A1') $f(x, \cdot) \in C^1(\mathbb{R})$ for every $x \in \Omega$. There exist $a_1 \in L^\infty(\Omega)$, $r \in (2, 2^*)$ such that

$$|f'_t(x, t)| \leq a_1(x)(1 + |t|^{r-2}) \quad \forall (x, t) \in \Omega \times \mathbb{R}.$$

Remark 4.3. An easy computation shows that (A1') implies (A6).

We are now in a position to establish a five-solutions existence result. It complements those previously obtained in [6, 14].

Theorem 4.4. *Let* (4.1), (A1'), (A2)–(A4) *be satisfied. Suppose also that*

(A7') *either*

$$a_{10} t^2 \leq f(x, t)t \leq \hat{\lambda}_3 t^2, \quad (x, t) \in \Omega \times [-\delta_3, \delta_3],$$

for some $a_{10} > \hat{\lambda}_2$ *and* $\delta_3 > 0$, *or*

$$\hat{\lambda}_m t^2 \leq f(x, t)t \leq \hat{\lambda}_{m+1} t^2, \quad (x, t) \in \Omega \times [-\delta_3, \delta_3],$$

where $m \geq 3$.

Then (1.4) *possesses at least five nontrivial solutions* $u_i \in C^1(\overline{\Omega})$, $i = 0, \dots, 4$, *with* u_0, u_1, u_2, u_3 *as in Theorem 3.19.*

Proof. Thanks to Remarks 3.20 and 4.3, the conclusion of Theorem 3.19 holds for the present framework. So, it remains to find a further solution $u_4 \in C^1(\overline{\Omega}) \setminus \{0\}$. Without loss of generality, we assume that u_0, u_3 are extremal (see Section 3.3), while a standard argument based on (A6) and (4.1) yields $u_3 \in \operatorname{int}_{C^1(\overline{\Omega})}([u_2, u_0])$; vide, e.g., [14, Theorem 3.2]. Still we write $\hat{f}$ for the function defined in (3.27) but with v_* and u_* replaced by u_2 and u_0, respectively. [6, Lemma 2.1] provides $\hat{a}, \hat{b} > 0$ fulfilling

$$\mathcal{E}(u) + \hat{a}\|u\|_2^2 \geq \hat{b}\|u\|^2 \quad \forall u \in X.$$

Pick any $\bar{a} \geq \hat{a}$ and consider the functional $\hat{\varphi}$ given by (3.28). The same reasoning adopted in the proof of Theorem 3.17 ensures here that $C_k(\hat{\varphi}, u_3) = C_k(\varphi, u_3)$. Thus

$$C_1(\varphi, u_3) \neq 0,$$

because u_3 is a mountain-pass type critical point for $\hat{\varphi}$; cf. [17, Corollary 6.81]. By (A1') one has $\varphi \in C^2(X)$ as well as

$$\langle \varphi''(u_3)u, v\rangle = \int_\Omega (\nabla u \cdot \nabla v + auv)dx + \int_{\partial\Omega} \beta uv\, d\sigma - \int_\Omega f_t'(x, u_3)uvdx, \tag{4.4}$$

for $u, v \in X$. Hence, if the Morse index of u_3 is zero, then

$$\|\nabla u\|_2^2 + \int_{\partial\Omega} \beta u^2 d\sigma \geq \int_\Omega [f_t'(x, u_3) - a]u^2 dx \quad \forall u \in X. \tag{4.5}$$

Write $\alpha := [f_t'(x, u_3) - a]^+$ and observe that $\alpha \in L^s(\Omega)$. Two situations may occur.

(1) $\alpha = 0$. Due to (4.4), for every $u \in \ker\varphi''(u_3)$ we get

$$\|\nabla u\|_2^2 + \int_{\partial\Omega} \beta(x)u(x)^2 d\sigma \leq 0,$$

which implies u constant.

(2) $\alpha \neq 0$. From (4.5) it follows $\hat{\lambda}_1(\alpha) \geq 1$ and by (4.4) the assertion $\ker\varphi''(u_3) \neq \{0\}$ forces $\hat{\lambda}_1(\alpha) = 1$, whence $\dim \ker\varphi''(u_3) = 1$.

In both cases we arrive at $\dim \ker\varphi''(u_3) \leq 1$. So, on account of [17, Proposition 6.101],

$$C_k(\varphi, u_3) = \delta_{k,1}\mathbb{Z} \quad \forall k \in \mathbb{N}_0. \tag{4.6}$$

Next, we define

$$\varphi_+(u) := \frac{1}{2}\mathcal{E}(u) - \int_\Omega F_+(x, u(x))\, dx, \quad u \in X,$$

where $F_+(x, \xi) := \int_0^\xi f(x, t)^+\, dt$. Assumption (A7) easily leads to $\varphi\lfloor_{C_+} = \varphi_+\lfloor_{C_+}$, which entails

$$C_k(\varphi\lfloor_{C^1(\overline{\Omega})}, u_1) = C_k(\varphi_+\lfloor_{C^1(\overline{\Omega})}, u_1)$$

because $u_1 \in \operatorname{int}(C_+)$; see Theorem 3.10. By denseness one has $C_k(\varphi, u_1) = C_k(\varphi_+, u_1)$. Now, observe that $\varphi_+ = \varphi_0 + c$, with appropriate $c > 0$ and φ_0 as in (3.8), on a neighbourhood of u_1. Consequently, $C_k(\varphi_+, u_1) = C_k(\varphi_0, u_1)$. Since u_1 is a mountain-pass type critical point for φ_0 (cf. the proof of Theorem 3.10), the same argument made above gives

$$C_k(\varphi, u_1) = \delta_{k,1}\mathbb{Z}, \quad k \in \mathbb{N}_0. \tag{4.7}$$

Gathering Theorem 3.10 and [17, Proposition 6.95], we derive

$$C_k(\varphi, u_0) = \delta_{k,0}\mathbb{Z} \quad \forall k \in \mathbb{N}_0. \tag{4.8}$$

Likewise,

$$C_k(\varphi, u_2) = \delta_{k,0}\mathbb{Z}, \quad \forall k \in \mathbb{N}_0, \tag{4.9}$$

while Lemmas 4.1–4.2 yield

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z}, \quad C_k(\varphi, \infty) = 0 \quad \forall k \in \mathbb{N}_0. \tag{4.10}$$

Finally, if $K(\varphi) = \{0, u_0, u_1, u_2, u_3\}$ then (2.1), with $t = -1$, and (4.6)–(4.10) would imply

$$(-1)^{d_m} + 2(-1)^0 + 2(-1)^1 = 0,$$

which is impossible. Thus, there exists $u_4 \in K(\varphi) \setminus \{0, u_0, u_1, u_2, u_3\}$, i.e., a fifth nontrivial solution to (1.1). Standard regularity results [24] ensure that $u_4 \in C^1(\overline{\Omega})$. □

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References

[1] S. Aizicovici, N. S. Papageorgiou, V. Staicu; *Degree Theory for Operators of Monotone Type and Nonlinear Elliptic Equations with Inequality Constraints*, Mem. Amer. Math. Soc., **196** (2008).

[2] S. Aizicovici, N. S. Papageorgiou, V. Staicu; *Existence of multiple solutions with precise sign information for superlinear Neumann problems,* Ann. Mat. Pura Appl., **188** (2009), 679–710.

[3] D. Arcoya and D. Ruiz; *The Ambrosetti-Prodi problem for the p-Laplace operator,* Comm. Partial Differential Equations, **31** (2006), 849–865.

[4] D. Averna, N. S. Papageorgiou, E. Tornatore; *Positive solutions for nonlinear Robin problems,* Electron. J. Differential Equations, **2017**, Paper No. 204, 25 pp.

[5] J.-N. Corvellec, A. Hantoute; *Homotopical stability of isolated critical points of continuous functionals,* Set-Valued Anal., **10** (2002), 143–164.

[6] G. D'aguì, S. A. Marano, N. S. Papageorgiou; *Multiple soutions to a Robin problem with indefinite weight and asymmetric reaction,* J. Math. Anal. Appl., **433** (2016), 1821–1845.

[7] J. I. Diaz, J. E. Saa; *Existence et unicité de solutions positives pour certaines equations elliptiques quasilineaires,* C. R. Math. Acad. Sci., Paris **305** (1987), 521–524.

[8] M. Filippakis, N. S. Papageorgiou; *Multiple constant sign and nodal solutions for nonlinear elliptic equations with the p-Laplacian,* J. Differential Equations, **245** (2008), 1883–1922.

[9] G. Fragnelli, D. Mugnai, N. S. Papageorgiou; *Positive and nodal solutions for parametric nonlinear Robin problems with indefinite potential,* Discrete Contin. Dyn. Syst., **36** (2016), 6133–6166.

[10] L. Gasiński, N. S. Papageorgiou; *Nonlinear Analysis,* Ser. Math. Anal. Appl., **9**, Chapman and Hall/CRC Press, Boca Raton, 2006.

[11] G. Li, C. Yang; *The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti-Rabinowitz condition,* Nonlinear Anal., **72** (2010), 4602–4613.

[12] S. A. Marano, S. J. N. Mosconi, N. S. Papageorgiou; *Multiple solutions to (p,q)-Laplacian problems with resonant concave nonlinearity,* Adv. Nonlinear Stud., **16** (2016), 51–65.

[13] S. A. Marano, N. S. Papageorgiou; *Constant-sign and nodal solutions to a Dirichlet problem with p-Laplacian and nonlinearity depending on a parameter,* Proc. Edinburgh Math. Soc., **57** (2014), 521–532.

[14] S. A. Marano, N. S. Papageorgiou; *On a Robin problem with indefinite weight and asymmetric reaction superlinear at $+\infty$,* J. Math. Anal. Appl., **443** (2016), 123–145.

[15] S. A. Marano, N. S. Papageorgiou; *On a Robin problem with p-Laplacian and reaction bounded only from above,* Monatsh. Math., **180** (2016), 317–336.

[16] D. Motreanu, V. V. Motreanu, N. S. Papageorgiou; *On p-Laplace equations with concave terms and asymmetric perturbations,* Proc. Roy. Soc. Edinburgh Sect. A, **141** (2011), 171–192.

[17] D. Motreanu, V. V. Motreanu, N. S. Papageorgiou; *Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems,* Springer, New York, 2013.

[18] D. Mugnai, N. S. Papageorgiou; *Resonant nonlinear Neumann problems with an indefinite weight,* Ann. Sc. Norm. Super. Pisa Cl. Sci., (5) **11** (2012), 729–788.

[19] N. S. Papageorgiou, V. D. Radulescu; *Multiple solutions with precise sign for nonlinear parametric Robin problems,* J. Differential Equations, **256** (2014), 2449–2479.

[20] N. S. Papageorgiou, V. D. Radulescu; *Nonlinear nonhomogeneous Robin problems with superlinear reaction term,* Adv. Nonlinear Stud., **16** (2016), 737–764.

[21] N. S. Papageorgiou, P. Winkert; *Nonlinear Robin problems with reaction of arbitrary growth,* Ann. Mat. Pura Appl., **195** (2016), 1207–1235.

[22] P. Pucci, J. Serrin; *The Maximum Principle,* Birkhäuser, Basel, 2007.

[23] J. L. Vázquez; *A strong maximum principle for some quasilinear elliptic equations,* Appl. Math. Optim., **12** (1984), 191–202.

[24] X.-J. Wang; *Neumann problems of semilinear elliptic equations involving critical Sobolev exponents,* J. Differential Equations, **93** (1991), 283–310.

Salvatore A. Marano (corresponding author)
Department of Mathematics and Computer Sciences, University of Catania, Viale A. Doria 6, 95125 Catania, Italy
E-mail address: marano@dmi.unict.it

Nikolaos S. Papageorgiou
Department of Mathematics, National Technical University of Athens, Zografou Campus, Athens 15780, Greece
E-mail address: npapg@math.ntua.gr