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MULTIPOINT INITIAL-FINAL VALUE PROBLEMS FOR DYNAMICAL SOBOLEV-TYPE EQUATIONS IN THE SPACE OF NOISES

ANGELO FAVINI, SOPHIYA A. ZAGREBINA, GEORGY A. SVIRIDYUK

Abstract. We prove the existence of a unique solution for a linear stochastic Sobolev-type equation with a relatively p-bounded operator and a multipoint initial-final condition, in the space of "noises". We apply the abstract results to specific multipoint initial-final and boundary value problems for the linear Hoff equation which models I-beam bulging under random load.

1. Introduction

In the simplest setup, a linear stochastic differential equation is of the form
$$d\eta = (S\eta + \psi)dt + Ad\omega, \tag{1.1}$$
where S and A are linear operators specified below, $\psi = \psi(t)$ is a deterministic load external action and $\omega = \omega(t)$ is a stochastic external action, $\eta = \eta(t)$ is the required stochastic process. Originally $d\omega$ stood for the differential of the Wiener process $\omega = W(t)$, whose generalized derivative is traditionally treated as white noise. Ito began studying the ordinary differential equations of the form (1.1) and was joined later by Stratonovich and Skorokhod. The Ito-Stratonovich-Skorokhod approach in the finite-dimensional case is still popular [6]. Moreover, it has been extended successfully to the infinite-dimensional setup [7], and even to Sobolev-type equations [13]. In the framework of this direction, the linear stochastic Hoff equation with the initial-final condition was considered [9].

However, recently a new approach to linear stochastic equations arose [11] and is actively developing [8] in optimal measurement theory. Namely, instead of (1.1) we consider the linear stochastic Sobolev-type equation
$$L\overset{\circ}{\eta} = M\eta + N\omega, \tag{1.2}$$
where $\eta = \eta(t)$ is the required stochastic process and $\omega = \omega(t)$ is a prescribed stochastic process corresponding to external action, $\overset{\circ}{\eta}$ is the Nelson-Gliklikh derivative [4, 8, 11] of η, the operators L, M, and N are linear and continuous. By way of example, [4, 11] consider the "white noise" $\omega = \overset{\circ}{W}$, while, as shown previously [8], it is more adequate to the Einstein-Smoluchowski theory of Brownian motion

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than the traditional white noise $d\omega = dW$ in (1.1). (Here $W = W(t)$ is a Wiener K-process for a nuclear operator K).

Apart from the introduction, the conclusion, and the list of references, this article consists of three sections. The first one deals with the deterministic inhomogeneous linear Sobolev-type equation

$$L\dot{u} = Mu + f, \tag{1.3}$$

where the operator M is (L,p)-bounded with $p \in \{0\} \cup \mathbb{N}$. (Note that we use the term "Sobolev type equations" [10] as synonymous terms "degenerate equations" [5] and "equations not solvable with respect to the highest-order derivative" [1]).

We define multipoint initial-final conditions and state a theorem on the existence of a unique solution. We borrowed all results from [12, 13] and therefore give them without proofs. The second section extends the deterministic results of the first one to the stochastic setup by analogy with [11]; sketches of proofs complement the results. In the third section, by way of example, we consider the linear stochastic Hoff equation [9] which models I-beam bulging. In closing, we outline possible directions for further research. The list of references, not intended to be complete, reflects the authors' tastes and preferences.

2. Deterministic linear equations

Given two Banach spaces $\mathfrak{U}$ and $\mathfrak{F}$, take two operators: $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, that is, a linear and continuous one; and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$, that is, a linear, closed, and densely defined one. Set

$$\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$$

is called a L-resolvent set of an operator M. The set $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ is called L-spectrum of an operator M. It is easy to show [12, Chapter 4] that the L-resolvent set of the operator M is always open, and, consequently, the L-spectrum of the operator M is always closed. An operator M is called (L,σ)-bounded, if L-spectrum is a bounded set (for the terminology and results, see [12, Chapter 4]). So, if the operator M is (L,σ)-bounded, then there exist degenerate analytic groups of solving operators

$$U^t = \frac{1}{2\pi i}\int_\gamma R^L_\mu(M)e^{\mu t}d\mu \quad \text{and} \quad F^t = \frac{1}{2\pi i}\int_\gamma L^L_\mu(M)e^{\mu t}d\mu$$

defined on the spaces $\mathfrak{U}$ and $\mathfrak{F}$ respectively; moreover, $U^0 \equiv P$ and $F^0 \equiv Q$ are projections. Here γ is the contour bounding a domain D which contains the L-spectrum $\sigma^L(M)$ of the operator M; also, $R^L_\mu(M) = (\mu L - M)^{-1}L$ is the right L-resolvent of M, while $L^L_\mu(M) = L(\mu L - M)^{-1}$ is the left one. For a degenerate analytic group the concepts of kernel $\ker U^\cdot = \ker P = \ker U^t$ and the image $\operatorname{im} U^\cdot = \operatorname{im} P = \operatorname{im} U^t$ for all $t \in \mathbb{R}$ are well-defined. Put $\mathfrak{U}^0 = \ker U^\cdot$, $\mathfrak{U}^1 = \operatorname{im} U^\cdot$, $\mathfrak{F}^0 = \ker F^\cdot$, and $\mathfrak{F}^1 = \operatorname{im} F^\cdot$. Then $\mathfrak{U}^0 \oplus \mathfrak{U}^1 = \mathfrak{U}$ and $\mathfrak{F}^0 \oplus \mathfrak{F}^1 = \mathfrak{F}$. Denote also by L_k the restriction of L to $\mathfrak{U}^k$ and by M_k the restriction of M to $\operatorname{dom} M \cap \mathfrak{U}^k$, for $k = 0, 1$.

Theorem 2.1 (Splitting theorem [12, Chapter 4])**.** *If the operator M is (L,σ)-bounded then*

(i) $L_k \in \mathcal{L}(\mathfrak{U}^k;\mathfrak{F}^k)$ *for* $k = 0, 1$*;*
(ii) $M_0 \in \mathcal{C}l(\mathfrak{U}^0;\mathfrak{F}^0)$ *and* $M_1 \in \mathcal{L}(\mathfrak{U}^1;\mathfrak{F}^1)$*;*
(iii) *the operators* $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$ *and* $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$ *exist.*

Put $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ and $S = L_1^{-1}M_1 \in \mathcal{L}(\mathfrak{U}^1)$.

Corollary 2.2 ([12, Chapter 4]). *If the operator M is (L,σ)-bounded, then*

$$(\mu L - M)^{-1} = -\sum_{k=0}^{\infty} \mu^k H^k M_0^{-1}(\mathbb{I} - Q) + \sum_{k=1}^{\infty} \mu^{-k} S^{k-1} L_1^{-1} Q$$

for every $\mu \in \mathbb{C} \setminus \bar{D}$.

The operator M is called (L,p)-bounded with $p \in \{0\} \cup \mathbb{N}$ whenever $H^p \neq \mathbb{O}$ but $H^{p+1} = \mathbb{O}$.

We introduce the condition

(A1) $\sigma^L(M) = \bigcup_{j=0}^m \sigma_j^L(M)$ for $m \in \mathbb{N}$; furthermore, $\sigma_j^L(M) \neq \emptyset$, there exists a closed contour $\gamma_j \subset \mathbb{C}$, bounding a domain $D_j \supset \sigma_j^L(M)$, such that $\overline{D_j} \cap \sigma_0^L(M) = \emptyset$ and $\overline{D_k} \cap \overline{D_l} = \emptyset$ for all $j,k,l = \overline{1,m}$ with $k \neq l$.

Theorem 2.3 ([12]). *If the operator M is (L,σ)-bounded and condition* (A1) *is fulfilled then*

(i) *there exist degenerate analytic groups*

$$U_j^t = \frac{1}{2\pi i}\int_{\gamma_j} R_\mu^L(M) e^{\mu t} d\mu, \quad j = \overline{1,m}.$$

(ii) $U^t U_j^s = U_j^s U^t = U_j^{s+t}$ *for all s, $t \in \mathbb{R}$ and $j = \overline{1,m}$;*

(iii) $U_k^t U_l^s = U_l^s U_k^t = \mathbb{O}$ *for all s, $t \in \mathbb{R}$ and k, $l = \overline{1,m}$ with $k \neq l$.*

Put $U_0^t = U^t - \sum_{k=1}^m U_k^t$ for $t \in \mathbb{R}$.

Remark 2.4. Consider the identity elements $P_j \equiv U_j^0$ of the constructed degenerate analytic groups $\{U_j^t : t \in \mathbb{R}\}$, for $j = \overline{0,m}$. It is obvious that $PP_j = P_jP = P_j$ for $j = \overline{0,m}$, and $P_kP_l = P_lP_k = \mathbb{O}$ for k, $l = \overline{0,m}$ with $k \neq l$. Similarly, we can construct projectors $Q_j \in \mathcal{L}(\mathfrak{F})$ for $j = \overline{0,m}$ (see [12] for details) such that $QQ_j = Q_jQ = Q_j$ for $j = \overline{0,m}$ and $Q_kQ_l = Q_lQ_k = \mathbb{O}$ for k, $l = \overline{0,m}$ with $k \neq l$.

We refer to P_j and Q_j for $j = \overline{0,m}$ as *relatively spectral projectors.*

We introduce the subspaces $\mathfrak{U}^{1j} = \operatorname{im} P_j$ and $\mathfrak{F}^{1j} = \operatorname{im} Q_j$ for $j = \overline{0,m}$. By construction,

$$\mathfrak{U}^1 = \oplus_{j=0}^m \mathfrak{U}^{1j} \quad \text{and} \quad \mathfrak{F}^1 = \oplus_{j=0}^m \mathfrak{F}^{1j}.$$

We denote by L_{1j} the restriction of L to $\mathfrak{U}^{1j}$ and by M_{1j} the restriction of M to $\operatorname{dom} M \cap \mathfrak{U}^{1j}$, for $j = \overline{0,m}$. It is not difficult to show that $P_j\varphi \in \operatorname{dom} M$; therefore, if $\varphi \in \operatorname{dom} M$ then the domain $\operatorname{dom} M_{1j} = \operatorname{dom} M \cap \mathfrak{U}^{1j}$ is dense in $\mathfrak{U}^{1j}$, for $j = \overline{0,m}$.

Theorem 2.5 (Generalized spectral theorem [12]). *Suppose that $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ and $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$, operator M is (L,σ)-bounded, and condition* (A1) *is satisfied, then*

(i) $L_{1j} \in \mathcal{L}(\mathfrak{U}^{1j};\mathfrak{F}^{1j})$ *and* $M_{1j} \in \mathcal{L}(\mathfrak{U}^{1j};\mathfrak{F}^{1j})$ *for $j = \overline{0,m}$;*

(ii) *the operators $L_{1j}^{-1} \in \mathcal{L}(\mathfrak{F}^{1j};\mathfrak{U}^{1j})$ exist, for $j = \overline{0,m}$.*

Thus, we assume that condition (A1) is fulfilled. Fix $\tau_j \in \mathbb{R}$ with $\tau_j < \tau_{j+1}$, vectors $u_j \in \mathfrak{U}$ for $j = \overline{0,m}$, and vector-function $f \in C^\infty(\mathbb{R};\mathfrak{F})$. Consider the linear inhomogeneous Sobolev-type equation

$$L\dot{u} = Mu + f. \tag{2.1}$$

We refer to a vector-function $u \in C^\infty(\mathbb{R};\mathfrak{U})$ satisfying (2.1) as a *solution* to (2.1). We refer to a solution $u = u(t)$, for $t \in \mathbb{R}$, to (2.1) satisfying the conditions

$$P_j(u(\tau_j) - u_j) = 0, \quad j = \overline{0,m}, \tag{2.2}$$

as a *solution to the multipoint initial-final value problem* for (2.1).

Theorem 2.6 ([12]). *If the operator M is (L,p)-bounded for $p \in \{0\} \cup \mathbb{N}$ and condition (A1) holds then for all $f \in C^\infty(\mathbb{R};\mathfrak{F})$ and $u_j \in \mathfrak{U}$, for $j = \overline{0,m}$, there exists a unique solution to problem* (2.1), (2.2)*; furthermore, it is of the form*

$$\begin{aligned} u(t) = &-\sum_{q=0}^{p} H^q M_0^{-1}(\mathbb{I} - Q) f^{(q)}(t) \\ &+ \sum_{j=0}^{m} U_j^{t-\tau_j} u_j + \sum_{j=0}^{m} \int_{\tau_j}^{t} U_j^{t-\tau_j-s} L_{1j}^{-1} Q_j f(s) ds. \end{aligned} \tag{2.3}$$

An example is presented in Section 4 of this article.

3. Stochastic linear equations

For a real separable Hilbert space $\mathfrak{U} \equiv (\mathfrak{U}, \langle \cdot, \cdot \rangle)$, take an operator $K \in \mathcal{L}(\mathcal{U})$ whose spectrum $\sigma(K)$ is nonnegative, discrete, with finite multiplicities and accumulates only to zero. Denote by $\{\lambda_j\}$ the sequence of eigenvalues of K enumerated in the non-increasing order taking the multiplicities into account. The linear span of the set $\{\varphi_j\}$ of associated orthonormal eigenvectors of K is dense in $\mathfrak{U}$. Assume also that K is a nuclear operator, that is, its trace satisfies $\operatorname{Tr} K = \sum_{j=1}^\infty \lambda_j < +\infty$.

Take a sequence $\{\eta_j\}$ of independent stochastic processes $\eta_j : \Omega \times \mathcal{I} \to \mathbb{R}$, a complete probability space Ω, and an interval $\mathcal{I} \subset \mathbb{R}$. Equip $\mathbb{R}$ with the Borel σ-algebra. The set of random variables with zero mean and finite variances constitutes a Hilbert space with the inner product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$. Denote this Hilbert space by $\mathbf{L}_2$. Assume that the random variables $\eta_j(\omega, t) \in \mathbf{L}_2$ are Gaussian for all $\omega \in \mathcal{A}$ and $t \in \mathcal{I}$, where $\mathcal{A}$ is a σ-algebra on Ω. In addition, the sample trajectory $\eta_j(\omega, \cdot)$ is almost surely continuous, that is, $\eta_j \in \mathbf{CL}_2$. (For a detailed description of the spaces $\mathbf{C}^l\mathbf{L}_2$ for $l \in \{0\} \cup \mathbb{N}$, see [4, 11].) Define the $\mathfrak{U}$-valued stochastic K-process

$$\Theta_K(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\eta_j(t)\varphi_j \tag{3.1}$$

on assuming that the series (3.1) converges uniformly on every compact subset of $\mathcal{I}$. Observe that if $\{\eta_j\} \subset \mathbf{CL}_2$ then the existence of a stochastic K-process Θ_K implies that its trajectories are almost surely (a.s.) continuous. Introduce the Nelson-Gliklikh derivatives

$$\overset{\circ}{\Theta}{}_K^{(l)}(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\overset{\circ}{\eta}{}_j^{(l)}(t)\varphi_j \tag{3.2}$$

of the stochastic K-process on assuming that the derivatives in the right-hand side up to order l exist and all series converge uniformly on every compact subset of $\mathcal{I}$. (For a detailed description of the Nelson-Gliklikh derivative, see [4, 6, 11]). As in [4, 11] we introduce the *space of differentiable "noises"* $\mathbf{C}_K^l\mathbf{L}_2$ of stochastic K-processes whose trajectories are a.s. continuously differentiable on $\mathcal{I}$ in the sense of Nelson-Gliklikh up to order $l \in \{0\} \cup \mathbb{N}$.

As an example, let us present “black noise”, a stochastic K-process whose trajectories a.s. coincide with the zero (that is, absolute silence), as well as “white noise”

$$\overset{\circ}{W}_K(t) = \frac{W_K(t)}{2t}, \tag{3.3}$$

the Nelson-Gliklikh derivative of the Wiener K-process

$$W_K(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)\varphi_j, \quad t \in \overline{\mathbb{R}}_+.$$

Here $\beta_j = \beta_j(t)$ is the Brownian motion of the form

$$\beta_j(t) = \sum_{k=1}^{\infty} \xi_{jk} \sin \frac{\pi(2k+1)}{2} t, \quad t \in \overline{\mathbb{R}}_+,$$

where ξ_{jk} are pairwise independent Gaussian random variables such that $\mathbf{E}\xi_{jk} = 0$ and $\mathbf{D}\xi_{jk} = [\frac{\pi(2k+1)}{2}]^{-2}$, here $\xi_{jk} \in \mathbf{L}_2$, $\mathbf{E}$ is mathematical expectation and $\mathbf{D}$ is dispersion.

Having considered the deterministic equation (1.3) in the previous section, we now proceed to the stochastic equation (1.2). Assume that the operator M is (L,p)-bounded, with $p \in \{0\} \cup \mathbb{N}$, and condition (A1) is satisfied. Consider the linear stochastic Sobolev-type equation

$$L\overset{\circ}{\eta} = M\eta + N\omega, \tag{3.4}$$

where $\eta = \eta(t)$ is the required stochastic K-process and $\omega = \omega(t)$ is a known stochastic K-process, and the operator $N \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$.

Take $\tau_0 = 0$ and $\tau_j \in \mathbb{R}_+$ with $\tau_{j-1} < \tau_j$ for $j = \overline{1,m}$. Complement (3.4) with the multipoint initial-final conditions

$$P_j(\eta(\tau_j) - \xi_j) = 0, \quad j = \overline{0,m}, \tag{3.5}$$

where P_j are the relatively spectral projectors from Remark 2.4. Below, in view of (3.3), we also have to consider the *weak* (in the sense of S. Krein) *multipoint initial-final conditions*

$$\lim_{t\to\tau_0+} P_0(\eta\,(t) - \xi_0) = 0, \quad P_j(\eta(\tau_j) - \xi_j) = 0, \quad j = \overline{1,m}. \tag{3.6}$$

Here

$$\xi_j = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\xi_{jk}\varphi_k,\ j = \overline{0,m}, \tag{3.7}$$

where $\xi_{jk} \in \mathbf{L}_2$ is a Gaussian random variable such that series (3.7) is convergent. (For instance $\mathbf{D}\xi_{jk} \le C_j$, $k \in \mathbb{N}$, $j = \overline{0,m}$). Call a stochastic K-process $\eta \in \mathbf{C}^1_K\mathbf{L}_2$ a (*classical*) *solution* to (3.4) whenever a.s. all its trajectories satisfy (3.4) for some stochastic K-process $\omega \in \mathbf{C}_K\mathbf{L}_2$, some operator $N \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, and all $t \in \mathcal{I}$. (Here and henceforth $\mathcal{I} = (0,+\infty)$). Call a solution $\eta = \eta(t)$ to (3.4) a (*classical*) *solution* to problem (3.4), (3.5) (problem (3.4), (3.6)) whenever in addition condition (3.5) (condition (3.6)) is satisfied.

Theorem 3.1. *For $p \in \{0\} \cup \mathbb{N}$ take an (L,p)-bounded operator M and assume that condition (A1) holds. Given $\tau_j \in \mathbb{R}_+$ for $j = \overline{1,m}$, an operator $N \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, a nuclear operator $K \in \mathcal{L}(\mathfrak{U})$ with real spectrum $\sigma(K)$, a stochastic K-process $\omega = \omega(t)$ such that $(\mathbb{I}-Q)N\omega \in \mathbf{C}^{p+1}_K\mathbf{L}_2$ and $QN\omega \in \mathbf{C}_K\mathbf{L}_2$, and random variables*

$\xi_j \in \mathbf{L}_2$, for $j = \overline{0,m}$, such that (3.7) *are fulfilled, there exists a unique solution $\eta \in \mathbf{C}^1_K\mathbf{L}_2$ to problem* (3.4), (3.5)*; moreover, it is of the form*

$$\begin{aligned} \eta(t) = &- \sum_{q=0}^{p} H^q M_0^{-1}(\mathbb{I} - Q)\overset{\circ}{\omega}{}^{(q)}(t) \\ &+ \sum_{j=0}^{m} \Big[U_j^{t-\tau_j}\xi_j + \int_{\tau_j}^{t} U_j^{t-\tau_j-s} L_{1j}^{-1} Q_j N\omega(s)ds \Big], \quad t \in \mathcal{I}. \end{aligned} \tag{3.8}$$

Let us sketch the proof. It is straightforward to verify that (3.8) is a solution to problem (3.4), (3.5). To establish the uniqueness, reduce the problem to the equivalent system

$$L\overset{\circ}{\eta} = M\eta,\ P_j\,\eta^j(\tau_j) = 0, \quad j = \overline{0,m}.$$

By Theorem 2.1 the first equation here is equivalent to the system

$$H\overset{\circ}{\eta}{}^0 = \eta^0,\ \overset{\circ}{\eta}{}^1 = S\eta^1, \tag{3.9}$$

where $\eta^0 = (\mathbb{I} - P)\eta$ and $\eta^1 = P\eta$. Taking now the Nelson-Gliklikh derivative of the first equation and multiplying on the left by H we obtain in succession

$$0 = H^{p+1}\overset{\circ}{\eta}{}^{0(p+1)} = \ldots = H^2\overset{\circ}{\eta}{}^{0(2)} = \cdots = H\overset{\circ}{\eta}{}^0 = \eta^0.$$

By Theorem 2.3 and the initial-final conditions (3.5), the second equation of (3.9) yields $\eta^1 = \sum_{j=0}^{m} U^{t-\tau_j}0 = 0$.

In view of (3.3), problem (3.4), (3.5) is not solvable when the right-hand side of (3.4) is the “white noise” $\omega(t) = \overset{\circ}{W}_K(t)$. In this case instead of conditions (3.5) we should consider conditions (3.6).

Corollary 3.2. *If all the hypotheses of Theorem 3.1 hold and $\omega(t) = \overset{\circ}{W}_K(t)$ then, given random variables $\xi_j \in \mathbf{L}_2$ as in* (3.7)*, there exists a unique solution to problem* (3.4), (3.6)*; furthermore, it has the form*

$$\begin{aligned} \eta(t) = &\sum_{j=0}^{m} \Big[U_j^{t-\tau_j}\xi_j - S_jP_j \int_{\tau_j}^{t} U_j^{t-\tau_j-s} L_{1j}^{-1} Q_j N W_K(s)ds \\ &+ L_{1j}^{-1}Q_jNW_K(t) \Big] - \sum_{q=0}^{p} H^q M_0^{-1}(\mathbb{I} - Q)\ \overset{\circ}{W}{}_K^{(q+1)}(t), \quad t \in \overline{\mathbb{R}}_+. \end{aligned} \tag{3.10}$$

The proof of the above corollary is similar to that of Theorem 3.1. The difference in the additive terms is caused by an application of integration “by parts”,

$$\begin{aligned} &\int_{\tau_j}^{t} U_j^{t-\tau_j-s} L_{1j}^{-1} Q_j N \overset{\circ}{W}_K(s)ds \\ &= L_{1j}^{-1}Q_jN(W_K(t) - W_K(\tau_j)) - S_jP_j \int_{\tau_j}^{t} U_j^{t-\tau_j-s} L_{1j}^{-1} Q_j N W_K(s)ds, \end{aligned}$$

which follows from the properties of Nelson-Gliklikh derivative. Here $S_j = L_{1j}^{-1}M_{1j}$ for $j = \overline{0,m}$.

4. Linear Hoff equation with additive "white noise"

Consider a bounded domain $D \subset \mathbb{R}^d$ ($d \in \mathbb{N}$) with boundary ∂D of class C^∞. Denote by $\mathfrak{U}$ and $\mathfrak{F}$ the function spaces $\mathfrak{U} = \{u \in W_2^{l+2}(D) : u(x) = 0, x \in \partial D\}$ and $\mathfrak{F} = W_2^l(D)$, where $l \in \{0\} \cup \mathbb{N}$. Evidently, $\mathfrak{U}$ is a real separable Hilbert space densely and continuously embedded into $\mathfrak{F}$. Fixing $\alpha, \mu \in \mathbb{R}$, construct the operators $L = \mu\mathbb{I} + \Delta$ and $M = \alpha\mathbb{I}$, where Δ is the Laplace operator, and the symbol $\mathbb{I}$ stands for the embedding operator $\mathbb{I} : \mathfrak{U} \hookrightarrow \mathfrak{F}$; we also emphasize that here M is not invertible. Consider also the spectral problem

$$-\Delta u = \nu u \text{ in } D \text{ and } u(x) = 0 \text{ for } x \in \partial D. \tag{4.1}$$

Its solution is a family $\{\nu_j\} \subset \mathbb{R}_+$ of eigenvalues enumerated in the nondecreasing order taking their multiplicities into account and accumulating only to $+\infty$, as well as the associated orthonormal (in the sense of $\mathcal{U}$) family of eigenfunctions $\{\varphi_j\}$. It is not difficult to show (see [9] for instance) that for all $\mu \in \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$ the operator M is $(L, 0)$-bounded; moreover, its L-spectrum is

$$\sigma^L(M) = \big\{\mu_k = \frac{\alpha}{\mu - \nu_k},\ k \in \mathbb{N} \setminus \{l : \mu = \nu_l\}\big\} \cup \{0\}. \tag{4.2}$$

Furthermore, for $m \in \mathbb{N}$ construct the operator $\Lambda = (-\Delta)^m$ with

$$\operatorname{dom}\Lambda = \{u \in W_2^{l+2m}(D) : \Delta^k u(x) = 0,\ x \in \partial D,\ k = \overline{0, m-1}\}.$$

The family of eigenfunctions of Λ coincides with the family $\{\varphi_j\}$, while its family of eigenvalues is $\{\nu_j^m\}$. Since their asymptotics is $\nu_j^m \sim j^{\frac{2m}{d}} \to \infty$ as $j \to \infty$, we can choose $m \in \mathbb{N}$ so that, firstly, the dimension d of the domain D has some acceptable physical meaning, and secondly, the series $\sum_{j=1}^\infty \nu_j^{-1}$ converges. Then the Green operator of Λ is nuclear, and we take it as K. Therefore, consider the linear stochastic Hoff equation in the form

$$L\overset{\circ}{\eta} = M\eta + \overset{\circ}{W}_K, \tag{4.3}$$

where L and M are defined above, while N is the embedding operator $\mathbb{I} : \mathfrak{U} \hookrightarrow \mathfrak{F}$ and $\overset{\circ}{W}_K = \overset{\circ}{W}_K(t)$ is the Nelson-Gliklikh derivative of the $\mathfrak{U}$-valued Wiener K-process $W_K = W_K(t)$, for $t \in \mathbb{R}_+$.

To state initial-final conditions, we need relatively spectral projectors. In this example we confine the discussion, for the sake of simplicity, to just two initial-final conditions. Furthermore, here we present the initial-final conditions satisfying condition (A1), while in Remark 4.2 below we verify that in this case, thanks to the structure of $\sigma^L(M)$ in (4.2), we can avoid condition (A1). Thus, take the projectors

$$P(Q) = \begin{cases} \mathbb{I}_{\mathfrak{U}}(\mathbb{I}_{\mathfrak{F}}) & \text{if } \mu \neq \nu_j\ \forall j \in \mathbb{N}; \\ \mathbb{I}_{\mathfrak{U}} - \sum_{j:\mu=\nu_j} \langle \cdot, \varphi_j\rangle_{\mathfrak{U}}\varphi_j \Big(\mathbb{I}_{\mathfrak{F}} - \sum_{j:\mu=\nu_j} \langle \cdot, \psi_j\rangle_{\mathfrak{F}}\psi_j\Big), \end{cases}$$

where $\{\psi_j\}$ is a family of eigenfunctions $\{\varphi_j\}$ orthonormal in the sense of the inner product $\langle \cdot, \cdot\rangle_{\mathfrak{F}}$ in $\mathfrak{F}$. Furthermore, choose $h \in \mathbb{R}_+$ with $h < \max_{j\in\mathbb{N}}\{|\nu_j|\}$ and construct the projectors

$$\begin{gathered} P_1 = \mathbb{I}_{\mathfrak{U}} - \sum_{h<|\nu_j|} \langle \cdot, \varphi_j\rangle_{\mathfrak{U}}\varphi_j, \quad Q_1 = \mathbb{I}_{\mathfrak{F}} - \sum_{h<|\nu_j|} \langle \cdot, \psi_j\rangle_{\mathfrak{F}}\psi_j; \\ P_0 = P - P_1, \quad Q_0 = Q - Q_1. \end{gathered} \tag{4.4}$$

Observe that in the construction of these projectors condition (A1) holds because $\sigma_0^L(M) = \{\mu_j \in \sigma^L(M) : |\nu_j| \leq h\}$ and $\sigma_1^L(M) = \{\mu_j \in \sigma^L(M) : |\nu_j| > h\}$; hence,

$\sigma_0^L(M)\cap\sigma_1^L(M)=\emptyset$. Finally, choose $\tau_1\in\mathbb{R}_+$ as well as random variables ξ_0 and ξ_1 independent of each other and of stochastic K-processes η and pose the initial-final conditions

$$\lim_{t\to 0+} P_0(\eta\,(t)-\xi_0)=0,\quad P_1(\eta(\tau_1)-\xi_1)=0, \tag{4.5}$$

where

$$\xi_0=\sum_{k=1}^{\infty}\sqrt{\nu_k}\xi_{0k}\varphi_k,\quad \xi_1=\sum_{k=1}^{\infty}\sqrt{\nu_k}\xi_{1k}\varphi_k. \tag{4.6}$$

Applying the results of Section 2 to problem (4.3), (4.5), we obtain the following theorem.

Theorem 4.1. *If condition* (A1) *is satisfied then for all numbers $\mu\in\mathbb{R}$, $\alpha\in\mathbb{R}\backslash\{0\}$ and $\tau_1\in\mathbb{R}_+$, as well as random variables ξ_{0k} andξ_{1k} such as $\mathbf{D}\xi_{0k}\le C_0$ and $\mathbf{D}\xi_{1k}\le C_1$ for some C_0, $C_1\in\mathbb{R}_+$ there exists a unique solution $\eta=\eta(t)$, for $t\in\mathbb{R}_+$, to problem* (4.3), (4.5)*; furthermore, it is of the form*

$$\begin{aligned}\eta(t)&=(L_{10}^{-1}Q_0+L_{11}^{-1}Q_1)W_K(t)-L_{11}^{-1}Q_1W_K(\tau_1)\\&\quad-S_0P_0\int_0^t U_0^{t-s}L_{10}^{-1}Q_0W_K(s)ds+U_0^t\xi_0+U_1^{t-\tau_1}\xi_1\\&\quad-S_1P_1\int_{\tau_1}^t U_1^{t-\tau_1-s}L_{11}^{-1}Q_1W_K(s)ds-M_0^{-1}(\mathbb{I}-Q)N\overset{\circ}{W}_K(t),\end{aligned} \tag{4.7}$$

for $t\in\mathbb{R}_+$.

Here

$$\begin{gathered}U_0^t=\sum_{\nu_j\in\sigma_0^L(M)}e^{t\mu_j}\langle\cdot,\varphi_j\rangle_{\mathcal{U}}\varphi_j,\quad U_1^t=\sum_{\nu_j\in\sigma_1^L(M)}e^{t\mu_j}\langle\cdot,\varphi_j\rangle_{\mathcal{U}}\varphi_j,\\ L_{10}^{-1}=\sum_{\nu_j\in\sigma_0^L(M)}(\mu-\nu_j)^{-1}\langle\cdot,\varphi_j\rangle_{\mathcal{U}}\varphi_j,\\ L_{11}^{-1}=\sum_{\nu_j\in\sigma_1^L(M)}(\mu-\nu_j)^{-1}\langle\cdot,\varphi_j\rangle_{\mathcal{U}}\varphi_j,\\ S_{10}=\alpha\sum_{\nu_j\in\sigma_0^L(M)}(\mu-\nu_j)^{-1}\langle\cdot,\varphi_j\rangle_{\mathcal{U}}\varphi_j,\\ S_{11}=\alpha\sum_{\nu_j\in\sigma_1^L(M)}(\mu-\nu_j)^{-1}\langle\cdot,\varphi_j\rangle_{\mathcal{U}}\varphi_j,\\ M_0^{-1}=\alpha^{-1}\sum_{\nu_j=\mu}\langle\cdot,\psi_j\rangle_{\mathcal{F}}\psi_j.\end{gathered} \tag{4.8}$$

Remark 4.2. Verify that in this concrete case condition (A1) could not be satisfied; however, Theorem 4.1 remains valid. Let all eigenvalues be simple, put $\sigma_0^L(M)=\{\mu_j\in\sigma^L(M):\,j=2n\}$ and $\sigma_1^L(M)=\{\mu_j\in\sigma^L(M):\,j=2n-1\}$, $n\in\mathbb{N}$. Then $\sigma_0^L(M)\cap\sigma_1^L(M)=\emptyset$. Nevertheless, (4.4) and (4.8) remain valid, and so (4.7) holds. The uniqueness of this solution is proved in the standard fashion (see Section 2).

Conclusion. The next stage of our studies is to carry over the ideas and methods of the theory of multipoint initial-final problems for linear Sobolev-type equations from relatively p-bounded setup to relatively p-sectorial setup by analogy with [3, 11]. In addition, it would be interesting to extend these ideas and methods to

Sobolev-type equations of high order [4], and also apply them to inverse problems as in [2].

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Angelo Favini
Department of Mathematics, Bologna University, Piazza di Porta San Donato 5, 40126, Bologna (BO), Italy
E-mail address: angelo.favini@unibo.it

Sophiya A. Zagrebina
Department of Mathematical and Computer Modelling, South Ural State University, Lenin av., 76, Chelyabinsk, 454080, Russian Federation
E-mail address: zagrebinasa@susu.ru

Georgy A. Sviridyuk
Department of Mathematical Physics Equations, South Ural State University, Lenin av., 76, Chelyabinsk, 454080, Russian Federation
E-mail address: sviridyuk@susu.ru