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REMARK ON PERIODIC BOUNDARY-VALUE PROBLEM FOR SECOND-ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We establish conditions for the unique solvability of periodic boundary value problem for second-order linear equations. We make more precise a result proved in [3].

1. Introduction

Consider the periodic boundary-value problem

$$u'' = p(t)u + q(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{1.1}$$

where $p, q : [0, \omega] \to \mathbb{R}$ are Lebesgue integrable functions. By a solution of given in (1.1) equation, as usual, we understand a function $u \in AC^1([0, \omega])$ such that for almost all $t \in [0, \omega]$.

Definition 1.1. We say that the function $p \in L([0, \omega])$ belongs to the set $V^-(\omega)$ (resp. $V^+(\omega)$) if for every $u \in AC^1([0, \omega])$ satisfying

$$u''(t) \ge p(t)u(t) \quad \text{for a.e. } t \in [0, \omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega),$$

the inequality

$$u(t) \le 0 \quad \text{for } t \in [0, \omega] \quad (\text{resp. } u(t) \ge 0 \quad \text{for } t \in [0, \omega]) \tag{1.2}$$

is fulfilled.

It is clear that if $p \in V^-(\omega)$ (resp. $p \in V^+(\omega)$), then the homogeneous problem

$$u'' = p(t)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)$$

has no nontrivial solution. Consequently, by virtue of Fredholm's alternative, the problem (1.1) is uniquely solvable. Moreover, if $q(t) \ge 0$ for $t \in [0, \omega]$, then the unique solution u of the problem (1.1) satisfies (1.2).

It is also evident that if $p \in V^-(\omega)$ (resp. $p \in V^+(\omega)$) and the functions $u, v \in AC^1([0, \omega])$ satisfy differential inequalities

$$u''(t) \ge p(t)u(t), \quad v''(t) \le p(t)v(t) \quad \text{for a.e. } t \in [0, \omega]$$

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and boundary conditions

$$u^{(i)}(0) - u^{(i)}(\omega) = v^{(i)}(0) - v^{(i)}(\omega), \quad i = 0, 1,$$

then the inequality

$$u(t) \le v(t) \quad \text{for } t \in [0,\omega] \quad (\text{resp. } u(t) \ge v(t) \quad \text{for } t \in [0,\omega])$$

holds.

Properties of the sets $V^-(\omega)$ and $V^+(\omega)$ plays a crucial role in the theory of periodic boundary value problems for nonlinear equations (see, e. g., [3, 2]). Therefore, it is desirable to establish sufficient conditions for the inclusion $p \in V^-(\omega)$, resp. $p \in V^+(\omega)$. One can find several integral conditions in [3].

Theorem 1.2 ([3, Theorem 11.1])**.** *Let $p \not\equiv 0$ and*

$$\|[p]_-\|_1 \le \frac{\|[p]_+\|_1}{1 + \frac{\omega}{4}\,\|[p]_+\|_1}\,. \tag{1.3}$$

Then $p \in V^-(\omega)$.

The main result of this article makes more precise Theorem 1.2. In particular, it covers also the case when $\|[p]_-\|_1 \ge 4/\omega$.

Below we use the following notation: $\mathbb{R} =]-\infty, +\infty[$. For $x \in \mathbb{R}$, we put $[x]_+ = \frac{1}{2}(|x| + x)$ and $[x]_- = \frac{1}{2}(|x| - x)$.

Let $\omega > 0$ and $\lambda \in]0, \frac{1}{2}]$. Then

$$\Delta_\omega(\lambda) := \Big[\frac{1-2\lambda}{2\omega(1-\lambda)}\Big]^{\frac{1-\lambda}{\lambda}}.$$

I modified this sentence. Please check it

The set $AC^1([a,b])$ consists of absolutely continuous functions $u : [a,b] \to \mathbb{R}$ whose first derivative is also absolutely continuous on $[a,b]$. The set $L([a,b])$ consists of Lebesgue integrable functions $f : [a,b] \to \mathbb{R}$. If $f \in L([a,b])$ and $\lambda \in]0, \frac{1}{2}]$, then we put

$$\|f\|_\lambda = \Big(\int_a^b |f(s)|^\lambda \,\mathrm{d}s\Big)^{1/\lambda}.$$

By L_ω we denote the set of ω-periodic functions $f : \mathbb{R} \to \mathbb{R}$ such that $f \in L([0,\omega])$. Now we are able to formulate main results.

Theorem 1.3. *Let $p \not\equiv 0$, $\lambda \in]0, \frac{1}{2}[$, and*

$$\|[p]_-\|_1 < \frac{4}{\omega} + \Delta_\omega(\lambda)\|[p]_-\|_\lambda, \tag{1.4}$$

$$\begin{aligned} \|[p]_-\|_1 \le \|[p]_+\|_1 \Big(1 - \frac{\omega}{4}\,\|[p]_-\|_1 + \frac{\omega}{4}\,\Delta_\omega(\lambda)\|[p]_-\|_\lambda\Big) \\ + \frac{\omega}{4}\,\Delta_\omega(\lambda)\|[p]_+\|_\lambda\|[p]_-\|_1. \end{aligned} \tag{1.5}$$

Then the inclusion $p \in V^-(\omega)$ holds.

Remark 1.4. It is not difficult to verify that if (1.3) holds then (1.4) and (1.5) are fulfilled. Indeed, it follows from (1.3) that $\|[p]_-\|_1 < 4/\omega$. Hence, (1.4) holds. On the other hand, (1.3) is equivalent to the inequality $\|[p]_-\|_1 + \frac{\omega}{4}\|[p]_+\|_1\|[p]_-\|_1 \le \|[p]_+\|_1$, i. e., $\|[p]_-\|_1 \le \|[p]_+\|_1(1 - \frac{\omega}{4})\|[p]_-\|_1$ and consequently, (1.5) holds. Thus, Theorem 1.3 generalizes Theorem 1.2. On the other hand, since $\Delta_\omega(1/2) = 0$, conditions (1.4) and (1.5) with $\lambda = 1/2$ are equivalent to (1.3). In other words, one can regard Theorem 1.2 as "limit case" of Theorem 1.3.

Corollary 1.5. *Let $p \not\equiv 0$ and $\lambda \in]0, 1/2[$. Let, moreover, one of the following two items be fulfilled:*

(i) $\|[p]_-\|_1 \le 4/\omega$ *and* $\|[p]_+\|_1 \|[p]_-\|_\lambda + \frac{4}{\omega}\|[p]_+\|_\lambda \ge \frac{16}{\omega^2 \Delta_\omega(\lambda)}$ *;*

(ii) $\|[[p]_-]\|_1 < \frac{4}{\omega} + \Delta_\omega(\lambda)\|[p]_-\|_\lambda$ *and* $\|[p]_+\|_\lambda \ge \frac{4}{\omega\Delta_\omega(\lambda)}$ *.*

Then the inclusion $p \in V^-(\omega)$ holds.

To be more concrete, put $\lambda = 1/3$. Then $\Delta_\omega(\lambda) = 1/(16\omega^2)$ and conditions of Corollary 1.5 reads as follows:

(i) $\|[p]_-\|_1 \le 4/\omega$ and $\|[p]_+\|_1 \|[p]_-\|_{1/3} + \frac{4}{\omega}\|[p]_+\|_{1/3} \ge 16^2$;

(ii) $\|[[p]_-]\|_1 < \frac{4}{\omega} + \frac{1}{16\omega^2}\|[p]_-\|_{1/3}$ and $\|[p]_+\|_{1/3} \ge 64\omega$.

We postpone the proof of Theorem 1.3 until Section 3, after some auxiliary propositions stated in Section 2.

2. Auxiliary statements

First of all for convenience of the reader, we recall some known results.

Definition 2.1. We say that the function $p \in L_\omega$ belongs to the set $D(\omega)$ if the problem

$$u'' = p(t)u; \quad u(\alpha) = 0,\ u(\beta) = 0$$

has no nontrivial solution for any $\alpha < \beta$ satisfying $\beta - \alpha < \omega$.

Proposition 2.2 ([3, Theorem 9.3]). *Let $p \in L_\omega$,such that $p \not\equiv 0$, and $\int_0^\omega p(s)\,\mathrm{d}s \le 0$. Then $p \in V^+(\omega)$ if and only if $p \in D(\omega)$.*

Proposition 2.3 ([3, Lemma 2.7]). *Let $p \in V^+(\omega)$, $q \in L([0,\omega])$, $q(t) \ge 0$ for $t \in [0,\omega]$, and $q \not\equiv 0$. Then the (unique) solution u of the problem (1.1) satisfies $u(t) > 0$ for $t \in [0,\omega]$.*

Proposition 2.4 ([3, Theorem 8.3]). *Let $p \in L([0,\omega])$. Then the inclusion $p \in V^-(\omega)$ holds if and only if there exists a positive function $\gamma \in AC^1([0,\omega])$ satisfying*

$$\gamma''(t) \le p(t)\gamma(t) \quad \text{for a.e. } t \in [0,\omega], \quad \gamma(0) \ge \gamma(\omega), \quad \frac{\gamma'(\omega)}{\gamma(\omega)} \ge \frac{\gamma'(0)}{\gamma(0)},$$

and

$$\gamma(0) - \gamma(\omega) + \frac{\gamma'(\omega)}{\gamma(\omega)} - \frac{\gamma'(0)}{\gamma(0)} + \operatorname{meas}\{t \in [0,\omega] : \gamma''(t) < p(t)\gamma(t)\} > 0.$$

Let $f \in L([a, a+\omega])$, then we define

$$\begin{aligned} G_a(f)(t) &= (a+\omega-t)\int_a^t (s-a)f(s)\,\mathrm{d}s \\ &\quad + (t-a)\int_t^{a+\omega}(a+\omega-s)f(s)\,\mathrm{d}s \quad \text{for } t \in [a, a+\omega]. \end{aligned} \tag{2.1}$$

Proposition 2.5. *Let $\lambda \in]0, \frac{1}{2}[$, $f \in L([a, a+\omega])$, and $f(t) \ge 0$ for $t \in [a, a+\omega]$. Then we have the estimates*

$$G_a(f)(t) \le (t-a)(a+\omega-t)\Big(\|f\|_1 - \Delta_\omega(\lambda)\|f\|_\lambda\Big) \quad \text{for } t \in [a, a+\omega], \tag{2.2}$$

$$G_a(f)(t) \ge (t-a)(a+\omega-t)\Delta_\omega(\lambda)\|f\|_\lambda \quad \text{for } t \in [a, a+\omega]\,. \tag{2.3}$$

Proof. By Hölder's inequality, we have

$$\begin{aligned}\int_a^t f^\lambda(s)\,\mathrm{d}s &= \int_a^t \big[(s-a)f(s)\big]^\lambda (s-a)^{-\lambda}\,\mathrm{d}s\\ &\le \Big(\frac{1-\lambda}{1-2\lambda}\Big)^{1-\lambda}(t-a)^{1-2\lambda}\Big(\int_a^t (s-a)f(s)\,\mathrm{d}s\Big)^\lambda \quad \text{for } t\in[a,a+\omega].\end{aligned}$$

Hence,

$$\int_a^t (s-a)f(s)\,\mathrm{d}s \ge \Big(\frac{1-2\lambda}{1-\lambda}\Big)^{\frac{1-\lambda}{\lambda}}(t-a)^{-\frac{1-2\lambda}{\lambda}}\Big(\int_a^t f^\lambda(s)\,\mathrm{d}s\Big)^{1/\lambda}$$

for $t\in[a,a+\omega]$. Analogously,

$$\int_t^{a+\omega}(a+\omega-s)f(s)\,\mathrm{d}s \ge \Big(\frac{1-2\lambda}{1-\lambda}\Big)^{\frac{1-\lambda}{\lambda}}(a+\omega-t)^{-\frac{1-2\lambda}{\lambda}}\Big(\int_t^{a+\omega} f^\lambda(s)\,\mathrm{d}s\Big)^{1/\lambda}.$$

Consequently,

$$\begin{aligned}G_a(f)(t) &\ge \Big(\frac{1-2\lambda}{1-\lambda}\Big)^{\frac{1-\lambda}{\lambda}}(t-a)(a+\omega-t)\Big[\frac{1}{(t-a)^{\frac{1-\lambda}{\lambda}}}\Big(\int_a^t f^\lambda(s)\,\mathrm{d}s\Big)^{1/\lambda}\\ &\quad + \frac{1}{(a+\omega-t)^{\frac{1-\lambda}{\lambda}}}\Big(\int_t^{a+\omega} f^\lambda(s)\,\mathrm{d}s\Big)^{1/\lambda}\Big]\\ &\ge \Big(\frac{1-2\lambda}{\omega(1-\lambda)}\Big)^{\frac{1-\lambda}{\lambda}}(t-a)(a+\omega-t)\\ &\quad \times\Big[\Big(\int_a^t f^\lambda(s)\,\mathrm{d}s\Big)^{1/\lambda} + \Big(\int_t^{a+\omega} f^\lambda(s)\,\mathrm{d}s\Big)^{1/\lambda}\Big] \quad \text{for } t\in\,]a,a+\omega[\,.\end{aligned} \tag{2.4}$$

On the other hand, it is clear that

$$x^{1/\lambda} + (A-x)^{1/\lambda} \ge \frac{1}{2^{\frac{1-\lambda}{\lambda}}}\,A^{1/\lambda} \quad \text{for } x\in[0,A]. \tag{2.5}$$

Estimate (2.3) now follows from (2.4) in view of (2.5).

In the same way one can show that

$$H_a(f)(t) \ge (t-a)(a+\omega-t)\Delta_\omega(\lambda)\|f\|_\lambda \quad \text{for } t\in[a,a+\omega], \tag{2.6}$$

where

$$\begin{aligned}H_a(f)(t) &:= (a+\omega-t)\int_a^t (t-s)f(s)\,\mathrm{d}s\\ &\quad + (t-a)\int_t^{a+\omega}(s-t)f(s)\,\mathrm{d}s \quad \text{for } t\in[a,a+\omega].\end{aligned}$$

By direct calculations one can easily verify that

$$G_a(f)(t) = (t-a)(a+\omega-t)\|f\|_1 - H_a(f)(t) \quad \text{for } t\in[a,a+\omega].$$

Hence, in view of (2.6), we get (2.2). □

3. Proof of main result

Proof of Theorem 1.3. Extend the function p periodically and denote it by the same letter. Suppose that $[p]_- \not\equiv 0$ since otherwise it is known (see, e. g., Theorem 1.2) that $p \in V^-(\omega)$. In view of (1.4) and [1, Theorem 1.2], we have that $-[p]_- \in D(\omega)$. Hence, by virtue of Proposition 2.2, the inclusion $-[p]_- \in V^+(\omega)$ holds as well. Denote by γ a solution of the problem

$$\gamma'' = -[p(t)]_- \gamma + [p(t)]_+ ; \quad \gamma(0) = \gamma(\omega), \quad \gamma'(0) = \gamma'(\omega). \tag{3.1}$$

In view of (1.5), it is clear that $[p]_+ \not\equiv 0$ and consequently, by Proposition 2.3, we have

$$\gamma(t) > 0 \quad \text{for } t \in [0, \omega].$$

It is also evident that $\gamma \not\equiv Const$. Now we show that

$$\gamma(t) > 1 \quad \text{for } t \in [0, \omega]. \tag{3.2}$$

Put

$$m := \min\big\{\gamma(t) : t \in [0, \omega]\big\}, \quad M := \max\big\{\gamma(t) : t \in [0, \omega]\big\}.$$

Extend the function γ periodically and denote it by the same letter. Then there exists $a \in [0, \omega[$ such that

$$\gamma(a) = m, \quad \gamma(a + \omega) = m.$$

It is cleat that the function γ is a solution of the Dirichlet problem

$$\gamma'' = -[p(t)]_- \gamma + [p(t)]_+ ; \quad \gamma(a) = m, \quad \gamma(a + \omega) = m\,. \tag{3.3}$$

By direct calculations one can easily verify that

$$\gamma(t) = m + \frac{1}{\omega}\, G_a([p]_- \gamma)(t) - \frac{1}{\omega}\, G_a([p]_+)(t) \quad \text{for } t \in [a, a + \omega], \tag{3.4}$$

where G_a is defined by (2.1). By Proposition 2.5, we obtain

$$\begin{aligned} G_a([p]_- \gamma)(t) &\le M G_a([p]_-)(t) \\ &\le M(t-a)(a+\omega-t)\Big(\|[p]_-\|_1 - \Delta_\omega(\lambda)\|[p]_-\|_\lambda\Big) \quad \text{for } t \in [a, a+\omega] \end{aligned}$$

and

$$G_a([p]_+)(t) \ge \Delta_\omega(\lambda)(t-a)(a+\omega-t)\|[p]_+\|_\lambda \quad \text{for } t \in [a, a+\omega].$$

Hence, from (3.4) it follows that

$$\gamma(t) \le m + \frac{(t-a)(a+\omega-t)}{\omega}\Big(M\Big(\|[p]_-\|_1 - \Delta_\omega(\lambda)\|[p]_-\|_\lambda\Big) - \Delta_\omega(\lambda)\|[p]_+\|_\lambda\Big)$$

for $t \in [a, a+\omega]$. Taking now into account that $\gamma \not\equiv Const.$ we get from the latter inequality that

$$M\Big(\|[p]_-\|_1 - \Delta_\omega(\lambda)\|[p]_-\|_\lambda\Big) - \Delta_\omega(\lambda)\|[p]_+\|_\lambda > 0$$

and consequently

$$\gamma(t) < m + \frac{\omega}{4}\big(M\Big(\|[p]_-\|_1 - \Delta_\omega(\lambda)\|[p]_-\|_\lambda\Big) - \Delta_\omega(\lambda)\|[p]_+\|_\lambda\big) \tag{3.5}$$

for $t \in [a, a+\omega] \setminus \{t_0\}$, where $t_0 = a + \frac{\omega}{2}$.

In view of (1.5), it is clear that

$$m + \frac{\omega}{4}\Big(M\Big(\|[p]_-\|_1 - \Delta_\omega(\lambda)\|[p]_-\|_\lambda\Big) - \Delta_\omega(\lambda)\|[p]_+\|_\lambda\Big)$$

$$
\begin{aligned}
&= m - 1 + 1 - \frac{\omega}{4}\,\Delta_\omega(\lambda)\|[p]_+\|_\lambda + \frac{\omega}{4}\,M\Big(\|[p]_-\|_1 - \Delta_\omega(\lambda)\|[p]_-\|_\lambda\Big)\\
&\le m - 1 + \frac{\omega}{4}\,M\Big(\|[p]_-\|_1 - \Delta_\omega(\lambda)\|[p]_-\|_\lambda\Big)\\
&\quad + \frac{\|[p]_+\|_1}{\|[p]_-\|_1}\Big(1 - \frac{\omega}{4}\,\|[p]_-\|_1 + \frac{\omega}{4}\,\Delta_\omega(\lambda)\|[p]_-\|_\lambda\Big).
\end{aligned}
$$

From (3.5) it follows that

$$
\gamma(t) < m - 1 + \frac{\|[p]_+\|_1}{\|[p]_-\|_1} + \frac{\omega}{4}\Big(M - \frac{\|[p]_+\|_1}{\|[p]_-\|_1}\Big)\Big(\|[p]_-\|_1 - \Delta_\omega(\lambda)\|[p]_-\|_\lambda\Big) \tag{3.6}
$$

for $t \in [a, a+\omega] \setminus \{t_0\}$. On the other hand, (3.5) implies

$$
m \ge M\Big(1 - \frac{\omega}{4}\Big(\|[p]_-\|_1 - \Delta_\omega(\lambda)\|[p]_-\|_\lambda\Big)\Big) + \frac{\omega}{4}\,\Delta_\omega(\lambda)\|[p]_+\|_\lambda\,. \tag{3.7}
$$

From (3.1) it follows that

$$
\int_0^\omega [p(s)]_+\,\mathrm{d}s = \int_0^\omega [p(s)]_-\gamma(s)\,\mathrm{d}s \tag{3.8}
$$

and consequently

$$
M \ge \frac{\|[p]_+\|_1}{\|[p]_-\|_1}\,.
$$

If $M > \frac{\|[p]_+\|_1}{\|[p]_-\|_1}$ then, in view of (1.4) and (1.5), inequality (3.7) implies that $m > 1$ and consequently, (3.2) holds. Let now $M = \frac{\|[p]_+\|_1}{\|[p]_-\|_1}$. Then, in view of (3.6), we have

$$
\gamma(t) < m - 1 + \frac{\|[p]_+\|_1}{\|[p]_-\|_1} \quad \text{for } t \in [a, a+\omega] \setminus \{t_0\},
$$

which, together with (3.8) and the condition $[p]_- \not\equiv 0$, imply

$$
\|[p]_+\|_1 < (m-1)\|[p]_-\|_1 + \|[p]_+\|_1.
$$

Hence, $m > 1$. Thus, we have proved that (3.2) holds.

Now it follows from (3.1), in view of (3.2), that the function γ satisfies conditions of Propositions 2.4 and therefore, $p \in V^-(\omega)$. □

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