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MACLAURIN SERIES FOR $\sin_p$ WITH p AN INTEGER GREATER THAN 2

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Abstract. We find an explicit formula for the coefficients of the generalized Maclaurin series for $\sin_p$ provided $p > 2$ is an integer. Our method is based on an expression of the n-th derivative of $\sin_p$ in the form

$$\sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{p-1}(x) \cos_p^{2-p}(x)\,, \quad x \in (0, \frac{\pi_p}{2}),$$

where $\cos_p$ stands for the first derivative of $\sin_p$. The formula allows us to compute the nonzero coefficients

$$\alpha_n = \frac{\lim_{x\to 0+} \sin_p^{(np+1)}(x)}{(np+1)!}\,.$$

1. Introduction

Let us consider initial value problem

$$\begin{gathered} -(|u'|^{p-2}u')' - (p-1)|u|^{p-2}u = 0\,, \\ u(0) = 0\,, \quad u'(0) = 1\,, \end{gathered} \tag{1.1}$$

where $p > 1$ is a given parameter and $u\colon \mathbb{R} \to \mathbb{R}$ is a function such that $u \in C^1(\mathbb{R})$ and $|u'|^{p-2}u' \in C^1(\mathbb{R})$. It is known that the solution of (1.1) exists and is unique (see Elbert [9]). Since the pioneering work of del Pino, Elgueta and Manásevich [8], this solution is usually denoted by $\sin_p$. Note that it generalizes the *sine* function which is the unique solution of (1.1) for $p = 2$. Moreover, the function $\sin_p$ also satisfies the generalized trigonometric identity

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1\,, \quad x \in \mathbb{R}\,, \tag{1.2}$$

where $\cos_p(x) := \frac{\mathrm{d}}{\mathrm{d}x} \sin_p(x)$, which resembles the classical trigonometric identity for $p = 2$. We also define

$$\pi_p := 2\int_0^1 \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s = \frac{2\pi}{p\sin(\pi/p)}.$$

Let us note that the function $\sin_p$ is odd, $2\pi_p$-periodic, and $\sin_p(x) = \sin_p(\pi_p - x)$ (see, e.g., [9]). These properties are frequently used when the function $\sin_p$ is

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evaluated numerically. In fact, any evaluation of $\sin_p$ at an arbitrary point $x \in \mathbb{R}$ can be reduced to an evaluation of $\sin_p$ at a point in the interval $[0, \pi_p/2]$.

It turns out that the system of functions $\{\sin_p(k\pi_p\, x)\}_{k=1}^{+\infty}$ has applications in approximation theory, see Binding et al. [4] for pioneering work in this direction. Indeed, there exists $p_0 > 1$ such that, for $p > p_0$, $\{\sin_p(k\pi_p\, x)\}_{k=1}^{+\infty}$ forms a Riesz basis of $L^2(0,1)$ and a Schauder basis of $L^r(0,1)$ for any $1 < r < +\infty$. The approach from [4] was corrected and improved by Bushell and Edmunds [7] where the value p_0 was established as the solution of the transcendental equation

$$\frac{2\pi}{p_0 \sin(\pi/p_0)} = \frac{2\pi^2}{\pi^2 - 8}\,.$$

Boulton and Lord [5] use the basis $\{\sin_p(k\pi_p\, x)\}_{k=1}^{+\infty}$ in their numerical implementation of the Galerking method for finding an approximate solution to the boundary-initial value problem

$$\begin{gathered} \frac{\partial u}{\partial t}(x,t) - \frac{\partial}{\partial x}\Big(\big|\frac{\partial u}{\partial x}(x,t)\big|^{p-2}\frac{\partial u}{\partial x}(x,t)\Big) = g(x) \\ u(x,0) = 0\,, \quad x \in (0,1)\,, \\ u(0,t) = u(1,t) = 0 \quad t > 0\,, \end{gathered} \tag{1.3}$$

where $g \in L^2(0,1)$. It appears that this choice of basis leads to very accurate results using only few terms of this basis. However, a main drawback of the Galerkin method in [5] is the evaluation of the values of the function $\sin_p$ on $[0, \pi_p/2]$. In [5], the inverse function of $\sin_p$,

$$\operatorname{arcsin}_p(x) := \int_0^x \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s\,, \quad x \in [0,1]\,, \tag{1.4}$$

is used for that purpose. The function $\sin_p$ on $[0, \pi_p/2]$ is then evaluated using numerical inverse of the function arcsin_p, which is a very time consuming process. Since the problem (1.3) and its generalizations appear in various applications, see e.g. Smreker [23] (bulding of wells), Leibenson [15] (extraction of oil and natural gas), Wilkins [24] (bulding of rock-fill dams), Aronsson et al. [1], Evans et al. [10] (sandpile growth), Kuijper [13] (image analysis), and Bermejo et al. [3] (climatology), it is important to find a more efficient numerical implementations of $\sin_p$. Last but not least, the generalized Prüfer transform using $\sin_p$ and its derivative appears to be a very efficient theoretical tool in studying various initial and/or boundary value problems for quasilinears equation of the type (or some of its generalization)

$$-(|u'|^{p-2}u')' - q(x)|u|^{p-2}u = f(x)$$

(under various conditions on q and f) see, e.g., [9], Reichel and Walter [21], and/or Benedikt and Girg [2]. In Brown and Reichel [6], a numerical method based on the Prüfer transform was proposed. Again the main drawback the method was the lack of an efficient numerical implementation of $\sin_p$. To address the issue in this paper we obtain explicit formulas for coefficients of the Maclaurin series of $\sin_p$. This is very difficult task in general and we are not able to deal with this problem for all $p > 1$. As a starting point for further research in this direction, we provide such formulas for any integer p bigger than 2. Let us note that even this partial result can already be used in practical applications, since (1.3) with $p \to +\infty$ is considered as a model for sandpile growth (see [1] and [10] for more details).

More precisely, our goal is to find Maclaurin series for $\sin_p$ provided p is even and generalized Maclaurin series for $\sin_p$ provided p is odd. Generalized Maclaurin series is defined as

$$\sum_{n=0}^{+\infty} \alpha_n x|x|^{rn}, \quad r \geq 1 \,.$$

Peetre [20] conjectured that the radius of convergence of generalized Maclaurin series for $\sin_p$ is $\pi_p/2$ for any $p > 1$. Local convergence of generalized Maclaurin series was studied in Paredes and Uchiyama [19]. Peetre's conjecture [20] was proved in Girg and Kotrla [11] for when $p > 2$ is an integer. It remains to find the coefficients of the (generalized) Maclaurin series. One can employ (1.4) and follow the ideas presented in Lang and Edmunds [14]. Since

$$\operatorname{arcsin}_p(x) = \int_0^x \frac{1}{(1-s^p)^{1/p}} \, \mathrm{d}s = x \, {}_2F_1(\frac{1}{p}, \frac{1}{p}, 1 + \frac{1}{p}; x^p), \quad x \in [0,1) \,,$$

where ${}_2F_1(a, b, c; z)$ is Gauss's hypergeometric function,

$$\operatorname{arcsin}_p(x) = \sum_{k=0}^{+\infty} \frac{\Gamma(k+\frac{1}{p})}{(kp+1)\Gamma(\frac{1}{p})} \frac{x^{kp+1}}{k!} \,, \tag{1.5}$$

where Γ stands for the gamma function. We can obtain desired coefficients using the well-known procedure for inverting power series (see, e.g., Morse and Feshbach [18, p. 411 - 413]). Our aim is to derive the coefficients independently of the inverse function. It was shown in Girg and Kotrla [12] that the nonzero coefficients correspond only to the monomials x^{kp+1}, $k \in \mathbb{N}$. Then

$$\sin_p(x) = \sum_{n=0}^{+\infty} \frac{\sin_p^{(np+1)}(0)}{(np+1)!} x^{np+1} \quad x \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2}) \,,$$

for p even. In addition, it was proved in [12] that the series

$$\sum_{n=0}^{+\infty} \frac{\lim_{x\to 0+} \sin_p^{(np+1)}(x)}{(np+1)!} x^{np+1}$$

coincides on $[0, \pi_p/2)$ with the series obtained by formal inversion of (1.5) provided p odd. Hence, by the oddness of $\sin_p$,

$$\sin_p(x) = \sum_{n=0}^{+\infty} \frac{\lim_{x\to 0+} \sin_p^{(np+1)}(x)}{(np+1)!} x|x|^{np} \,, \quad x \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2}).$$

It remains then to find an explicit formula for

$$\alpha_n := \frac{1}{(np+1)!} \lim_{x\to 0+} \sin_p^{(np+1)}(x), \quad p \in \mathbb{N}, \ p > 2 \,.$$

Notation: In the presented paper, the symbol $\prod$ represents the product of a (possibly finite) sequence of terms as usual. In addition, we define

$$\prod_{i=j_1}^{j_2} b_i = 1$$

for any sequence b_i provided $j_1 = j_2 + 1$.

Theorem 1.1. *Let $p > 2$ be an integer and*

$$\sin_p(x) = \sum_{n=0}^{+\infty} \alpha_n x|x|^{np}, \quad x \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2}). \tag{1.6}$$

Then $\alpha_0 = 1$, $\alpha_1 = -\frac{1}{p(p+1)}$, and for $n \geq 2$,

$$\begin{aligned}
\alpha_n = \frac{(-1)^n}{(np+1)!} & \sum_{\substack{i_1=1\\ i_1\neq p-1}}^{p} \sum_{\substack{i_2=i_1+1\\ i_2\neq 2p-1}}^{2p} \dots \\
& \sum_{\substack{i_{n-1}=i_{n-2}+1\\ i_{n-1}\neq (n-1)p-1}}^{(n-1)p} \Big[\prod_{m_1=1}^{i_1-1} (p-1-(m_1-1))\Big]\big(1-(p-1-(i_1-1))\big) \\
& \times \Big[\prod_{m_2=i_1+1}^{i_2-1} (2(p-1)-(m_2-2))\Big](1-(2(p-1)-(i_2-2)))\dots \\
& \times \Big[\prod_{m_{n-1}=i_{n-2}+1}^{i_{n-1}-1} ((n-1)(p-1)-(m_{n-1}-(n-1)))\Big]\big(1-((n-1) \\
& \times (p-1)-(i_{n-1}-(n-1)))\big)[n(p-1)-(i_{n-1}-n+1)]!
\end{aligned} \tag{1.7}$$

The proof of Theorem 1.1 is based on a method of rewriting higher derivatives of $\sin_p$ introduced in [11]. The method is described again in Section 2 for the convenience of the reader. Theorem 1.1 is proved in Section 3.

Let us note that the above-mentioned definitions of $\sin_p$ and $\cos_p$ are not the only ones found in the literature (see, e.g., Lindqvist [16]).

2. Higher order derivatives

Let us state some basic notation from formal languages.

Definition 2.1. (Salomaa and Soittola [22], I.2, p. 4, and/or Manna [17], p. 2–3, p. 47, and p. 78) An *alphabet* (denoted by V) is a finite nonempty set of letters. A *word* (denoted by w) over an alphabet V is a finite string of zero or more letters from the alphabet V. The word consisting of zero letters is called the *empty word*. The set of all words over an alphabet V is denoted by V^* and the set of all nonempty words over an alphabet V is denoted by V^+. For strings w_1 and w_2 over V, their juxtaposition w_1w_2 is called *catenation* of w_1 and w_2, in operator notation $\mathrm{cat} : V^* \times V^* \to V^*$ and $\mathrm{cat}(w_1, w_2) = w_1w_2$. We also define the length of the word w, in operator notation $\mathrm{len} : V^* \to \mathbb{N} \cup \{0\}$, which for a given word w yields the number of letters in w when each letter is counted as many times as it occurs in w. We also use *reverse function* $\mathrm{rev} : V^* \to V^*$ which reverses the order of the letters in any word w (see [17, p. 47, p. 78]).

We consider the alphabet $V = \{0, 1\}$ and the set of all nonempty words V^+. Thus words in V^+ are, e.g.,

$$“\texttt{0}”, “\texttt{1}”, “\texttt{01}”, “\texttt{10}”, “\texttt{11}” \cdots .$$

For instance, cat(“1110”, “011”) = “1110011”, and

$$\mathrm{rev}(“\texttt{010011000}”) = “\texttt{000110010}”, \ \mathrm{len}(“\texttt{010011000}”) = 9 .$$

Table 1. Differentiability of $\sin_p(x)$

p, k	x in $(0,\pi_p/2)$	$(-\pi_p/2,\pi_p/2)$	$\mathbb{R}$
$p=2$	C^∞	C^∞	C^∞
$p=2k$, $k\in\mathbb{N}\setminus\{1\}$	C^∞	C^∞	C^1
$p=2k+1$, $k\in\mathbb{N}$	C^∞	C^p	C^1
$p\in\mathbb{R}\setminus\mathbb{N}$, $p>2$	C^∞	$C^{\lceil p\rceil}$	C^1
$p\in(1,2)$	C^∞	C^2	C^2

Let $m\in\mathbb{N}$, $k\in\mathbb{N}\cup\{0\}, 0\le k\le 2^{m-2}-1$ and $(k)_{2,n-2}$ be the string of bits of length $m-2$ which represents binary expansion of k (it means, e.g., for $k=3$ and $m=5$, $(3)_{2,5-2}=$ "`011`").

The differentiability of $\sin_p(x)$ at $x=0$ was studied in [11] leading to the results in Table 1.

In particular, $\sin_p(\cdot)\in C^\infty(0,\pi_p/2)$. Let

$$T:=\{a\sin_p^q(\cdot)\cos_p^{1-q}(\cdot):\ a,q\in\mathbb{R}\}\,,$$

and $\mathrm{D_s}\colon T\to T$ and $\mathrm{D_c}\colon T\to T$ be defined as follows:

$$\mathrm{D_s}\,a\sin_p^q(\cdot)\cos_p^{1-q}(\cdot)=\begin{cases}aq\sin_p^{q-1}(\cdot)\cos_p^{1-(q-1)}(\cdot)\,, & q\neq 0\,,\\ 0\,, & q=0\,,\end{cases}\tag{2.1}$$

and

$$\mathrm{D_c}\,a\sin_p^q(\cdot)\cos_p^{1-q}(\cdot)=\begin{cases}-a(1-q)\sin_p^{q+p-1}(\cdot)\cos_p^{1-(q+p-1)}(\cdot)\,, & q\neq 1\,,\\ 0\,, & q=1\,.\end{cases}\tag{2.2}$$

Finally, we define $\mathrm{D}_{k,m}$ in two steps.

Step 1 We create an ordered $(m-2)$-tuple $d_{k,m-2}\in\{\mathrm{D_s},\mathrm{D_c}\}^{m-2}$ (cartesian product of sets $\{\mathrm{D_s},\mathrm{D_c}\}$ of length $m-2$) from $\operatorname{rev}((k)_{2,m-2})$ such that for $1\le i\le m-2$, $d_{k,m-2}$ contains $\mathrm{D_s}$ on the i-th position if $\operatorname{rev}((k)_{2,n-2})$ contains "`0`" on the i-th position, and $d_{k,m}$ contains $\mathrm{D_c}$ on the i-th position if $\operatorname{rev}((k)_{2,m-2})$ contains "`1`" on the i-th position (it means, e.g., for $k=3$, and $m=5$, we obtain $d_{3,5-2}=(\mathrm{D_c},\mathrm{D_c},\mathrm{D_s})$).

Step 2 We define $\mathrm{D}_{k,m}$ as the composition of operators $\mathrm{D_s},\mathrm{D_c}$ in the order they appear in the ordered $m-2$-tuple $d_{k,m-2}$ (it means, e.g., for $k=3$, and $m=5$, we obtain $\mathrm{D}_{3,5}=(\mathrm{D_c}\circ\mathrm{D_c}\circ\mathrm{D_s})$).

Let us point out that it is possible to recover the index k from the positions of $\mathrm{D_c}$ in $\mathrm{D}_{k,m}$. We will denote by $j(k)\ge 0$ the number of $\mathrm{D_c}$ in $\mathrm{D}_{k,m}$ and, if $j(k)\neq 0$, we denote by $i_1,i_2,\dots,i_{j(k)}$ its positions counted from back (i.e., in the order of application of $\mathrm{D_s}$ and/or $\mathrm{D_c}$). Then

$$k=2^{m-2-(i_1-1)}+2^{m-2-(i_2-1)}+\ldots+2^{m-2-(i_{j(k)}-1)}\,.\tag{2.3}$$

If $j(k)=0$, $k=0$.

Definition 2.1 and the definition of $\mathrm{D}_{k,m}$ are taken from [11] in almost unchanged form for the convenience of the reader who is not familiar with our previous work. However, the rewriting diagrams in [11], where the construction of $\mathrm{D}_{k,m}$ is visualized, are not included here.

It follows from the first derivative of the p-trigonometric identity (1.2) that

$$\sin_p^{(2)}(x) = -\sin_p^{p-1}(x)\cos_p^{2-p}(x)\,, \quad x \in (0, \frac{\pi_p}{2})\,. \tag{2.4}$$

Note that $\sin_p(x) > 0$ and $\cos_p(x) > 0$ for $x \in (0, \pi_p/2)$. Hence, we can use $\mathrm{D}_{k,n}$ to express

$$\begin{aligned} \sin_p^{(m)}(x) &= \sum_{k=0}^{2^{m-2}-1} \mathrm{D}_{k,m} \sin_p^{(2)}(x) \\ &= \sum_{k=0}^{2^{m-2}-1} \mathrm{D}_{k,m}(-1)\sin_p^{p-1}(x)\cos_p^{2-p}(x)\,, \quad x \in (0, \frac{\pi_p}{2})\,, \end{aligned} \tag{2.5}$$

for $m > 2$ be a positive integer. Let us explain the procedure for $m = 3$ at first. In that case

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}(-1)\sin_p^{p-1}(x)\cos_p^{2-p}(x) \\ &= (-1)(p-1)\sin_p^{p-2}(x)\cos_p^{3-p}(x) \\ &\quad + (-1)(2-p)\sin_p^{p-1}(x)\cos_p^{1-p}(x)\sin_p^{(2)}(x) \\ &= (-1)(p-1)\sin_p^{p-2}(x)\cos_p^{3-p}(x) \\ &\quad + (-1)(1-(p-1))\sin_p^{p-1+p-1}(x)\cos_p^{1-(p-1+p-1)}(x) \\ &= \mathrm{D_S}\sin_p^{(2)}(x) + \mathrm{D_c}\sin_p^{(2)}(x) \end{aligned}$$

for any $x \in (0, \pi_p/2)$ by the definition of $\mathrm{D_S}$ and $\mathrm{D_c}$. The proof of (2.5), which proceeds by induction, can be found in [11, Lemma 4.5, p. 110].

There are two special cases in composing the symbolic operators for $p \in \mathbb{N}, p > 2$, which can be used for reducing of terms in (2.5).

Case 1 Assume that there exists $k \in \mathbb{N} \cup \{0\}$, $k \le 2^{m-2} - 1$ such that

$$\mathrm{D}_{k,m}\sin_p^{(2)}(\cdot) = a\sin_p(\cdot)\cos_p^0(\cdot)\,. \tag{2.6}$$

The further application of $\mathrm{D_c}$ is meaningless since it produce 0 by (2.2). The situation (2.6) occurs, e.g., after $p-2$ applications of $\mathrm{D_S}$ on $\sin_p^{(2)}(\cdot)$.

Case 2 If there exists $k \in \mathbb{N}$, $k \le 2^{m-2} - 1$, such that

$$\mathrm{D}_{k,m}\sin_p^{(2)}(\cdot) = a\sin_p^0(\cdot)\cos_p^1(\cdot)\,, \tag{2.7}$$

then the application of $\mathrm{D_s}$ produces 0, see (2.1). The situation (2.7) occurs, e.g., after $p-1$ applications of $\mathrm{D_S}$ on $\sin_p^{(2)}(\cdot)$. This is the essential argument in the proof that the exponent q is always nonnegative, see [11, Lemma 4.6, p.113] for more details.

3. Proof of main result

Proof of Theorem 1.1. It follows from [12, Theorem 6, p. 3] that

$$\alpha_n = \frac{1}{(np+1)!}\lim_{x\to 0+}\sin_p^{(np+1)}(x) \tag{3.1}$$

for p odd, and it is obvious that (3.1) is valid for p even, since $\sin_p(\cdot)$ belongs to $C^\infty(-\pi_p/2, \pi_p/2)$ in this case. We obtain $\alpha_0 = \lim_{x\to 0+}\cos_p(x) = 1$ for $p \in \mathbb{N}$,

$p > 2$. Let $n \in \mathbb{N}$ and $x \in (0, \pi_p/2)$. By [11, Lemma 4.5, p. 110]

$$\begin{aligned}\sin_p^{(np+1)}(x) &= \sum_{k=0}^{2^{np-1}-1} - \mathrm{D}_{k,np+1} \sin_p^{p-1}(x) \cos_p^{2-p}(x)\\ &= \sum_{k=0}^{2^{np-1}-1} a_{k,np+1} \sin_p^{q_{k,np+1}}(x) \cos_p^{1-q_{k,np+1}}(x)\,,\end{aligned}$$

where $a_{k,np+1} \in \mathbb{R}$ and $q_{k,np+1} \in \mathbb{N} \cup \{0\}$. It follows that

$$\begin{aligned}\lim_{x\to 0+} \sin_p^{(np+1)}(x) &= \sum_{k=0}^{2^{np-1}-1} a_{k,np+1} \lim_{x\to 0+} \sin_p^{q_{k,np+1}}(x) \cos_p^{1-q_{k,np+1}}(x)\\ &= \sum_{\substack{k=0\\ q_{k,np+1}=0}}^{2^{np-1}-1} a_{k,np+1}\,.\end{aligned} \tag{3.2}$$

Our first aim is to describe $k \in \mathbb{N} \cup \{0\}$, $0 \le k \le 2^{np-1} - 1$ such that $q_{k,n} = 0$. We use the alphabet $V = \{0, 1\}$ introduced in Definition 2.1 for this purpose and we employ the formula

$$q_{k,np+1} = j(k)(p-1) + (np - 1 - j(k))(-1) + p - 1 \tag{3.3}$$

proved in [11, Lemma 4.5, p. 11)]. Let us recall that $j(k)$ is the number of occurrences of $\mathrm{D_c}$ in $\mathrm{D}_{k,np+1}$. It follows from the condition $q_{k,n} = 0$ that $j(k) = n-1$. Then $k = 0$ for $n = 1$ which implies

$$\begin{aligned}\lim_{x\to 0+} \sin_p^{(p+1)}(x) &= -\lim_{x\to 0+} \mathrm{D}_{0,p+1} \sin_p^{p-1}(x) \cos_p^{2-p}(x)\\ &= -\lim_{x\to 0+} (p-1)! \sin_p^{0}(x) \cos_p^{1}(x) = -(p-1)!\end{aligned} \tag{3.4}$$

by (2.1), the definition of $\mathrm{D_s}$. Substituting (3.4) into (3.1) we obtain

$$\alpha_1 = -\frac{1}{p(p+1)}\,.$$

We will assume $n \ge 2$ in the rest of the proof. Then

$$k = 2^{np-1-(i_1-1)} + 2^{np-1-(i_2-1)} + \ldots + 2^{np-1-(i_{n-1}-1)}$$

by (2.3). Moreover,

$$\forall s \in \mathbb{N},\ 1 \le s \le n-1:\ i_s \le sp\,. \tag{3.5}$$

Indeed, let there exist $s_0 \in \mathbb{N}$, $1 \le s_0 \le n-1$: $i_{s_0} > s_0 p$ and let

$$k_1 := \begin{cases} 0 & \text{for } s_0 = 1\,,\\ 2^{np-1-(i_1-1)} + 2^{np-1-(i_2-1)} + \ldots + 2^{np-1-(i_{s_0-1}-1)} & \text{for } s_0 \ge 2\,.\end{cases}$$

The binary expansion $(k_1)_{2,i_{s_0}-1}$ of k_1 defines $\mathrm{D}_{k_1,i_{s_0}+1}$ by the composition of the symbolic operators $\mathrm{D_s}$ and/or $\mathrm{D_c}$ taking the first $i_{s_0} - 1$ operators from $\mathrm{D}_{k,np+1}$ (in the order of its application). The exponent $q_{k_1,i_{s_0}+1}$ in $\mathrm{D}_{k_1,i_{s_0}+1} \sin_p^{(2)}(\cdot)$ satisfies

$$q_{k_1,i_{s_0}+1} = (s_0 - 1)(p-1) + (i_{s_0} - 1 - s_0 + 1)(-1) + p - 1 = s_0 p - i_{s_0} < 0$$

by (3.3) and the assumption $i_{s_0} > s_0 p$. Since $q_{k,np+1} \geq 0$ for any $n \in \mathbb{N} \cup \{0\}$ and all $0 \leq k \leq 2^{np-1} - 1$ provided $p > 1$ be an integer, we get the contradiction. Hence,

$$\alpha_n = \frac{1}{(np+1)!} \sum_{i_1=1}^{p} \sum_{i_2=i_1+1}^{2p} \cdots \sum_{i_{n-1}=i_{n-2}+1}^{(n-1)p} a_{k_0,np+1}\,, \tag{3.6}$$

where $k_0 = 2^{np-1-(i_1-1)} + 2^{np-1-(i_2-1)} + \ldots + 2^{np-1-(i_{n-1}-1)}$.

It remains to express $a_{k_0,np+1}$ as the polynomial in p. We will apply $\mathrm{D_s}$ and/or $\mathrm{D_c}$ on $\sin_p^{(2)}(\cdot)$ recursively. Let us denote by a_i the coefficient and q_i the exponent obtained by i steps of recursion. The base cases are $a_0 = -1$ and $q_0 = p - 1$ by (2.4) and inductive clauses are given by (2.1) and (2.2), i.e.,

$$a_{i+1} = \begin{cases} q_i \cdot a_i & \text{if } \mathrm{D_s} \text{ is applied}\,, \\ -(1-q_i)a_i & \text{if } \mathrm{D_c} \text{ is applied}\,, \end{cases} \tag{3.7}$$

and

$$q_{i+1} = \begin{cases} q_i - 1 & \text{if } \mathrm{D_s} \text{ is applied}\,, \\ q_i + p - 1 & \text{if } \mathrm{D_c} \text{ is applied}\,. \end{cases} \tag{3.8}$$

It follows from the definition of $\mathrm{D}_{k_0,np+1}$ that the operator $\mathrm{D_s}$ is applied in the first $i_1 - 1$ steps of recursion. It means that

$$a_{i_1-1} = -(p-1)(p-2)\cdots(p-1-(i_1-2)) \quad \text{and} \quad q_{i_1-1} = p-1-(i_1-1)\,.$$

by (2.1). Applying the operator $\mathrm{D_c}$ on the next position we have

$$\begin{gathered} a_{i_1} = -(p-1)(p-2)\cdots(p-1-(i_1-2))(-1)(1-(p-1-(i_1-1))), \\ q_{i_1} = 2(p-1) - (i_1-1)\,. \end{gathered}$$

Applying $i_2 - 1 - i_1$ times the operator $\mathrm{D_s}$ and we obtain

$$\begin{aligned} a_{i_2-1} = &-(p-1)(p-2)\cdots(p-1-(i_1-2))(-1)(1-(p-1-(i_1-1))) \\ &\times (2(p-1)-(i_1-1))\cdots(2(p-1)-(i_2-3)) \end{aligned}$$

and

$$q_{i_2-1} = 2(p-1) - (i_2-2)$$

(provided $i_2 > i_1 + 1$). The application of $\mathrm{D_c}$ leads to

$$\begin{aligned} a_{i_2} = &-(p-1)(p-2)\cdots(p-1-(i_1-2))(-1)(1-(p-1-(i_1-1))) \\ &\times (2(p-1)-(i_1-1))\cdots(2(p-1)-(i_2-3))(-1)(1-(2(p-1)-(i_2-2))) \end{aligned}$$

and

$$q_{i_2} = 3(p-1) - (i_2-2)\,.$$

It follows by the recursive application of $\mathrm{D_s}$ and/or $\mathrm{D_c}$ that

$$\begin{aligned} a_{i_{n-1}} = (-1)&\Big[\prod_{m_1=1}^{i_1-1}(p-1-(m_1-1))\Big](-1)\big(1-(p-1-(i_1-1))\big) \\ &\times \Big[\prod_{m_2=i_1+1}^{i_2-1}(2(p-1)-(m_2-2))\Big](-1)(1-(2(p-1)-(i_2-2)))\cdots \\ &\times \Big[\prod_{m_{n-1}=i_{n-2}+1}^{i_{n-1}-1}((n-1)(p-1)-(m_{n-1}-(n-1)))\Big] \end{aligned}$$

$$\times (-1)(1-((n-1)(p-1)-(i_{n-1}-(n-1))))$$

and

$$q_{i_{n-1}} = n(p-1)-(i_{n-1}-n+1)\,,$$

where i_{n-1} is the last position of $\mathrm{D_c}$. Since the remaining symbolic operators in $\mathrm{D}_{k_0,np+1}$ are $\mathrm{D_s}$ and $q_{k_0,np+1}=0$ by (3.2), we finally get

$$\begin{aligned}
&a_{k_0,np+1}\\
&= (-1)\Big[\prod_{m_1=1}^{i_1-1}(p-1-(m_1-1))\Big](-1)\big(1-(p-1-(i_1-1))\big)\\
&\quad\times\Big[\prod_{m_2=i_1+1}^{i_2-1}(2(p-1)-(m_2-2))\Big](-1)(1-(2(p-1)-(i_2-2)))\cdots\\
&\quad\times\Big[\prod_{m_{n-1}=i_{n-2}+1}^{i_{n-1}-1}((n-1)(p-1)-(m_{n-1}-(n-1)))\Big]\\
&\quad\times(-1)(1-((n-1)(p-1)-(i_{n-1}-(n-1))))\\
&\quad\times\Big[n(p-1)-(i_{n-1}-n+1)\Big]!
\end{aligned}\tag{3.9}$$

Substituting (3.9) into (3.6) we obtain desired formula (1.7). The positions $i_s = sp-1$ are excluded in (1.7) since it produce zero due to the terms $1-(s(p-1)-(i_s-s))$ in product (3.9) (see Case 1 in Section 2). □

4. Concluding remarks

Remark 4.1. The proof of Theorem 1.1 provides a procedure to generate any coefficient α_n, $n\ge 2$ of Maclaurin series (1.6) for $\sin_p$, when $p>2$ is an integer. It is convenient to generate all vectors $v\in\{0,1\}^{np-1}$ with exactly $n-1$ occurrences of "1"s, which satisfy condition (3.5), i.e.,

$$\forall s\in\mathbb{N},\ 1\le s\le n-1\ :\ i_s\le sp\,.$$

Let us note that i_s is the position of "1" in v. Then the recursions (3.7) with $a_0=-1$ and (3.8) with $q_0=p-1$ can to applied by all possible vectors v to obtain the coefficient $a_v\in\mathbb{R}$. Let us remind that zero and one means that $\mathrm{D_s}$ and $\mathrm{D_c}$ is applied, respectively, and the order of application $\mathrm{D_s}$ and/or $\mathrm{D_c}$ is reversed. Finally, the resulting coefficient α_n is given as sum of all a_v which is divided by $(np+1)!$.

Remark 4.2. The coefficients α_n, $n\ge 2$, can be also computed recursively by the formula

$$\begin{aligned}
\alpha_{n+1} &= (-1)\Big[\frac{(p-1)!}{((n+1)p+1)((n+1)p)\cdots(np+2)}\Big]\alpha_n\\
&\quad+\frac{(-1)^{n+1}}{((n+1)p+1)!}\sum_{\substack{i_1=1\\ i_1\ne p-1}}^{p}\ \sum_{\substack{i_2=i_1+1\\ i_2\ne 2p-1}}^{2p}\cdots\\
&\quad\sum_{i_{n-1}=i_{n-2}+1}^{np-\mathbf{2}}\Big[\prod_{m_1=1}^{i_1-1}(p-1-(m_1-1))\Big]\big(1-(p-1-(i_1-1))\big)
\end{aligned}$$

$$\times \Big[\prod_{m_2=i_1+1}^{i_2-1} (2(p-1)-(m_2-2)) \Big] (1-(2(p-1)-(i_2-2))) \cdots$$
$$\times \Big[\prod_{m_n=i_{n-1}+1}^{i_n-1} (n(p-1)-(m_n-n)) \Big]$$
$$\times (1-(n(p-1)-(i_n-n))))[n(p-1)-(i_n-n)]!$$

with $\alpha_1 = -1/(p(p+1))$.

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