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EXISTENCE AND UNIQUENESS OF THE GENERALIZED POISEUILLE SOLUTION FOR NONSTATIONARY MICROPOLAR FLOW IN AN INFINITE CYLINDER

MICHAL BENEŠ, IGOR PAŽANIN, MARKO RADULOVIĆ

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Abstract. We consider the nonstationary motion of a viscous incompressible micropolar fluid having a prescribed flux in an infinite cylinder. The global existence and uniqueness result for the generalized time-dependent Poiseuille solution is provided by means of semidiscretization in time and by passing to the limit from discrete approximations.

1. Introduction

It is well known that the Navier-Stokes model has a serious limitation because it does not take into account the microstructure of the fluid. Among various non-Newtonian models aiming to overcome this issue, micropolar fluids (proposed by Eringen [5]) seems to be the most appropriate. The mathematical model of micropolar fluid is based on the introduction of a new vector field, the angular velocity field of rotation of particles, taking into account the microrotation of the fluid particles. Consequently, one new vector equation is added to the Navier-Stokes system resulting from the conservation of the angular momentum. The coupled nonlinear system of PDEs obtained in such way is suitable for describing the behavior of numerous real fluids (e.g. liquid crystals, muddy fluids, polymeric suspensions, animal blood etc.) that cannot be represented by classical Navier-Stokes equations. For that reason, micropolar fluid flows have been extensively studied and one can find many results throughout the mathematical literature. Let us just mention that a comprehensive survey of the mathematical theory underlying the micropolar fluid model can be found in the monograph by Lukaszewicz [11].

In this article, we study a nonstationary flow of a micropolar fluid through an infinite cylinder with a prescribed flux. Our research has been inspired by the results on classical Newtonian flow provided by Pileckas [16]. More precisely, the existence of the standard nonstationary Poiseuille solution in an infinite cylinder $\Pi = \{x = (x', x_3) \in \mathbb{R}^3 \colon x_3 \in \mathbb{R},\ x' = (x_1, x_2) \in \sigma\}$ has been brought in [13] in Hölder spaces (see also [14] investigating the asymptotic behavior of the Poiseuille

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solution as $t \to \infty$). In [15], Pileckas has considered a *generalized time-dependent Poiseuille flow* in Π by assuming that the solution $(\mathbf{u}, p)$ has the form

$$\begin{gathered}\mathbf{u}(x,t) = (u_1(x',t), u_2(x',t), u_3(x',t)),\\ p(x,t) = \hat{p}(x',t) - q(t)x_3 + p_0(t),\end{gathered}$$

where $p_0(t)$ is an arbitrary function of time. The solvability of such problem in Sobolev spaces has been established by constructing the Galerkin approximations of the solution. Our goal here is to generalize this result for a micropolar setting, i.e. to prove the global existence and uniqueness result for a *generalized nonstationary micropolar Poiseuille solution*. In view of that, the paper is organized as follows. In the rest of this section we introduce the micropolar equations and suppose that the solution is of general micropolar Poiseuille form. We then decompose the problem, obtaining a classical 2D micropolar problem and a micropolar inverse problem. The existence of the 2D micropolar problem is addressed in Section 2, following [12]. In Section 3, we prove the existence of the micropolar inverse problem by semidiscretization in time, proving the existence of the discrete problem, deriving a-priori estimates for the discrete approximations, using the compactness method and treating the case for $T = \infty$. In Section 4, we address the existence and uniqueness of the solution to the original coupled problem for $T \in \langle 0, \infty]$. Finally, in the Appendix, for the sake of reader's convenience, we discuss the solvability of parabolic systems in Hilbert spaces.

To conclude the introduction part, let us provide few more bibliographic remarks. In [19], the author has proved the existence of weak solutions to the initial boundary value problem for incompressible micropolar fluids, in the absence of body forces and moments and with homogeneous Dirichlet boundary conditions. In [20], the existence and uniqueness of a global solution for micropolar fluid equations has been established with periodic boundary conditions and with external forces and moments independent of the longitudinal coordinate x_3. Quite recently, local-in-time existence and uniqueness of strong solutions for the incompressible micropolar fluid equations in bounded or unbounded domains of $\mathbb{R}^3$ has been shown in [4]. The micropolar Poiseuille solution has been employed in [18] for the purpose of studying the stationary micropolar Leray problem. Most recently, the asymptotic behavior of the (standard) nonstationary micropolar Poiseuille solution in a thin pipe has been investigated in [3] by the authors of this paper. Using the two-scale expansion method with respect to the pipe's thickness, the effective flow has been found and rigorously justified.

1.1. **Micropolar equations.** We consider an infinite cylinder $\Pi = \{x \in \mathbb{R}^3 \colon x_3 \in \mathbb{R},\ x' = (x_1, x_2) \in \sigma\}$, where σ is a bounded open set of class $\mathcal{C}^2$ in $\mathbb{R}^2$. We denote the Cartesian coordinates $x = ((x_1, x_2), x_3) \equiv (x', x_3)$, with x_3 being the direction coinciding with the axis of the cylinder. We consider the initial boundary value problem for the nonstationary micropolar fluid flow in an infinite cylinder Π:

$$\begin{gathered}\partial_t \mathbf{u} - (\nu + \nu_r)\Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 2\nu_r \operatorname{rot} \mathbf{w} + \mathbf{f},\\ \operatorname{div} \mathbf{u} = 0,\\ \partial_t \mathbf{w} - (c_a + c_d)\Delta \mathbf{w} + (\mathbf{u} \cdot \nabla)\mathbf{w} - (c_0 + c_d - c_a)\nabla \operatorname{div} \mathbf{w} + 4\nu_r \mathbf{w}\\ = 2\nu_r \operatorname{rot} \mathbf{u} + \mathbf{g},\end{gathered} \tag{1.1}$$

with the boundary and initial conditions

$$\mathbf{u}|_{\partial\Pi} = \mathbf{0}, \quad \mathbf{w}|_{\partial\Pi} = \mathbf{0} \tag{1.2}$$

and

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{w}(x,0) = \mathbf{w}_0(x) \tag{1.3}$$

along with the flux condition with the given flow rate $F(t)$,

$$\int_\sigma u_3(x',t)dx' = F(t). \tag{1.4}$$

Here $\mathbf{u}(x',x_3,t) = (u_1(x',x_3,t), u_2(x',x_3,t), u_3(x',x_3,t))$ stands for the velocity field, $\mathbf{w}(x',x_3,t) = (w_1(x',x_3,t), w_2(x',x_3,t), w_3(x',x_3,t))$ is the angular velocity of rotation of the fluid particles (the microrotation field), while $p(x',x_3,t)$ is the pressure. The positive constants are the Newtonian viscosity ν, the microrotation viscosity ν_r, while c_0, c_a and c_d are coefficients of angular viscosities. The external sources of linear and angular momentum are given with functions $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_2, g_3)$, respectively. Throughout the paper, we assume that the nonstationary solution of the problem (1.1)–(1.4) has the generalized Poiseuille form

$$\mathbf{u}(x,t) = (u_1(x',t), u_2(x',t), u_3(x',t)), \tag{1.5}$$

$$\mathbf{w}(x,t) = (w_1(x',t), w_2(x',t), w_3(x',t)), \tag{1.6}$$

$$p(x,t) = \hat{p}(x',t) - q(t)x_3 + p_0(t), \tag{1.7}$$

where $p_0(t)$ is an arbitrary function in t. We also assume that

$$\begin{gathered}
\mathbf{u}_0(x) = (u_{01}(x'), u_{02}(x'), u_{03}(x')),\\
\mathbf{w}_0(x) = (w_{01}(x'), w_{02}(x'), w_{03}(x')),\\
\mathbf{f}(x,t) = (f_1(x',t), f_2(x',t), f_3(x',t)),\\
\mathbf{g}(x,t) = (g_1(x',t), g_2(x',t), g_3(x',t))
\end{gathered}$$

are independent of x_3 and that it holds the necessary compatibility condition

$$\int_\sigma u_{03}(x')dx' = F(0).$$

To formulate the resulting problem in a more compact form, we introduce the following notation:

$$\begin{gathered}
\hat{\mathbf{u}}(x',t) = (u_1(x',t), u_2(x',t)), \quad \hat{\mathbf{u}}_0(x') = (u_{01}(x'), u_{02}(x')),\\
\hat{\mathbf{f}}(x',t) = (f_1(x',t), f_2(x',t)), \quad \omega(x',t) = w_3(x',t), \quad \omega_0(x') = w_{03}(x'),\\
v(x',t) = u_3(x',t), \quad v_0(x') = u_{03}(x'), \quad f(x',t) = f_3(x',t),\\
\hat{\mathbf{w}}(x',t) = (w_1(x',t), w_2(x',t)), \quad \hat{\mathbf{w}}_0(x') = (w_{01}(x'), \quad w_{02}(x')),\\
\hat{\mathbf{g}}(x',t) = (g_1(x',t), g_2(x',t)), \quad g(x',t) = g_3(x',t).
\end{gathered}$$

Further, from now on, we denote

$$\begin{gathered}
\operatorname{rot}_{x'} \boldsymbol{\phi} = \frac{\partial\phi_2}{\partial x_1} - \frac{\partial\phi_1}{\partial x_2}, \quad \operatorname{div}_{x'} \boldsymbol{\phi} = \frac{\partial\phi_1}{\partial x_1} + \frac{\partial\phi_2}{\partial x_2}, \quad \nabla^\perp_{x'}\phi = \Big(\frac{\partial\phi}{\partial x_2}, -\frac{\partial\phi}{\partial x_1}\Big),\\
\Delta_{x'}\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2}, \quad \nabla_{x'}\phi = \Big(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}\Big)
\end{gathered}$$

for any sufficiently smooth scalar function ϕ and a vector function $\boldsymbol{\phi} = (\phi_1, \phi_2)$.

Taking the generalized Poiseuille solution (1.5)–(1.7), plugging it into the system (1.1)–(1.4) and decomposing the obtained system of equations we obtain the following two problems set on the cross-section σ:

$$\begin{gathered}
\frac{\partial \hat{\mathbf{u}}}{\partial t} - (\nu+\nu_r)\Delta_{x'}\hat{\mathbf{u}} + (\hat{\mathbf{u}}\cdot\nabla_{x'})\hat{\mathbf{u}} + \nabla_{x'}\hat{p} = 2\nu_r\nabla_{x'}^{\perp}\omega + \hat{\mathbf{f}},\\
\nabla_{x'}\cdot\hat{\mathbf{u}} = 0,\\
\frac{\partial \omega}{\partial t} - (c_a+c_d)\Delta_{x'}\omega + (\hat{\mathbf{u}}\cdot\nabla_{x'})\omega + 4\nu_r\omega = 2\nu_r \operatorname{rot}_{x'}\,\hat{\mathbf{u}} + g,\\
\hat{\mathbf{u}}(x',t)|_{\partial\sigma} = \mathbf{0}, \quad \omega(x',t)|_{\partial\sigma} = 0,\\
\hat{\mathbf{u}}(x',0) = \hat{\mathbf{u}}_0(x'), \quad \omega(x',0) = \omega_0(x')
\end{gathered} \tag{1.8}$$

and

$$\begin{gathered}
\frac{\partial v}{\partial t} - (\nu+\nu_r)\Delta_{x'}v + (\hat{\mathbf{u}}\cdot\nabla_{x'})v - q(t) = 2\nu_r\operatorname{rot}_{x'}\,\hat{\mathbf{w}} + f,\\
\begin{aligned}
&\frac{\partial \hat{\mathbf{w}}}{\partial t} - (c_a+c_d)\Delta_{x'}\hat{\mathbf{w}} + (\hat{\mathbf{u}}\cdot\nabla_{x'})\hat{\mathbf{w}}\\
&- (c_0+c_d-c_a)\nabla_{x'}\operatorname{div}_{x'}\hat{\mathbf{w}} + 4\nu_r\hat{\mathbf{w}}\\
&= 2\nu_r\nabla_{x'}^{\perp}v + \hat{\mathbf{g}},
\end{aligned}\\
v|_{\partial\sigma} = 0, \quad \hat{\mathbf{w}}|_{\partial\sigma} = \mathbf{0},\\
v(x',0) = v_0(x'), \quad \hat{\mathbf{w}}(x',0) = \hat{\mathbf{w}}_0(x').
\end{gathered} \tag{1.9}$$

The system (1.9) is completed with the flux condition

$$\int_\sigma v(x',t)dx' = F(t). \tag{1.10}$$

1.2. **Basic notation and function spaces.** Vectors and vector functions are denoted by boldface letters. Unless specified otherwise, we use Einstein's summation convention for indices running from 1 to 2. Throughout the paper, we will always use positive constants C, c, c_1, c_2, $\dots$, which are not specified and may differ from line to line. Moreover, we suppose that $r, s, r' \in [1,\infty]$, where r' denotes the conjugate exponent to $r > 1$, $1/r + 1/r' = 1$. Let us introduce some functions spaces for functions defined on σ or $\sigma\times(0,T)$, $0 < T \le \infty$. $L^r(\sigma)$ denotes the usual Lebesgue space equipped with the norm $\|\cdot\|_{L^r(\sigma)}$ and $W^{k,r}(\sigma)$, $k \ge 0$ (k need not to be an integer, see [7]), denotes the usual Sobolev-Slobodecki space with the norm $\|\cdot\|_{W^{k,r}(\sigma)}$. Recall that $W^{0,r}(\sigma) := L^r(\sigma)$. Let E be the Banach space. By $L^r(0,T;E)$ we denote the Bochner space (see [1]). Further, $C([0,T];E)$ represents the space of continuous functions on the internal $[0,T]$, with values in the Banach space E, with the usual norm. Moreover, let $\tilde{V} := \{\boldsymbol{\phi} \in C_0^\infty(\sigma)^2;\ \boldsymbol{\phi} = (\phi_1,\phi_2),\ \operatorname{div}_{x'}\boldsymbol{\phi} = 0 \text{ in } \sigma\}$. Let the linear space V and H, respectively, be closures of $\tilde{V}$ in the norm of $W^{1,2}(\sigma)^2$ and $L^2(\sigma)^2$.

To simplify mathematical formulations we introduce the following notation:

$$a(\boldsymbol{\phi},\boldsymbol{\psi}) := \int_\sigma \frac{\partial\phi_i}{\partial x_j}\frac{\partial\psi_i}{\partial x_j}dx', \tag{1.11}$$

$$b(\boldsymbol{\phi},\boldsymbol{\psi},\boldsymbol{\varphi}) := \int_\sigma \phi_j\frac{\partial\psi_i}{\partial x_j}\varphi_i dx', \tag{1.12}$$

$$d(\boldsymbol{\phi},\psi,\varphi) := \int_\sigma \phi_j\frac{\partial\psi}{\partial x_j}\varphi dx', \tag{1.13}$$

$$((\boldsymbol{\phi}, \boldsymbol{\psi})) := \int_\sigma \phi_i \psi_i dx', \tag{1.14}$$

$$(\psi, \varphi) := \int_\sigma \psi \varphi dx'. \tag{1.15}$$

In (1.11)–(1.15) all functions $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, $\boldsymbol{\varphi}$, ψ and φ are regular enough, such that all integrals on the right-hand sides make sense.

2. Solvability of problem (1.8)

In this section we recall some well-known results concerning the existence, regularity and uniqueness for micropolar incompressible fluid flows in two-dimensional bounded domains. First, we introduce notions of weak solutions to the problem (1.8).

Definition 2.1. (i) Let $T \in (0, \infty)$ and suppose that

$$\hat{\mathbf{f}} \in L^2(0, T; H), \quad g \in L^2(0, T; L^2(\sigma)), \quad \hat{\mathbf{u}}_0 \in H, \quad \omega_0 \in L^2(\sigma).$$

By a weak solution of the problem (1.8) on $(0, T)$ we mean a pair $[\hat{\mathbf{u}}, \omega]$ such that

$$\hat{\mathbf{u}} \in L^2(0, T; V) \cap C([0, T]; H),$$
$$\omega \in L^2(0, T; W_0^{1,2}(\sigma)) \cap C([0, T]; L^2(\sigma))$$

and the system

$$\begin{aligned} &\frac{d}{dt}((\hat{\mathbf{u}}(t), \boldsymbol{\psi})) + (\nu + \nu_r) a(\hat{\mathbf{u}}(t), \boldsymbol{\psi}) + b(\hat{\mathbf{u}}(t), \hat{\mathbf{u}}(t), \boldsymbol{\psi}) \\ &= 2\nu_r((\nabla_{x'}^{\perp} \omega(t), \boldsymbol{\psi})) + ((\hat{\mathbf{f}}(t), \boldsymbol{\psi})) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\frac{d}{dt}(\omega(t), \varphi) + (c_a + c_d)((\nabla \omega(t), \nabla \varphi)) + d(\hat{\mathbf{u}}(t), \omega(t), \varphi) + 4\nu_r(\omega(t), \varphi) \\ &= 2\nu_r(\operatorname{rot}_{x'} \hat{\mathbf{u}}(t), \varphi) + (g(t), \varphi) \end{aligned} \tag{2.2}$$

holds for every $[\boldsymbol{\psi}, \varphi] \in V \times W_0^{1,2}(\sigma)$ in the sense of scalar distributions on $(0, T)$ and

$$\hat{\mathbf{u}}(x', 0) = \hat{\mathbf{u}}_0(x') \quad \text{in } \sigma, \tag{2.3}$$

$$\omega(x', 0) = \omega_0(x') \quad \text{in } \sigma. \tag{2.4}$$

(ii) Let $T = +\infty$ and suppose that $\hat{\mathbf{f}} \in L^2(0, \infty; H)$, $g \in L^2(0, \infty; L^2(\sigma))$, $\hat{\mathbf{u}}_0 \in H$ and $\omega_0 \in L^2(\sigma)$. By a weak solution of the problem (1.8) on $(0, +\infty)$ we mean a pair $[\hat{\mathbf{u}}, \omega]$ such that $\hat{\mathbf{u}} \in L^2(0, \infty; V) \cap C([0, \infty); H)$ $\omega \in L^2(0, \infty; W_0^{1,2}(\sigma)) \cap C([0, \infty); L^2(\sigma))$, $\hat{\mathbf{u}}(x', 0) = \hat{\mathbf{u}}_0(x')$, $\omega(x', 0) = \omega_0(x')$ in σ and the system (2.1)–(2.2) holds for every $[\boldsymbol{\psi}, \varphi] \in V \times W_0^{1,2}(\sigma)$ in the sense of scalar distributions on $(0, +\infty)$.

Theorem 2.2 ([12, 19])**.** *There exists a unique solution of the problem* (1.8) *in the sense of Definition 2.1.*

Theorem 2.3 ([21])**.** *Let $T \in (0, +\infty]$ and $[\hat{\mathbf{u}}, \omega]$ be the solution of the problem* (1.8) *in the sense of Definition 2.1. In addition, let $\hat{\mathbf{u}}_0 \in V$ and $\omega_0 \in W_0^{1,2}(\sigma)$. Then*

$$\partial_t \hat{\mathbf{u}} \in L^2(0, T; H), \quad \hat{\mathbf{u}} \in L^2(0, T; W^{2,2}(\sigma)^2) \cap L^\infty(0, T; V), \tag{2.5}$$

$$\partial_t \omega \in L^2(0,T;L^2(\sigma)), \quad \omega \in L^2(0,T;W^{2,2}(\sigma)) \cap L^\infty(0,T;W_0^{1,2}(\sigma)). \tag{2.6}$$

Proof. Let $[\hat{\mathbf{u}}, \omega]$ be the weak solution of the problem (1.8). Then for the right hand side of (2.1) we have

$$((2\nu_r \nabla_{x'}^{\perp} \omega, \cdot)) + ((\hat{\mathbf{f}}, \cdot)) \in L^2(0,T;H).$$

Now, assuming $\hat{\mathbf{u}}_0 \in V$, (2.5) follows from [21, Theorem 3.10, Chapter 3]. Finally, (2.6) can be proved by similar arguments. □

3. Solvability of problem (1.9)–(1.10)

Definition 3.1. Let $T \in (0,\infty]$ and suppose that

$$\hat{\mathbf{u}} \in L^2(0,T;W^{2,2}(\sigma)^2) \cap L^\infty(0,T;V), \tag{3.1}$$

$$\hat{\mathbf{g}} \in L^2(0,T;L^2(\sigma)^2),\ f \in L^2(0,T;L^2(\sigma)),\ F \in W^{1,2}((0,T)), \tag{3.2}$$

$$\hat{\mathbf{w}}_0 \in W_0^{1,2}(\sigma)^2,\ v_0 \in W_0^{1,2}(\sigma). \tag{3.3}$$

The weak solution of problem (1.9)–(1.10) is a triplet $[v, \hat{\mathbf{w}}, q]$ such that

$$\begin{gathered} v \in L^\infty(0,T;W_0^{1,2}(\sigma)) \cap W^{1,2}(0,T;L^2(\sigma)), \\ \hat{\mathbf{w}} \in L^\infty(0,T;W_0^{1,2}(\sigma)^2) \cap W^{1,2}(0,T;L^2(\sigma)^2), \\ q \in L^2((0,T)), \\ v(x',0) = v_0(x'), \quad \hat{\mathbf{w}}(x',0) = \hat{\mathbf{w}}_0(x') \end{gathered} \tag{3.4}$$

and the following equalities hold:

$$\begin{aligned} &\frac{d}{dt}(v(t),\varphi) + (\nu+\nu_r)((\nabla_{x'} v(t), \nabla_{x'}\varphi)) + d(\hat{\mathbf{u}}(t), v(t), \varphi) \\ &= q(t)(1,\varphi) + 2\nu_r(\mathrm{rot}\ \hat{\mathbf{w}}(t), \varphi) + (f(t),\varphi) \end{aligned} \tag{3.5}$$

for all $\varphi \in W_0^{1,2}(\sigma)$,

$$\begin{aligned} &\frac{d}{dt}((\hat{\mathbf{w}}(t), \boldsymbol{\psi})) + (c_a + c_d) a(\hat{\mathbf{w}}(t), \boldsymbol{\psi}) + b(\hat{\mathbf{u}}(t), \hat{\mathbf{w}}(t), \boldsymbol{\psi}) \\ &+ (c_0 + c_d - c_a)(\operatorname{div} \hat{\mathbf{w}}(t), \operatorname{div} \boldsymbol{\psi}) + 4\nu_r((\hat{\mathbf{w}}(t), \boldsymbol{\psi})) \\ &= 2\nu_r((\nabla_{x'}^{\perp} v(t), \boldsymbol{\psi})) + ((\hat{\mathbf{g}}(t), \boldsymbol{\psi})) \end{aligned} \tag{3.6}$$

for all $\boldsymbol{\psi} \in W_0^{1,2}(\sigma)^2$ and for almost every $t \in (0,T)$, and

$$\int_\sigma v(x',t)dx' = F(t) \quad \text{for almost every } t \in (0,T). \tag{3.7}$$

Theorem 3.2. *There exists a solution of problem* (1.9)–(1.10) *in the sense of Definition 3.1.*

The detailed proof of Theorem 3.2 is split into several steps.

Approximations on $(0,T)$, $T \in (0,+\infty)$. Let $T \in (0,+\infty)$, fix $n \in \mathbb{N}$ and let $h := T/n$ be a time step. Further, let us consider

$$\begin{aligned} f_n^i(x') &:= \frac{1}{h}\int_{(i-1)h}^{ih} f(x',s)\mathrm{d}s, \quad i = 1,\dots,n, \\ \mathbf{g}_n^i(x') &:= \frac{1}{h}\int_{(i-1)h}^{ih} \hat{\mathbf{g}}(x',s)\mathrm{d}s, \quad i = 1,\dots,n, \end{aligned}$$

$$\mathbf{u}_n^i(x') := \frac{1}{h}\int_{(i-1)h}^{ih} \hat{\mathbf{u}}(x',s)\mathrm{d}s, \quad i=1,\dots,n,$$
$$F_n^i := \frac{1}{h}\int_{(i-1)h}^{ih} F(s)\mathrm{d}s, \quad i=1,\dots,n,$$
$$\mathbf{w}_n^0(x') := \hat{\mathbf{w}}_0(x'),$$
$$v_n^0(x') := v_0(x')$$

a.e. in σ.

First, note that, in view of (3.1), we have

$$\mathbf{u}_n^i \in W^{2,2}(\sigma)^2 \quad \text{and} \quad h\sum_{i=1}^n \|\mathbf{u}_n^i\|^2_{W^{2,2}(\sigma)^2} \le C, \tag{3.8}$$

where C is independent of n (cf. [17, page 206, (8.28) and Lemma 8.7]) and by the Sobolev embedding we can write

$$\|\mathbf{u}_n^i\|_{L^4(\sigma)^2} \le c_1\|\mathbf{u}_n^i\|_{W_0^{1,2}(\sigma)^2} \le c_2 \tag{3.9}$$

with c_1 and c_2 independent of i and n. Further, by the Sobolev embedding and (3.8) we also have

$$\mathbf{u}_n^i \in L^\infty(\sigma)^2 \quad \text{and} \quad h\sum_{i=1}^n \|\mathbf{u}_n^i\|^2_{L^\infty(\sigma)^2} \le C, \tag{3.10}$$

where C is independent of n.

Now we are ready to approximate the evolution problem by an implicit time discretization scheme. Then we define, in each time step, $[v_n^i, \mathbf{w}_n^i, q_n^i]$ as a solution of the following recurrence steady problem: for a given couple $[v_n^{i-1}, \mathbf{w}_n^{i-1}] \in W_0^{1,2}(\sigma)\times W_0^{1,2}(\sigma)^2\times\mathbb{R}$ find a triple $[v_n^i, \mathbf{w}_n^i, q_n^i] \in W_0^{1,2}(\sigma)\times W_0^{1,2}(\sigma)^2\times\mathbb{R}$, $i=1,\dots,n$, such that

$$\begin{aligned}
&\frac{1}{h}\big(v_n^i - v_n^{i-1}, \varphi\big) + (\nu+\nu_r)((\nabla_{x'}v_n^i, \nabla_{x'}\varphi)) + d(\mathbf{u}_n^i, v_n^i, \varphi)\\
&= q_n^i(1,\varphi) + 2\nu_r(\operatorname{rot}_{x'}\mathbf{w}_n^i, \varphi) + (f_n^i, \varphi)
\end{aligned} \tag{3.11}$$

for all $\varphi \in W_0^{1,2}(\sigma)$,

$$\begin{aligned}
&\frac{1}{h}((\mathbf{w}_n^i - \mathbf{w}_n^{i-1}, \boldsymbol{\psi})) + (c_a+c_d)a(\mathbf{w}_n^i, \boldsymbol{\psi}) + b(\mathbf{u}_n^i, \mathbf{w}_n^i, \boldsymbol{\psi})\\
&+ (c_0+c_d-c_a)(\operatorname{div}_{x'}\mathbf{w}_n^i, \operatorname{div}_{x'}\boldsymbol{\psi}) + 4\nu_r((\mathbf{w}_n^i, \boldsymbol{\psi}))\\
&= 2\nu_r((\nabla_{x'}^\perp v, \boldsymbol{\psi})) + ((\mathbf{g}_n^i, \boldsymbol{\psi}))
\end{aligned} \tag{3.12}$$

for all $\boldsymbol{\psi} \in W_0^{1,2}(\sigma)^2$ and

$$\int_\sigma v_n^i \, dx' = F_n^i. \tag{3.13}$$

Theorem 3.3. *Let $[v_n^{i-1}, \mathbf{w}_n^{i-1}] \in W_0^{1,2}(\sigma)\times W_0^{1,2}(\sigma)^2$ and $\mathbf{u}_n^i \in V$ be given. Then there exists the triple $[v_n^i, \mathbf{w}_n^i, q_n^i] \in W_0^{1,2}(\sigma)\times W_0^{1,2}(\sigma)^2\times\mathbb{R}$, the solution to the discrete problem* (3.11)–(3.13).

Proof. Denote $U=(v,\mathbf{w})$ and $V=(\varphi,\boldsymbol{\psi})$ and define

$$\begin{aligned}
\mathcal{B}(U,V) &= (\nu+\nu_r)((\nabla_{x'}v, \nabla_{x'}\varphi)) - 2\nu_r(\operatorname{rot}_{x'}\mathbf{w}, \varphi)\\
&\quad + (c_a+c_d)a(\mathbf{w}, \boldsymbol{\psi}) + (c_0+c_d-c_a)(\operatorname{div}_{x'}\mathbf{w}, \operatorname{div}_{x'}\boldsymbol{\psi})
\end{aligned}$$

$$+ 4\nu_r((\mathbf{w},\boldsymbol{\psi})) - 2\nu_r((\nabla^{\perp}_{x'} v,\boldsymbol{\psi})).$$

In [18] it is shown that

$$\mathcal{B}(U,V) \le c\|U\|_{W^{1,2}(\sigma)^3}\|V\|_{W^{1,2}(\sigma)^3}$$

and

$$c\|U\|^2_{W^{1,2}(\sigma)^3} \le \mathcal{B}(U,U) \tag{3.14}$$

for all $U,V \in W^{1,2}(\sigma)^3$. Now, it is easy to show that the form $\mathcal{A}$, defined by

$$\mathcal{A}(U,V) = \mathcal{B}(U,V) + \frac{1}{h}(v,\varphi) + d(\mathbf{u}^i_n, v,\varphi) + \frac{1}{h}((\mathbf{w},\boldsymbol{\psi})) + b(\mathbf{u}^i_n,\mathbf{w},\boldsymbol{\psi}), \tag{3.15}$$

is continuous. Moreover, applying the interpolation and Young's inequality we have

$$\begin{aligned} \Big|\int_\sigma (\mathbf{u}^i_n\cdot\nabla_{x'})v\ v\ dx'\Big| &\le c\|\mathbf{u}^i_n\|_{L^4(\sigma)^2}\|v\|_{W^{1,2}(\sigma)}\|v\|_{L^4(\sigma)} \\ &\le C(\varepsilon)\|\mathbf{u}^i_n\|^4_{L^4(\sigma)^2}\|v\|^2_{L^2(\sigma)} + \varepsilon\|v\|^2_{W^{1,2}(\sigma)} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \Big|\int_\sigma (\mathbf{u}^i_n\cdot\nabla_{x'})\mathbf{w}\cdot\mathbf{w}\,dx'\Big| &\le c\|\mathbf{u}^i_n\|_{L^4(\sigma)^2}\|\mathbf{w}\|_{W^{1,2}(\sigma)^2}\|\mathbf{w}\|_{L^4(\sigma)^2} \\ &\le C(\varepsilon)\|\mathbf{u}^i_n\|^4_{L^4(\sigma)^2}\|\mathbf{w}\|^2_{L^2(\sigma)^2} + \varepsilon\|\mathbf{w}\|^2_{W^{1,2}(\sigma)^2}. \end{aligned} \tag{3.17}$$

Taking $V = U$ in (3.15), using (3.9), (3.14), (3.16) and (3.17) and taking h and ε small enough we can write

$$\begin{aligned} \mathcal{A}(U,U) &\ge \mathcal{B}(U,U) + \frac{1}{h}\|v\|^2_{L^2(\sigma)} + \frac{1}{h}\|\mathbf{w}\|^2_{L^2(\sigma)^2} \\ &\quad - \Big|\int_\sigma (\mathbf{u}^i_n\cdot\nabla_{x'})v\ v\ dx'\Big| - \Big|\int_\sigma (\mathbf{u}^i_n\cdot\nabla_{x'})\mathbf{w}\cdot\mathbf{w}\,dx'\Big| \\ &\ge \mathcal{B}(U,U) + \frac{1}{h}\|v\|^2_{L^2(\sigma)} + \frac{1}{h}\|\mathbf{w}\|^2_{L^2(\sigma)^2} \\ &\quad - C(\varepsilon)\|\mathbf{u}^i_n\|^4_{L^4(\sigma)^2}\|v\|^2_{L^2(\sigma)} - \varepsilon\|v\|^2_{W^{1,2}(\sigma)} \\ &\quad - C(\varepsilon)\|\mathbf{u}^i_n\|^4_{L^4(\sigma)^2}\|\mathbf{w}\|^2_{L^2(\sigma)^2} - \varepsilon\|\mathbf{w}\|^2_{W^{1,2}(\sigma)^2} \\ &\ge c\|U\|^2_{W^{1,2}(\sigma)^3}. \end{aligned} \tag{3.18}$$

Hence, there exists $h_0 > 0$ (small enough) such that for all $h \le h_0$, the form $\mathcal{A}$, defined by the equation (3.15), is continuous and coercive. By the Lax-Milgram theorem, there exists $(v_R,\mathbf{w}_R)$ such that

$$\begin{aligned} &\frac{1}{h}(v_R,\varphi) + (\nu+\nu_r)((\nabla_{x'}v_R,\nabla_{x'}\varphi)) + d(\mathbf{u}^i_n, v_R,\varphi) - 2\nu_r(\operatorname{rot}_{x'}\mathbf{w}_R,\varphi) \\ &= \frac{1}{h}(v^{i-1}_n,\varphi) + (f^i_n,\varphi) \end{aligned}$$

for all $\varphi \in W^{1,2}_0(\sigma)$ and

$$\begin{aligned} &\frac{1}{h}((\mathbf{w}_R,\boldsymbol{\psi})) + (c_a+c_d)a(\mathbf{w}_R,\boldsymbol{\psi}) + (c_0+c_d-c_a)(\operatorname{div}_{x'}\mathbf{w}_R,\operatorname{div}_{x'}\boldsymbol{\psi}) \\ &+ b(\mathbf{u}^i_n,\mathbf{w}_R,\boldsymbol{\psi}) + 4\nu_r((\mathbf{w}_R,\boldsymbol{\psi})) - 2\nu_r((\nabla^{\perp}_{x'}v_R,\boldsymbol{\psi})) \\ &= \frac{1}{h}((\mathbf{w}^{i-1}_n,\boldsymbol{\psi})) + ((\mathbf{g}^i_n,\boldsymbol{\psi})) \end{aligned}$$

for all $\boldsymbol{\psi} \in W_0^{1,2}(\sigma)^2$. Similarly, there exists $(\widetilde{v}_F, \widetilde{\mathbf{w}}_F)$ such that

$$\begin{aligned} &\frac{1}{h}(\widetilde{v}_F,\varphi) + (\nu+\nu_r)((\nabla_{x'}\widetilde{v}_F, \nabla_{x'}\varphi)) + d(\mathbf{u}_n^i, \widetilde{v}_F, \varphi) - 2\nu_r(\operatorname{rot}_{x'}\widetilde{\mathbf{w}}_F, \varphi) \\ &= (1,\varphi) \end{aligned} \tag{3.19}$$

for all $\varphi \in W_0^{1,2}(\sigma)$ and

$$\begin{aligned} &\frac{1}{h}((\widetilde{\mathbf{w}}_F, \boldsymbol{\psi})) + (c_a+c_d)a(\widetilde{\mathbf{w}}_F, \boldsymbol{\psi}) + (c_0+c_d-c_a)(\operatorname{div}_{x'}\widetilde{\mathbf{w}}_F, \operatorname{div}_{x'}\boldsymbol{\psi}) \\ &+ b(\mathbf{u}_n^i, \tilde{\mathbf{w}}_F, \boldsymbol{\psi}) + 4\nu_r((\widetilde{\mathbf{w}}_F, \boldsymbol{\psi})) - 2\nu_r((\nabla_{x'}^{\perp}\widetilde{v}_F, \boldsymbol{\psi})) = 0 \end{aligned} \tag{3.20}$$

for all $\boldsymbol{\psi} \in W_0^{1,2}(\sigma)^2$. Using $\varphi = \widetilde{v}_F$ and $\boldsymbol{\psi} = \widetilde{\mathbf{w}}_F$ in (3.19) and (3.20), respectively, we verify (in view of coercivity of $\mathcal{A}$)

$$\int_\sigma \widetilde{v}_F \, dx' \neq 0.$$

Now, let

$$\widetilde{C}_F := \int_\sigma \widetilde{v}_F \, dx' \quad \text{and} \quad C_R := \int_\sigma v_R \, dx'.$$

Further, by the same arguments (Lax-Milgram) we have $[v_F, \mathbf{w}_F]$, the solution to the problem

$$\begin{aligned} &\frac{1}{h}(v_F,\varphi) + (\nu+\nu_r)((\nabla_{x'}v_F, \nabla_{x'}\varphi)) + d(\mathbf{u}_n^i, v_F, \varphi) - 2\nu_r(\operatorname{rot}_{x'}\mathbf{w}_F, \varphi) \\ &= \frac{F_n^i - C_R}{\widetilde{C}_F}(1,\varphi) \end{aligned} \tag{3.21}$$

for all $\varphi \in W_0^{1,2}(\sigma)$ and

$$\begin{aligned} &\frac{1}{h}((\mathbf{w}_F, \boldsymbol{\psi})) + (c_a+c_d)a(\mathbf{w}_F, \boldsymbol{\psi}) + (c_0+c_d-c_a)(\operatorname{div}\mathbf{w}_F, \operatorname{div}\boldsymbol{\psi}) \\ &+ b(\mathbf{u}_n^i, \mathbf{w}_F, \boldsymbol{\psi}) + 4\nu_r((\mathbf{w}_F, \boldsymbol{\psi})) - 2\nu_r((\nabla^{\perp}v_F, \boldsymbol{\psi})) = 0 \end{aligned} \tag{3.22}$$

for all $\boldsymbol{\psi} \in W_0^{1,2}(\sigma)^2$. Now, comparing (3.19)–(3.20) and (3.21)–(3.22) we can write

$$v_F = \widetilde{v}_F \frac{F_n^i - C_R}{\widetilde{C}_F} \quad \text{and} \quad \mathbf{w}_F = \widetilde{\mathbf{w}}_F \frac{F_n^i - C_R}{\widetilde{C}_F}.$$

Finally, let us set

$$v_n^i = v_F + v_R, \quad \mathbf{w}_n^i = \mathbf{w}_F + \mathbf{w}_R, q_n^i = \frac{F_n^i - C_R}{\widetilde{C}_F}.$$

It is easy to see that v_n^i, q_n^i and $\mathbf{w}_n^i$ solve (3.11) and (3.12) and v_n^i has the correct net flux which can be verified as

$$\begin{aligned} \int_\sigma v_n^i \, dx' &= \int_\sigma v_F + v_R \, dx' \\ &= \frac{F_n^i - C_R}{\widetilde{C}_F} \int_\sigma \widetilde{v}_F \, dx' + C_R \\ &= \frac{F_n^i - C_R}{\widetilde{C}_F} \widetilde{C}_F + C_R = F_n^i. \end{aligned}$$

The proof of Theorem 3.3 is complete. $\square$

A-priori estimates for discrete approximations. Using $\varphi = (v_n^i - v_n^{i-1})/h$ as a test function in (3.11) we obtain

$$
\begin{aligned}
&\|\frac{v_n^i - v_n^{i-1}}{h}\|_{L^2(\sigma)}^2 + \frac{(\nu+\nu_r)}{2h}\|\nabla_{x'} v_n^i\|_{L^2(\sigma)^2}^2 - \frac{(\nu+\nu_r)}{2h}\|\nabla_{x'} v_n^{i-1}\|_{L^2(\sigma)^2}^2 \\
&+ \frac{(\nu+\nu_r)}{2h}\|\nabla_{x'} v_n^i - \nabla_{x'} v_n^{i-1}\|_{L^2(\sigma)^2}^2 \\
&\le \frac{2\nu_r}{h}(\mathrm{rot}_{x'}\,\mathbf{w}_n^i, v_n^i - v_n^{i-1}) - \frac{1}{h} d(\mathbf{u}_n^i, v_n^i, v_n^i - v_n^{i-1}) \\
&\quad + \varepsilon |q_n^i|^2 + \frac{c}{\varepsilon}\Big|\frac{F_n^i - F_n^{i-1}}{h}\Big|^2 + \varepsilon \|\frac{v_n^i - v_n^{i-1}}{h}\|_{L^2(\sigma)}^2 + \frac{c}{\varepsilon}\|f_n^i\|_{L^2(\sigma)}^2 .
\end{aligned} \tag{3.23}
$$

For the second term on the right-hand side in (3.23) we can write, using (3.9)–(3.10),

$$
\begin{aligned}
\frac{1}{h}|d(\mathbf{u}_n^i, v_n^i, v_n^i - v_n^{i-1})| &\le \frac{1}{h}\|\mathbf{u}_n^i\|_{L^4(\sigma)^2}\|v_n^i - v_n^{i-1}\|_{W^{1,2}(\sigma)}\|v_n^i - v_n^{i-1}\|_{L^4(\sigma)} \\
&\quad + \|\mathbf{u}_n^i\|_{L^\infty(\sigma)^2}\|v_n^{i-1}\|_{W^{1,2}(\sigma)}\|\frac{v_n^i - v_n^{i-1}}{h}\|_{L^2(\sigma)} .
\end{aligned} \tag{3.24}
$$

For the first term on the right-hand side in (3.24) we have

$$
\begin{aligned}
&\frac{1}{h}\|\mathbf{u}_n^i\|_{L^4(\sigma)^2}\|v_n^i - v_n^{i-1}\|_{W^{1,2}(\sigma)}\|v_n^i - v_n^{i-1}\|_{L^4(\sigma)} \\
&\le \frac{c}{h}\|\mathbf{u}_n^i\|_{L^4(\sigma)^2}\|v_n^i - v_n^{i-1}\|_{W^{1,2}(\sigma)}^{3/2}\|v_n^i - v_n^{i-1}\|_{L^2(\sigma)}^{1/2} \\
&\le \frac{\varepsilon}{h}\|v_n^i - v_n^{i-1}\|_{W^{1,2}(\sigma)}^2 + \frac{C(\varepsilon)}{h}\|\mathbf{u}_n^i\|_{L^4(\sigma)^2}^4\|v_n^i - v_n^{i-1}\|_{L^2(\sigma)}^2 .
\end{aligned} \tag{3.25}
$$

For the second term on the right-hand side in (3.24) we have

$$
\begin{aligned}
&\|\mathbf{u}_n^i\|_{L^\infty(\sigma)^2}\|v_n^{i-1}\|_{W^{1,2}(\sigma)}\|\frac{v_n^i - v_n^{i-1}}{h}\|_{L^2(\sigma)} \\
&\le \varepsilon\|\frac{v_n^i - v_n^{i-1}}{h}\|_{L^2(\sigma)}^2 + C(\varepsilon)\|\mathbf{u}_n^i\|_{L^\infty(\sigma)^2}^2\|v_n^{i-1}\|_{W^{1,2}(\sigma)}^2 .
\end{aligned} \tag{3.26}
$$

Now, combining (3.24), (3.25) and (3.26) together with (3.23) we obtain

$$
\begin{aligned}
&(1-2\varepsilon)\|\frac{v_n^i - v_n^{i-1}}{h}\|_{L^2(\sigma)}^2 + \frac{(\nu+\nu_r)}{2h}\|\nabla_{x'} v_n^i\|_{L^2(\sigma)^2}^2 \\
&- \frac{(\nu+\nu_r)}{2h}\|\nabla_{x'} v_n^{i-1}\|_{L^2(\sigma)^2}^2 + \frac{(\nu+\nu_r)}{2h}\|\nabla_{x'} v_n^i - \nabla_{x'} v_n^{i-1}\|_{L^2(\sigma)^2}^2 \\
&\le \frac{2\nu_r}{h}(\mathrm{rot}_{x'}\,\mathbf{w}_n^i, v_n^i - v_n^{i-1}) + \frac{\varepsilon}{h}\|v_n^i - v_n^{i-1}\|_{W^{1,2}(\sigma)}^2 \\
&\quad + \frac{C(\varepsilon)}{h}\|\mathbf{u}_n^i\|_{L^4(\sigma)^2}^4\|v_n^i - v_n^{i-1}\|_{L^2(\sigma)}^2 + \varepsilon|q_n^i|^2 + \frac{c}{\varepsilon}\Big|\frac{F_n^i - F_n^{i-1}}{h}\Big|^2 \\
&\quad + \frac{c}{\varepsilon}\|f_n^i\|_{L^2(\sigma)}^2 + C(\varepsilon)\|\mathbf{u}_n^i\|_{L^\infty(\sigma)^2}^2\|v_n^{i-1}\|_{W^{1,2}(\sigma)}^2 .
\end{aligned} \tag{3.27}
$$

Likewise, using $\boldsymbol{\psi} = (\mathbf{w}_n^i - \mathbf{w}_n^{i-1})/h$ in (3.12) we arrive at

$$
\begin{aligned}
&\|\frac{\mathbf{w}_n^i - \mathbf{w}_n^{i-1}}{h}\|_{L^2(\sigma)^2}^2 \\
&+ \frac{(c_a+c_d)}{2h} a(\mathbf{w}_n^i, \mathbf{w}_n^i) - \frac{(c_a+c_d)}{2h} a(\mathbf{w}_n^{i-1}, \mathbf{w}_n^{i-1}) \\
&+ \frac{(c_a+c_d)}{2h} a(\mathbf{w}_n^i - \mathbf{w}_n^{i-1}, \mathbf{w}_n^i - \mathbf{w}_n^{i-1})
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(c_0+c_d-c_a)}{2h}\|\operatorname{div}_{x'}\mathbf{w}_n^i\|_{L^2(\sigma)}^2 - \frac{(c_0+c_d-c_a)}{2h}\|\operatorname{div}_{x'}\mathbf{w}_n^{i-1}\|_{L^2(\sigma)}^2 \\
&+ \frac{(c_0+c_d-c_a)}{2h}\|\operatorname{div}_{x'}\mathbf{w}_n^i - \operatorname{div}_{x'}\mathbf{w}_n^{i-1}\|_{L^2(\sigma)}^2 \\
&+ \frac{4\nu_r}{2h}\|\mathbf{w}_n^i\|_{L^2(\sigma)^2}^2 - \frac{4\nu_r}{2h}\|\mathbf{w}_n^{i-1}\|_{L^2(\sigma)^2}^2 + \frac{4\nu_r}{2h}\|\mathbf{w}_n^i - \mathbf{w}_n^{i-1}\|_{L^2(\sigma)^2}^2 \\
&\le \frac{2\nu_r}{h}((\nabla_{x'}^{\perp}v_n^i, \mathbf{w}_n^i - \mathbf{w}_n^{i-1})) + \varepsilon\|\frac{\mathbf{w}_n^i - \mathbf{w}_n^{i-1}}{h}\|_{L^2(\sigma)^2}^2 \\
&\quad + \frac{c}{\varepsilon}\|\mathbf{g}_n^i\|_{L^2(\sigma)^2}^2 - \frac{1}{h}b(\mathbf{u}_n^i, \mathbf{w}_n^i, \mathbf{w}_n^i - \mathbf{w}_n^{i-1}).
\end{aligned} \tag{3.28}
$$

Adding first terms on the right hand sides in (3.23) and (3.28) we deduce

$$
\begin{aligned}
&\frac{2\nu_r}{h}(\operatorname{rot}_{x'}\mathbf{w}_n^i, v_n^i - v_n^{i-1}) + \frac{2\nu_r}{h}((\nabla_{x'}^{\perp}v_n^i, \mathbf{w}_n^i - \mathbf{w}_n^{i-1})) \\
&= \frac{2\nu_r}{h}((\mathbf{w}_n^i, \nabla_{x'}^{\perp}v_n^i)) - \frac{2\nu_r}{h}((\mathbf{w}_n^{i-1}, \nabla_{x'}^{\perp}v_n^{i-1})) \\
&\quad + \frac{2\nu_r}{h}((\mathbf{w}_n^i - \mathbf{w}_n^{i-1}, \nabla_{x'}^{\perp}(v_n^i - v_n^{i-1}))).
\end{aligned} \tag{3.29}
$$

By Young's inequality we have

$$
\begin{aligned}
&\frac{2\nu_r}{h}((\mathbf{w}_n^i - \mathbf{w}_n^{i-1}, \nabla_{x'}^{\perp}(v_n^i - v_n^{i-1}))) \\
&\le \frac{2\nu_r}{h}\frac{1}{4}\|\nabla_{x'}\left(v_n^i - v_n^{i-1}\right)\|_{L^2(\sigma)^2}^2 + \frac{2\nu_r}{h}\|\mathbf{w}_n^i - \mathbf{w}_n^{i-1}\|_{L^2(\sigma)^2}^2.
\end{aligned} \tag{3.30}
$$

For the last term on the right-hand side in (3.28) we can write

$$
\begin{aligned}
&\frac{1}{h}b(\mathbf{u}_n^i, \mathbf{w}_n^i, \mathbf{w}_n^i - \mathbf{w}_n^{i-1}) \\
&\le \frac{1}{h}\|\mathbf{u}_n^i\|_{L^4(\sigma)^2}\|\mathbf{w}_n^i - \mathbf{w}_n^{i-1}\|_{W^{1,2}(\sigma)^2}\|\mathbf{w}_n^i - \mathbf{w}_n^{i-1}\|_{L^4(\sigma)^2} \\
&\quad + \|\mathbf{u}_n^i\|_{L^\infty(\sigma)^2}\|\mathbf{w}_n^{i-1}\|_{W^{1,2}(\sigma)^2}\|\frac{\mathbf{w}_n^i - \mathbf{w}_n^{i-1}}{h}\|_{L^2(\sigma)^2}.
\end{aligned} \tag{3.31}
$$

For the first term on the right-hand side in (3.31) we have

$$
\begin{aligned}
&\frac{1}{h}\|\mathbf{u}_n^i\|_{L^4(\sigma)^2}\|\mathbf{w}_n^i - \mathbf{w}_n^{i-1}\|_{W^{1,2}(\sigma)^2}\|\mathbf{w}_n^i - \mathbf{w}_n^{i-1}\|_{L^4(\sigma)^2} \\
&\le \frac{c}{h}\|\mathbf{u}_n^i\|_{L^4(\sigma)^2}\|\mathbf{w}_n^i - \mathbf{w}_n^{i-1}\|_{W^{1,2}(\sigma)^2}^{3/2}\|\mathbf{w}_n^i - \mathbf{w}_n^{i-1}\|_{L^2(\sigma)^2}^{1/2} \\
&\le \frac{\varepsilon}{h}\|\mathbf{w}_n^i - \mathbf{w}_n^{i-1}\|_{W^{1,2}(\sigma)^2}^2 + \frac{C(\varepsilon)}{h}\|\mathbf{u}_n^i\|_{L^4(\sigma)^2}^4\|\mathbf{w}_n^i - \mathbf{w}_n^{i-1}\|_{L^2(\sigma)^2}^2.
\end{aligned} \tag{3.32}
$$

For the second term on the right-hand side in (3.31) we have

$$
\begin{aligned}
&\|\mathbf{u}_n^i\|_{L^\infty(\sigma)^2}\|\mathbf{w}_n^{i-1}\|_{W^{1,2}(\sigma)^2}\|\frac{\mathbf{w}_n^i - \mathbf{w}_n^{i-1}}{h}\|_{L^2(\sigma)^2} \\
&\le \varepsilon\|\frac{\mathbf{w}_n^i - \mathbf{w}_n^{i-1}}{h}\|_{L^2(\sigma)^2}^2 + C(\varepsilon)\|\mathbf{u}_n^i\|_{L^\infty(\sigma)^2}^2\|\mathbf{w}_n^{i-1}\|_{W^{1,2}(\sigma)^2}^2.
\end{aligned} \tag{3.33}
$$

Now, summing (3.27) together with (3.28) and using (3.29)–(3.33) we arrive at

$$
\begin{aligned}
&(1-2\varepsilon)\|\frac{v_n^i-v_n^{i-1}}{h}\|^2_{L^2(\sigma)}+(1-2\varepsilon)\|\frac{\mathbf{w}_n^i-\mathbf{w}_n^{i-1}}{h}\|^2_{L^2(\sigma)^2}\\
&+\frac{(\nu+\nu_r)}{2h}\|\nabla_{x'}v_n^i\|^2_{L^2(\sigma)^2}-\frac{(\nu+\nu_r)}{2h}\|\nabla_{x'}v_n^{i-1}\|^2_{L^2(\sigma)^2}\\
&+\frac{(c_a+c_d)}{2h}a(\mathbf{w}_n^i,\mathbf{w}_n^i)-\frac{(c_a+c_d)}{2h}a(\mathbf{w}_n^{i-1},\mathbf{w}_n^{i-1})\\
&+\frac{\nu}{2h}\|\nabla_{x'}v_n^i-\nabla_{x'}v_n^{i-1}\|^2_{L^2(\sigma)}+\frac{(c_a+c_d)}{2h}a(\mathbf{w}_n^i-\mathbf{w}_n^{i-1},\mathbf{w}_n^i-\mathbf{w}_n^{i-1})\\
&+\frac{(c_0+c_d-c_a)}{2h}\|\operatorname{div}_{x'}\mathbf{w}_n^i\|^2_{L^2(\sigma)}-\frac{(c_0+c_d-c_a)}{2h}\|\operatorname{div}_{x'}\mathbf{w}_n^{i-1}\|^2_{L^2(\sigma)}\\
&+\frac{(c_0+c_d-c_a)}{2h}\|\operatorname{div}_{x'}\mathbf{w}_n^i-\operatorname{div}_{x'}\mathbf{w}_n^{i-1}\|^2_{L^2(\sigma)}\\
&+\frac{4\nu_r}{2h}\|\mathbf{w}_n^i\|^2_{L^2(\sigma)^2}-\frac{4\nu_r}{2h}\|\mathbf{w}_n^{i-1}\|^2_{L^2(\sigma)^2}\\
&\le\frac{\varepsilon}{h}\|v_n^i-v_n^{i-1}\|^2_{W^{1,2}(\sigma)}+\frac{C(\varepsilon)}{h}\|\mathbf{u}_n^i\|^4_{L^4(\sigma)^2}\|v_n^i-v_n^{i-1}\|^2_{L^2(\sigma)}\\
&\quad+\frac{\varepsilon}{h}\|\mathbf{w}_n^i-\mathbf{w}_n^{i-1}\|^2_{W^{1,2}(\sigma)^2}+\frac{C(\varepsilon)}{h}\|\mathbf{u}_n^i\|^4_{L^4(\sigma)^2}\|\mathbf{w}_n^i-\mathbf{w}_n^{i-1}\|^2_{L^2(\sigma)^2}\\
&\quad+\varepsilon|q_n^i|^2+\frac{c}{\varepsilon}\Big|\frac{F_n^i-F_n^{i-1}}{h}\Big|^2+\frac{c}{\varepsilon}\|f_n^i\|^2_{L^2(\sigma)}\\
&\quad+C(\varepsilon)\|\mathbf{u}_n^i\|^2_{L^\infty(\sigma)^2}\|v_n^{i-1}\|^2_{W^{1,2}(\sigma)}+C(\varepsilon)\|\mathbf{u}_n^i\|^2_{L^\infty(\sigma)^2}\|\mathbf{w}_n^{i-1}\|^2_{W^{1,2}(\sigma)^2}\\
&\quad+\frac{c}{\varepsilon}\|\mathbf{g}_n^i\|^2_{L^2(\sigma)^2}+\frac{2\nu_r}{h}((\mathbf{w}_n^i,\nabla^\perp_{x'}v_n^i))-\frac{2\nu_r}{h}((\mathbf{w}_n^{i-1},\nabla^\perp_{x'}v_n^{i-1})).
\end{aligned}
\tag{3.34}
$$

Summing (3.34) for $i=1,2,\dots,k$ we obtain

$$
\begin{aligned}
&(1-2\varepsilon)\sum_{i=1}^k\|\frac{v_n^i-v_n^{i-1}}{h}\|^2_{L^2(\sigma)}+(1-2\varepsilon)\sum_{i=1}^k\|\frac{\mathbf{w}_n^i-\mathbf{w}_n^{i-1}}{h}\|^2_{L^2(\sigma)^2}\\
&+\frac{(\nu+\nu_r)}{2h}\|\nabla_{x'}v_n^k\|^2_{L^2(\sigma)^2}+\frac{\nu}{2h}\sum_{i=1}^k\|\nabla_{x'}v_n^i-\nabla_{x'}v_n^{i-1}\|^2_{L^2(\sigma)^2}\\
&+\frac{(c_a+c_d)}{2h}a(\mathbf{w}_n^k,\mathbf{w}_n^k)+\frac{(c_a+c_d)}{2h}\sum_{i=1}^k a(\mathbf{w}_n^i-\mathbf{w}_n^{i-1},\mathbf{w}_n^i-\mathbf{w}_n^{i-1})\\
&+\frac{4\nu_r}{2h}\|\mathbf{w}_n^k\|^2_{L^2(\sigma)^2}+\frac{(c_0+c_d-c_a)}{2h}\|\operatorname{div}_{x'}\mathbf{w}_n^k\|^2_{L^2(\sigma)}\\
&+\frac{(c_0+c_d-c_a)}{2h}\sum_{i=1}^k\|\operatorname{div}_{x'}\mathbf{w}_n^i-\operatorname{div}_{x'}\mathbf{w}_n^{i-1}\|^2_{L^2(\sigma)}\\
&\le\frac{\varepsilon}{h}\sum_{i=1}^k\|v_n^i-v_n^{i-1}\|^2_{W^{1,2}(\sigma)}+\frac{C(\varepsilon)}{h}\sum_{i=1}^k\|\mathbf{u}_n^i\|^4_{L^4(\sigma)^2}\|v_n^i-v_n^{i-1}\|^2_{L^2(\sigma)}\\
&\quad+\frac{\varepsilon}{h}\sum_{i=1}^k\|\mathbf{w}_n^i-\mathbf{w}_n^{i-1}\|^2_{W^{1,2}(\sigma)^2}+\frac{C(\varepsilon)}{h}\sum_{i=1}^k\|\mathbf{u}_n^i\|^4_{L^4(\sigma)^2}\|\mathbf{w}_n^i-\mathbf{w}_n^{i-1}\|^2_{L^2(\sigma)^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon\sum_{i=1}^{k}|q_n^i|^2+\frac{c}{\varepsilon}\sum_{i=1}^{k}\Big|\frac{F_n^i-F_n^{i-1}}{h}\Big|^2+\frac{c}{\varepsilon}\sum_{i=1}^{k}\|f_n^i\|^2_{L^2(\sigma)}\\
&+C(\varepsilon)\sum_{i=1}^{k}\|\mathbf{u}_n^i\|^2_{L^\infty(\sigma)^2}\|v_n^{i-1}\|^2_{W^{1,2}(\sigma)}+C(\varepsilon)\sum_{i=1}^{k}\|\mathbf{u}_n^i\|^2_{L^\infty(\sigma)^2}\|\mathbf{w}_n^{i-1}\|^2_{W^{1,2}(\sigma)^2}\\
&+\frac{c}{\varepsilon}\sum_{i=1}^{k}\|\mathbf{g}_n^i\|^2_{L^2(\sigma)^2}+\frac{2\nu_r}{h}((\mathbf{w}_n^k,\nabla_{x'}^{\perp}v_n^k))-\frac{2\nu_r}{h}((\mathbf{w}_n^0,\nabla_{x'}^{\perp}v_n^0))\\
&+\frac{2\nu_r}{h}\|\mathbf{w}_n^0\|^2_{L^2(\sigma)^2}+\frac{(\nu+\nu_r)}{2h}\|\nabla_{x'}v_n^0\|^2_{L^2(\sigma)^2}+\frac{(c_a+c_d)}{2h}a(\mathbf{w}_n^0,\mathbf{w}_n^0)\\
&+\frac{(c_0+c_d-c_a)}{2h}\|\operatorname{div}_{x'}\mathbf{w}_n^0\|^2_{L^2(\sigma)}.
\end{aligned}\tag{3.35}
$$

Again, applying Young's inequality we can write

$$
\frac{2\nu_r}{h}((\mathbf{w}_n^k,\nabla_{x'}^{\perp}v_n^k))\le\frac{2\nu_r}{h}\frac{1}{4}\|\nabla_{x'}v_n^k\|^2_{L^2(\sigma)^2}+\frac{2\nu_r}{h}\|\mathbf{w}_n^k\|^2_{L^2(\sigma)^2}.\tag{3.36}
$$

By the Friedrichs inequality we have

$$
\frac{\varepsilon}{h}\sum_{i=1}^{k}\|v_n^i-v_n^{i-1}\|^2_{W^{1,2}(\sigma)}\le C\frac{\varepsilon}{h}\sum_{i=1}^{k}\|\nabla_{x'}v_n^i-\nabla_{x'}v_n^{i-1}\|^2_{L^2(\sigma)^2}\tag{3.37}
$$

and

$$
\frac{\varepsilon}{h}\sum_{i=1}^{k}\|\mathbf{w}_n^i-\mathbf{w}_n^{i-1}\|^2_{W^{1,2}(\sigma)^2}\le C\frac{\varepsilon}{h}\sum_{i=1}^{k}a(\mathbf{w}_n^i-\mathbf{w}_n^{i-1},\mathbf{w}_n^i-\mathbf{w}_n^{i-1})\tag{3.38}
$$

and finally, in view of (3.9), we have

$$
\frac{C(\varepsilon)}{h}\sum_{i=1}^{k}\|\mathbf{u}_n^i\|^4_{L^4(\sigma)^2}\|v_n^i-v_n^{i-1}\|^2_{L^2(\sigma)}\le hcC(\varepsilon)\sum_{i=1}^{k}\|\frac{v_n^i-v_n^{i-1}}{h}\|^2_{L^2(\sigma)}\tag{3.39}
$$

and

$$
\frac{C(\varepsilon)}{h}\sum_{i=1}^{k}\|\mathbf{u}_n^i\|^4_{L^4(\sigma)^2}\|\mathbf{w}_n^i-\mathbf{w}_n^{i-1}\|^2_{L^2(\sigma)^2}\le hcC(\varepsilon)\sum_{i=1}^{k}\|\frac{\mathbf{w}_n^i-\mathbf{w}_n^{i-1}}{h}\|^2_{L^2(\sigma)^2}.\tag{3.40}
$$

Hence, using (3.36)–(3.40), the inequality (3.35) can be further simplified as

$$
\begin{aligned}
&(1-2\varepsilon-hcC(\varepsilon))\sum_{i=1}^{k}\|\frac{v_n^i-v_n^{i-1}}{h}\|^2_{L^2(\sigma)}+\frac{\nu}{2h}\|\nabla_{x'}v_n^k\|^2_{L^2(\sigma)^2}\\
&+\frac{1}{h}\Big(\frac{\nu}{2}-C\varepsilon\Big)\sum_{i=1}^{k}\|\nabla_{x'}v_n^i-\nabla_{x'}v_n^{i-1}\|^2_{L^2(\sigma)^2}\\
&\quad+(1-2\varepsilon-hcC(\varepsilon))\sum_{i=1}^{k}\|\frac{\mathbf{w}_n^i-\mathbf{w}_n^{i-1}}{h}\|^2_{L^2(\sigma)^2}+\frac{(c_a+c_d)}{2h}a(\mathbf{w}_n^k,\mathbf{w}_n^k)\\
&+\frac{1}{h}\Big(\frac{c_a+c_d}{2}-C\varepsilon\Big)\sum_{i=1}^{k}a(\mathbf{w}_n^i-\mathbf{w}_n^{i-1},\mathbf{w}_n^i-\mathbf{w}_n^{i-1})\\
&\le\varepsilon\sum_{i=1}^{k}|q_n^i|^2+C(\varepsilon)\sum_{i=1}^{k}\|\mathbf{u}_n^i\|^2_{L^\infty(\sigma)^2}\Big(\|v_n^{i-1}\|^2_{W^{1,2}(\sigma)}+\|\mathbf{w}_n^{i-1}\|^2_{W^{1,2}(\sigma)^2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{c}{\varepsilon}\sum_{i=1}^{k}\Big|\frac{F_n^i-F_n^{i-1}}{h}\Big|^2+\frac{c}{\varepsilon}\sum_{i=1}^{k}\|f_n^i\|^2_{L^2(\sigma)}+\frac{c}{\varepsilon}\sum_{i=1}^{k}\|\mathbf{g}_n^i\|^2_{L^2(\sigma)^2}\\
&-\frac{2\nu_r}{h}(\mathbf{w}_n^0,\nabla^{\perp}_{x'}v_n^0)+\frac{2\nu_r}{h}\|\mathbf{w}_n^0\|^2_{L^2(\sigma)^2}+\frac{(\nu+\nu_r)}{2h}\|\nabla_{x'}v_n^0\|^2_{L^2(\sigma)^2}\\
&+\frac{(c_a+c_d)}{2h}a(\mathbf{w}_n^0,\mathbf{w}_n^0)+\frac{(c_0+c_d-c_a)}{2h}\|\operatorname{div}_{x'}\mathbf{w}_n^0\|^2_{L^2(\sigma)}.
\end{aligned}\tag{3.41}
$$

What remains is to handle the first term on the right hand side in (3.41). Here we follow the ideas used in [16]. Let V_0 be the solution of the following Dirichlet problem for the Poisson equation:

$$
-(\nu+\nu_r)\Delta_{x'}V_0=1\quad\text{in }\sigma,\tag{3.42}
$$

$$
V_0=0\quad\text{on }\partial\sigma.\tag{3.43}
$$

Using $\varphi=V_0$ as a test function in (3.11) we obtain

$$
\begin{aligned}
&\frac{1}{h}(v_n^i-v_n^{i-1},V_0)+(\nu+\nu_r)((\nabla_{x'}v_n^i,\nabla_{x'}V_0))+d(\mathbf{u}_n^i,v_n^i,V_0)\\
&=q_n^i(1,V_0)+2\nu_r(\operatorname{rot}_{x'}\mathbf{w}_n^i,V_0)+(f_n^i,V_0).
\end{aligned}\tag{3.44}
$$

From (3.42)–(3.43) and (3.13) we have

$$
(\nu+\nu_r)((\nabla_{x'}v_n^i,\nabla_{x'}V_0))=\int_\sigma v_n^i\,dx'=F_n^i.\tag{3.45}
$$

Hence, combining (3.44) with (3.45) we obtain

$$
\begin{aligned}
&\frac{1}{h}(v_n^i-v_n^{i-1},V_0)+F_n^i+d(\mathbf{u}_n^i,v_n^i,V_0)\\
&=q_n^i\int_\sigma V_0\,dx'+2\nu_r(\operatorname{rot}_{x'}\mathbf{w}_n^i,V_0)+(f_n^i,V_0).
\end{aligned}\tag{3.46}
$$

Furthermore, we have

$$
d(\mathbf{u}_n^i,v_n^i,V_0)\le c\|\mathbf{u}_n^i\|_{L^4(\sigma)^2}\|v_n^i\|_{W^{1,2}(\sigma)}\|V_0\|_{L^4(\sigma)}.
$$

From (3.46) we deduce

$$
\begin{aligned}
|q_n^i|^2\Big|\int_\sigma V_0\,dx'\Big|^2\le{}& c\Big\|\frac{v_n^i-v_n^{i-1}}{h}\Big\|^2_{L^2(\sigma)}\|V_0\|^2_{L^2(\sigma)}+c(F_n^i)^2\\
&+c\|\mathbf{u}_n^i\|^2_{L^4(\sigma)^2}\|v_n^i\|^2_{W^{1,2}(\sigma)}\|V_0\|^2_{L^4(\sigma)}\\
&+c(2\nu_r)^2\|\operatorname{rot}_{x'}\mathbf{w}_n^i\|^2_{L^2(\sigma)}\|V_0\|^2_{L^2(\sigma)}\\
&+c\|f_n^i\|^2_{L^2(\sigma)}\|V_0\|^2_{L^2(\sigma)}.
\end{aligned}\tag{3.47}
$$

Using Friedrichs' inequality

$$
\int_\sigma|V_0|^2\,dx'\le C\int_\sigma|\nabla_{x'}V_0|^2\,dx'
$$

and (3.42)–(3.43) we obtain

$$
\int_\sigma|V_0|^2\,dx'\le C\int_\sigma|\nabla_{x'}V_0|^2\,dx'=\frac{C}{(\nu+\nu_r)}\int_\sigma V_0\,dx'=\frac{C\kappa_0}{(\nu+\nu_r)},\tag{3.48}
$$

where $\kappa_0=\int_\sigma V_0\,dx'$. Moreover, by the Sobolev embedding theorem [7] we have

$$
\|V_0\|^2_{L^4(\sigma)}\le c_1\|V_0\|^2_{W^{1,2}(\sigma)}\le C\int_\sigma|\nabla_{x'}V_0|^2\,dx'=\frac{C\kappa_0}{(\nu+\nu_r)}.\tag{3.49}
$$

Now, combining (3.47) with (3.48)–(3.49) we deduce

$$\begin{aligned} |q_n^i|^2 \le \frac{c|F_n^i|^2}{(\kappa_0)^2} + \frac{C}{\kappa_0(\nu+\nu_r)}\Big(\|\frac{v_n^i - v_n^{i-1}}{h}\|^2_{L^2(\sigma)} + \|\mathbf{u}_n^i\|^2_{L^4(\sigma)^2}\|v_n^i\|^2_{W^{1,2}(\sigma)} \\ + (2\nu_r)^2\|\operatorname{rot}_{x'}\mathbf{w}_n^i\|^2_{L^2(\sigma)} + \|f_n^i\|^2_{L^2(\sigma)}\Big). \end{aligned} \tag{3.50}$$

Note that

$$\|\operatorname{rot}_{x'}\mathbf{w}_n^i\|^2_{L^2(\sigma)} \le C\|\mathbf{w}_n^i\|^2_{W^{1,2}(\sigma)^2}.$$

Combining (3.41) with (3.50) we arrive at the estimate

$$\begin{aligned}
&\Big(1 - 2\varepsilon - hcC(\varepsilon) - \varepsilon\frac{C}{\kappa_0(\nu+\nu_r)}\Big)\sum_{i=1}^k \|\frac{v_n^i - v_n^{i-1}}{h}\|^2_{L^2(\sigma)} \\
&+ \frac{\nu}{2h}\|\nabla_{x'}v_n^k\|^2_{L^2(\sigma)^2} + \frac{1}{h}\big(\frac{\nu}{2} - C\varepsilon\big)\sum_{i=1}^k\|\nabla_{x'}v_n^i - \nabla_{x'}v_n^{i-1}\|^2_{L^2(\sigma)^2} \\
&+ (1 - 2\varepsilon - hcC(\varepsilon))\sum_{i=1}^k\|\frac{\mathbf{w}_n^i - \mathbf{w}_n^{i-1}}{h}\|^2_{L^2(\sigma)^2} + \frac{(c_a+c_d)}{2h}\int_\sigma |\nabla_{x'}\mathbf{w}_n^k|^2\,dx' \\
&+ \frac{1}{h}\Big(\frac{c_a+c_d}{2} - C\varepsilon\Big)\sum_{i=1}^k\int_\sigma|\nabla_{x'}\mathbf{w}_n^i - \nabla_{x'}\mathbf{w}_n^{i-1}|^2 dx' \\
&\le \varepsilon\frac{C_1}{\kappa_0(\nu+\nu_r)}\sum_{i=1}^k\Big(\|\mathbf{u}_n^i\|^2_{L^4(\sigma)^2}\|v_n^i\|^2_{W^{1,2}(\sigma)} + C_2\nu_r^2\|\mathbf{w}_n^i\|^2_{W^{1,2}(\sigma)^2}\Big) \\
&\quad + C(\varepsilon)\sum_{i=1}^k\|\mathbf{u}_n^i\|^2_{L^\infty(\sigma)^2}\Big(\|v_n^{i-1}\|^2_{W^{1,2}(\sigma)} + \|\mathbf{w}_n^{i-1}\|^2_{W^{1,2}(\sigma)^2}\Big) \\
&\quad + \frac{\varepsilon c}{(\kappa_0)^2}\sum_{i=1}^k|F_n^i|^2 + \frac{c}{\varepsilon}\sum_{i=1}^k\big|\frac{F_n^i - F_n^{i-1}}{h}\big|^2 \\
&\quad + \Big(\frac{c}{\varepsilon} + \varepsilon\frac{C}{\kappa_0(\nu+\nu_r)}\Big)\sum_{i=1}^k\|f_n^i\|^2_{L^2(\sigma)} + \frac{c}{\varepsilon}\sum_{i=1}^k\|\mathbf{g}_n^i\|^2_{L^2(\sigma)^2} \\
&\quad - \frac{2\nu_r}{h}((\mathbf{w}_n^0, \nabla^\perp_{x'}v_n^0)) + \frac{2\nu_r}{h}\|\mathbf{w}_n^0\|^2_{L^2(\sigma)^2} + \frac{(\nu+\nu_r)}{2h}\|\nabla_{x'}v_n^0\|^2_{L^2(\sigma)^2} \\
&\quad + \frac{(c_a+c_d)}{2h}\int_\sigma|\nabla_{x'}\mathbf{w}_n^0|^2\,dx' + \frac{(c_0+c_d-c_a)}{2h}\|\operatorname{div}_{x'}\mathbf{w}_n^0\|^2_{L^2(\sigma)}.
\end{aligned}$$

Finally, taking $\varepsilon > 0$ small enough such that

$$\big(\frac{\nu}{2} - C\varepsilon\big) > 0, \quad \big(\frac{c_a+c_d}{2} - C\varepsilon\big) > 0$$

and then taking $h_0 > 0$ small enough such that (for all $h < h_0$)

$$\Big(1 - 2\varepsilon - hcC(\varepsilon) - 2\varepsilon\frac{C}{\kappa_0(\nu+\nu_r)}\Big) > 0 \quad \text{and} \quad (1 - 2\varepsilon - hcC(\varepsilon)) > 0,$$

we arrive at: for $k = 1, 2, \dots, n$,

$$
\begin{aligned}
&\|v_n^k\|_{W^{1,2}(\sigma)}^2 + \|\mathbf{w}_n^k\|_{W^{1,2}(\sigma)^2} \\
&+ h \sum_{i=1}^{k} \|\frac{v_n^i - v_n^{i-1}}{h}\|_{L^2(\sigma)}^2 + h \sum_{i=1}^{k} \|\frac{\mathbf{w}_n^i - \mathbf{w}_n^{i-1}}{h}\|_{L^2(\sigma)^2}^2 \\
&\le C_1 + C_2 h \sum_{i=1}^{k} \|\mathbf{u}_n^i\|_{L^\infty(\sigma)^2}^2 \Big(\|v_n^{i-1}\|_{W^{1,2}(\sigma)}^2 + \|\mathbf{w}_n^{i-1}\|_{W^{1,2}(\sigma)^2}^2 \Big) \\
&\quad + C_3 h \sum_{i=1}^{k} \big(\|v_n^i\|_{W^{1,2}(\sigma)}^2 + \|\mathbf{w}_n^i\|_{W^{1,2}(\sigma)^2}^2 \big).
\end{aligned}
\tag{3.51}
$$

From the latter estimate we can write

$$
\begin{aligned}
&(1 - hC_3)\Big(\|v_n^k\|_{W^{1,2}(\sigma)}^2 + \|\mathbf{w}_n^k\|_{W^{1,2}(\sigma)^2}^2 \Big) \\
&\le C_1 + C_2 h \|\mathbf{u}_n^1\|_{L^\infty(\sigma)^2}^2 \Big(\|v_n^0\|_{W^{1,2}(\sigma)}^2 + \|\mathbf{w}_n^0\|_{W^{1,2}(\sigma)^2}^2 \Big) \\
&\quad + h \sum_{i=1}^{k-1} \big(C_2 \|\mathbf{u}_n^{i+1}\|_{L^\infty(\sigma)^2}^2 + C_3 \big) \Big(\|v_n^i\|_{W^{1,2}(\sigma)}^2 + \|\mathbf{w}_n^i\|_{W^{1,2}(\sigma)^2}^2 \Big).
\end{aligned}
$$

Now, assuming $h_0 > 0$ small enough so that $h_0 < 1/C_3$, we can write (for all $h < h_0$)

$$
\|v_n^k\|_{W^{1,2}(\sigma)}^2 + \|\mathbf{w}_n^k\|_{W^{1,2}(\sigma)^2}^2 \le c_1 + c_2 h \sum_{i=1}^{k-1} A_i \Big(\|v_n^i\|_{W^{1,2}(\sigma)}^2 + \|\mathbf{w}_n^i\|_{W^{1,2}(\sigma)^2}^2 \Big),
$$

where

$$
A_i = C_2 \|\mathbf{u}_n^{i+1}\|_{L^\infty(\sigma)^2}^2 + C_3.
$$

Note that, in view of (3.10), we have

$$
h \sum_{i=1}^{k-1} A_i < C,
$$

where C is independent of h. Now, we can use directly the discrete version of the Gronwall inequality (see [17, Theorem 1.46]). In such a way, we obtain

$$
\|v_n^k\|_{W^{1,2}(\sigma)}^2 \le C, \quad k = 1, 2, \dots, n, \tag{3.52}
$$

$$
\|\mathbf{w}_n^k\|_{W^{1,2}(\sigma)^2}^2 \le C, \quad k = 1, 2, \dots, n, \tag{3.53}
$$

and from (3.51) we obtain also the estimates

$$
h \sum_{i=1}^{k} \|\frac{v_n^i - v_n^{i-1}}{h}\|_{L^2(\sigma)}^2 \le C, \quad k = 1, 2, \dots, n, \tag{3.54}
$$

$$
h \sum_{i=1}^{k} \|\frac{\mathbf{w}_n^i - \mathbf{w}_n^{i-1}}{h}\|_{L^2(\sigma)^2}^2 \le C, \quad k = 1, 2, \dots, n. \tag{3.55}
$$

Temporal interpolants and uniform estimates. For each fixed time step h, we define the piecewise constant interpolants

$$\bar{\varphi}_n(t) = \varphi_n^i$$

for $t \in ((i-1)h, ih]$ and, in addition, we extend $\bar{\varphi}_n$ for $t \le 0$ by $\bar{\varphi}_n(t) = \varphi_0$ for $t \in (-h, 0]$. Furthermore, we define the piecewise linear time interpolants $(i = 1, 2, \dots, n)$ with

$$\phi_n(t) = \phi_n^{i-1} + \frac{t-(i-1)h}{h}(\phi_n^i - \phi_n^{i-1})$$

for $t \in ((i-1)h, ih]$. As a consequence of the estimates (3.52)–(3.55) we have

$$\|\bar{v}_n(t)\|^2_{W^{1,2}(\sigma)} \le C \quad \text{for all } t \in [0,T], \tag{3.56}$$

$$\|\bar{\mathbf{w}}_n(t)\|^2_{W^{1,2}(\sigma)^2} \le C \quad \text{for all } t \in [0,T], \tag{3.57}$$

$$\int_0^T \|\partial_t v_n(t)\|^2_{L^2(\sigma)} \mathrm{d}t \le C, \tag{3.58}$$

$$\int_0^T \|\partial_t \mathbf{w}_n(t)\|^2_{L^2(\sigma)^2} \mathrm{d}t \le C. \tag{3.59}$$

Finally, in view of (3.50), we also have

$$\int_0^T |\bar{q}_n(t)|^2 \, dt \le C. \tag{3.60}$$

Passage to the limit. By (3.11)–(3.13), the time interpolants

$$\bar{v}_n \in L^\infty(0,T; W_0^{1,2}(\sigma)), \quad \bar{\mathbf{w}}_n \in L^\infty(0,T; W_0^{1,2}(\sigma)^2),$$
$$v_n \in W^{1,2}(0,T; L^2(\sigma)), \quad \mathbf{w}_n \in W^{1,2}(0,T; L^2(\sigma)^2), \quad \bar{q}_n \in L^\infty((0,T)),$$

and satisfy the equations

$$\begin{aligned} &\frac{d}{dt}(v_n(t), \varphi) + (\nu + \nu_r)((\nabla_{x'}\bar{v}_n(t), \nabla_{x'}\varphi)) + d(\bar{\mathbf{u}}_n(t), \bar{v}_n(t), \varphi) \\ &= \bar{q}_n(t)(1,\varphi) + 2\nu_r(\mathrm{rot}_{x'}\,\bar{\mathbf{w}}_n(t), \varphi) + (\bar{f}_n(t), \varphi) \end{aligned} \tag{3.61}$$

for all $\varphi \in W_0^{1,2}(\sigma)$,

$$\begin{aligned} &\frac{d}{dt}((\mathbf{w}_n(t), \boldsymbol{\psi})) + (c_a + c_d)a(\bar{\mathbf{w}}_n(t), \boldsymbol{\psi}) + b(\bar{\mathbf{u}}_n(t), \bar{\mathbf{w}}_n(t), \boldsymbol{\psi}) \\ &+ (c_0 + c_d - c_a)(\mathrm{div}_{x'}\,\bar{\mathbf{w}}_n(t), \mathrm{div}_{x'}\,\boldsymbol{\psi}) + 4\nu_r((\bar{\mathbf{w}}_n(t), \boldsymbol{\psi})) \\ &= 2\nu_r((\nabla^\perp_{x'}\bar{v}_n(t), \boldsymbol{\psi})) + ((\bar{\mathbf{g}}_n(t), \boldsymbol{\psi})) \end{aligned} \tag{3.62}$$

for all $\boldsymbol{\psi} \in W_0^{1,2}(\sigma)^2$ and for almost every $t \in (0,T)$ and the flux condition

$$\int_\sigma \bar{v}_n(t) \, dx' = \bar{F}_n(t) \quad \text{for all } t \in (0,T). \tag{3.63}$$

The a priori estimates (3.56)–(3.60) allow us to conclude that there exist $v \in L^2(0,T; W_0^{1,2}(\sigma))$, $\hat{\mathbf{w}} \in L^2(0,T; W_0^{1,2}(\sigma)^2)$ and $q \in L^2((0,T))$ such that, letting $n \to +\infty$ (along a selected subsequence),

$$\bar{v}_n \rightharpoonup v \quad \text{weakly* in } L^\infty(0,T; W_0^{1,2}(\sigma)), \tag{3.64}$$

$$\bar{\mathbf{w}}_n \rightharpoonup \hat{\mathbf{w}} \quad \text{weakly* in } L^\infty(0,T; W_0^{1,2}(\sigma)^2), \tag{3.65}$$

$$\partial_t v_n \rightharpoonup \partial_t v \quad \text{weakly in } L^2(0,T; L^2(\sigma)), \tag{3.66}$$

$$\partial_t \mathbf{w}_n \rightharpoonup \partial_t \hat{\mathbf{w}} \quad \text{weakly in } L^2(0,T;L^2(\sigma)^2), \tag{3.67}$$

$$\bar{q}_n \rightharpoonup q \quad \text{weakly in } L^2((0,T)). \tag{3.68}$$

The above established convergences (3.64)–(3.68) are sufficient for taking the limit $n \to \infty$ in (3.61), (3.62) and (3.63) (along a selected subsequence) to get the weak solution of the system (1.9)–(1.10) in the sense of Definition 3.1 on $(0,T)$, $T \in (0,+\infty)$.

Solvability of problem (1.9)–(1.10) **on** $(0,+\infty)$**.** Using $\varphi = v$ as a test function in equation (3.5), $\boldsymbol{\psi} = \hat{\mathbf{w}}$ as a test function in equation (3.6) and integrating from 0 to s we obtain, in particular,

$$\begin{aligned} &\frac{1}{2}\|v(s)\|^2_{L^2(\sigma)} + (\nu+\nu_r)\int_0^s \|\nabla_{x'} v(t)\|^2_{L^2(\sigma)^2} dt + \int_0^s d(\hat{\mathbf{u}}(t), v(t), v(t))\,dt \\ &= \frac{1}{2}\|v_0\|^2_{L^2(\sigma)} + \int_0^s q(t)F(t)\,dt + 2\nu_r \int_0^s (\operatorname{rot}_{x'} \hat{\mathbf{w}}(t), v(t))\,dt \\ &\quad + \int_0^s (f(t), v(t))\,dt \end{aligned} \tag{3.69}$$

and

$$\begin{aligned} &\frac{1}{2}\|\hat{\mathbf{w}}(s)\|^2_{L^2(\sigma)^2} + (c_a+c_d)\int_0^s a(\hat{\mathbf{w}}(t), \hat{\mathbf{w}}(t))\,dt + \int_0^s b(\hat{\mathbf{u}}(t), \hat{\mathbf{w}}(t), \hat{\mathbf{w}}(t))\,dt \\ &+ (c_0+c_d-c_a)\int_0^s \|\operatorname{div}_{x'} \hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)}\,dt + 4\nu_r \int_0^s \|\hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)^2}\,dt \\ &= \frac{1}{2}\|\hat{\mathbf{w}}_0\|^2_{L^2(\sigma)^2} + 2\nu_r \int_0^s ((\nabla^{\perp}_{x'} v(t), \hat{\mathbf{w}}(t)))\,dt + \int_0^s ((\hat{\mathbf{g}}(t), \hat{\mathbf{w}}(t)))\,dt. \end{aligned} \tag{3.70}$$

Recall that

$$\int_0^s ((\operatorname{rot}_{x'} \hat{\mathbf{w}}(t), v(t)))\,dt = \int_0^s ((\nabla^{\perp}_{x'} v(t), \hat{\mathbf{w}}(t)))\,dt \tag{3.71}$$

and that

$$\Big|\int_0^s ((\nabla^{\perp}_{x'} v(t), \hat{\mathbf{w}}(t)))\,dt\Big| \le \frac{1}{4}\int_0^s \|\nabla_{x'} v(t)\|^2_{L^2(\sigma)^2} dt + \int_0^s \|\hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)^2} dt. \tag{3.72}$$

Furthermore, we have

$$\begin{aligned} &\Big|\int_0^s d(\hat{\mathbf{u}}(t), v(t), v(t))\,dt\Big| \\ &\le c\int_0^s \|\hat{\mathbf{u}}(t)\|_{L^4(\sigma)^2} \|v(t)\|_{W^{1,2}(\sigma)} \|v(t)\|_{L^4(\sigma)}\,dt \\ &\le C(\varepsilon)\int_0^s \|\hat{\mathbf{u}}(t)\|^4_{L^4(\sigma)^2} \|v(t)\|^2_{L^2(\sigma)}\,dt + \varepsilon \int_0^s \|v(t)\|^2_{W^{1,2}(\sigma)}\,dt \end{aligned} \tag{3.73}$$

and

$$\begin{aligned} &\Big|\int_0^s b(\hat{\mathbf{u}}(t), \hat{\mathbf{w}}(t), \hat{\mathbf{w}}(t))\,dt\Big| \\ &\le c\int_0^s \|\hat{\mathbf{u}}(t)\|_{L^4(\sigma)^2} \|\hat{\mathbf{w}}(t)\|_{W^{1,2}(\sigma)^2} \|\hat{\mathbf{w}}(t)\|_{L^4(\sigma)^2} dt \\ &\le C(\varepsilon)\int_0^s \|\hat{\mathbf{u}}(t)\|^4_{L^4(\sigma)^2} \|\hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)^2} dt + \varepsilon \int_0^s \|\hat{\mathbf{w}}(t)\|^2_{W^{1,2}(\sigma)^2}\,dt. \end{aligned} \tag{3.74}$$

Combining (3.69)–(3.74) we obtain

$$
\begin{aligned}
&\frac{1}{2}\|v(s)\|^2_{L^2(\sigma)} + \frac{1}{2}\|\hat{\mathbf{w}}(s)\|^2_{L^2(\sigma)^2} \\
&+ (\nu+\nu_r)\int_0^s \|\nabla_{x'} v(t)\|^2_{L^2(\sigma)^2}\,dt + (c_a+c_d)\int_0^s a(\hat{\mathbf{w}}(t),\hat{\mathbf{w}}(t))\,dt \\
&+ (c_0+c_d-c_a)\int_0^s \|\operatorname{div}_{x'}\hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)}\,dt + 4\nu_r\int_0^s \|\hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)^2}dt \\
&\le \frac{1}{2}\|v_0\|^2_{L^2(\sigma)} + \frac{1}{2}\|\hat{\mathbf{w}}_0\|^2_{L^2(\sigma)^2} \\
&\quad + \nu_r\int_0^s \|\nabla_{x'} v(t)\|^2_{L^2(\sigma)^2}\,dt + 4\nu_r\int_0^s \|\hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)^2}\,dt \\
&\quad + \xi\int_0^s \|v(t)\|^2_{W^{1,2}(\sigma)}dt + c_1(\xi)\int_0^s \|\hat{\mathbf{u}}(t)\|^4_{L^4(\sigma)^2}\|v(t)\|^2_{L^2(\sigma)}\,dt \\
&\quad + \xi\int_0^s \|\hat{\mathbf{w}}(t)\|^2_{W^{1,2}(\sigma)^2}\,dt + c_2(\xi)\int_0^s \|\hat{\mathbf{u}}(t)\|^4_{L^4(\sigma)^2}\|\hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)^2}\,dt \\
&\quad + \xi\int_0^s |q(t)|^2dt + c_3(\xi)\int_0^s |F(t)|^2dt \\
&\quad + \xi\Big(\int_0^s \|\hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)^2}\,dt + \int_0^s \|v(t)\|^2_{L^2(\sigma)}dt\Big) \\
&\quad + c_4(\xi)\Big(\int_0^s \|\hat{\mathbf{g}}(t)\|^2_{L^2(\sigma)^2}dt + \int_0^s \|f(t)\|^2_{L^2(\sigma)}\,dt\Big),
\end{aligned}
$$

where ξ is an "arbitrarily small" positive real number. Applying the Friedrichs inequality, the latter estimate can be further simplified as

$$
\begin{aligned}
&\|v(s)\|^2_{L^2(\sigma)} + \|\hat{\mathbf{w}}(s)\|^2_{L^2(\sigma)^2} + \int_0^s \|v(t)\|^2_{W^{1,2}(\sigma)}\,dt + \int_0^s \|\hat{\mathbf{w}}(t)\|^2_{W^{1,2}(\sigma)^2}\,dt \\
&\le c_1(\xi)\Big(\|v_0\|^2_{L^2(\sigma)} + \|\hat{\mathbf{w}}_0\|^2_{L^2(\sigma)^2}\Big) \\
&\quad + c_2(\xi)\int_0^s \|\hat{\mathbf{u}}(t)\|^4_{L^4(\sigma)^2}\big(\|v(t)\|^2_{L^2(\sigma)} + \|\hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)^2}\big)\,dt \\
&\quad + \xi\int_0^s |q(t)|^2dt + c_3(\xi)\int_0^s |F(t)|^2dt \\
&\quad + c_4(\xi)\Big(\int_0^s \|\hat{\mathbf{g}}(t)\|^2_{L^2(\sigma)^2}\,dt + \int_0^s \|f(t)\|^2_{L^2(\sigma)}\,dt\Big).
\end{aligned} \tag{3.75}
$$

Moreover, in view of [16, Theorem 2.7, eq. (2.74)], we have

$$
\begin{aligned}
&\|\partial_t v\|^2_{L^2(0,s;L^2(\sigma))} + \|v\|^2_{L^\infty(0,s;W^{1,2}(\sigma))} + \|v\|^2_{L^2(0,s;W^{2,2}(\sigma))} + \|q\|^2_{L^2((0,s))} \\
&\le c\Big(\|\operatorname{rot}_{x'}\hat{\mathbf{w}}\|^2_{L^2(0,s;L^2(\sigma))} + \|(\hat{\mathbf{u}}\cdot\nabla_{x'})v\|^2_{L^2(0,s;L^2(\sigma))} + \|f\|^2_{L^2(0,s;L^2(\sigma))} \\
&\quad + \|F\|^2_{W^{1,2}((0,s))} + \|v_0\|^2_{W^{1,2}(\sigma)}\Big),
\end{aligned} \tag{3.76}
$$

where c does not depend on s.

Using the Sobolev embedding and the interpolation inequality [7, 10] we have

$$
\|v\|_{W^{1,4}(\sigma)} \le c_1\|v\|_{W^{3/2,2}(\sigma)} \le c_2\|v\|^{1/2}_{W^{1,2}(\sigma)}\|v\|^{1/2}_{W^{2,2}(\sigma)}.
$$

Furthermore, we can write

$$\begin{aligned}
&\|(\hat{\mathbf{u}}\cdot\nabla_{x'})v\|^2_{L^2(0,s;L^2(\sigma))}\\
&\le c_1(\sigma)\int_0^s \|\hat{\mathbf{u}}(t)\|^2_{L^4(\sigma)^2}\|v(t)\|^2_{W^{1,4}(\sigma)}\mathrm{d}t\\
&\le c_1(\sigma)\int_0^s \|\hat{\mathbf{u}}(t)\|^2_{L^4(\sigma)^2}\Big(c_2(\delta)\|v(t)\|^2_{W^{1,2}(\sigma)}+\delta\|v(t)\|^2_{W^{2,2}(\sigma)}\Big)\mathrm{d}t\\
&\le c_1(\sigma)\|\hat{\mathbf{u}}\|^2_{L^\infty(0,s;L^4(\sigma)^2)}\Big(c_2(\delta)\|v\|^2_{L^2(0,s;W^{1,2}(\sigma))}+\delta\|v\|^2_{L^2(0,s;W^{2,2}(\sigma))}\Big).
\end{aligned}\tag{3.77}$$

Note that

$$\|\hat{\mathbf{u}}\|^2_{L^\infty(0,s;L^4(\sigma)^2)}\le c\|\hat{\mathbf{u}}\|^2_{L^\infty(0,s;W^{1,2}(\sigma)^2)}\le c\|\hat{\mathbf{u}}\|^2_{L^\infty(0,\infty;W^{1,2}(\sigma)^2)}\le C,$$

where C is independent of s. Therefore, using (3.77) in (3.76) and taking δ small enough we deduce

$$\begin{aligned}
\|q\|^2_{L^2((0,s))}\le c\Big(&\|\hat{\mathbf{w}}\|^2_{L^2(0,s;W^{1,2}(\sigma)^2)}+\|v\|^2_{L^2(0,s;W^{1,2}(\sigma))}\\
&+\|f\|^2_{L^2(0,s;L^2(\sigma))}+\|F\|^2_{W^{1,2}((0,s))}+\|v_0\|^2_{W^{1,2}(\sigma)}\Big).
\end{aligned}\tag{3.78}$$

Now, substituting (3.78) into (3.75) and taking ξ "small" enough we obtain

$$\begin{aligned}
&\|v(s)\|^2_{L^2(\sigma)}+\|\hat{\mathbf{w}}(s)\|^2_{L^2(\sigma)^2}+\|v\|^2_{L^2(0,s;W^{1,2}(\sigma))}+\|\hat{\mathbf{w}}\|^2_{L^2(0,s;W^{1,2}(\sigma)^2)}\\
&\le c_1\Big(\|v_0\|^2_{L^2(\sigma)}+\|v_0\|^2_{W^{1,2}(\sigma)}+\|\hat{\mathbf{w}}_0\|^2_{L^2(\sigma)^2}\Big)\\
&\quad+c_2\Big(\|f\|^2_{L^2(0,s;L^2(\sigma))}+\|\hat{\mathbf{g}}\|^2_{L^2(0,s;L^2(\sigma)^2)}+\|F\|^2_{W^{1,2}((0,s))}\Big)\\
&\quad+c_3\int_0^s\|\hat{\mathbf{u}}(t)\|^4_{L^4(\sigma)^2}\Big(\|v(t)\|^2_{L^2(\sigma)}+\|\hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)^2}\Big)dt.
\end{aligned}\tag{3.79}$$

Note that (3.79) holds for all $s\ge 0$. Further, introducing the notation

$$\begin{gathered}
C_1=c_1\Big(\|v_0\|^2_{L^2(\sigma)}+\|v_0\|^2_{W^{1,2}(\sigma)}+\|\hat{\mathbf{w}}_0\|^2_{L^2(\sigma)^2}\Big),\\
\chi(s)=c_2\Big(\|f\|^2_{L^2(0,s;L^2(\sigma))}+\|\hat{\mathbf{g}}\|^2_{L^2(0,s;L^2(\sigma)^2)}+\|F\|^2_{W^{1,2}((0,s))}\Big),\\
C_2=C_1+\chi(+\infty),
\end{gathered}$$

the inequality (3.79) can be simplified as

$$\begin{aligned}
&\|v(s)\|^2_{L^2(\sigma)}+\|\hat{\mathbf{w}}(s)\|^2_{L^2(\sigma)^2}\\
&\le C_2+\int_0^s c_3\|\hat{\mathbf{u}}(t)\|^4_{L^4(\sigma)^2}\Big(\|v(t)\|^2_{L^2(\sigma)}+\|\hat{\mathbf{w}}(t)\|^2_{L^2(\sigma)^2}\Big)dt.
\end{aligned}$$

Applying the Gronwall inequality we arrive at

$$\|v(s)\|^2_{L^2(\sigma)}+\|\hat{\mathbf{w}}(s)\|^2_{L^2(\sigma)^2}\le C_2\exp\int_0^s c_3\|\hat{\mathbf{u}}(t)\|^4_{L^4(\sigma)^2}dt.\tag{3.80}$$

Recall that we assume $\hat{\mathbf{u}}\in L^2(0,\infty;W^{2,2}(\sigma)^2)\cap L^\infty(0,\infty;V)$, see (3.1). Raising and integrating the interpolation inequality [1, Theorem 5.8]

$$\|\hat{\mathbf{u}}(t)\|_{L^4(\sigma)^2}\le c\|\hat{\mathbf{u}}(t)\|^{1/2}_{W^{1,2}(\sigma)^2}\|\hat{\mathbf{u}}(t)\|^{1/2}_{L^2(\sigma)^2},$$

from 0 to s we obtain

$$\begin{aligned}\Big(\int_0^s \|\hat{\mathbf{u}}(t)\|^4_{L^4(\sigma)^2}\mathrm{d}t\Big)^{1/4} &\le c\Big(\int_0^s \|\hat{\mathbf{u}}(t)\|^2_{L^2(\sigma)^2}\|\hat{\mathbf{u}}(t)\|^2_{W^{1,2}(\sigma)^2}\mathrm{d}t\Big)^{1/4}\\ &\le c\|\hat{\mathbf{u}}\|^{1/2}_{L^2(0,s;L^2(\sigma)^2)}\|\hat{\mathbf{u}}\|^{1/2}_{L^\infty(0,s;W^{1,2}(\sigma)^2)}\\ &\le c\|\hat{\mathbf{u}}\|^{1/2}_{L^2(0,s;W^{2,2}(\sigma)^2)}\|\hat{\mathbf{u}}\|^{1/2}_{L^\infty(0,s;W^{1,2}(\sigma)^2)}\end{aligned} \tag{3.81}$$

where $c = c(\sigma)$. Now, letting $s \to \infty$ we obtain $\hat{\mathbf{u}} \in L^4(0,\infty;L^4(\sigma)^2)$ and from (3.80) we have

$$\|v(s)\|^2_{L^2(\sigma)} + \|\hat{\mathbf{w}}(s)\|^2_{L^2(\sigma)^2} \le c$$

for all s and c does not depend on s. Hence, from (3.79) we further deduce

$$\|v\|^2_{L^2(0,s;W^{1,2}(\sigma))} + \|\hat{\mathbf{w}}\|^2_{L^2(0,s;W^{1,2}(\sigma)^2)} \le C_2 + c\int_0^s \|\hat{\mathbf{u}}(t)\|^4_{L^4(\sigma)^2}dt$$

and, finally, letting $s \to +\infty$,

$$\|v\|^2_{L^2(0,\infty;W_0^{1,2}(\sigma))} + \|\hat{\mathbf{w}}\|^2_{L^2(0,\infty;W_0^{1,2}(\sigma)^2)} \le c. \tag{3.82}$$

Now, in view of (2.5) (with $T = +\infty$) and (3.82) we have

$$2\nu_r(\mathrm{rot}_{x'}\,\hat{\mathbf{w}},\cdot) + (f,\cdot) - d(\hat{\mathbf{u}},v,\cdot) \in L^2(0,\infty;L^2(\sigma)).$$

Moreover, using [16, Theorem 2.7, eq. (2.74)], we deduce

$$\begin{aligned}&\|\partial_t v\|^2_{L^2(0,\infty;L^2(\sigma))} + \|v\|^2_{L^\infty(0,\infty;W_0^{1,2}(\sigma))} + \|q(\tau)\|^2_{L^2((0,\infty))}\\ &\le c\Big(\|f\|^2_{L^2(0,\infty;L^2(\sigma))} + \|\mathrm{rot}_{x'}\,\hat{\mathbf{w}}\|^2_{L^2(0,\infty;L^2(\sigma))}\\ &\quad + \|d(\hat{\mathbf{u}},v,\cdot)\|^2_{L^2(0,\infty;L^2(\sigma))} + \|F\|^2_{W^{1,2}((0,\infty))} + \|v_0\|^2_{W_0^{1,2}(\sigma)}\Big).\end{aligned}$$

On the other hand, with $v \in L^2(0,\infty;W_0^{1,2}(\sigma))$ and $\hat{\mathbf{w}} \in L^2(0,\infty;W_0^{1,2}(\sigma)^2)$ in hand, we rewrite (3.6) as

$$\begin{aligned}&\frac{d}{dt}((\hat{\mathbf{w}}(t),\boldsymbol{\psi})) + (c_a+c_d)a(\hat{\mathbf{w}}(t),\boldsymbol{\psi}) + (c_0+c_d-c_a)(\mathrm{div}_{x'}\,\hat{\mathbf{w}}(t),\mathrm{div}_{x'}\,\boldsymbol{\psi})\\ &= 2\nu_r((\nabla^\perp_{x'}v(t),\boldsymbol{\psi})) + ((\hat{\mathbf{g}}(t),\boldsymbol{\psi})) - 4\nu_r((\hat{\mathbf{w}}(t),\boldsymbol{\psi})) - b(\hat{\mathbf{u}}(t),\hat{\mathbf{w}}(t),\boldsymbol{\psi})\end{aligned}$$

for all $\boldsymbol{\psi} \in W_0^{1,2}(\sigma)^2$ and for almost every $t \in (0,T)$ and $\hat{\mathbf{w}}(x',0) = \hat{\mathbf{w}}_0(x')$.

In view of (2.5), (3.2) and (3.82) we have

$$2\nu_r((\nabla^\perp_{x'}v,\cdot)) + ((\hat{\mathbf{g}},\cdot)) - 4\nu_r((\hat{\mathbf{w}},\cdot)) - b(\hat{\mathbf{u}},\hat{\mathbf{w}},\cdot) \in L^2(0,\infty;L^2(\sigma)^2).$$

Note that the bilinear form $\gamma(\cdot,\cdot)$, defined by the equation

$$\gamma(\boldsymbol{\phi},\boldsymbol{\psi}) := (c_a+c_d)a(\boldsymbol{\phi},\boldsymbol{\psi}) + (c_0+c_d-c_a)(\mathrm{div}_{x'}\,\boldsymbol{\phi},\mathrm{div}_{x'}\,\boldsymbol{\psi})$$

for all $\boldsymbol{\phi},\boldsymbol{\psi} \in W_0^{1,2}(\sigma)^2$, is symmetric and positive definite. Hence, we have

$$\partial_t\hat{\mathbf{w}} \in L^2(0,\infty;L^2(\sigma)^2), \quad \hat{\mathbf{w}} \in L^\infty(0,\infty;W_0^{1,2}(\sigma)^2),$$

such that

$$\begin{aligned}&\|\partial_t\hat{\mathbf{w}}\|^2_{L^2(0,\infty;L^2(\sigma)^2)} + \|\hat{\mathbf{w}}\|^2_{L^\infty(0,\infty;W_0^{1,2}(\sigma)^2)}\\ &\le c\Big(\|\nabla^\perp_{x'}v\|^2_{L^2(0,\infty;L^2(\sigma)^2)} + \|\hat{\mathbf{w}}\|^2_{L^2(0,\infty;L^2(\sigma)^2)}\\ &\quad + \|b(\hat{\mathbf{u}}(t),\hat{\mathbf{w}}(t),\cdot)\|^2_{L^2(0,\infty;L^2(\sigma)^2)} + \|\hat{\mathbf{g}}\|^2_{L^2(0,\infty;L^2(\sigma)^2)} + \|\hat{\mathbf{w}}_0\|^2_{W_0^{1,2}(\sigma)^2}\Big)\\ &\le C,\end{aligned}$$

see Theorem 5.1. The proof of Theorem 3.2 is thus complete.

4. Existence and uniqueness for the coupled problem (1.8)–(1.10)

Theorem 4.1. *Let $T \in (0, \infty]$ and suppose that*

$$\begin{gathered} \hat{\mathbf{f}} \in L^2(0,T;H), \quad g \in L^2(0,T;L^2(\sigma)), \\ \hat{\mathbf{u}}_0 \in V, \quad \omega_0 \in W_0^{1,2}(\sigma), \\ \hat{\mathbf{g}} \in L^2(0,T;L^2(\sigma)^2), \quad f \in L^2(0,T;L^2(\sigma)), \quad F \in W^{1,2}((0,T)), \\ \hat{\mathbf{w}}_0 \in W_0^{1,2}(\sigma)^2, \quad v_0 \in W_0^{1,2}(\sigma). \end{gathered}$$

Then there exist a pair $[\hat{\mathbf{u}}, \omega]$, such that

$$\hat{\mathbf{u}} \in L^\infty(0,T;V) \cap W^{1,2}(0,T;H), \tag{4.1}$$

$$\omega \in L^\infty(0,T;W_0^{1,2}(\sigma)) \cap W^{1,2}(0,T;L^2(\sigma)) \tag{4.2}$$

and a triplet $[v, \hat{\mathbf{w}}, q]$, such that

$$\begin{gathered} v \in L^\infty(0,T;W_0^{1,2}(\sigma)) \cap W^{1,2}(0,T;L^2(\sigma)), \\ \hat{\mathbf{w}} \in L^\infty(0,T;W_0^{1,2}(\sigma)^2) \cap W^{1,2}(0,T;L^2(\sigma)^2), \\ q \in L^2((0,T)), \end{gathered}$$

satisfying (2.1)–(2.4) *and* (3.4)–(3.7)*, respectively, for almost every $t \in (0,T)$.*
The solution to the coupled problem (1.8)–(1.10) *is also globally unique.*

Proof. The existence of $[\hat{\mathbf{u}}, \omega]$ satisfying (4.1) and (4.2) follows directly from Theorem 2.2 and Theorem 2.3. With $[\hat{\mathbf{u}}, \omega]$ in hand, the existence of $[v, \hat{\mathbf{w}}, q]$ follows from Theorem 3.2.

Note that the uniqueness result for the two-dimensional system of Navier-Stokes equations is a classical result, see e.g. [21]. The uniqueness of the weak solution $[\hat{\mathbf{u}}, \omega]$ to the problem (1.8) can be found in [12]. Now, suppose that there are two solutions $[v_1, \hat{\mathbf{w}}_1, q_1]$ and $[v_2, \hat{\mathbf{w}}_2, q_2]$ of the problem (3.4)–(3.7) on $(0, +\infty)$. Denote $v_{12} = v_1 - v_2$, $\hat{\mathbf{w}}_{12} = \hat{\mathbf{w}}_1 - \hat{\mathbf{w}}_2$ and $q_{12} = q_1 - q_2$. Then it holds $v_{12}(x', 0) = 0$ and $\hat{\mathbf{w}}_{12}(x', 0) = \mathbf{0}$, with v_{12}, $\hat{\mathbf{w}}_{12}$ and q_{12} satisfying the equations

$$\begin{aligned} &\frac{d}{dt}(v_{12}(t), \varphi) + (\nu + \nu_r)((\nabla_{x'} v_{12}(t), \nabla_{x'} \varphi)) + d(\hat{\mathbf{u}}(t), v_{12}(t), \varphi) \\ &= q_{12}(t)(1, \varphi) + 2\nu_r(\operatorname{rot}_{x'} \hat{\mathbf{w}}_{12}(t), \varphi) \end{aligned} \tag{4.3}$$

for all $\varphi \in W_0^{1,2}(\sigma)$,

$$\begin{aligned} &\frac{d}{dt}((\hat{\mathbf{w}}_{12}(t), \boldsymbol{\psi})) + (c_a + c_d) a(\hat{\mathbf{w}}_{12}(t), \boldsymbol{\psi}) + b(\hat{\mathbf{u}}(t), \hat{\mathbf{w}}_{12}(t), \boldsymbol{\psi}) \\ &+ (c_0 + c_d - c_a)(\operatorname{div}_{x'} \hat{\mathbf{w}}_{12}(t), \operatorname{div}_{x'} \boldsymbol{\psi}) + 4\nu_r((\hat{\mathbf{w}}_{12}(t), \boldsymbol{\psi})) \\ &= 2\nu_r((\nabla_{x'}^{\perp} v_{12}(t), \boldsymbol{\psi})) \end{aligned} \tag{4.4}$$

for all $\boldsymbol{\psi} \in W_0^{1,2}(\sigma)^2$ and for a.e. $t \in (0, +\infty)$; as well as the flux condition

$$\int_\sigma v_{12}(x', t)\, dx' = 0 \quad \text{on } (0, +\infty).$$

Hence substituting $\varphi = v_{12}$ and $\boldsymbol{\psi} = \hat{\mathbf{w}}_{12}$ in relations (4.3)–(4.4) and integrating from 0 to s, we obtain

$$\begin{aligned} &\frac{1}{2}\|v_{12}(t)\|^2_{L^2(\sigma)} + (\nu+\nu_r)\int_0^s \|\nabla_{x'} v_{12}(t)\|^2_{L^2(\sigma)^2}\,dt \\ &+ \int_0^s d(\hat{\mathbf{u}}(t), v_{12}(t), v_{12}(t))\,dt \\ &= \frac{1}{2}\|v_{12}(0)\|^2_{L^2(\sigma)} + \int_0^s q_{12}(t)\underbrace{\int_\sigma v_{12}\ dx'}_{=0}\ dt \\ &\quad + 2\nu_r\int_0^s (\mathrm{rot}_{x'}\,\hat{\mathbf{w}}_{12}(t), v_{12}(t))\,dt \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} &\frac{1}{2}\|\hat{\mathbf{w}}_{12}(t)\|^2_{L^2(\sigma)^2} - \frac{1}{2}\|\hat{\mathbf{w}}_{12}(0)\|^2_{L^2(\sigma)^2} + (c_a+c_d)\int_0^s\int_\sigma |\nabla_{x'}\hat{\mathbf{w}}_{12}|^2\,dx'\,dt \\ &+ \int_0^s b(\hat{\mathbf{u}}(t), \hat{\mathbf{w}}_{12}(t), \hat{\mathbf{w}}_{12}(t))\,dt + (c_0+c_d-c_a)\int_0^s \|\,\mathrm{div}_{x'}\,\hat{\mathbf{w}}_{12}(t)\|^2_{L^2(\sigma)^2}\,dt \\ &+ 4\nu_r\int_0^s \|\hat{\mathbf{w}}_{12}(t)\|^2_{L^2(\sigma)^2}\,dt \\ &= 2\nu_r\int_0^s ((\nabla^{\perp}_{x'} v_{12}(t), \hat{\mathbf{w}}_{12}(t)))\,dt. \end{aligned} \tag{4.6}$$

Now, combining (4.5) and (4.6) and using (3.18) we obtain

$$\begin{aligned} &\|v_{12}(t)\|^2_{L^2(\sigma)} + \int_0^s \|v_{12}(t)\|^2_{W_0^{1,2}(\sigma)}\,dt \\ &+ \|\hat{\mathbf{w}}_{12}(t)\|^2_{L^2(\sigma)^2} + \int_0^s \|\hat{\mathbf{w}}_{12}(t)\|^2_{W_0^{1,2}(\sigma)^2}\,dt \\ &\le C\Big(\|v_{12}(0)\|^2_{L^2(\sigma)} + \|\hat{\mathbf{w}}_{12}(0)\|^2_{W_0^{1,2}(\sigma)^2}\Big) \end{aligned}$$

on $(0,+\infty)$. Now the uniqueness follows from $v_{12}(0) = 0$ and $\hat{\mathbf{w}}_{12}(0) = \mathbf{0}$. The proof is thus complete. $\square$

5. Appendix: Solvability of parabolic systems in Hilbert spaces

In this appendix, we recall, for the convenience of the reader, the well-known result concerning the solvability and L^2-regularity of parabolic problems.

Theorem 5.1. *Let Ω be a bounded domain in $\mathbb{R}^2$, $\Omega \in C^{0,1}$, $T \in (0,+\infty]$. Let $\mathbf{f} \in L^2(0,T;L^2(\Omega)^2)$ and $\mathbf{v}_0 \in W_0^{1,2}(\Omega)^2$. Let $\mathfrak{a}$ be a continuous, coercive and symmetric bilinear form on $W_0^{1,2}(\Omega)^2$. Let the form $((\cdot,\cdot))$ be defined by (1.14). Then there exists the unique $\mathbf{v} \in L^\infty(0,T;W_0^{1,2}(\Omega)^2) \cap W^{1,2}(0,T;L^2(\Omega)^2)$ such that*

$$((\mathbf{v}'(t),\boldsymbol{\psi})) + \mathfrak{a}\,(\mathbf{v}(t),\boldsymbol{\psi}) = ((\mathbf{f}(t),\boldsymbol{\psi})) \tag{5.1}$$

for every $\boldsymbol{\psi} \in W_0^{1,2}(\Omega)^2$ and for almost every $t \in (0,T)$ and

$$\mathbf{v}(0) = \mathbf{v}_0. \tag{5.2}$$

Moreover,

$$\begin{aligned} &\|\mathbf{v}\|_{L^\infty(0,T;W_0^{1,2}(\Omega)^2)} + \|\mathbf{v}'\|_{L^2(0,T;L^2(\Omega)^2)} \\ &\le c\Big(\|\mathbf{f}\|_{L^2(0,T;L^2(\Omega)^2)} + \|\mathbf{v}_0\|_{W_0^{1,2}(\Omega)^2}\Big), \end{aligned} \tag{5.3}$$

where c is independent of T.

Proof. We follow [21, Chapter 3], see also [2, Section 3, Proof of Theorem 3.4]. It can be shown as in [21, Chapter I, 2.6] that there exist functions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \dots, \boldsymbol{\phi}_k, \dots \in W_0^{1,2}(\Omega)^2 \subset L^2(\Omega)^2$ and real positive numbers $\lambda_1, \lambda_2, \dots \lambda_k, \dots \to \infty$ for $k \to \infty$, such that

$$\mathfrak{a}(\boldsymbol{\phi}_k, \boldsymbol{\psi}) = \lambda_k((\boldsymbol{\phi}_k, \boldsymbol{\psi}))$$

for every $\boldsymbol{\psi} \in W_0^{1,2}(\Omega)^2$. $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \dots$ is a system which is complete in both $L^2(\Omega)^2$ and $W_0^{1,2}(\Omega)^2$, orthonormal in $L^2(\Omega)^2$ and orthogonal in $W_0^{1,2}(\Omega)^2$.

Since $\mathbf{f} \in L^2(0,T;L^2(\Omega)^2)$ and $\mathbf{v}_0 \in W_0^{1,2}(\Omega)^2$, we have

$$\mathbf{f} = \sum_{k=1}^\infty \alpha_k(t)\boldsymbol{\phi}_k, \quad \mathbf{v}_0 = \sum_{k=1}^\infty a_k \boldsymbol{\phi}_k,$$

where

$$\sum_{k=1}^\infty \int_0^T \alpha_k(t)^2\,dt + \sum_{k=1}^\infty a_k^2 < \infty.$$

Let y_k be a solution of the ordinary differential equation

$$y_k'(t) + \lambda_k y_k(t) = \alpha_k(t) \tag{5.4}$$

(which holds for almost every $t \in (0,T)$) with the initial condition

$$y_k(0) = a_k \tag{5.5}$$

for $k = 1, 2, \dots$. Then it holds

$$y_k(t) = \int_0^t e^{\lambda_k(s-t)}\alpha_k(s)ds + a_k e^{-\lambda_k t}$$

for every $t \in (0,T)$. Hence $y_k \in W^{1,2}((0,t))$. Multiplying (5.4) by $2y_k'$ and integrating over $(0,t)$ we obtain

$$\begin{aligned} 2\int_0^t {y_k'}^2(s)ds + \lambda_k y_k^2(t) &= \lambda_k {y_k}^2(0) + 2\int_0^t \alpha_k(s){y_k}'(s)ds \\ &\le \lambda_k y_k^2(0) + \int_0^t {y_k'}^2(s)ds + \int_0^t \alpha_k^2(s)ds \end{aligned}$$

for $k = 1, 2, \dots$ and for every $t \in (0,T)$; therefore

$$\int_0^t {y_k'}^2(s)ds + \lambda_k y_k^2(t) \le \lambda_k y_k^2(0) + \int_0^t \alpha_k^2(s)ds. \tag{5.6}$$

Thus (5.6) yields

$$\begin{aligned} \sum_{k=1}^\infty \int_0^t {y_k'}^2(s)ds + \sum_{k=1}^\infty \lambda_k y_k^2(t) &\le \sum_{k=1}^\infty \int_0^T {y_k'}^2(s)ds + \sum_{k=1}^\infty \lambda_k y_k^2(t) \\ &\le 2\sum_{k=1}^\infty \lambda_k y_k^2(0) + 2\sum_{k=1}^\infty \int_0^T \alpha_k^2(s)ds \end{aligned}$$

for every $t \in (0,T)$ and therefore we have

$$\mathbf{v} = \sum_{k=1}^{\infty} y_k(t)\boldsymbol{\phi}_k \in L^\infty(0,T;W_0^{1,2}(\Omega)^2), \quad \mathbf{v}' \in L^2(0,T;L^2(\Omega)^2)$$

and $\mathbf{v}$, the solution of (5.1), satisfies the estimate (5.3).

Finally, suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are solutions of this problem for given data $\mathbf{f}$ and $\mathbf{v}_0$. Denote $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$. Then

$$((\mathbf{v}_{12}'(t), \boldsymbol{\psi})) + \mathfrak{a}\left(\mathbf{v}_{12}(t), \boldsymbol{\psi}\right) = 0 \tag{5.7}$$

for every $\boldsymbol{\psi} \in W_0^{1,2}(\Omega)^2$ and for almost every $t \in (0,T)$ and

$$\mathbf{v}_{12}(0) = \mathbf{0}.$$

Using $\boldsymbol{\psi} = \mathbf{v}_{12}(t)$ in (5.7) and integrating over $(0,T)$ we obtain

$$\|\mathbf{v}_{12}(T)\|^2_{L^2(\Omega)^2} + \int_0^T \mathfrak{a}\left(\mathbf{v}_{12}(t), \mathbf{v}_{12}(t)\right)\, dt = 0.$$

Therefore $\mathbf{v}_{12} = \mathbf{0}$ and consequently $\mathbf{v}_1 = \mathbf{v}_2$. This completes the proof. □

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Michal Beneš

Department of Mathematics, Faculty of Civil Engineering, Czech Technical University in Prague, Thákurova 7, 166 29 Prague 6, Czech Republic

E-mail address: `michal.benes@cvut.cz`

Igor Pažanin

Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia

E-mail address: `pazanin@math.hr`

Marko Radulović

Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia

E-mail address: `mradul@math.hr`