Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 150, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A SECOND-ORDER ITERATIVE BOUNDARY-VALUE PROBLEM

ERIC R. KAUFMANN

Abstract. We consider the existence and uniqueness of solutions to the second-order iterative boundary-value problem

$$x''(t) = f(t, x(t), x^{[2]}(t)), \quad a \le t \le b,$$

where $x^{[2]}(t) = x(x(t))$, with solutions satisfying one of the boundary conditions $x(a) = a$, $x(b) = b$ or $x(a) = b$, $x(b) = a$. The main tool employed to establish our results is the Schauder fixed point theorem.

1. Introduction

The study of iterative differential equations can be traced back to papers by Petuhov [9] and Eder [4]. In 1965 Petuhov [9] considered the existence of solutions to the functional differential equation $x'' = \lambda x(x(t))$ under the condition that $x(t)$ maps the interval $[-T, T]$ into itself and that $x(0) = x(T) = \alpha$. He obtained conditions on λ and α for the existence and uniqueness of solutions. In 1984, Eder [4] studied solutions of the first order equation $x'(t) = x(x(t))$. The author proved that every solution either vanishes identically or is strictly monotonic. The author established conditions for the existence, uniqueness, analyticity, and analytic dependence of solutions on initial data. In 1990, using Schauder's fixed point theorem Wang [10] obtained a solution of $x' = f(x(x(t))), x(a) = a$, where a is one endpoint of the interval of existence. In 1993, Fečkan showed the existence of local solutions via the Contraction Mapping Principle for the initial value problem for the iterative differential equation $x'(t) = f(x(x(t))), x(0) = 0$. For more on iterative differential equations see the papers [1, 2] [5]-[8], [11]-[14] and references therein.

In this paper we consider the existence and uniqueness of solutions to the second-order iterative boundary-value problem

$$x''(t) = f(t, x(t), x^{[2]}(t)), \quad a < t < b, \tag{1.1}$$

where $x^{[2]}(t) = x(x(t))$, with solutions satisfying one of the following boundary conditions:

$$x(a) = a, \quad x(b) = b; \tag{1.2}$$

$$x(a) = b, \quad x(b) = a. \tag{1.3}$$

2010 *Mathematics Subject Classification.* 34B15, 34K10, 39B05.

Key words and phrases. Iterative differential equation; Schauder fixed point theorem; contraction mapping principle.

Submitted September 20 2017. Published August 8, 2018.

We assume throughout that $f\colon [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous. Due to the iterative term $x^{[2]}(t)$, in order for solutions to be well-defined, we require that the image of x be in the interval $[a,b]$; that is, we need $a\le x(t)\le b$ for all $t\in[a,b]$.

In Section 2, we first rewrite (1.1), (1.2) as an integral equation and then state a condition under which solutions of the integral equation will be solutions of the boundary value problem. We also state properties of the kernel that will be needed in the sequel. In Section 3, we state and prove theorems on the existence and uniqueness of solutions for the boundary value problems (1.1), (1.2) and (1.1), (1.3). We provide an example to demonstrate our results.

2. Preliminaries

Our goals in this section are to convert the boundary value (1.1), (1.2) to a fixed point problem and to state theorems we will need to prove the existence and uniqueness. To this end, let $x\in C^2[a,b]$ be a solution of

$$\begin{gathered} x''(t)=f(t,x(t),x^{[2]}(t)),\quad a<t<b,\\ x(a)=a,\quad x(b)=b. \end{gathered}$$

We begin by integrating the equation $x''(t)=f(t,x(t),x^{[2]}(t))$ twice.

$$x(t)=a+x'(a)(t-a)+\int_a^t (t-s)f(s,x(s),x^{[2]}(s))\,ds. \tag{2.1}$$

After applying the boundary condition $x(b)=b$, we can solve for $x'(a)$ to obtain,

$$x'(a)=1-\frac{1}{b-a}\int_a^b (b-s)f(s,x(s),x^{[2]}(s))\,ds.$$

Now substitute this expression for $x'(a)$ into (2.1).

$$x(t)=t-\frac{(t-a)}{b-a}\int_a^b (b-s)f(s,x(s),x^{[2]}(s))\,ds+\int_a^t (t-s)f(s,x(s),x^{[2]}(s))\,ds.$$

We can rewrite this equation in the form

$$\begin{aligned} x(t)=t&-\frac{1}{b-a}\int_t^b (t-a)(b-s)f(s,x(s),x^{[2]}(x))\,ds\\ &-\frac{1}{b-a}\int_a^t (t-a)(b-s)f(s,x(s),x^{[2]}(s))\,ds\\ &+\frac{1}{b-a}(t-s)f(s,x(s),x^{[2]}(s))\,ds. \end{aligned}$$

Finally, we combine the last two integrals and simplify the integrand.

$$\begin{aligned} x(t)=t&+\frac{1}{b-a}\int_t^b (t-a)(s-b)f(s,x(s),x^{[2]}(x))\,ds\\ &+\frac{1}{b-a}\int_a^t (t-b)(s-a)f(s,x(s),x^{[2]}(s))\,ds. \end{aligned}$$

Thus, if $x\in C^2[a,b]$ is a solution of

$$\begin{gathered} x''(t)=f(t,x(t),x^{[2]}(t)),\quad a<t<b,\\ x(a)=a,\quad x(b)=b, \end{gathered}$$

then $x \in C[a,b]$ must satisfy the integral equation

$$x(t) = t + \int_a^b G(t,s) f(s, x(s), x^{[2]}(s))\, ds, \quad a \le t \le b, \tag{2.2}$$

where

$$G(t,s) = \frac{1}{b-a} \begin{cases} (t-b)(s-a), & a \le s \le t \le b, \\ (t-a)(s-b), & a \le t \le s \le b. \end{cases}$$

Define the operator $T_1 : C[a,b] \to C[a,b]$ by

$$(T_1 x)(t) = t + \int_a^b G(t,s) f(s, x(s), x^{[2]}(s))\, ds.$$

Note that $(T_1 x)(a) = a$ and $(T_1 x)(b) = b$. Also,

$$\begin{aligned} (T_1 x)'(t) = 1 &+ \frac{1}{b-a} \int_a^t (s-a) f(s, x(s), x^{[2]}(s))\, ds \\ &- \frac{1}{b-a} \int_t^b (b-s) f(s, x(s), x^{[2]}(s))\, ds, \end{aligned}$$

and

$$(T_1 x)''(t) = f(s, x(s), x^{[2]}(s)).$$

Recall that in order for the solution of (1.1), (1.2) to be well-defined we need $a \le x(t) \le b$, for all $a \le t \le b$. As such, if $x \in C[a,b]$ is a fixed point of T_1 such that $a \le (T_1 x)(t) \le b$ for all $t \in [a,b]$, then x is a solution of (1.1), (1.2). We have the following lemma.

Lemma 2.1. *The function x is a solution of* (1.1), (1.2) *if and only if $a \le (T_1 x)(t) \le b$ and x is a fixed point of T_1.*

To establish our uniqueness results we will need the following results concerning the kernel of (2.2). The proof of this lemma is straight forward and hence omitted.

Lemma 2.2. *The function*

$$G(t,s) = \frac{1}{b-a} \begin{cases} (t-b)(s-a), & a \le s \le t \le b, \\ (t-a)(s-b), & a \le t \le s \le b \end{cases}$$

satisfies

$$|G(t,s)| \le |G(s,s)|, \quad t, s \in [a,b] \times [a,b],$$
$$\int_a^b |G(s,s)|\, ds = \frac{1}{6}(b-a)^2.$$

We conclude this section with Schauder's fixed point theorem [3].

Theorem 2.3 (Schauder)**.** *Let A be a nonempty compact convex subset of a Banach space and let $T : A \to A$ be continuous. Then T has a fixed point in A.*

3. Existence and uniqueness of solutions

We present our main results in this section. From Lemma 2.1 we note that we need $a \leq (T_1x)(t) \leq b$ for all $t \in [a, b]$. The following condition will be used to control the range of T_1x.

(H1) There exists constants $K, L > 0$ such that $-K \leq f(t, u, v) \leq L$ for all $t \in [a, b]$, $u, v \in \mathbb{R}$ and $1 - \frac{b-a}{2}(K + L) > 0$.

We are now ready to state our first result.

Theorem 3.1. *Suppose that condition* (H1) *holds. The there exists a solution of the boundary-value problem* (1.1), (1.2).

Proof. Consider the Banach space $\Phi = (C[a, b], \|\cdot\|)$ with the norm defined by $\|x\| = \max_{t\in[a,b]} |x(t)|$. Let $m = \max\{|a|, |b|\}$ and let $\Phi_m = \{x \in \Phi : \|x\| \leq m\}$. Since (H1) holds,

$$\begin{aligned}
(T_1x)'(t) &= 1 + \frac{1}{b-a}\int_a^t (s-a)f(s, x(s), x^{[2]}(s))\,ds \\
&\quad - \frac{1}{b-a}\int_t^b (b-s)f(s, x(s), x^{[2]}(s))\,ds \\
&\geq 1 - \frac{K}{b-a}\int_a^t (s-a)\,ds - \frac{L}{b-a}\int_t^b (b-s)\,ds \\
&\geq 1 - \frac{b-a}{2}(K+L) > 0.
\end{aligned}$$

Consequently T_1x is increasing. Since $(T_1x)(a) = a$ and $(T_1x)(b) = b$, then $a \leq (T_1x)(t) \leq b$ for all $t \in [a, b]$. An application of Schauder's theorem yields a fixed point x of T_1 and the proof is complete. $\square$

By Lemma 2.1 the function x is a solution of (1.1), (1.2).

Using the same technique as in Section 2, we can show that the boundary-value problem (1.1), (1.3) is equivalent to the integral equation

$$(T_2x)(t) = (b+a) - t + \int_a^b G(t, s)f(s, x(s), x^{[2]}(s))\,ds$$

provided $a \leq (T_2x)(t) \leq b$.

Theorem 3.2. *Suppose that condition* (H1) *holds. The there exists a solution of the boundary-value problem* (1.1), (1.3).

Proof. As in the proof of Theorem 3.1, we first show that T_2x is monotone. From condition (H1) we have

$$\begin{aligned}
(T_2x)'(t) &= -1 + \frac{1}{b-a}\int_a^t (s-a)f(s, x(s), x^{[2]}(s))\,ds \\
&\quad - \frac{1}{b-a}\int_t^b (b-s)f(s, x(s), x^{[2]}(s))\,ds \\
&\leq -1 + \frac{b-a}{2}(K+L) < 0.
\end{aligned}$$

The rest of the proof is the same as in Theorem 3.1. $\square$

Example 3.3. Consider the following boundary-value problem with parameter k.

$$x''(t) = k\cos(x^{[2]}(t)) \tag{3.1}$$

$$x(0) = 0,\ x(\pi) = \pi. \tag{3.2}$$

Here, $f(t,u,v) = k\cos v$. Since $-|k| \le k\cos v \le |k|$, then $1-\frac{b-a}{2}(K+L) = 1-\frac{\pi}{2}|k|$. By Theorem 3.1 there exists a solution of (3.1), (3.2) for all values of k such that $|k| < \frac{2}{\pi}$.

We now consider uniqueness of solutions of (1.1), (1.2) and (1.1), (1.3). To this end, we need the following condition.

(H2) There exists $M, N > 0$ such that $|f(t,u_1,v_1) - f(t,u_2,v_2)| \le M|u_1-u_2| + N|v_1-v_2|$ for all $t \in [a,b]$, $u_1,u_2,v_1,v_2 \in \mathbb{R}$.

Theorem 3.4. *Suppose that* (H1) *and* (H2) *hold. Assume that*

$$\frac{1}{6}(M+N)(b-a)^2 < 1.$$

Then there exists a unique solution of (1.1), (1.2).

Proof. Since (H1) holds, then there exists a fixed point x of T_1. Suppose that x_1 and x_2 are two distinct fixed points of T_1. Then for all $t \in [a,b]$ we have,

$$\begin{aligned}
|x_1(t) - x_2(t)| &= |(T_1x_1)(t) - (T_1x_2)(t)| \\
&= \Big|\int_a^b G(t,s)\big(f(s,x_1(s),x_1^{[2]}(s)) - f(s,x_2(s),x_2^{[2]}(s))\big)\,ds\Big| \\
&\le \int_a^b |G(t,s)|\big|f(s,x_1(s),x_1^{[2]}(s)) - f(s,x_2(s),x_2^{[2]}(s))\big|\,ds \\
&\le \int_a^b |G(s,s)|\Big(M|x_1(s)-x_2(s)| + N|x_1^{[2]}(s) - x_2^{[2]}(s)|\Big) \\
&\le \frac{1}{6}(M+N)(b-a)^2\|x_1-x_2\| \\
&< \|x_1-x_2\|.
\end{aligned}$$

Thus, $\|x_1-x_2\| < \|x_1-x_2\|$ and we have a contradiction. Consequently, the fixed point x of T_1 is unique. By Lemma 2.1 x is the unique solution of (1.1), (1.2) and the proof is complete. $\square$

In a similar manner we can prove the following theorem.

Theorem 3.5. *Suppose that* (H1) *and* (H2) *hold. Assume that*

$$\frac{1}{6}(M+N)(b-a)^2 < 1.$$

Then there exists a unique solution of (1.1), (1.3).

Example 3.6. We again consider the boundary-value problem (3.1), (3.2),

$$x''(t) = k\cos(x^{[2]}(t))$$

$$x(0) = 0, \quad x(\pi) = \pi.$$

By the Mean Value Theorem we know there exists a $\xi \in [0,\pi]$ such that

$$|k\cos v_1 - k\cos v_2| = |k||\sin\xi||v_1-v_2| \le |k||v_1-v_2|.$$

We have $\frac{1}{6}(M+N)(b-a)^2 = |k|\pi^2/6$. By Theorem 3.4 there exists a unique solution of (3.1), (3.2) for all values of k such that $|k| < 6/\pi^2$.

Note that the results in this paper can be extended to boundary-value problems of the form

$$\begin{gathered} x'' = f\big(t, x(t), x^{[2]}(t), \dots, x^{[n]}(t)\big), \\ x(a) = a, \quad x(b) = b, \end{gathered}$$

as well as boundary-value problems of the form

$$\begin{gathered} x'' = f\big(t, x(t), x^{[2]}(t), \dots, x^{[n]}(t)\big), \\ x(a) = b, \quad x(b) = a. \end{gathered}$$

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Eric R. Kaufmann

Department of Mathematics & Statistics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA

E-mail address: erkaufmann@ualr.edu