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FRACTIONAL p-LAPLACIAN EQUATIONS ON RIEMANNIAN MANIFOLDS

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Abstract. In this article we establish the theory of fractional Sobolev spaces on Riemannian manifolds. As a consequence we investigate some important properties, such as the reflexivity, separability, the embedding theorem and so on. As an application, we consider fractional p-Laplacian equations with homogeneous Dirichlet boundary conditions

$$\begin{gathered}(-\Delta_g)^s_p u(x) = f(x,u) \quad \text{in } \Omega,\\ u = 0 \quad \text{in } M \setminus \Omega,\end{gathered}$$

where $N > ps$ with $s \in (0,1)$, $p \in (1,\infty)$, $(-\Delta_g)^s_p$ is the fractional p-Laplacian on Riemannian manifolds, (M,g) is a compact Riemannian $N-$manifold, Ω is an open bounded subset of M with smooth boundary $\partial\Omega$, and f is a Carathéodory function satisfying the Ambrosetti-Rabinowitz type condition. By using variational methods, we obtain the existence of nontrivial weak solutions when the nonlinearity f satisfies sub-linear or super-linear growth conditions.

1. Introduction

Recently, great attention has been paid on the study of problem involving fractional and non-local operators. This type of problem arises in many applications, such as, continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes, see [3, 7, 19] and the references therein. Here we would like to point out some interesting models involving the fractional Laplacian, such as, the fractional Lane-Emden equation (see [11]), the fractional Schrödinger equation (see [37, 38]), the fractional Kirchhoff equation (see [12, 27, 28, 36]), the fractional Cahn-Hilliard, Allen-Cahn and porous medium equations (see [2, 33]), the fractional Yamabe problem (see [9]) and so on, have attracted recently considerable attention. Indeed, the literature on non-local operators and their applications is very interesting and quite large, we refer the interested reader to [5, 8, 21, 22] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the interested reader to [10, 24].

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In this article we deal with the fractional p–Laplace problem

$$\begin{gathered}(-\Delta_g)^s_p u(x) = f(x,u) \quad \text{in } \Omega,\\ u = 0 \quad \text{in } M\setminus\Omega,\end{gathered} \tag{1.1}$$

where $N > ps$ with $s \in (0,1)$, $p \in (1,\infty)$, (M,g) is a compact Riemannian $N-$manifold, $\Omega \subset M$ is an open bounded set with smooth boundary $\partial\Omega$, $f : \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function and $(-\Delta_g)^s_p u(x)$ is the fractional p-Laplace operator which (up to normalization factors) may be defined as

$$(-\Delta_g)^s_p u(x) = 2\lim_{\varepsilon\to0^+}\int_{M\setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{(d_g(x,y))^{N+ps}}\, d\mu_g(y)$$

for $x \in M$, where $B_x(\varepsilon)$ denotes the geodesic ball of M of center x and radius ε and $d_g(x,y)$ defines a distance on M whose topology coincides with the original one of M, see Section 2 for more details.

In the Euclidean case, problem (1.1) reduces to the fractional Laplacian problem as $p = 2$:

$$\begin{gathered}(-\Delta)^s u(x) = f(x,u) \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^N\setminus\Omega.\end{gathered} \tag{1.2}$$

One typical feature of problem (1.2) is the nonlocality, in the sense that the value of $(-\Delta)^s u(x)$ at any point $x \in \Omega$ depends not only on Ω, but actually on the entire space $\mathbb{R}^N$. The functional framework that takes into account problem (1.2) with Dirichlet boundary condition was introduced in [29, 30]. It is well known that problem (1.1) has been used to model some physical phenomena occurring in nonlocal reaction-diffusion problems, non-Newtonian fluid, non-Newtonian filtration and turbulent flows of a gas in a porous medium, and so on. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p > 2$ are called dilatant fluid and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids. Concerning the fractional Sobolev spaces in $\mathbb{R}^N$ and its applications to the qualitative analysis of solutions for problem (1.2), we refer to [18, 17, 20, 25, 23, 31, 35] and the references therein for further details.

In recent years, the conformal fractional Laplacian has received a lot of attention. More precisely, the conformal fractional Laplacian is defined on the boundary of a Poincaré-Einstein manifold in view of scattering theory, see [14] for all the necessary background. Caffarelli and Silvestre [8] presented a construction for the standard fractional Laplacian $(-\Delta_{\mathbb{R}^N})^s$ as a Dirichlet-to-Neumann operator of a uniformly degenerate elliptic boundary value problem. In the manifold case, Chang and González [9] linked the original definition of the conformal fractional Laplacian coming from scattering theory to a Dirichlet-to-Neumann operator for a related elliptic extension problem, thus allowing for an analytic treatment of Yamabe-type problems in the non-local setting, see [15]. As for several definitions of fractional Laplace operator and their interrelation, we refer to [10] for more details.

Inspired by the above works, we are interested in considering the integral definition of fractional Laplacian from $\mathbb{R}^N$ to Riemannian manifolds. To our best knowledge, there is no result along this line. It is worth to point out that our definition seems easier to be understood than the conformal fractional Laplacian, just from the analytic points of view. In the mean time, our definition would be convenient to generalize some related existence results on fractional Laplace equations exploited by variational methods to those of Riemannian manifolds.

As an application of the fractional Sobelev spaces on Riemannian manifolds, we will consider the existence of weak solutions for problem (1.1). For this purpose, we assume that $\Omega \subset M$ is a open bounded set and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following:

(A1) There exist $a > 0$ and $1 < q < p_s^* = Np/(N - ps)$ such that
$$|f(x, \eta)| \le a(1 + |\eta|^{q-1}),$$
for a.e. $x \in \Omega$, $\eta \in \mathbb{R}$;

(A2) There exist $\gamma > p$ and $r > 0$ such that for a.e. $x \in \Omega$ and $r \in \mathbb{R}$, $|\xi| \ge r$,
$$0 < \gamma F(x, \xi) \le \xi f(x, \xi),$$
where $F(x, \xi) = \int_0^\xi f(x, \tau)d\tau$;

(A3) It holds
$$\lim_{\zeta \to 0} \frac{f(x, \zeta)}{|\zeta|^{p-1}} = 0 \text{ uniformly for a. e. } x \in \Omega;$$

(A4) There exist $a_1 > 0$ and an open bounded set $\Omega_0 \subset \Omega$ such that
$$|f(x, \rho)| \ge a_1|\rho|^{q-1} \quad \text{for a.e. } x \in \Omega_0 \text{ and all } \rho \in \mathbb{R}.$$

Now, we give the definition of weak solutions for problem (1.1).

Definition 1.1. We say that $u \in W_0^{s,p}(\Omega)$ is a weak solution of problem (1.1), if
$$\begin{aligned}
&\iint_{M \times M} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{(d_g(x, y))^{N+ps}} \, d\mu_g(x)d\mu_g(y) \\
&= \int_\Omega f(x, u(x))\varphi(x)d\mu_g(x),
\end{aligned}$$
for any $\varphi \in W_0^{s,p}(M)$, where space $W_0^{s,p}(M)$ will be introduced in Section 2.

Then, by variational methods, we can get the following existence results for problem (1.1).

Theorem 1.2. *Let* (A1) *and* (A4) *hold. If* $1 < q < p$*, then the problem* (1.1) *has a nontrivial weak solution in* $W_0^{s,p}(M)$.

Theorem 1.3. *Let* (A1)–(A3) *hold. If* $p < q < p_s^*$*, then problem* (1.1) *has a nontrivial weak solution in* $W_0^{s,p}(M)$.

Remark 1.4. Theorems 1.2 and 1.3 can be viewed as the counterpart of [34, Theorems 1.1 and 1.2] on compact Riemannian N-manifold in the non-Kirchhoff case.

This article is organized as follows. In Section 2, we will present some necessary definitions and properties of space $W_0^{s,p}(M)$. In Section 3, using variational methods, we obtain the existence of weak solutions for problem (1.1) in two cases: $1 < q < p$ and $p < q < p_s^*$.

2. Fractional Sobolev space on Riemannian manifolds

Let we first recall some basic material on Riemannian geometry (see [4, 16]). Let (M, g) be a smooth Riemannian $N-$manifold, and let ∇ be the Levi-Civita connection. For $u \in C^\infty(M)$, then $\nabla^k u$ denotes the $k-$th covariant derivative of u. In local coordinates, the pointwise norm of $\nabla^k u$ is given by

$$|\nabla^k u| = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1 i_2 \dots i_k} (\nabla^k u)_{j_1 j_2 \dots j_k}$$

When $k = 1$, the components of ∇u in local coordinates are given by $(\nabla u)_i = \nabla^i u$. By definition one has that

$$|\nabla u| = \sum_{i,j=1}^{\infty} g^{ij} \nabla^i u \nabla^j u$$

Given (M, g) a smooth Riemannian $N-$manifold, and $\gamma : [a, b] \to M$ a curve of class C^1, the length of γ is

$$L(\gamma) = \int_a^b \sqrt{g\big(\gamma(t)\big)\Big(\big(\frac{d\gamma}{dt}\big)(t), \big(\frac{d\gamma}{dt}\big)(t)\Big)} d\mu.$$

For $x, y \in M$, let $C^1_{x,y}$ be the space of piecewise C^1 curves $\gamma : [a, b] \to M$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then $d_g(x, y) = \inf_{C^1_{x,y}} L(\gamma)$ defines a distance on M whose topology coincides with the original one of M. In particular, by Stine's theorem, a smooth Riemannian manifold is paracompact. By definition, d_g is the distance associated to g.

Given (M, g) a smooth Riemannian N-manifold, one can define a natural positive Radon measure on M. In particular, the theory of the Lebesgue integral can be applied. For $(\Omega_i, \varphi_i)_{i\in I}$ some atlas of M, we shall say that a family $(\Omega_j, \varphi_j, \eta_j)_{j\in J}$ is a partition of unity subordinate to $(\Omega_i, \varphi_i)_{i\in I}$. As one can easily check, for any atlas $(\Omega_i, \varphi_i)_{i\in I}$ of M, there exists a partition of unity $(\Omega_j, \varphi_j, \eta_j)_{j\in J}$ subordinate $(\Omega_i, \varphi_i)_{i\in I}$. Then we can define the Riemannian measure as follows: given $u : M \to \mathbb{R}$ is continuous with compact support, and given $(\Omega_i, \varphi_i)_{i\in I}$ is an atlas of M,

$$\int_M u(x) d\mu_g(x) = \sum_{k\in J} \int_{\varphi_k(\Omega_k)} \Big(\sqrt{\det(g_{ij})}\eta_k u\Big) \circ \varphi_k^{-1}(x) dx,$$

where $(\Omega_j, \varphi_j, \eta_j)_{j\in J}$ is a partition of unity subordinate to $(\Omega_i, \varphi_i)_{i\in I}$, $d\mu_g(x) = \sqrt{\det(g_{ij})}dx$ is the Riemannian volume element on (M, g), where the g_{ij} are the components of the Riemannian metric g in the chart and dx is the Lebesgue volume element of $\mathbb{R}^N$.

In what follows, we give some basic results that will be used in the next section. In the Euclidean case, we refer to [13, 29, 30, 34] for related results. Let $0 < s < 1 < p < \infty$ be real numbers and the fractional critical exponent p_s^* be defined as

$$p_s^* = \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N \\ \infty & \text{if } sp \geq N. \end{cases}$$

This section is devoted to the definition of the fractional Sobolev spaces on Riemannian manifolds. We start by fixing the fractional exponent s in $(0, 1)$. For any $p \in [1, +\infty)$, we define $W^{s,p}(M)$ as follows:

$$W^{s,p}(M) = \Big\{u \in L^p(M) : \frac{|u(x) - u(y)|}{(d_g(x, y))^{\frac{n}{p}+s}} \in L^p(M \times M)\Big\}$$

i.e, an intermediary Banach space between $L^p(M)$ and $W^{s,p}(M)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(M)} = \Big(\int_M |u(x)|^p d\mu_g(x) + [u]^p_{W^{s,p}(M)} \Big)^{1/p},$$

where the term

$$[u]_{W^{s,p}(M)} = \Big(\iint_{M\times M} \frac{|u(x)-u(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \Big)^{1/p}.$$

is the so-called Gagliardo (semi)norm of u.

It is easy to prove that $\|\cdot\|_{W^{s,p}(M)}$ is a norm on $W^{s,p}(M)$. We will work in the closed linear subspace

$$W^{s,p}_0(M) = \big\{ u \in W^{s,p}(M) : \mathrm{supp}(u) \text{ is a compact subset of } M \big\},$$

where $\mathrm{supp}(u) = \overline{\{x \in M : u(x) \neq 0\}}$.

Lemma 2.1. *Let (M, d_g) be a complete Riemannian $N-$manifold with finite volume, then $C^\infty_0(M) \subset W^{s,p}_0(M)$.*

Proof. For $v \in C^\infty_0(M)$, we only need to check that

$$\iint_{M\times M} \frac{|v(x)-v(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) < \infty.$$

Notice that

$$\begin{gathered} |v(x)-v(y)| \le \|\nabla v\|_{L^\infty(M)} d_g(x,y), \\ |v(x)-v(y)| \le 2\|v\|_{L^\infty(M)} \end{gathered}$$

for all $x, y \in M$. Thus,

$$|v(x)) - v(y))|^p \le (2\|v\|_{C^1(M)})^p \min\{(d_g(x,y))^p, 1\}.$$

Therefore,

$$\begin{aligned} &\iint_{M\times M} \frac{|(\eta_s v)(x) - (\eta_s v)(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\ &\le \mathrm{Vol}(M) (2\|v\|_{C^1(M)})^p \iint_{M\times M} \frac{\min\{(d_g(x,y))^p, 1\}}{(d_g(x,y))^{N+sp}} d\mu_g(x) d\mu_g(y) < \infty. \end{aligned}$$

Consequently, for $v \in C^\infty_0(M)$ we have

$$\iint_{M\times M} \frac{|v(x)-v(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) < \infty.$$

This implies $v \in W^{s,p}_0(M)$. □

Remark 2.2. Lemma 2.1 and the fact that $C^\infty_0(M)$ is dense in $L^p(M)$ (see for example [16]), imply that $C^\infty_0(M)$ is dense also in $W^{s,p}(M)$.

Remark 2.3. The space $W^{s,p}_0(M)$ is the closure of $C^\infty_0(M)$ in $W^{s,p}(M)$.

Lemma 2.4. *Let (M, d_g) be a compact Riemannian $N-$manifold. Then*

(1) *there exists a positive constant $C_1 = C_1(N, p, q, s)$ such that for any $v \in W^{s,p}_0(M)$ and $1 \le q \le p^*_s$,*

$$\|v\|^p_{L^q(M)} \le C_1 \iint_{M\times M} \frac{|v(x)-v(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y).$$

(2) *there exists a constant* $\widetilde{C} = \widetilde{C}(N, p, q, s)$ *such that for any* $v \in W_0^{s,p}(M)$,

$$\begin{aligned}
&\iint_{M\times M} \frac{|v(x)-v(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\
&\le \|v\|^p_{W^{s,p}(M)} \\
&\le \widetilde{C} \iint_{M\times M} \frac{|v(x)-v(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y).
\end{aligned}$$

Proof. Let $v \in W_0^{s,p}(M)$. Since M is compact, M can be covered by a finite number of charts

$$(B_{x_k}(r), \varphi_k)_{k=1,2,\dots,m}$$

satisfying

$$B_0(r/2) \subset \varphi_k(B_{x_k}(r)) \subset B_0(2r) \quad \text{and} \quad \frac{1}{Q}\delta_{ij} \le g^s_{ij} \le Q\delta_{ij}, \tag{2.1}$$

where g^s_{ij} are bilinear forms, $Q > 1$ is given, $B_{x_k}(r)$ denotes the ball of M of center x_k and radius r, $B_0(2r)$ denotes the Euclidean ball of $\mathbb{R}^N$ of center 0 and radius $2r$. Moreover, we have

$$\frac{1}{C}|\varphi_k^{-1}(y_1) - \varphi_k^{-1}(y_2)| \le d_g(y_1, y_2) \le C|\varphi_k^{-1}(y_1) - \varphi_k^{-1}(y_2)|, \tag{2.2}$$

for $y_1, y_2 \in B_{x_k}(r)$ where $C > 1$ is given.

Let (η_k) be a smooth partition of unity subordinate to the covering $B_{x_k}(r)$. For any k, using [10, Theorem 6.5], we obtain

$$\begin{aligned}
&\|v\|^p_{L^q(M)} \le 2^p \sum_{k=1}^m \|\eta_k v\|^p_{L^q(M)} \\
&\le 2^{\frac{pq+pN}{q}} \sum_{k=1}^m \|(\eta_k v)\circ \varphi^{-1}\|^p_{L^q(\mathbb{R}^N)} \\
&\le C_0 2^{\frac{pq+pN}{q}} \sum_{k=1}^m \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|(\eta_k v)(\varphi_k^{-1}(x)) - (\eta_k v)(\varphi_k^{-1}(y))|^p}{|x-y|^{N+ps}}\, dx\, dy \\
&\le CC_0 2^{\frac{pq+pN}{q}+2N} \sum_{k=1}^m \iint_{M\times M} \frac{|(\eta_k v)(\bar{x}) - (\eta_k v)(\bar{y})|^p}{(d_g(\bar{x},\bar{y}))^{N+ps}} d\mu_g(\bar{x}) d\mu_g(\bar{y}) \\
&\le mCC_0 2^{\frac{pq+pN}{q}+2N} \iint_{M\times M} \frac{|v(\bar{x}) - v(\bar{y})|^p}{(d_g(\bar{x},\bar{y}))^{N+ps}} d\mu_g(\bar{x}) d\mu_g(\bar{y}) \\
&= C_1 \iint_{M\times M} \frac{|v(\bar{x}) - v(\bar{y})|^p}{(d_g(\bar{x},\bar{y}))^{N+ps}} d\mu_g(\bar{x}) d\mu_g(\bar{y}),
\end{aligned} \tag{2.3}$$

where $C_1 = mCC_0 2^{\frac{pq+pN}{q}+2N}$ is a positive constant depending only on N, s, p, q. Thus, we obtain the assertion (1). The assertion (2) easily follows by combining the definition of norm of $W^{s,p}(M)$ with (2.3). $\square$

Remark 2.5. By Lemma 2.4, we obtain an equivalent norm on $W_0^{s,p}(M)$ defined as

$$\|v\|_{W_0^{s,p}(M)} = \Big(\iint_{M\times M} \frac{|u(x)-u(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y)\Big)^{1/p},$$

for all $v \in W_0^{s,p}(M)$.

Lemma 2.6. *Let (M, d_g) be a compact Riemannian N-manifold, $p \in [1,\infty]$ and $s \in (0,1)$. Then*
$$\|u\|_{W^{s,p}(M)} \leq \|u\|_{W^{1,p}(M)}$$
for some suitable positive constant $C = C(N,s,p) \geq 1$. In particular,
$$W^{1,p}(M) \subseteq W^{s,p}(M).$$

Proof. Let $\gamma : [0,1] \to M$ be the minimizing geodesic from x and y, where $x, y \in M$. Then for $v \in W^{s,p}(M)$ we have
$$\begin{aligned}
&\int_M \int_{M\cap\{d_g(x,y)<1\}} \frac{|v(x)-v(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x)d\mu_g(y)\\
&\leq \int_M \int_{M\cap\{d_g(x,y)<1\}} \int_0^1 \frac{|\nabla v(\gamma(t))|^p}{(d_g(x,y))^{N+ps-p}} dt d\mu_g(y)d\mu_g(x)\\
&\leq \int_M \frac{\|\nabla v\|_{L^p(M)}}{(d_g(x,y))^{N+ps-p}} d\mu_g(y)\\
&\leq C(N,s,p)\|\nabla v\|_{L^p(M)}
\end{aligned} \tag{2.4}$$
and
$$\begin{aligned}
&\int_M \int_{M\cap\{d_g(x,y)\geq 1\}} \frac{|v(x)-v(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x)d\mu_g(y)\\
&\leq 2^{p-1}\int_M \int_{M\cap\{d_g(x,y)\geq 1\}} \frac{|v(x)|^p-|v(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x)d\mu_g(y)\\
&\leq C(N,p)\|v\|_{L^p(M)}.
\end{aligned} \tag{2.5}$$
From (2.4) and (2.5) it follows that
$$\|v\|_{W^{s,p}(M)} \leq C(N,s,p)\|v\|_{W^{1,p}(M)}.$$
Thus the proof is complete. □

Remark 2.7. Remark 2.2 and Lemma 2.6, imply that $W^{1,p}(M)$ is dense also in $W^{s,p}(M)$.

Lemma 2.8. *Let (M, d_g) be a compact Riemannian $N-$manifold. Then $W^{s,p}(M)$ is separable.*

Proof. Since $W^{1,p}(M)$ is a separable Banach space (see [16]), there exists a countable dense subset $\mathfrak{A}$ of $W^{1,p}(M)$. We claim that $\mathfrak{A}$ is also dense in $W^{s,p}(M)$. For each $u \in W^{s,p}(M)$, there exists a sequence $\{u_n\}_n$ in $W^{1,p}(M)$ such that $u_n \to u$ strongly in $W^{s,p}(M)$, by the density of $W^{1,p}(M)$ in $W^{s,p}(M)$. Hence, for each $n \geq 1$, there exists a sequence $\{u_{m,n}\}_m$ in $\mathfrak{A}$ such that
$$\lim_{m\to\infty} \|u_{m,n} - u_n\|_{W^{1,p}(M)} = 0.$$
By the standard diagonal process, there exists a sequence $\{u_{m_n,n}\}_n \subseteq \{u_{m,n}\}_m$ such that
$$\lim_{n\to\infty} \|u_{m_n,n} - u_n\|_{W^{1,p}(M)} = 0.$$
Therefore, Lemma 2.6 yields
$$\begin{aligned}
\|u_{m_n,n} - u\|_{W^{s,p}(M)} &\leq \|u_{m_n,n} - u_n\|_{W^{s,p}(M)} + \|u_n - u\|_{W^{s,p}(M)}\\
&\leq C\|u_{m_n,n} - u_n\|_{W^{1,p}(M)} + \|u_n - u\|_{W^{s,p}(M)}.
\end{aligned}$$

This implies that $u_{m_n,n} \to u$ strongly in $W^{s,p}(M)$ as $n \to \infty$. Hence $\mathfrak{A}$ is dense in $W^{s,p}(M)$. This, together with the countability of $\mathfrak{A}$, completes the proof. $\square$

Lemma 2.9. *If (M, d_g) is a complete Riemannian N-manifold, then $W_0^{s,p}(M)$ is a Banach space.*

Proof. We only need to check that $W_0^{s,p}(M)$ is complete with respect to the norm $\|\cdot\|_{W_0^{s,p}(M)}$. Let $\{u_t\}$ be a cauchy sequence in $W_0^{s,p}(M)$. Thus, for any $\varepsilon > 0$ there exists N_ε such that if $n, m \geq N_\varepsilon$, then

$$\|u_n - u_m\|^p_{L^p(M)} \leq \|u_n - u_m\|^p_{W_0^{s,p}(M)} < \varepsilon. \tag{2.6}$$

Let $\{G_l\}$ be a sequence of compact sets such that $G_l \subset G_{l+1} \subset M$ for $l \in \mathbb{N}$ and $M = \cup_{l=1}^\infty G_l$. Then the sequence $\{u_t\}$ is Cauchy in each $L^p(G_l)$ for $l \in \mathbb{N}$. By induction we may find subsequences $\{u_t^{(l)}\}_t$ and $u^{(l)} \in L^p(G_l)$ such that $u_t^{(l)} \to u^{(l)}$ a.e. on G_l for $l \in \mathbb{N}$, and $u^{(l+1)}\chi_{G_l} = u^{(l)}$. Thus, $\lim_{\tau\to\infty} u_\tau^{(\tau)} = \lim_{\tau\to\infty} u^{(\tau)}\chi_{G_\tau} = u$ a.e. on M. Therefore, by the Fatou Lemma and the second inequality in (2.6) with $\varepsilon = 1$, we have

$$\begin{aligned}
&\iint_{M\times M} \frac{|u(x) - u(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\
&\leq \liminf_{\tau\to\infty} \iint_{M\times M} \frac{|u_\tau^{(\tau)}(x) - u_\tau^{(\tau)}(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\
&\leq \liminf_{\tau\to\infty} \Big(\|u_\tau^{(\tau)} - u_{\mu_1}\|_{W_0^{s,p}(M)} + \|u_{\mu_1}\|_{W_0^{s,p}(M)}\Big)^p \\
&\leq \Big(1 + \|u_{\mu_1}\|_{W_0^{s,p}(M)}\Big)^p < \infty.
\end{aligned}$$

Thus, $u \in W_0^{s,p}(M)$. Let $t \geq \mu_\varepsilon$, by the second inequality in (2.6) and Fatou's lemma, we obtain

$$\|u_t - u\|^p_{W_0^{s,p}(M)} \leq \liminf_{\tau\to\infty} \|u_t - u_\tau^{(\tau)}\|^p_{W_0^{s,p}(M)} \leq \varepsilon,$$

that is, $u_n \to u$ strongly in $W_0^{s,p}(M)$ as $n \to \infty$. $\square$

Lemma 2.10. *Let (M, d_g) be a complete Riemannian N-manifold. Then $W_0^{s,p}(M)$ is uniformly convex.*

Proof. Let $u, v \in W_0^{s,p}(M)$ satisfy $\|u\|_{W_0^{s,p}(M)} = \|v\|_{W_0^{s,p}(M)} = 1$ and $\|u-v\|_{W_0^{s,p}(M)} \geq \varepsilon$, where $\varepsilon \in (0, 2)$.

Case $p \geq 2$**.** By [1, inequality (28)], we have

$$\begin{aligned}
&\|\frac{u+v}{2}\|^p_{W_0^{s,p}(M)} + \|\frac{u-v}{2}\|^p_{W_0^{s,p}(M)} \\
&\leq \frac{1}{2} \iint_{M\times M} \frac{|u(x) - u(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\
&\quad + \frac{1}{2} \iint_{M\times M} \frac{|v(x) - v(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x) d\mu_g(y) \\
&= \frac{1}{2}\|u\|^p_{W_0^{s,p}(M)} + \frac{1}{2}\|v\|^p_{W_0^{s,p}(M)} = 1.
\end{aligned} \tag{2.7}$$

From (2.7) it follows that $\|\frac{u+v}{2}\|^p_{W_0^{s,p}(M)} \leq 1 - (\varepsilon/2)^p$. Taking $\delta = \delta(\varepsilon)$ such that $1 - (\varepsilon/2)^p = (1-\delta)^p$, we obtain $\|\frac{u+v}{2}\|_{W_0^{s,p}(M)} \leq (1 - \delta)$.

Case $1 < p < 2$. Note that

$$\|u\|^{p'}_{W_0^{s,p}(M)} = \Big[\iint_{M\times M} \Big(\Big(\frac{|u(x)-u(y)|}{(d_g(x,y))^{\frac{N}{p}+s}} \Big)^{p'} \Big)^{p-1} d\mu_g(x) d\mu_g(y) \Big]^{\frac{1}{p-1}},$$

where $p' = p/(p-1)$. With the help of the reverse Minkowski inequality (see [1, Theorem 2.13]) and the inequality (27) in [1], we obtain

$$\begin{aligned}
&\|\frac{u+v}{2}\|^{p'}_{W_0^{s,p}(M)} + \|\frac{u-v}{2}\|^{p'}_{W_0^{s,p}(M)} \\
&\le \Big\{ \iint_{M\times M} \Big[\Big(|\frac{(u(x)-u(y)) + (v(x)-v(y))}{2(d_g(x,y))^{\frac{N}{p}+s}}| \Big)^{p'} \\
&\quad + \Big(|\frac{(u(x)-u(y)) - (v(x)-v(y))}{2(d_g(x,y))^{\frac{N}{p}+s}}| \Big)^{p'} \Big]^{p-1} d\mu_g(x) d\mu_g(y) \Big\}^{\frac{1}{p-1}} \\
&\le \Big(\frac{1}{2}\|u\|^p_{W_0^{s,p}(M)} + \frac{1}{2}\|v\|^p_{W_0^{s,p}(M)} \Big)^{p'-1} = 1.
\end{aligned} \tag{2.8}$$

By (2.8), we have

$$\|\frac{u+v}{2}\|^{p'}_{W_0^{s,p}(M)} \le 1 - \frac{\varepsilon^{p'}}{2^{p'}}.$$

Taking $\delta = \delta(\varepsilon)$ such that $1 - (\varepsilon/2)^{p'} = (1-\delta)^{p'}$, we obtain the desired conclusion. $\square$

Remark 2.11. According to [1, Theorem 1.21], $W_0^{s,p}(M)$ is a reflexive Banach space.

Lemma 2.12. *Let (M, d_g) be a compact Riemannian N-manifold and $\{v_j\}$ be a bounded sequence in $W_0^{s,p}(M)$. Then, there exists $v \in L^q(M)$ such that up to a subsequence,*

$$v_j \to v \quad \textit{strongly in } L^q(M), \textit{ as } j \to \infty,$$

for any $q \in [1, p_s^*)$.

Proof. For any $\{v_j\}$, which is a bounded sequence in $W_0^{s,p}(M)$. Since M is compact, M can be covered by a finite number of charts $(\Omega_k, \varphi_k)_{k=1,2,\dots,m}$ such that for any k the components g^k_{ij} of g in (Ω_k, φ_k) satisfying

$$\frac{1}{2}\delta_{ij} \le g^k_{ij} \le 2\delta_{ij}$$

are bilinear forms. Let (η_k) be a smooth partition of unity subordinate to the covering (Ω_k). By means of Corollary 7.2 in [10], for any k, there exists $\omega_k \in L^q(\mathbb{R}^N)$ such that

$$(\eta_k v_j) \circ \varphi^{-1} \to \omega_k \quad \text{strongly in } L^q(\varphi_k(\Omega_k)), \text{ as } j \to \infty.$$

Then

$$\eta_k v_j \to \omega_k \circ \varphi = u_k \quad \text{strongly in } L^q(\Omega_k), \text{ as } j \to \infty.$$

Furthermore, we can define $v = \sum_{k=1}^m u_s \in L^q(M)$ satisfying

$$v_j \to v \quad \text{strongly in } L^q(M), \text{ as } j \to \infty.$$

Thus, the proof is complete. $\square$

3. Proofs of main restults

Following the approach of [34], we will give the proofs of Theorems 1.2 and 1.3. For the reader's convenience, here we give a detailed treatment. For $u \in W_0^{s,p}(M)$, we define

$$J(u) = \frac{1}{p} \iint_{M\times M} \frac{|u(x)-u(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x)d\mu_g(y), \quad H(u) = \int_\Omega F(x,u)d\mu_g(x),$$
$$I(u) = J(u) - H(u).$$

Obviously, the energy functional $I : W_0^{s,p}(M) \to \mathbb{R}$ associated with problem (1.1) is well defined.

Lemma 3.1. *If f satisfies* (A1)*, then the functional $H \in C^1(W_0^{s,p}(M), \mathbb{R})$ and*

$$\langle H'(u), v\rangle = \int_\Omega f(x,u)vd\mu_g(x) \quad \text{for all } u,\ v \in W_0^{s,p}(\Omega).$$

Proof. (i) H is Gâteaux-differentiable in $W_0^{s,p}(M)$. Let $u, v \in W_0^{s,p}(M)$. For each $x \in \Omega$ and $0 < |t| < 1$, by the mean value theorem, there exits $0 < \delta < 1$,

$$\begin{aligned} \frac{1}{t}(F(x,u+tv) - F(x,u)) &= \frac{1}{t}\int_0^{u+tv} f(x,s)ds - \frac{1}{t}\int_0^u f(x,s)ds \\ &= \frac{1}{t}\int_u^{u+tv} f(x,s)ds \\ &= f(x,u+\delta tv)v. \end{aligned}$$

Combining (A1) with Young's inequality, we obtain

$$\begin{aligned} |f(x,u+\delta tv)v| &\le a(|v| + |u+\delta tv|^{q-1}|v|) \\ &\le a(2|v|^q + |u+\delta tv|^q + 1) \le a2^q(|v|^q + |u|^q + 1). \end{aligned}$$

Since $1 < q < p_s^*$, by Lemma 2.4 we have $u, v \in L^q(M)$. Moreover, the Lebesgue's dominated convergence theorem implies

$$\begin{aligned} \lim_{t\to 0} \frac{1}{t}(H(u+tv) - H(u)) &= \lim_{t\to 0} \int_\Omega f(x,u+\delta tv)vd\mu_g(x) \\ &= \int_\Omega \lim_{t\to 0} f(x,u+\delta tv)vd\mu_g(x) = \int_\Omega f(x,u)vd\mu_g(x). \end{aligned}$$

(ii) The continuity of Gateaux-derivative. Let $\{u_n\} \subset W_0^{s,p}(M), u \in W_0^{s,p}(M)$ such that $u_n \to u$ strongly in $W_0^{s,p}(M)$ as $n \to \infty$. Without loss of generality, we assume that $u_n \to u$ a.e. in Ω. In view of (A1), for any measurable subset $U \subset \Omega$,

$$\int_U |f(x,u_n)|^{q'} d\mu_g(x) \le 2^{\frac{q+1}{q-1}} a^{\frac{q}{q-1}} \Big(\int_U |u_n|^q d\mu_g(x) + \mu(U) \Big),$$

where $\mu(U)$ denotes the N dimensional Radon measure of set U. Since $1 < q < p_s^*$, by Lemma 2.4 and Hölder's inequality, we have

$$\begin{aligned} \int_U |f(x,u_n)|^{q'} d\mu_g(x) &\le 2^{\frac{q+1}{q-1}} a^{\frac{q}{q-1}} \Big(\| |u_n|^q \|_{L^{\frac{p_s^*}{q}}(U)} \|1\|_{L^{\frac{p_s^*}{p_s^*-q}}(U)} + \mu(U) \Big) \\ &\le C(\mu(U))^{\frac{p_s^*-q}{p_s^*}} + C\mu(U). \end{aligned} \tag{3.1}$$

It follows from (3.1) that the sequence $\{|f(x,u_n)-f(x,u)|^{q'}\}$ is uniformly bounded and equi-integrable in $L^1(\Omega)$. The Vitali convergence theorem implies

$$\lim_{n\to\infty}\int_\Omega |f(x,u_n)-f(x,u)|^{q'}d\mu_g(x)=0.$$

Thus, by Hölder's inequality and Lemma 2.4(1), we obtain

$$\begin{aligned}\|H'(u_n)-H'(u)\| &\le \|f(x,u_n)-f(x,u)\|_{L^{q'}(\Omega)}\|\varphi\|_{L^q(\Omega)}\\ &\le C_1^{1/p}\|f(x,u_n)-f(x,u)\|_{L^{q'}(\Omega)}\to 0,\end{aligned}$$

as $n\to\infty$. Hence, we complete the proof. □

Using the same strategy as in Lemma 3.1, we have

Lemma 3.2. *The functional $J\in C^1(W_0^{s,p}(M),\mathbb{R})$ and*

$$\langle J'(u),v\rangle=\iint_{M\times M}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{(d_g(x,y))^{N+ps}}d\mu_g(x)d\mu_g(y),$$

for all $u,v\in W_0^{s,p}(M)$. Moreover, for each $u\in W_0^{s,p}(M)$, $J'(u)\in W_0^{s,p}(M)^*$, where $W_0^{s,p}(M)^*$ denotes the dual space of $W_0^{s,p}(M)$.

Proof. Firstly, it is easy to see that

$$\begin{aligned}&\langle J'(u),v\rangle\\ &=\iint_{M\times M}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{(d_g(x,y))^{N+ps}}d\mu_g(x)d\mu_g(y),\end{aligned}\tag{3.2}$$

for all $u,\ v\in W_0^{s,p}(M)$. It follows from (3.2) that for each $u\in W_0^{s,p}(M)$, $J'(u)\in W_0^{s,p}(M)^*$.

Next, we prove that $J\in C^1(W_0^{s,p}(M),\mathbb{R})$. Let $\{u_n\}\subset W_0^{s,p}(M)$, $u\in W_0^{s,p}(M)$ with $u_n\to u$ strongly in $W_0^{s,p}(M)$ as $n\to\infty$. By Lemma 2.12 there exists a subsequence of $\{u_n\}$ still denoted by $\{u_n\}$ such that $u_n\to u$ a.e. in Ω. Then the sequence

$$\Big\{\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{(d_g(x,y))^{\frac{N+ps}{p'}}}\Big\}_n\quad\text{is bounded in } L^{p'}(\Omega\times\Omega),$$

and

$$\begin{aligned}\mathcal{M}_n(x,y)&:=\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{(d_g(x,y))^{\frac{N+ps}{p'}}}\\ &\to \mathcal{M}(x,y):=\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{(d_g(x,y))^{\frac{N+ps}{p'}}}\end{aligned}$$

a.e. in $M\times M$. Thus, the Brézis-Lieb Lemma (see [6]) implies

$$\begin{aligned}&\lim_{n\to\infty}\iint_{M\times M}(\mathcal{M}_n(x,y)-\mathcal{M}(x,y))^{p'}d\mu_g(x)d\mu_g(y)\\ &=\lim_{n\to\infty}\iint_{M\times M}([u_n]_{W^{s,p}(M)}-[u]_{W^{s,p}(M)})d\mu_g(x)d\mu_g(y).\end{aligned}\tag{3.3}$$

The fact that $u_n\to u$ strongly in $W_0^{s,p}(M)$ implies

$$\lim_{n\to\infty}\iint_{M\times M}(\mathcal{M}_n(x,y)-\mathcal{M}(x,y))^{p'}d\mu_g(x)d\mu_g(y)=0.\tag{3.4}$$

Combining (3.4) with the Hölder inequality, we have

$$\|J'(u_n)-J'(u)\| = \sup_{v\in W_0^{s,p}(\Omega),\, \|v\|_{W_0^{s,p}(\Omega)}\le 1} |\langle J'(u_n)-J'(u), v\rangle| \to 0,$$

as $n\to\infty$. □

Combining Lemmas 3.1 and 3.2, we obtain that $I\in C^1(W_0^{s,p}(M),\mathbb{R})$ and

$$\begin{aligned}\langle I'(u), v\rangle &= \iint_{M\times M} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{(d_g(x,y))^{N+ps}} d\mu_g(x)d\mu_g(y)\\ &\quad - \int_\Omega f(x,u)v\,d\mu_g(x),\end{aligned}$$

for all $u,v\in W_0^{s,p}(\Omega)$.

Case 1: $1<q<p$. In this subsection, we prove the existence of weak solutions of problem (1.1), where the growth exponent q of function f satisfies $1<q<p$.

Lemma 3.3. *Let* (A1) *be satisfied. Then the functional $I\in C^1(W_0^{s,p}(M),\mathbb{R})$ is weakly lower semi-continuous.*

Proof. Firstly, we notice that the map $v\mapsto \|v\|^p_{W_0^{s,p}(M)}$ is lower semi-continuous in the weak topology of $W_0^{s,p}(M)$. Indeed, we define a functional $\psi: W_0^{s,p}(M)\to\mathbb{R}$ as

$$\psi(v) = \iint_{M\times M} \frac{|v(x)-v(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x)d\mu_g(y).$$

Similar to Lemma 3.2, we obtain $\psi\in C^1(W_0^{s,p}(M))$ and

$$\begin{aligned}&\langle \psi'(w), v\rangle\\ &= p\iint_{M\times M} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))(v(x)-v(y))}{(d_g(x,y))^{N+ps}} d\mu_g(x)d\mu_g(y),\end{aligned}$$

for all $w,v\in W_0^{s,p}(M)$. Note that

$$\begin{aligned}\psi\big(\frac{w+v}{2}\big) &\le \iint_{M\times M} \frac{2^{-1}|w(x)-w(y)|^p + 2^{-1}|v(x)-v(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x)d\mu_g(y)\\ &= \frac{1}{2}\psi(w)+\frac{1}{2}\psi(v).\end{aligned}$$

Thus, ψ is a convex functional in $W_0^{s,p}(M)$. Furthermore, ψ is subdifferentiable and the subdifferential denoted by $\partial\psi$ satisfies $\partial\psi(u)=\{\psi'(u)\}$ for each $u\in W_0^{s,p}(M)$ (see [26, Proposition 1.1]). Now, let $\{v_n\}\subset W_0^{s,p}(M)$, $v\in W_0^{s,p}(M)$ with $v_n\rightharpoonup v$ weakly in $W_0^{s,p}(M)$ as $n\to\infty$. Then it follows from the definition of subdifferential that

$$\psi(v_n)-\psi(v)\ge \langle \psi'(v), v_n - v\rangle.$$

Hence, we obtain $\psi(v)\le \liminf_{n\to\infty}\psi(v_n)$, that is, the map $v\mapsto \|v\|^p_{W_0^{s,p}(M)}$ is weakly lower semi-continuous.

Let $u_n\rightharpoonup u$ weakly in $W_0^{s,p}(M)$. By assumption (H1) and Lemma 2.12, up to a subsequence, $u_n\to u$ strongly in $L^q(\Omega)$. Without loss of generality, we assume that $u_n\to u$ a.e. in Ω. Assumption (A1) implies

$$F(x,t)\le a\left(|t|+q^{-1}|t|^q\right)\le 2a(|t|^q+1).$$

Thus, for any measurable subset $U \subset \Omega$,

$$\int_U |F(x,u_n)|d\mu_g(x) \le 2a\int_U |u_n|^q dx + 2a\mu(U).$$

From $1<q<p_s^*$, Lemma 2.4 and Hölder's inequality, we have

$$\begin{aligned}\int_U |F(x,u_n)|d\mu_g(x) &\le 2a\||u_n|^q\|_{L^{\frac{p_s^*}{q}}(U)}\|1\|_{L^{\frac{p_s^*}{p_s^*-q}}(U)} + 2a\mu(U)\\ &\le 2aC\|u_n\|^q_{W_0^{s,p}(M)}(\mu(U))^{\frac{p_s^*-q}{p_s^*}} + 2a\mu(U).\end{aligned}$$

Similar to the proof of Lemma 3.1, we obtain

$$\lim_{n\to\infty}\int_\Omega F(x,u_n)d\mu_g(x) = \int_\Omega F(x,u)d\mu_g(x).$$

Thus, the functional H is weakly continuous. Furthermore, we obtain that I is weakly lower semi-continuous. □

Proof of Theorem 1.2. By (A1), we have $|F(x,t)| \le 2a(|t|^p+1)$. Thus, by Lemma 2.4, we obtain

$$\begin{aligned}I(u) &\ge \frac{1}{p}\iint_{M\times M}\frac{|u(x)-u(y)|^p}{(d_g(x,y))^{N+ps}}d\mu_g(x)d\mu_g(y) - 2a\int_\Omega |u|^q d\mu_g(x) - 2a\mu(\Omega)\\ &\ge \frac{1}{p}\|u\|^p_{W_0^{s,p}(M)} - 2aC_1^{\frac{q}{p}}\|u\|^q_{W_0^{s,p}(M)} - 2a\mu(\Omega).\end{aligned}$$

Since $q<p$, we have $I(u)\to\infty$ as $\|u\|_{W_0^{s,p}(M)}\to\infty$. By Lemma 3.3, I is weakly lower semi-continuous on $W_0^{s,p}(M)$. So the functional I has a minimum point u_0 in $W_0^{s,p}(\Omega)$ (see [32, Theorem 1.2]) and $u_0 \in W_0^{s,p}(M)$ is a weak solution of problem (1.1). □

Case 2: $p<q<p_s^*$**.**

Lemma 3.4. *Let f satisfy* (A1) *and* (A3). *If $p<q<p_s^*$, then there exist $\rho>0$ and $\alpha>0$ such that*

$$I(u)\ge\alpha>0,$$

for any $u\in W_0^{s,p}(M)$ with $\|u\|_{W_0^{s,p}(M)}=\rho$.

Proof. In view of (A1) and (A3), for any $\varepsilon>0$, there exists $C(\varepsilon)>0$ such that for any $\xi\in\mathbb{R}$ and a.e. $x\in\Omega$, we have

$$|F(x,\xi)| \le \varepsilon|\xi|^p + C(\varepsilon)|\xi|^q. \tag{3.5}$$

Let $u\in W_0^{s,p}(M)$. By (3.5) and Lemma 2.4, we obtain

$$\begin{aligned}I(u) &\ge \frac{1}{p}\iint_{M\times M}\frac{|u(x)-u(y)|^p}{(d_g(x,y))^{N+ps}}d\mu_g(x)d\mu_g(y) - \varepsilon\int_\Omega |u(x)|^p d\mu_g(x)\\ &\quad - C(\varepsilon)\int_\Omega |u(x)|^q d\mu_g(x)\\ &\ge \frac{1}{p}\|u\|^p_{W_0^{s,p}(M)} - \varepsilon C_1\|u\|^p_{W_0^{s,p}(M)} - C(\varepsilon)C_1^{\frac{q}{p}}\|u\|^q_{W_0^{s,p}(M)}.\end{aligned}\tag{3.6}$$

Choosing $\varepsilon = 1/(2pC_1)$, from (3.6) we have

$$I(u) \ge \frac{1}{2p}\|u\|^p_{W_0^{s,p}(M)} - C\|u\|^q_{W_0^{s,p}(M)} \ge \|u\|^p_{W_0^{s,p}(M)}\Big(\frac{1}{2p} - C\|u\|^{q-p}_{W_0^{s,p}(M)}\Big),$$

where C is a constant only depending on N, s, p. Now, let $\|u\|_{W_0^{s,p}(M)} = \rho > 0$. Since $q > p$, we can choose ρ sufficiently small such that $1/(2p) - C\rho^{q-p} > 0$, so that

$$I(u) \geq \rho^p\Big(\frac{1}{2p} - C\rho^{q-p}\Big) =: \alpha > 0.$$

As desired. $\square$

Lemma 3.5. *Let f satisfies* (A1)–(A3). *If $p < q < p_s^*$, then there exists $e \in C_0^\infty(\Omega)$ such that $\|e\|_{W_0^{s,p}(M)} \geq \rho$ and $I(\rho) < \alpha$, where ρ and α are given in Lemma 3.4.*

Proof. From (A2) it follows that

$$F(x,\xi) \geq r^{-\gamma} \min\{F(x,r), F(x,-r)\}|\xi|^\gamma, \tag{3.7}$$

for all $|\xi| > r$ and a.e. $x \in \Omega$. Thus, by (3.7) and $F(x,\xi) \leq \max_{|\xi|\leq r} F(x,\xi)$ for all $|\xi| \leq r$, we obtain

$$\begin{aligned} F(x,\xi) \geq\ & r^{-\gamma} \min\{F(x,r), F(x,-r)\}|\xi|^\gamma - \max_{|\xi|\leq r} F(x,\xi) \\ & - \min\{F(x,r), F(x,-r)\}, \end{aligned} \tag{3.8}$$

for any $\xi \in \mathbb{R}$ and a.e. $x \in \Omega$.

By Lemma 2.1, we can fix $u_0 \in C_0^\infty(\Omega)$ such that $\|u_0\|_{W_0^{s,p}(M)} = 1$. Now, let $t \geq 1$. By (3.8), we have

$$\begin{aligned} I(tu_0) &= \frac{1}{p}\iint_{M\times M} \frac{|tu_0(x) - tu_0(y)|^p}{(d_g(x,y))^{N+ps}} d\mu_g(x)d\mu_g(y) - \int_\Omega F(x, tu_0(x))d\mu_g(x) \\ &\leq \frac{t^p}{p} - r^{-\gamma}t^\gamma \int_\Omega \min\{F(x,r), F(x,-r)\}|u_0(x)|^\gamma d\mu_g(x) \\ &\quad + \int_\Omega \max_{|\xi|\leq r} F(x,\xi) + \min\{F(x,r), F(x,-r)\} d\mu_g(x). \end{aligned}$$

Using (A1) and (A2), we obtain that $0 < F(x,\xi) \leq a(|r| + |r|^q)$ for $|\xi| \leq r$ a. e. $x \in \Omega$. Thus, $0 < \min\{F(x,r), F(x,-r)\} < a(|r| + |r|^q)$ a.e. $x \in \Omega$. Since $\gamma > p$ by assumption (A2), passing to the limit as $t \to \infty$, we obtain that $I(tu_0) \to -\infty$. Thus, the assertion follows by taking $e = Tu_0$ with T sufficiently large. $\square$

Definition 3.6. We say that I satisfies (PS) condition in $W_0^{s,p}(M)$, if for any sequence $\{u_n\} \subset W_0^{s,p}(M)$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, there exists a convergent subsequence of $\{u_n\}$.

Lemma 3.7. *Let f satisfy* (A1)–(A3). *If $p < q < p_s^*$, then the functional I satisfies the* (PS) *condition.*

Proof. For any sequence $\{u_n\} \subset W_0^{s,p}(M)$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, there exits $C > 0$ such that $|\langle I'(u_n), u_n\rangle| \leq C\|u_n\|_{W_0^{s,p}(M)}$ and $|I(u_n)| \leq C$. By (A1), we have

$$\begin{aligned} &\Big|\int_{\Omega\bigcap\{|u_n|\leq r\}} (F(x,u_n) - \gamma^{-1}f(x,u_n)u_n)d\mu_g(x)\Big| \\ &\leq (a + \gamma^{-1})(r + r^q)|\Omega| \leq C, \end{aligned} \tag{3.9}$$

where $\{|u_n| \leq r\} = \{x \in \Omega : |u_n(x)| \leq r\}$. Thus, by (H2) and (3.9), we obtain

$$C + C\|u_n\|_{W_0^{s,p}(M)} \geq I(u_n) - \frac{1}{\gamma}\langle I'(u_n), u_n\rangle$$

$$\begin{aligned}
&\geq \Big(\frac{1}{p}-\frac{1}{\gamma}\Big)\iint_{M\times M}\frac{|u_n(x)-u_n(y)|^p}{(d_g(x,y))^{N+ps}}d\mu_g(x)d\mu_g(y)\\
&\quad -\int_{\Omega\bigcap\{|u_n|\leq r\}}\big(F(x,u_n)-\gamma^{-1}f(x,u_n)u_n\big)\,d\mu_g(x)\\
&\geq \Big(\frac{1}{p}-\frac{1}{\gamma}\Big)\iint_{M\times M}\frac{|u_n(x)-u_n(y)|^p}{(d_g(x,y))^{N+ps}}d\mu_g(x)d\mu_g(y)-C,
\end{aligned}$$

where C denotes various positive constants. Hence, $\{u_n\}$ is bounded in $W_0^{s,p}(M)$. Since $W_0^{s,p}(M)$ is a reflexive Banach space, up to a subsequence, still denoted by $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in $W_0^{s,p}(M)$. Then $\langle I'(u_n), u_n-u\rangle \to 0$.

For each $\varphi \in W_0^{s,p}(M)$ we define a functional $T: W_0^{s,p}(M)\to (W_0^{s,p}(M))'$ by

$$\langle T(\varphi), v\rangle = \iint_{M\times M}\frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{(d_g(x,y))^{N+ps}}(v(x)-v(y))d\mu_g(x)d\mu_g(y),$$

for all $v \in W_0^{s,p}(M)$. Clearly, by the Hölder inequality, $T(\varphi)$ is also continuous, being

$$|\langle T(\varphi), v\rangle| \leq \|\varphi\|^{p-1}_{W_0^{s,p}(M)}\|v\|_{W_0^{s,p}(M)} \quad \text{for all } v\in W_0^{s,p}(M).$$

Thus, we have

$$\langle I'(u_n), u_n-u\rangle = \langle T(u_n), u_n-u\rangle - \int_\Omega f(x,u_n)(u_n-u)d\mu_g(x) \to 0 \tag{3.10}$$

as $n\to\infty$. Moreover, by Lemma 2.12, up to a subsequence,

$$u_n \to u \quad \text{strongly in } L^q(\Omega) \text{ and a.e. in } \Omega.$$

Thus, $f(x,u_n)(u_n-u)\to 0$ a.e. in Ω as $n\to\infty$. It is easy to check that sequence $\{f(x,u_n)(u_n-u)\}$ is uniformly bounded and equi-integrable in $L^1(\Omega)$. Hence, the Vitali convergence theorem implies

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u)d\mu_g(x)=0.$$

Therefore, from (3.10) it follows that

$$\lim_{n\to\infty}\langle T(u_n), u_n-u\rangle = 0.$$

Furthermore, by the weak convergence of $\{u_n\}$ in $W_0^{s,p}(M)$, we obtain

$$\lim_{n\to\infty}\langle T(u_n)-T(u), u_n-u\rangle = 0.$$

Let us recall the well–known vector inequalities:

$$\begin{gathered}
\big(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\big)\cdot(\xi-\eta)\geq C_p|\xi-\eta|^p, \quad p\geq 2;\\
\big(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\big)\cdot(\xi-\eta)\geq \widetilde{C}_p\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}}, \quad 1<p<2,
\end{gathered}$$

for all $\xi,\eta\in\mathbb{R}^N$, where $C_p, \widetilde{C}_p$ are constants depending only on p. From which it is easy to verify that for $p>2$ and $1<p<2$, we have

$$\iint_{M\times M}\frac{|u_n(x)-u_n(y)-u(x)+u(y)|^p}{(d_g(x,y))^{N+ps}}d\mu_g(x)d\mu_g(y)\to 0, \tag{3.11}$$

as $n\to\infty$. Hence, from (3.11) we obtain that $u_n\to u$ strongly in $W_0^{s,p}(M)$ as $n\to\infty$. Therefore, the proof is complete. $\square$

Proof of Theorem 1.3. According to Lemmas 3.4–3.7, the Mountain Pass Theorem [32, Theorem 6.1] implies that there exists a critical point $u \in W_0^{s,p}(M)$ for problem (1.1). $\square$

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