Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 157, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF GROUND STATE SOLUTIONS FOR QUASILINEAR SCHRÖDINGER EQUATIONS WITH VARIABLE POTENTIALS AND ALMOST NECESSARY NONLINEARITIES

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Communicated by Binlin Zhang

Abstract. In this article we prove the existence of ground state solutions for the quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = g(u), \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $V \in \mathcal{C}^1(\mathbb{R}^N, [0,\infty))$ satisfies mild decay conditions and $g \in \mathcal{C}(\mathbb{R},\mathbb{R})$ satisfies Berestycki-Lions conditions which are almost necessary. In particular, we introduce some new inequalities and techniques to overcome the lack of compactness.

1. Introduction

We study the existence of ground state solutions for the quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \Delta(u^2)u = g(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $V : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

- (A1) $V \in \mathcal{C}(\mathbb{R}^N, [0,\infty))$ and $V(x) \leq V_\infty := \lim_{|y|\to\infty} V(y)$ for all $x \in \mathbb{R}^N$;
- (A2) $g \in \mathcal{C}(\mathbb{R},\mathbb{R})$, $\lim_{|t|\to 0} g(t)/t = 0$ and $\lim_{|t|\to\infty} |g(t)|/|t|^{2\cdot 2^*-1} = 0$, where $2^* = 2N/(N-2)$ and $2 \cdot 2^*$ is the critical exponent for (1.1);
- (A3) there exists $s_0 \in \mathbb{R}$ such that $G(s_0) > \frac{1}{2}V_\infty s_0^2$, where $G(t) = \int_0^t g(s)\mathrm{d}s$.

This type of equation has been introduced in [3, 13] to study a model of self-trapped electrons in quadratic or hexagonal lattices (see also [2]). After the work of Poppenberg [17], equations like (1.1) have received much attention in mathematical analysis and applications in recent years, see e.g. [6, 11, 16, 18, 19, 28].

Observe that formally (1.1) is the Euler-Lagrange equation associated to the following functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (1+2u^2)|\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V(x)u^2 \mathrm{d}x - \int_{\mathbb{R}^N} G(u)\mathrm{d}x. \tag{1.2}$$

Since J is not well defined in general in $H^1(\mathbb{R}^N)$, we employ an argument developed by Colin and Jeanjean [11], and make the change of variables by $v = f^{-1}(u)$, where

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2010 *Mathematics Subject Classification.* 35J20, 35Q55.
Key words and phrases. Quasilinear Schrödinger equation; ground state solution; Berestycki-Lions conditions.

Submitted March 27, 2018. Published August 29, 2018.

f is defined by

$$f'(t) = \frac{1}{\sqrt{1+2|f(t)|^2}} \quad \text{on } [0,+\infty), \quad f(-t) = -f(t) \quad \text{on } (-\infty, 0].$$

After the change of variables from J, we obtain the functional

$$\begin{aligned} I(v) &= J(u) = J(f(v)) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V(x) f^2(v)\mathrm{d}x - \int_{\mathbb{R}^N} G(f(v))\mathrm{d}x. \end{aligned} \tag{1.3}$$

Note that

$$|f(t)| \le |t|, \quad |f(t)| \le 2^{1/4}|t|^{1/2}, \quad f(t)/2 \le f'(t)t \le f(t), \quad \forall t \in \mathbb{R}. \tag{1.4}$$

Under assumptions (A1) and (A2), we have $I \in \mathcal{C}^1(H^1(\mathbb{R}^N), \mathbb{R})$, and critical points of I are solutions of the semi-linear equation

$$-\Delta v + V(x) f(v) f'(v) = g(f(v)) f'(v), \quad x \in \mathbb{R}^N, \tag{1.5}$$

moreover, v is a solution of (1.5) if and only if $u = f(v)$ solves (1.1), see [11, 16]. A solution is called a ground state solution if its energy is minimal among all nontrivial solutions. For more related semi-linear problems, we refer to [5, 4, 7, 9, 10, 21, 20, 23, 28, 29] and so on.

Under assumptions (A1) and (A2), almost all of the previous works on (1.1) required g satisfies a global growth condition, for example, $g(t) = |t|^{p-2}t$ with $2 < p \le 4$, see [19, 25]; or

(AR) there exists $\mu \ge 4$ such that $g(t)t \ge \mu G(t) \ge 0$, $\quad \forall t \in \mathbb{R}$;

or

(Ne) $g(t)/|t|^3$ is nondecreasing for $t \in \mathbb{R} \setminus \{0\}$,

see [26, 27] and so on. In these papers, (AR) or (Ne) seems essential for the application of the mountain pass type theorem or the Nehari technique. In particular, with the aid of the Pohožaev manifold, Ruiz and Siciliano [19] and Wu and Wu [25] proved the existence of ground state solutions by assuming additional conditions on V, respectively:

(A4) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^+)$ and $t \mapsto t^{(N+2)/(N+p)} V(t^{1/(N+p)}x)$ is concave for any $x \in \mathbb{R}^N$;

(A5) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^+)$, $V(x) = V(|x|)$ and $t^{3-p}\nabla V(tx) \cdot x$ is non-increasing on $t \in (0,\infty)$ for any $x \in \mathbb{R}^N$.

We would like to point out that the strategies used in [26, 27] rely heavily on the form $g(t) = |t|^{p-2}t$. Different from previous works, we shall establish the existence of ground state solutions for (1.1) in presence of a Berestycki-Lions nonlinearity, that is g satisfies (A2) and (A3). This type of nonlinearity was introduced by Berestycki and Lions [1] for the study of the Schrödinger equation

$$-\Delta v + v = g(v), \quad x \in \mathbb{R}^N.$$

However, the approach used in [1] does not work for (1.1) because of the term $\Delta(u^2)u$ and $V(x) \not\equiv$ constant. These difficulties enforce the implementation of new ideas and techniques. To the best of our knowledge, there seem to be no similar results for (1.1).

To state our results, we introduce the following decay assumptions on ∇V:

(A6) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$ and there exists $\theta \in [0, 1)$ such that $\nabla V(x) \cdot x \le \frac{(N-2)^2\theta}{2|x|^2}$ for all $x \in \mathbb{R}^N \setminus \{0\}$;

(A7) $V \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R})$ and $\|\max\{\nabla V(x) \cdot x, 0\}\|_{N/2} < 2S$, where
$$S = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \|\nabla u\|_2^2 / \|u\|_{2^*}^2.$$

By an argument as [1, Proposition 1], we conclude the Pohožave type identity corresponding to (1.1)
$$\begin{aligned} \mathcal{P}(v) :=& \frac{N-2}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} [NV(x) + \nabla V(x) \cdot x]|f(v)|^2 \mathrm{d}x \\ & - N \int_{\mathbb{R}^N} G(f(v))\mathrm{d}x = 0\,, \end{aligned} \tag{1.6}$$
which is also used in [25, Lemma 2.1]. Let
$$\mathcal{M} := \{v \in H^1(\mathbb{R}^N) \setminus \{0\} : \mathcal{P}(v) = 0\}. \tag{1.7}$$

To recover the compactness of Sobolev spaces embeddings in $\mathbb{R}^N$, different from [25], we study the "limiting problem":
$$-\Delta u + V_\infty u - \Delta(u^2)u = g(u) \quad \text{in } \mathbb{R}^N, \tag{1.8}$$
and compare the critical level of (1.1) with the one of (1.8), instead of using radial compactness. Corresponding to (1.3) and (1.7), we define
$$I^\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}x + \frac{V_\infty}{2}\int_{\mathbb{R}^N} |f(v)|^2 \mathrm{d}x - \int_{\mathbb{R}^N} G(f(v))\mathrm{d}x, \tag{1.9}$$
for all $v \in H^1(\mathbb{R}^N)$, and
$$\mathcal{M}^\infty := \{v \in H^1(\mathbb{R}^N) \setminus \{0\} : \mathcal{P}^\infty(v) = 0\}, \tag{1.10}$$
where
$$\mathcal{P}^\infty(v) := \frac{N-2}{2}\|\nabla v\|_2^2 + \frac{NV_\infty}{2}\int_{\mathbb{R}^N} |f(v)|^2 \mathrm{d}x - N\int_{\mathbb{R}^N} G(f(v))\mathrm{d}x. \tag{1.11}$$
We state our main results as follows.

Theorem 1.1. *Assume that* $V_\infty > 0$, (A2) *and* (A3) *hold. Then problem* (1.8) *has a ground state solution* $u^\infty = f(v^\infty)$ *such that*
$$I^\infty(v^\infty) = \inf_{\mathcal{M}^\infty} I^\infty = \inf_{v \in \Lambda} \max_{t>0} I^\infty(v_t) > 0,$$
where $v_t(x) := v(x/t)$ *and*
$$\Lambda := \big\{v \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \big[\frac{1}{2}V_\infty |f(v)|^2 - G(f(v))\big]\mathrm{d}x < 0\big\}.$$

Theorem 1.2. *Assume that* (A1)–(A3) *and* (A6) *hold. Then* (1.1) *has a ground state solution.*

Theorem 1.3. *Assume that* (A1)–(A3) *and* (A7) *hold. Then* (1.1) *has a ground state solution.*

To prove Theorem 1.1, we must show that $m^\infty := \inf_{\mathcal{M}^\infty} I^\infty$ is achieved without global compactness and any information on $(I^\infty)'$. To do this, we use the scaling technique and some new inequalities related to $I^\infty(v), \mathcal{P}^\infty(v)$ and $I^\infty(v_t)$ (see (2.4) and Lemma 2.5). To prove Theorems 1.2 and 1.3, following the idea of Jeanjean and

Toland [15], we first construct a special bounded (PS) sequence (see (3.13)), then we prove that this bounded sequence converges weakly to a non trivial critical point of I by showing a crucial inequality $c_\lambda < m_\lambda^\infty$ (see Lemmas 3.4 and 3.5) inspired by [8, 22]. Unlike the existing literature, it is not required that v^∞ obtained in Theorem 1.1 is positive in the proof of the above inequality.

Remark 1.4. In Theorem 1.1 we establish the minimax characterization of m^∞ which is much simpler than the usual characterizations related to the Mountain Pass level. Our results complement and extend the existing ones on (1.1) in the literature.

The rest of the paper is organized as follows. In Section 2, we study the existence of ground state solutions for the limiting problem (1.8) by using the Pohožaev manifold, and give the proof of Theorem 1.1. In Section 3, based on the idea of Jeanjean and Toland [15], that is an approximation procedure to obtain a bounded (PS)-sequence for I, we show the existence of ground state solutions for (1.1), and complete the proofs of Theorems 1.2 and 1.3.

Throughout this paper, we denote the usual norms of $H^1(\mathbb{R}^N)$ and $L^s(\mathbb{R}^N)$ by

$$\|u\| = \Big(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\mathrm{d}x\Big)^{1/2}, \quad \|u\|_s = \Big(\int_{\mathbb{R}^N} |u|^s \mathrm{d}x\Big)^{1/s}, \quad s \in [2, \infty)$$

respectively, and positive constants possibly different in different places, by $C_1, C_2, \dots$.

2. Ground state solutions for the limiting problem (1.8)

In this section, we assume that $V_\infty > 0$, and give the proof of Theorem 1.1.

Lemma 2.1. *Assume that* (A2) *holds. Then*

$$I^\infty(v) = I^\infty(v_t) + \frac{1-t^N}{N}\mathcal{P}^\infty(v) + \frac{2 - Nt^{N-2} + (N-2)t^N}{2N}\|\nabla v\|_2^2, \tag{2.1}$$

for all $v \in H^1(\mathbb{R}^N)$ and $t > 0$.

Proof. Note that

$$I^\infty(v_t) = \frac{t^{N-2}}{2}\|\nabla v\|_2^2 + \frac{t^N}{2}V_\infty\|f(v)\|_2^2 - t^N \int_{\mathbb{R}^N} G(f(v))\mathrm{d}x. \tag{2.2}$$

From (1.9), (1.11) and (2.2), we deduce that (2.1) holds. $\square$

By a simple calculation, we can verify that

$$\mathfrak{g}(t) := 2 - Nt^{N-2} + (N-2)t^N > \mathfrak{g}(1) = 0, \quad \forall t \in [0,1) \cup (1, +\infty). \tag{2.3}$$

Thus, it follows from (2.1) and (2.3) that

$$I^\infty(v) > I^\infty(v_t) + \frac{1-t^N}{N}\mathcal{P}^\infty(v), \quad \forall u \in H^1(\mathbb{R}^N),\ t \in (0,1) \cup (1,\infty). \tag{2.4}$$

Corollary 2.2. *Assume that* (A2) *holds. Then*

$$I^\infty(v) = \max_{t>0} I^\infty(v_t), \quad \forall v \in \mathcal{M}^\infty.$$

Lemma 2.3. *Assume that* (A2) *and* (A3) *hold. Then $\Lambda \neq \emptyset$, and for any $v \in \Lambda$, there exists a unique $t_v > 0$ such that $v_{t_v} \in \mathcal{M}^\infty$.*

Proof. In view of the proof of [1, Theorem 2] and the property of f, (A3) implies $\Lambda \neq \emptyset$. Let $v \in \Lambda$ be fixed and define a function $\zeta(t) := I^\infty(v_t)$ on $(0,\infty)$. Clearly, by (1.11) and (2.2), we have

$$\begin{aligned} \zeta'(t) = 0 \Leftrightarrow \ & \frac{N-2}{2} t^{N-2} \|\nabla v\|_2^2 + \frac{NV_\infty}{2} t^N \int_{\mathbb{R}^N} |f(v)|^2 \mathrm{d}x \\ & - Nt^N \int_{\mathbb{R}^N} G(f(v)) \mathrm{d}x = 0 \\ & \Leftrightarrow \mathcal{P}^\infty(v_t) = 0 \ \Leftrightarrow \ v_t \in \mathcal{M}^\infty. \end{aligned} \tag{2.5}$$

By (A2), (A3), (2.2) and the definition of Λ, we get $\lim_{t\to 0} \zeta(t) = 0$, $\zeta(t) > 0$ for $t > 0$ small and $\zeta(t) < 0$ for t large. Therefore $\max_{t\in(0,\infty)} \zeta(t)$ is achieved at some $t_v > 0$ so that $\zeta'(t_v) = 0$ and $v_{t_v} \in \mathcal{M}^\infty$.

Next we claim that t_v is unique for any $v \in \Lambda$. For any given $v \in \Lambda$, if there exist two positive constants $t_1 \neq t_2$ such that $v_{t_1}, v_{t_2} \in \mathcal{M}^\infty$, i.e. $\mathcal{P}^\infty(v_{t_1}) = \mathcal{P}^\infty(v_{t_2}) = 0$, then (2.4) implies

$$\begin{aligned} I^\infty(v_{t_1}) &> I^\infty(v_{t_2}) + \frac{t_1^N - t_2^N}{Nt_1^N} \mathcal{P}^\infty(v_{t_1}) = I^\infty(v_{t_2}) \\ &> I^\infty(v_{t_1}) + \frac{t_2^N - t_1^N}{Nt_2^N} \mathcal{P}^\infty(v_{t_2}) = I^\infty(v_{t_1}). \end{aligned} \tag{2.6}$$

This contradiction shows $t_v > 0$ is unique for any $v \in \Lambda$. □

Lemma 2.4. *Assume that* (A2) *and* (A3) *hold. Then*

$$m^\infty = \inf_{\mathcal{M}^\infty} I^\infty = \inf_{v\in\Lambda} \max_{t>0} I^\infty(v_t) > 0.$$

Proof. It follows from Lemma 2.1 and Corollary 2.2 that

$$m^\infty = \inf_{\mathcal{M}^\infty} I^\infty = \inf_{v\in\Lambda} \max_{t>0} I^\infty(v_t).$$

Next we prove $m^\infty > 0$. By (A2), (1.4) and Sobolev embedding inequality, one has

$$\begin{aligned} \int_{\mathbb{R}^N} G(f(v)) \mathrm{d}x &\le \frac{V_\infty}{4} \|f(v)\|_2^2 + C_1 \|f(v)\|_{2\cdot 2^*}^{2\cdot 2^*} \\ &\le \frac{V_\infty}{4} \|f(v)\|_2^2 + C_2 \|\nabla v\|_2^{2^*}, \quad \forall v \in H^1(\mathbb{R}^N). \end{aligned} \tag{2.7}$$

By (1.11) and (2.7), one has

$$\begin{aligned} \frac{N-2}{2} \|\nabla v\|_2^2 + \frac{NV_\infty}{2} \|f(v)\|_2^2 &= N \int_{\mathbb{R}^N} G(f(v)) \mathrm{d}x \\ &\le \frac{NV_\infty}{4} \|f(v)\|_2^2 + NC_2 \|\nabla v\|_2^{2^*}, \end{aligned} \tag{2.8}$$

for all $v \in \mathcal{M}^\infty$, which implies there exists $\rho_0 > 0$ such that $\|\nabla v\|_2 \ge \rho_0$ for all $v \in \mathcal{M}^\infty$. By (1.9) and (1.11), we have

$$I^\infty(v) = \frac{1}{N} \mathcal{P}^\infty(v) + \frac{1}{N} \|\nabla v\|_2^2 \ge \frac{1}{N} \rho_0^2, \quad \forall v \in \mathcal{M}^\infty.$$

This shows that $m^\infty = \inf_{\mathcal{M}^\infty} I^\infty > 0$. □

Lemma 2.5. *Assume that* (A2) *and* (A3) *hold. Then* $m^\infty = \inf_{\mathcal{M}^\infty} I^\infty$ *is achieved.*

Proof. Let $\{v_n\} \subset \mathcal{M}^\infty$ be such that $I^\infty(v_n) \to m^\infty > 0$. By (1.9) and (1.11), one has

$$m^\infty + o(1) = I^\infty(v_n) - \frac{1}{N}\mathcal{P}^\infty(v_n) = \frac{1}{N}\|\nabla v_n\|_2^2, \tag{2.9}$$

which yields that $\{\|\nabla v_n\|_2\}$ is bounded. Next, we prove that $\{\|v_n\|_2\}$ is bounded. By (1.11) and (2.7), one has

$$\begin{aligned}\frac{N-2}{2}\|\nabla v_n\|_2^2 + \frac{NV_\infty}{2}\|f(v_n)\|_2^2 &= N\int_{\mathbb{R}^N} G(f(v_n))\mathrm{d}x \\ &\le \frac{NV_\infty}{4}\|f(v_n)\|_2^2 + NC_2\|\nabla v_n\|_2^{2^*},\end{aligned} \tag{2.10}$$

which implies $\{\|f(v_n)\|_2\}$ is bounded. Then it follows from (1.4) and Sobolev embedding inequality that

$$\begin{aligned}\int_{\mathbb{R}^N} v_n^2\mathrm{d}x &= \int_{|v_n|\le 1} v_n^2\mathrm{d}x + \int_{|v_n|>1} v_n^2\mathrm{d}x \\ &\le C_3\int_{|v_n|\le 1}|f(v_n)|^2\mathrm{d}x + \int_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x \\ &\le C_3\|f(v_n)\|^2 + S^{-2^*/2}\|\nabla v_n\|_2^{2^*}.\end{aligned} \tag{2.11}$$

Hence, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. By (A2) and (1.4), for some $p \in (2, 2^*)$ and any $\varepsilon > 0$, one has

$$|G(f(t))| \le \varepsilon\Big(|f(t)|^2 + |t|^{2^*}\Big) + C_\varepsilon|t|^p, \quad \forall t \in \mathbb{R}. \tag{2.12}$$

Since $\mathcal{P}^\infty(v_n) = 0$ and $\|\nabla v_n\|_2 \ge \rho_0$ by (2.8), from (1.11), (2.12) and Lions' concentration compactness principle [24, Lemma 1.21], one can easily prove that there exist $\delta > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)}|v_n|^2\mathrm{d}x > \delta/2$. Let $\hat{v}_n(x) = v_n(x + y_n)$. Then

$$I^\infty(\hat{v}_n) \to m^\infty, \quad \mathcal{P}^\infty(\hat{v}_n) = 0, \tag{2.13}$$

and there exists $v^\infty \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\hat{v}_n \rightharpoonup v^\infty$ in $H^1(\mathbb{R}^N)$, $\hat{v}_n \to v^\infty$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $s \in [1, 2^*)$, $\hat{v}_n \to v^\infty$ a.e. on $\mathbb{R}^N$. Let $w_n = \hat{v}_n - v^\infty$. By a standard argument (see [24]), we have

$$I^\infty(\hat{v}_n) = I^\infty(v^\infty) + I^\infty(w_n) + o(1), \quad \mathcal{P}^\infty(\hat{v}_n) = \mathcal{P}^\infty(v^\infty) + \mathcal{P}^\infty(w_n) + o(1). \tag{2.14}$$

From (1.9), (1.11), (2.13) and (2.14), one has

$$\frac{1}{N}\|\nabla w_n\|_2^2 = m^\infty - \frac{1}{N}\|\nabla v^\infty\|_2^2 + o(1), \quad \mathcal{P}^\infty(w_n) = -\mathcal{P}^\infty(v^\infty) + o(1). \tag{2.15}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then we have

$$I^\infty(v^\infty) = m^\infty, \quad \mathcal{P}^\infty(v^\infty) = 0, \tag{2.16}$$

which implies that Lemma 2.5 holds. Next, we assume that $w_n \ne 0$. We claim that $\mathcal{P}^\infty(v^\infty) \le 0$. Otherwise, if $\mathcal{P}^\infty(v^\infty) > 0$, then (2.15) implies $\mathcal{P}^\infty(w_n) < 0$, and so $w_n \in \Lambda$ for large n. In view of Lemma 2.4, there exists $t_n > 0$ such that $(w_n)_{t_n} \in \mathcal{M}^\infty$. From (1.9), (1.11), (2.1), (2.3) and (2.15), we deduce

$$\begin{aligned}m^\infty - \frac{1}{N}\|\nabla v^\infty\|_2^2 + o(1) = \frac{1}{N}\|\nabla w_n\|_2^2 &= I^\infty(w_n) - \frac{1}{N}\mathcal{P}^\infty(w_n) \\ &\ge I^\infty\left((w_n)_{t_n}\right) - \frac{t_n^N}{N}\mathcal{P}^\infty(w_n) \ge m^\infty,\end{aligned}$$

which is a contradiction due to $v^\infty \neq 0$. Thus $\mathcal{P}^\infty(v^\infty) \le 0$, and so $v^\infty \in \Lambda$. In view of Lemma 2.4, there exists $\hat{t} > 0$ such that $(v^\infty)_{\hat{t}} \in \mathcal{M}^\infty$. From (1.9), (1.11), (2.1), (2.3), (2.13) and the weak semicontinuity of the norm, one has

$$\begin{aligned} m^\infty &= \lim_{n\to\infty} \big[I^\infty(\hat{v}_n) - \frac{1}{N}\mathcal{P}^\infty(\hat{v}_n)\big] = \frac{1}{N}\lim_{n\to\infty} \|\nabla \hat{v}_n\|_2^2 \\ &\ge \frac{1}{N}\|\nabla v^\infty\|_2^2 = I^\infty(v^\infty) - \frac{1}{N}\mathcal{P}^\infty(v^\infty) \\ &\ge I^\infty\left((v^\infty)_{\hat{t}}\right) - \frac{\hat{t}^N}{N}\mathcal{P}^\infty(v^\infty) \ge m^\infty, \end{aligned}$$

which implies that (2.16) holds. □

In the same way as in [22, Lemma 2.13], we can obtain the following lemma.

Lemma 2.6. *Assume that* (A2) *and* (A3) *hold. Then minimizers of* $\inf_{\mathcal{M}^\infty} I^\infty$ *are critical points of* I^∞.

Proof of Theorem 1.1. In view of Lemmas 2.4-2.6, there exists $v^\infty \in \mathcal{M}^\infty$ such that $(I^\infty)'(v^\infty) = 0$ and $I^\infty(v^\infty) = m^\infty = \inf_{v\in\Lambda}\max_{t>0} I^\infty(v_t) > 0$. □

3. Ground state solutions for (1.1)

In this section, we give the proofs of Theorems 1.2 and 1.3. To find a bounded (PS) sequence of I, we use the following result due to Jeanjean and Toland [15].

Proposition 3.1. *Let X be a Banach space and let $J \subset \mathbb{R}^+$ be an interval, and*

$$\Phi_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in J,$$

be a family of $\mathcal{C}^1$-functional on X such that

(i) *either $A(u) \to +\infty$ or $B(u) \to +\infty$, as $\|u\| \to \infty$;*
(ii) *B maps every bounded set of X into a set of $\mathbb{R}$ bounded below;*
(iii) *there are two points v_1, v_2 in X such that*

$$\tilde{c}_\lambda := \inf_{\gamma\in\tilde{\Gamma}} \max_{t\in[0,1]} \Phi_\lambda(\gamma(t)) > \max\{\Phi_\lambda(v_1), \Phi_\lambda(v_2)\}, \tag{3.1}$$

where

$$\tilde{\Gamma} = \{\gamma \in \mathcal{C}([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\lambda \in J$, there exists a sequence $\{u_n(\lambda)\}$ such that

(i) *$\{u_n(\lambda)\}$ is bounded in X;*
(ii) *$\Phi_\lambda(u_n(\lambda)) \to c_\lambda$;*
(iii) *$\Phi'_\lambda(u_n(\lambda)) \to 0$ in X^*, where X^* is the dual of X.*

To apply Proposition 3.1, for $\lambda \in [1/2, 1]$, we introduce two families of functional defined by

$$I_\lambda(v) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)|f(v)|^2\right) dx - \lambda \int_{\mathbb{R}^N} G(f(v)) dx, \tag{3.2}$$

$$I_\lambda^\infty(v) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V_\infty|f(v)|^2\right) dx - \lambda \int_{\mathbb{R}^N} G(f(v)) dx, \tag{3.3}$$

for all $v \in H^1(\mathbb{R}^N)$.

If $I'_\lambda(\bar{v}) = 0$ and $(I^\infty_\lambda)'(v^\infty) = 0$, then $\bar{v}$ and v^∞ satisfy the Pohožaev type identities $\mathcal{P}_\lambda(\bar{v}) = 0$ and $\mathcal{P}^\infty_\lambda(v^\infty) = 0$ respectively, where

$$\begin{aligned}\mathcal{P}_\lambda(v) = \frac{N-2}{2}\|\nabla v\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} [NV(x) + \nabla V(x)\cdot x]\,|f(v)|^2 \mathrm{d}x \\ - N\lambda \int_{\mathbb{R}^N} G(f(v))\mathrm{d}x,\end{aligned} \tag{3.4}$$

$$\mathcal{P}^\infty_\lambda(v) = \frac{N-2}{2}\|\nabla v\|_2^2 + \frac{NV_\infty}{2}\int_{\mathbb{R}^N} |f(v)|^2 \mathrm{d}x - N\lambda \int_{\mathbb{R}^N} G(f(v))\mathrm{d}x. \tag{3.5}$$

By Lemma 2.1, we have the following lemma.

Lemma 3.2. *Assume that* (A2) *holds. Then*

$$I^\infty_\lambda(v) = I^\infty_\lambda(v_t) + \frac{1-t^N}{N}\mathcal{P}^\infty_\lambda(v) + \frac{2 - Nt^{N-2} + (N-2)t^N}{2N}\|\nabla v\|_2^2, \tag{3.6}$$

for all $v \in H^1(\mathbb{R}^N)$ *and* $t > 0$.

In view of Theorem 1.1, $I^\infty_1 = I^\infty$ has a minimizer v^∞_1 on $\mathcal{M}^\infty_1 = \mathcal{M}^\infty$, i.e.

$$v^\infty_1 \in \mathcal{M}^\infty_1, \quad (I^\infty_1)'(v^\infty_1) = 0 \quad \text{and} \quad m^\infty_1 = I^\infty_1(v^\infty_1), \tag{3.7}$$

where

$$\mathcal{M}^\infty_\lambda = \left\{v \in H^1(\mathbb{R}^N)\setminus\{0\} : \mathcal{P}^\infty_\lambda(v) = 0\right\}, \quad m^\infty_\lambda = \inf_{\mathcal{M}^\infty_\lambda} I^\infty_\lambda, \quad \forall \lambda \in [1/2, 1].$$

Since (1.8) is autonomous, $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and $V(x) \le V_\infty$ but $V(x) \not\equiv V_\infty$, then there exist $\bar{x} \in \mathbb{R}^N$ and $\bar{r} > 0$ such that

$$V_\infty - V(x) > 0, \quad |v^\infty_1(x)| > 0 \quad \text{a.e. } |x - \bar{x}| \le \bar{r}. \tag{3.8}$$

Lemma 3.3. *Assume that* (A1)–(A3) *hold. Then*

- (i) *there exists* $T > 0$ *independent of* λ *such that* $I_\lambda\left((v^\infty_1)_T\right) < 0$ *for all* $\lambda \in [1/2, 1]$*;*
- (ii) *there exists a positive constant* κ_0 *independent of* λ *such that for all* $\lambda \in [1/2, 1]$,
$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) \ge \kappa_0 > \max\left\{I_\lambda(0), I_\lambda\left((v^\infty_1)_T\right)\right\},$$
where
$$\Gamma = \left\{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(1) = (v^\infty_1)_T\right\};$$
- (iii) c_λ *is bounded for* $\lambda \in [1/2, 1]$*;*
- (iv) m^∞_λ *is non-increasing on* $\lambda \in [1/2, 1]$*;*
- (v) $\limsup_{\lambda\to\lambda_0} c_\lambda \le c_{\lambda_0}$ *for* $\lambda_0 \in (1/2, 1]$.

Since $m^\infty_\lambda = I^\infty_\lambda(v^\infty_\lambda)$ and $\int_{\mathbb{R}^N} G(f(v^\infty_\lambda))\mathrm{d}x > 0$, then the proof of Lemma 3.3 is standard (see [14, Lemma 2.3]), so we omit it.

Lemma 3.4. *Assume that* (A1)–(A3) *hold. Then there exists* $\bar{\lambda} \in [1/2, 1)$ *such that* $c_\lambda < m^\infty_\lambda$ *for* $\lambda \in (\bar{\lambda}, 1]$.

Proof. The proof is similar to [22, Lemma 4.5], and we give the outlines for the completeness. It is easy to check that $I_\lambda\left((v^\infty_1)_t\right)$ is continuous on $t \in (0, \infty)$.

Hence for any $\lambda \in [1/2, 1]$, we can choose $t_\lambda \in (0, T)$ such that $I_\lambda\left((v_1^\infty)_{t_\lambda}\right) = \max_{t\in(0,T]} I_\lambda\left((v_1^\infty)_t\right)$. Setting

$$\gamma_0(t) = \begin{cases} (v_1^\infty)_{(tT)}, & \text{for } t > 0, \\ 0, & \text{for } t = 0. \end{cases}$$

Then

$$\gamma_0 \in \Gamma, \quad I_\lambda\left((v_1^\infty)_{t_\lambda}\right) = \max_{t\in[0,1]} I_\lambda\left(\gamma_0(t)\right) \geq c_\lambda, \tag{3.9}$$

where Γ and c_λ are defined by Lemma 3.3 (ii). Let

$$\zeta_0 := \min\{3\bar{r}/8(1+|\bar{x}|), 1/4\}. \tag{3.10}$$

Then it follows from (3.8) and (3.10) that

$$|x - \bar{x}| \leq \frac{\bar{r}}{2} \quad \text{and} \quad s \in [1-\zeta_0, 1+\zeta_0] \Rightarrow |sx - \bar{x}| \leq \bar{r}. \tag{3.11}$$

Since $\mathcal{P}^\infty(v_1^\infty) = 0$, we have $\int_{\mathbb{R}^N} G(f(v_1^\infty))\mathrm{d}x > 0$. Let

$$\begin{aligned} \bar{\lambda} := \max\Big\{ & \frac{1}{2}, 1 - \frac{(1-\zeta_0)^N \min_{s\in[1-\zeta_0,1+\zeta_0]} \int_{\mathbb{R}^N} [V_\infty - V(sx)]|f(v_1^\infty)|^2\mathrm{d}x}{2T^N \int_{\mathbb{R}^N} G(f(v_1^\infty))\mathrm{d}x}, \\ & 1 - \frac{\min\{\mathfrak{g}(1-\zeta_0), \mathfrak{g}(1+\zeta_0)\}\|\nabla v_1^\infty\|_2^2}{2NT^N \int_{\mathbb{R}^N} G(f(v_1^\infty))\mathrm{d}x} \Big\}, \end{aligned} \tag{3.12}$$

where $\mathfrak{g}$ is defined by (2.3). Then (2.3), (3.8) and (3.11) imply that $1/2 \leq \bar{\lambda} < 1$. We distinguish two cases:

Case i: $t_\lambda \in [1-\zeta_0, 1+\zeta_0]$. From (3.2), (3.3), (3.6)-(3.9), (3.11), (3.12) and Lemma 3.3 (iv), we have

$$\begin{aligned} m_\lambda^\infty \geq m_1^\infty &= I_1^\infty(v_1^\infty) \geq I_1^\infty\left((v_1^\infty)_{t_\lambda}\right) \\ &= I_\lambda\left((v_1^\infty)_{t_\lambda}\right) - (1-\lambda)t_\lambda^N \int_{\mathbb{R}^N} G(f(v_1^\infty))\mathrm{d}x \\ &\quad + \frac{t_\lambda^N}{2} \int_{\mathbb{R}^N} [V_\infty - V(t_\lambda x)]|f(v_1^\infty)|^2\mathrm{d}x \\ &\geq c_\lambda - (1-\lambda)T^N \int_{\mathbb{R}^N} G(f(v_1^\infty))\mathrm{d}x \\ &\quad + \frac{(1-\zeta_0)^N}{2} \min_{s\in[1-\zeta_0,1+\zeta_0]} \int_{\mathbb{R}^N} [V_\infty - V(sx)]\,|f(v_1^\infty)|^2\mathrm{d}x \\ &> c_\lambda, \quad \forall \lambda \in (\bar{\lambda}, 1]. \end{aligned}$$

Case ii: $t_\lambda \in (0, 1-\zeta_0) \cup (1+\zeta_0, T)$. From (A1), (2.3), (3.2), (3.3), (3.6), (3.7), (3.9), (3.12) and Lemma 3.3 (iv), we have

$$\begin{aligned} m_\lambda^\infty \geq m_1^\infty &= I_1^\infty(v_1^\infty) \geq I_1^\infty\left((v_1^\infty)_{t_\lambda}\right) + \frac{\mathfrak{g}(t_\lambda)\|\nabla v_1^\infty\|_2^2}{2N} \\ &= I_\lambda\left((v_1^\infty)_{t_\lambda}\right) - (1-\lambda)t_\lambda^N \int_{\mathbb{R}^N} G(f(v_1^\infty))\mathrm{d}x \\ &\quad + \frac{t_\lambda^N}{2} \int_{\mathbb{R}^N} [V_\infty - V(t_\lambda x)]|f(v_1^\infty)|^2\mathrm{d}x + \frac{\mathfrak{g}(t_\lambda)\|\nabla v_1^\infty\|_2^2}{2N} \end{aligned}$$

$$
\begin{aligned}
&\geq c_\lambda - (1-\lambda)T^N \int_{\mathbb{R}^N} G(f(v_1^\infty))\mathrm{d}x + \frac{\min\{\mathfrak{g}(1-\zeta_0), \mathfrak{g}(1+\zeta_0)\}\|\nabla v_1^\infty\|_2^2}{2N} \\
&> c_\lambda, \quad \forall \lambda \in (\bar{\lambda}, 1].
\end{aligned}
$$

In both cases, we obtain that $c_\lambda < m_\lambda^\infty$ for $\lambda \in (\bar{\lambda}, 1]$. □

Lemma 3.5. *Under the assumptions of Theorem 1.2 or Theorem 1.3, for almost every $\lambda \in (\bar{\lambda}, 1]$, there exists $v_\lambda \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $I'_\lambda(v_\lambda) = 0$ and $I_\lambda(v_\lambda) = c_\lambda$.*

Proof. In view of Proposition 3.1 and Lemma 3.3, for almost every $\lambda \in (\bar{\lambda}, 1]$, there exists a bounded sequence $\{v_n(\lambda)\} \subset H^1(\mathbb{R}^N)$ (for simplicity, we denote it by $\{v_n\}$) such that

$$
I_\lambda(v_n) \to c_\lambda > 0, \quad I'_\lambda(v_n) \to 0. \tag{3.13}
$$

By a splitting lemma [12, Lemma 3.3], there exist a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, and $v_\lambda \in H^1(\mathbb{R}^N)$, an integer $l \in \mathbb{N} \cup \{0\}$, and $w^1, \dots, w^l \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $v_n \rightharpoonup v_\lambda$ in $H^1(\mathbb{R}^N)$, $I'_\lambda(v_\lambda) = 0$,

$$
(I_\lambda^\infty)'(w^k) = 0, \quad I_\lambda^\infty(w^k) \geq m_\lambda^\infty, \quad 1 \leq k \leq l, \tag{3.14}
$$

$$
c_\lambda = I_\lambda(v_\lambda) + \sum_{k=1}^{l} I_\lambda^\infty(w^k). \tag{3.15}
$$

Since $I'_\lambda(v_\lambda) = 0$, then we have $\mathcal{P}_\lambda(v_\lambda) = 0$. Since $\|v_n\| \nrightarrow 0$, we deduce from (3.14) and (3.15) that if $v_\lambda = 0$, then

$$
l \geq 1, \quad c_\lambda = I_\lambda(v_\lambda) + \sum_{k=1}^{l} I_\lambda^\infty(w^k) \geq m_\lambda^\infty,
$$

which contradicts Lemma 3.4. Thus $v_\lambda \neq 0$. By (1.4), (3.2) and (3.4), one has

$$
I_\lambda(v_\lambda) = I_\lambda(v_\lambda) - \frac{1}{N}\mathcal{P}_\lambda(v_\lambda) = \frac{1}{N}\|\nabla v_\lambda\|_2^2 - \frac{1}{2N}\int_{\mathbb{R}^N} \nabla V(x) \cdot x|f(v_\lambda)|^2 \mathrm{d}x. \tag{3.16}
$$

If (A6) holds, then it follows from (1.4) and Hardy inequality that

$$
\int_{\mathbb{R}^N} \nabla V(x) \cdot x|f(v_\lambda)|^2 \mathrm{d}x \leq \frac{\theta(N-2)^2}{2} \int_{\mathbb{R}^N} \frac{v_\lambda^2}{|x|^2}\mathrm{d}x \leq 2\theta\|\nabla v_\lambda\|_2^2,
$$

which, together with (3.16), imply

$$
I_\lambda(v_\lambda) \geq \frac{1-\theta}{N}\|\nabla v_\lambda\|_2^2 > 0. \tag{3.17}
$$

If (A7) holds, then it follows from (1.4) and Sobolev embedding inequality that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \nabla V(x) \cdot x|f(v_\lambda)|^2 \mathrm{d}x \\
&\leq \Big(\int_{\mathbb{R}^N} |\max\{\nabla V(x) \cdot x, 0\}|^{N/2}\mathrm{d}x\Big)^{2/N} \Big(\int_{\mathbb{R}^N} |v_\lambda|^{2N/(N-2)}\mathrm{d}x\Big)^{(N-2)/N} \\
&\leq \frac{\|\max\{\nabla V(x) \cdot x, 0\}\|_{N/2}}{S}\|\nabla v_\lambda\|_2^2,
\end{aligned}
$$

this and (3.16), imply

$$
I_\lambda(v_\lambda) \geq \frac{2S - \|\max\{\nabla V(x) \cdot x, 0\}\|_{N/2}}{2NS}\|\nabla v_\lambda\|_2^2 > 0. \tag{3.18}
$$

Thus, by (3.15) and either of (3.17) and (3.18), one has

$$c_\lambda = I_\lambda(v_\lambda) + \sum_{k=1}^{l} I_\lambda^\infty(w^k) > l m_\lambda^\infty \quad \text{for } \lambda \in (\bar{\lambda}, 1],$$

which, together with Lemma 3.4, imply that $l = 0$ and $I_\lambda(v_\lambda) = c_\lambda$. □

Lemma 3.6. *Under the assumptions in Theorem 1.2 or Theorem 1.3, there exists $\bar{v} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $I'(\bar{v}) = 0$ and $0 < I(\bar{v}) < c_1$.*

Proof. From Lemma 3.5, there exist $\{\lambda_n\} \subset [\bar{\lambda}, 1]$ and $\{v_{\lambda_n}\} \subset H^1(\mathbb{R}^N) \setminus \{0\}$ (denoted it by $\{v_n\}$), such that

$$\lambda_n \to 1, \quad c_{\lambda_n} \to c_*, \quad I'_{\lambda_n}(v_n) = 0, \quad 0 < I_{\lambda_n}(v_n) \le c_{\lambda_n} \le c_1 + o(1). \tag{3.19}$$

From (3.19) and either of (3.17) and (3.18), we can deduce that $\{\|\nabla v_n\|_2\}$ is bounded. Next, we prove that $\{\|v_n\|_2\}$ is bounded. To this end, it suffices to show that $\{\|f(v_n)\|_2\}$ is bounded due to (2.11). Let

$$\varphi_n = f(v_n)/f'(v_n) = \sqrt{1 + 2f^2(v_n)}\, f(v_n), \quad \forall n \in \mathbb{N}.$$

Note that

$$|\nabla(f(v_n))| = \frac{|\nabla v_n|}{\sqrt{1 + 2f^2(v_n)}}. \tag{3.20}$$

By (1.4) and (3.20), one has

$$|\varphi_n| \le 2|v_n|, \quad |\nabla \varphi_n| = \Big(1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}\Big)|\nabla v_n| \le 2|\nabla v_n|, \quad \forall n \in \mathbb{N},$$

which implies $\varphi_n \in H^1(\mathbb{R}^N)$ for all $n \in \mathbb{N}$. Using (A1), it is easy to check that there exists a constant $\gamma_0 > 0$ such that

$$\int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)u^2]\mathrm{d}x \ge \gamma_0 \|u\|_2^2, \quad \forall u \in H^1(\mathbb{R}^N). \tag{3.21}$$

Then it follows from (3.19), (3.20) and (3.21) that

$$\begin{aligned} 0 &= \langle I'_{\lambda_n}(v_n), \varphi_n \rangle \\ &= \int_{\mathbb{R}^N} \Big(1 + \frac{2f^2(v_n)}{1 + 2f^2(v_n)}\Big)|\nabla v_n|^2 \mathrm{d}x + \int_{\mathbb{R}^N} V(x) f^2(v_n)\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} g(f(v_n)) f(v_n)\mathrm{d}x \\ &\ge \int_{\mathbb{R}^N} [|\nabla(f(v_n))|^2 + V(x) f^2(v_n)]\mathrm{d}x - \int_{\mathbb{R}^N} g(f(v_n)) f(v_n)\mathrm{d}x \\ &\ge \gamma_0 \|f(v_n)\|_2^2 - \int_{\mathbb{R}^N} g(f(v_n)) f(v_n)\mathrm{d}x, \end{aligned} \tag{3.22}$$

which, together with (A2), (1.4) and Sobolev embedding inequality imply

$$\begin{aligned} \gamma_0 \|f(v_n)\|_2^2 &\le \int_{\mathbb{R}^N} g(f(v_n)) f(v_n) \le \frac{\gamma_0}{2}\|f(v_n)\|_2^2 + C_5 \|v_n\|_{2^*}^{2^*} \\ &\le \frac{\gamma_0}{2}\|f(v_n)\|_2^2 + C_5 S^{-2^*/2} \|\nabla v_n\|_2^{2^*}. \end{aligned} \tag{3.23}$$

This shows that $\{\|f(v_n)\|_2\}$ is bounded due to the boundedness of $\{\|\nabla v_n\|_2\}$. Hence, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. The rest of the proof is similar to that of Lemma 3.5, so we omit it. □

Proof of Theorems 1.2 and 1.3. Let

$$\mathcal{K} := \{v \in H^1(\mathbb{R}^N) \setminus \{0\} : I'(v) = 0\}, \quad \bar{m} := \inf_{v \in \mathcal{K}} I(v).$$

Then Lemma 3.6 shows that $\mathcal{K} \neq \emptyset$ and $\bar{m} \leq c_1$. Similar to the proofs of (3.17) and (3.18), we have $I(v) = I_1(v) \geq 0$ for all $v \in \mathcal{K}$, and so $\bar{m} \geq 0$. Let $\{v_n\} \subset \mathcal{K}$ be such that $I'(v_n) = 0$ and $I(v_n) \to \bar{m}$. In the same way as the one of Lemma 3.6, we can prove that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. By Lemma 3.4, we have $\bar{m} \leq c_1 < m_1^\infty$. Similar to the proof of Lemma 3.5, we can deduce that there exists $\bar{v} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $I'(\bar{v}) = 0$ and $I(\bar{v}) = \bar{m}$. □

Acknowledgments. S. T. Chen is supported by the Hunan Provincial Innovation Foundation for Postgraduate (No. CX2017B041). X. H. Tang is supported by the National Natural Science Foundation of China (No. 11571370). The authors are grateful to the referees for their valuable comments and suggestions, which improved this article.

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